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Fixed Points Theorems for Mappings with Non-compact and Non-convex Domains

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This note gives some fixed point theorems for lower and upper semi-continuous mappings and mappings with open lower sections defined on non-compact and non-convex sets. It will be noted that the conditions of our theorems are not only sufficient but also necessary. Also our theorems generalize some well-known fixed point theorems such as the Kakutani fixed point theorem and the Brouwer–Schauder fixed point theorem by relaxing the compactness and convexity conditions. © 1991 Academic Press, Inc.

1. INTRODUCTION

Fixed point theorems have wide applications to problems in optimization, game theory, and economics (say, e.g., Arrow and Debreu [1], Border [4], Debreu [6], Nash [16], Shafer and Sonnenschein [18], and others). In particular, it is a key mathematical tool in proving the existence of equilibrium. However, most of the fixed point theorems in the literature (such as those in Border [4], Istrăţescu [11], Joshi and Bose [12], Smart [19], and the references cited therein) are proved upon compact convex sets and only give sufficient conditions for the existence of fixed points of mappings. Also, the fixed point theorems obtained by Halpern [8, 9], Browder [5], Fan [7], Petryshyn and Fitzpatrick [17], and Lim [14] among others assumed that the (weakly) inward mappings are upper semi-continuous or upper demi-continuous. To my best knowledge, no results are variable for lower semi-continuous inward mappings. The purpose of this note is to present some necessary and sufficient conditions for the existence of fixed points of lower semi-continuous and upper semi-continuous correspondences and correspondences with open lower sections defined on non-compact and non-convex sets in the Fréchet topological vector space and locally convex Hausdorff topological vector space. It will be noted that these results generalize the Kakutani's fixed point theorem

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and the Brouwer–Schauder fixed point theorem by relaxing the compactness and convexity conditions. As an application, Tian [20, 21] recently used the results obtained in this note to prove the existence theorems on abstract economies with a non-compact strategy space. Though this note only considers the existence of fixed points of the lower and upper semi-continuous mappings and mappings with lower open sections, we think some of the techniques developed in the note can be applied to extend other fixed point theorems for, say, nonexpansive and weakly inward mappings with non-closed and non-convex domains.

2. Notion and Definitions

Let X and Y be two topological spaces, and let 2^Y be the set of all subsets of Y. A correspondence $G: X \to 2^Y$ is said to be *upper semi-continuous* (in short, u.s.c.) if the set $\{x \in X: G(x) \subset V\}$ is open in X for every open subset V of Y. A correspondence $G: X \to 2^Y$ is said to be *lower semi-continuous* (in short, l.s.c.) if the set $\{x \in X: G(x) \cap V \neq \varnothing\}$ is open in X for every open subset V of Y. A correspondence $G: X \to 2^Y$ is said to have *open upper sections* if, for every $x \in X$, $G(x)$ is open in Y. A correspondence $G: X \to 2^Y$ is said to have *open lower sections* if the set $G^{-1}(y) = \{x \in X: y \in G(x)\}$ is open in X for every $y \in Y$. Let D be a non-empty convex subset of the locally convex Hausdorff topological vector space E and let $I_D(x) = \{y \in E: \lambda x + (1-\lambda) y \in D \text{ and } \lambda \in [0, 1)\}$. A correspondence $G: D \to 2^E$ is said to be *inward* on D if $G(x) \cap I_D(x) \neq \varnothing$ for all $x \in D$, or equivalently, for every $x \in D$, there exists a $y \in G(x)$ and $\lambda \in [0, 1)$ such that $\lambda x + (1-\lambda) y \in D$. A correspondence $G: D \to 2^E$ is said to be *weakly inward* on D if $G(x) \cap \operatorname{cl} I_D(x) \neq \varnothing$.

3. The Existence of Fixed Points of Mappings

Before proceeding to the main theorems, we state some technical lemmas which were due to Michael [15, Proposition 2.5] and Yannelis and Prabhakar [22, Fact 6.1].

Lemma 1. *Let X and Y be two topological spaces and let $\phi: X \to 2^Y$, and $\psi: X \to 2^Y$ be correspondences such that*

- (i) *ϕ is l.s.c. and has open upper sections,*
- (ii) *ψ is l.s.c.,*
- (iii) *for all $x \in X$, $\phi(x) \cap \psi(x) \neq \varnothing$.*

Then the correspondence $\theta: X \to 2^Y$ defined by $\theta(x) = \phi(x) \cap \psi(x)$ is l.s.c.

Lemma 2. *Let X and Y be two topological spaces and $\phi: X \to 2^Y$, and $\psi: X \to 2^Y$ be correspondences having open lower sections. Then the correspondence $\theta: X \to 2^Y$ defined by $\theta(x) = \phi(x) \cap \psi(x)$ has open lower sections.*

Our main results in the following give necessary and sufficient conditions for the existence of fixed points of mappings.

Theorem 1. *Let X be a non-empty subset in a Fréchet topological vector space E. Suppose that $F: X \to 2^E$ is a lower semi-continuous correspondence with non-empty closed convex values. Then the necessary and sufficient condition for the existence of a fixed point $x^* \in F(x^*)$ is that there exists a non-empty compact convex subset $C \subset X$ such that*

$$F(x) \cap C \neq \varnothing \qquad \forall x \in C. \tag{1}$$

Proof. Necessity. Suppose F has a fixed point $x^* \in F(x^*)$. Let $C = \{x^*\}$. Then the singleton set C is clearly compact and convex and $F(x) \cap C \neq \varnothing$ for all $x \in C$.

Sufficiency. Suppose there exists a non-empty compact convex set C such that $F(x) \cap C \neq \varnothing$ for all $x \in C$.

Define a correspondence $K: C \to 2^C$ by, for each $x \in C$,

$$K(x) = F(x) \cap C. \tag{2}$$

Since the correspondence $G: C \to 2^C$ defined by $G(x) = C$ for all $x \in C$ is clearly l.s.c. and has open upper sections in C and F is l.s.c., K is l.s.c. by Lemma 1. Note that also $K(x)$ is non-empty, compact, and convex for all $x \in C$. Therefore, by the Michael selection theorem (cf. Aubin [2, Theorem 15.3.5]), there exists a continuous function $f: C \to C$ such that $f(x) \in K(x)$ for all $x \in C$. Now, by applying the Brouwer–Schauder fixed point theorem (cf. Aubin [2, p. 284]), there exists a point $x^* \in C$ such that $x^* = f(x^*) \in K(x^*) \subset F(x^*)$ and thus F has a fixed point $x^* \in F(x^*)$. ▮

Remark 1. Even though F is l.s.c. and C is a non-empty compact and convex subset, the correspondence $K: C \to 2^C$ defined by $K(x) = F(x) \cap C$ for each $x \in C$ need not be l.s.c. on C if it is not true $K(x) \neq \varnothing$ for all $x \in C$. The following simple example illustrates this.

Example 1. Let $X = [0, 3]$ and $Y = [1, 5]$. Let $F: X \to 2^Y$ defined by $F(x) = [1 + x, 2 + x]$. If we let $C = [0, 2]$, then K is not l.s.c. on C since $\{x \in C: F(x) \cap C \neq \varnothing\} = [0, 1]$ is not open in C.

Remark 2. Note that the condition (1) implies that the correspondence F is inward on C since $C \subset I_C(x)$ for all $x \in C$.

Theorem 1 requires that X be a non-empty subset in a Fréchet topological vector space. We can extend the above theorem to a non-empty paracompact subset in a locally convex Hausdorff topological space if F is strengthened to have open lower sections (cf. Yannelis and Prabhakar [22, p. 237]).

THEOREM 2. *Let X be a non-empty paracompact subset in a locally convex Hausdorff topological vector space E. Suppose that $F: X \rightarrow 2^E$ has open lower sections such that $F(x)$ is non-empty, closed, and convex for all $x \in X$. Then the necessary and sufficient condition for the existence of a fixed point $x^* \in F(x^*)$ is that there exists a non-empty compact convex subset $C \subset X$ such that*

$$F(x) \cap C \neq \varnothing \qquad \forall x \in C. \tag{3}$$

Proof. The proof of necessity is the same. The proof of sufficiency is similar to the above. Suppose there exists a non-empty compact convex set C such that $F(x) \cap C \neq \varnothing$ for all $x \in C$.

Define a correspondence $K: C \rightarrow 2^C$ by, for each $x \in C$,

$$K(x) = F(x) \cap C. \tag{4}$$

Then, by Lemma 2, K has open lower sections by noting that the correspondence $G: C \rightarrow 2^C$ defined by $G(x) = C$ for all $x \in C$ has open lower sections. And $K(x)$ is non-empty, compact, and convex for all $x \in C$. Therefore, by the Theorem 3.1 in Yannelis and Prabhakar [22], there exists a continuous function $f: C \rightarrow C$ such that $f(x) \in K(x)$ for all $x \in C$. Hence, by the Brouwer–Schauder fixed point theorem, there exists a point $x^* \in C$ such that $x^* = f(x^*) \in K(x^*) \subset F(x^*)$ and thus F has a fixed point $x^* \in F(x^*)$. ▮

We now give a similar theorem for the upper semi-continuous correspondence.

THEOREM 3. *Let X be a non-empty subset in a locally convex Hausdorff topological vector space E. Suppose that $F: X \rightarrow 2^E$ is an upper semi-continuous correspondence with non-empty closed convex values. Then the necessary and sufficient condition for the existence of a fixed point $x^* \in F(x^*)$ is that there exists a non-empty compact convex subset $C \subset X$ such that*

$$F(x) \cap C \neq \varnothing \qquad \forall x \in C. \tag{5}$$

Proof. This theorem can be proved by applying Theorem 2 in Halpern [9] by noting $C \subset I_C(x)$ for all $x \in C$. Here we give a direct proof. We first show that the correspondence $K: C \rightarrow 2^C$ by $K(x) = F(x) \cap C$ for each $x \in C$ is u.s.c. In fact, since F is an upper semi-continuous correspondence with non-empty closed convex values, it is closed (i.e., its graph is closed) by Proposition 3.7 of Aubin and Ekeland [3]. Thus, by Theorem 8 of Aubin and Ekeland [3], K is an upper semi-continuous correspondence with non-empty compact values. Therefore, by Kakutani's fixed point theorem (cf. Aubin [2, p. 284]), there exists $x^* \in C$ such that $x^* \in K(x^*)$ and thus F has a fixed point $x^* \in F(x^*)$. ▮

Theorem 3 above can be extended to the following theorem which generalizes the fixed point theorems of Browder [5], Halpern [8, 9], and Halpern and Bergman [10] by relaxing the compactness and convexity sets.

THEOREM 4. *Let X be a non-empty subset in a locally convex Hausdorff topological vector space E. Suppose that $F: X \rightarrow 2^E$ is an upper semi-continuous correspondence with non-empty closed convex values. Then the necessary and sufficient condition for the existence of a fixed point $x^* \in F(x^*)$ is that there exists a non-empty compact convex subset $C \subset X$ such that F is weakly inward on C.*

Proof. We only need to show the sufficiency. Since F is weakly inward on the non-empty compact convex set C and $F: C \rightarrow 2^E$ is an upper semi-continuous correspondence with closed convex values, by Theorem 2 in Halpern [9], there exists a points $x^* \in C$ such that $x^* \in F(x^*)$. ▮

Remark 3. Observe that in case that X is a non-empty compact convex subset and F is a mapping from X into X, the sufficiency conditions of Theorems 1–3 are satisfied by $C = X$. So Theorems 2 and 3 generalize Theorem 3.2 in Yannelis and Prabhakar [22] and the Kalutani fixed theorem to non-compact and non-convex sets.

Remark 4. From the proofs of Theorems 1 and 3, we can see that there still exists a fixed point of F if the condition that F has non-empty closed convex values on X is weakened to the condition that F has non-empty closed convex values on C.

When a correspondence becomes a single-valued function, we have the following corollary by applying Theorem 3:

COROLLARY 1. *Let X be a non-empty subset in a locally convex Hausdorff topological vector space E. Suppose that $f: X \rightarrow E$ is a continuous function. Then the necessary and sufficient condition for the existence of a fixed point $x^* = f(x^*)$ is that there exists a compact convex subset $C \subset X$ such that*

$$f(x) \subset C \qquad \forall x \in C. \tag{6}$$

Thus, the above corollary generalizes the Brouwer–Schauder fixed point theorem by relaxing the compactness and convexity conditions.

Our theorems can prove the existence of a fixed point of a mapping which may have empty, non-compact, or non-convex values and whose domain may be non-compact and non-convex. The following simple example illustrates this.

EXAMPLE 2. Let $X=(-\infty, a)\cup(b, +\infty)\subset\mathbb{R}$, which is non-compact and non-convex. Here $0<a<b$. Define an upper semi-continuous correspondence $F\colon X\to\mathbb{R}$ by, for all $x\in X$,

$$F(x)=(-\infty, x-a]\cup[b, +\infty),$$

which is non-compact and non-convex. So we cannot apply, say, the Kakutani fixed point theorem to prove the existence of the fixed point of F. However, if we take $C=[b, b+2]$, then C is a compact and convex interval, F is an upper semi-continuous correspondence with non-empty compact convex values on C, and

$$F(x)\cap C\neq\varnothing \qquad \forall x\in C.$$

Thus, by Theorem 3 and Remark 4, F has a fixed point.

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