Dynamic resource allocation in fuzzy coalitions : a game theoretic model

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Dynamic Resource Allocation in Fuzzy Coalitions : A Game Theoretic Model*

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Abstract

We introduce an efficient and dynamic resource allocation mechanism within the framework of a cooperative game with fuzzy coalitions. A fuzzy coalition in a resource allocation problem can be so defined that membership grades of the players in it, are proportional to the fractions of their total resources. We call any distribution of the resources possessed by the players, among a prescribed number of coalitions, a fuzzy coalition structure and every membership grade (equivalently fraction of the total resource), a resource investment. It is shown that this resource investment is influenced by satisfaction of the players in regards to better performance under a cooperative setup. Our model is based on the real life situations, where possibly one or more players compromise on their resource investments in order to help forming a coalition.

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1 Introduction

Theory of cooperative games, since its inception by Neumann and Morgenstern [30] in 1953, has been successfully explaining many complex decision making situations. Physically, the idea revolves around situations where self interested players- representing companies or individuals can

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achieve more by forming coalitions than participating individually. Here, humane characteristics viz: players' satisfaction, efficiency, conflicts, capacity to work in a group or as individuals etc. have so far been given little consideration. Nevertheless, it is observed that those factors contribute to a great deal of effecting the output of a joint endeavor. It is further interesting to note that, satisfaction is an intrinsic ingredient to all those factors.(Interested reader may look at [6], where we have developed a model to show how satisfaction of players can be incorporated in payoff allocation).

In this paper, we have considered the problem of obtaining a suitable resource investment allocation matrix (fuzzy coalition structure) in an n-person cooperative game with fuzzy coalitions. The goal of our study is to provide a systematic treatment of satisfaction level as a basis for negotiation among rational agents, capable of participating in different fuzzy coalitions with possibly varied rate of memberships simultaneously. The negotiation among the players, is carried out through a consensus mediator. We assume that the worth of (or profit from) a fuzzy coalition is evolved dynamically as opposed to its static behavior considered in most of the literature so far. Let $N = \{1, 2, ..., n\}$ be the set of players or agents. Any subset of N is called a crisp coalition. In a crisp coalition, participation of a player is full or nil i.e. either a player invests her full resource (or puts her full power) to a coalition or she does not give it at all. The set of all crisp coalitions of N is denoted by 2^N. A cooperative game is defined as a real valued function $v : 2^N \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $v(\emptyset) = 0$. For each crisp coalition $c \in 2^N$ the real number $v(c)$ is known as the worth of the coalition c (or profit incurred from c). However, if a player, with her resource (or power) in hand, wants to participate in various coalitions simultaneously, it would be practically impossible for her to provide full resource(power) to all of them. This leads to the notion of fuzzy coalitions and fuzzy coalition structures. A fuzzy coalition is defined as a fuzzy set of N, and represented by an n-tuple, where its ith component represents the degree of participation (or fraction of the resource) of player i in it. Thus a fuzzy coalition is formed when the participating players invest fractions of their total resources. Resource investment can therefore be made synonymous with the formation of a fuzzy coalition. Similarly a fuzzy coalition structure may be defined as a resource investment matrix, in which the (i, j)th entry represents the fraction of resource of the ith player in the jth fuzzy coalition. Formally, a fuzzy coalition structure is a class of fuzzy coalitions formed simultaneously by the players providing their partial participations. Therefore, from a player's perspective, fuzzy coalitions (in a fuzzy coalition structure) may be termed as "investment avenues". Aubin[1], Butnariu[8], Branzei et al. [7] contributed to a great deal of the theory of fuzzy cooperative games and thereby justified the fuzzification in terms of the players' participation in a coalition. In both crisp or fuzzy environment, the challenge in the study of cooperative games rests in finding a suitable resource investment matrix for the players and a suitable distribution of the worth or profit to its players thereafter. Shapley values, core, minimum norm solutions, compromise values etc. are some of the solution concepts hitherto found in the literature those deal with distribution of worth [1, 8, 20, 21, 22, 29, 7].

In [6], we have mentioned that behavioral psychologists often advocate for various means for ensuring collective and efficient coordination among the agents/players. It is seen that, satisfaction of the participating players, as an efficient mean, can enhance the overall performance of a coalitional activity. It has the ability to increase the worth of (profit from) a coalition by encouraging group performance so that the players would end up with getting more from it. Moreover, all players in a cooperative game do not possess capacities *at par*(by capacity, here, we mean a quantification of the resources owned by a player). Therefore, satisfaction of a player in forming a coalition is indicative of her capacity in terms of her resources. When a coalition is formed with a fraction of the player's resource, it would be a matter of her primary concern to notice whether she is satisfied with it, up to a desired level. Moreover, players with fractional resources and multiple assignments usually find it difficult to put their efforts individually in a single assignment and rather would search for some companions to form a coalition. For them, it is somewhat beneficial to compromise with the resources they provide for the sake of forming a coalition. Furthermore, players' investment preferences, if allowed to choose of their own, range from getting involved in a large number of coalitions with smaller rates of participations (small fragments of resources) to accumulating them for a fewer ones. In game theoretic terminologies, this means that some players are inclined to have a large number of fuzzy coalitions with smaller membership grades, while others like fewer ones with larger membership grades. This idea is well explained by the notion of risk analysis in investment problems, where the members in the former group of investors (players) are less interested in taking risk (return is small but probability is high) and the later ones are risk takers (return may be large but probability is small). Therefore, negotiation and compromise among the players in forming coalitions is highly relevant in a cooperative game theoretic setup. This motivates us to model a mediator imposed identification of an efficient allocation of resource investments for our game. Existing literature, to the best of our knowledge, has not considered such inter-coalitional strategies in regards to forming a coalition structure.

In what follows, formation of coalition for cooperative games is broadly divided into two categories, namely static and dynamic. The static coalitions do not, in general reflect the cooperations among the players explicitly. In crisp cooperative games, Dieckmann et al[11], Ray et al[25, 26, 27] have contributed a lot to the dynamic coalition formation. However, in fuzzy environment,a little work has been carried out in this direction. In [4, 5, 6], some mathematical models (both probabilistic and deterministic) are proposed for allocation of payoffs among the players assuming that each provides the membership value of satisfaction upon receiving a proposal offered by a mediator.

As already mentioned, along with a dynamic payoff allocation procedure among the members of a fuzzy coalition, it is equally (or at least not less) important to devise a mechanism in allocating resource investments of players in different coalitions to form a fuzzy coalition structure. For a meaningful coalition, the members need to be supportive to one another. However, every player desires to invest more to gain more. Therefore, it is natural to expect all the players to be satisfied *at par* with a resource investment allocation matrix in a cooperative environment. Thus an aggregated satisfaction value over a particular resource investment allocation to an individual player within the coalition can be derived to meet the requirements. A solution in this paradigm should be such that every player is *almost equally* satisfied with it.

An example

The state of Assam in India has been known to the world by her tea industries and other natural resources such as coal, petroleum, limestones etc. It is the home to the world famous one horn rhinoceros. The Kaziranga National Park has been declared a world heritage site by the UNESCO for providing natural shelter to the one horn rhinoceros. The area lies in the terrain of Himalayan range on the north and eastern side and the Barail range on the south and western part. The region attracts many migratory birds during the winter. A number of tourists flock to the various tourist places scattered in the region, however the tourism industry is still in the neonatal stage. There is little government initiatives and therefore, scope of initiations by private firms towards tourism industry is enormous. Tea industry in the state, on the other hand, has witnessed a paradigm shift from big corporate houses functioning from outside the state to the small tea growers, who are mainly local enthusiasts. However, it is observed that the small tea planters (who do not have tea factories) are not getting proper value of their raw products (green tea) due to the dominance of big tea industries (who have tea factories). Furthermore, the region is equally famous for its wide varieties of flora and fauna. Hundreds of species of wild Orchids are found in the nearby forests. Mass level cultivation of orchids can also be a promising industry in the state. Nevertheless, most of these ventures get disturbed by occasional as well as frequently occurring natural and man made hazards. As the area is influenced by a number of subtropical phenomena, such as heavy rainfall, flood, landslides along with other natural and man made disasters like earthquakes, insurgency and social unrest etc., it is more natural to put resources in different endeavours (to form fuzzy coalitions) by distributing risk of loss and carry out the tasks all together rather than sticking to a particular industry. Suppose, three persons P_1 , P_2 and P_3 together took a collective decision to-

(a) set up a tea factory for the small tea planters of the area.

(b) cultivate orchids on commercial basis in order to cater to the needs of the local and global markets.

(c) set up an agency for venturing organized tourism in the area including tea tourism.

It is important to note that all the above three possible enterprises have interlinks, so that each is expected to grow hand in hand along with the remaining two. We may further assume that the players are unevenly skilled to these three endeavors. They finally have to submit their proposals to the concerned departments of the state government(Department of Industry, Tourism and Agriculture etc.) for necessary approval and other legalities. The problem now rests on how much resource of each player can be judiciously allocated to each such enterprise so that the total resource is exhausted. In game theoretic terminology, this would resort to finding that how we can make three simultaneous fuzzy coalitions with the membership grades provided by the three players. Ordinary static solution concepts can be employed to have one, nevertheless for creating a better work environment producing optimum synergy, we may incorporate the satisfaction levels of each of the players in the solution searching procedure as mentioned above.

The two fold allocation process

Initially all the players would inform the mediator about their available resources for investment and would jointly fix a number of possible coalitions for the game. Consequently the players would announce the total resources for each coalition to work with. Indeed each coalition can have different resource options (depending on the same announced by the players) with different levels of risk. We accept that the risk level of a coalition to work with large resource is less than the one with small resource. The mediator will then find the optimal fuzzy coalition structure and the corresponding optimal total resource allocation in such a way that the sum of the total resources allocated for all coalitions is equal to the total resource in her hand. The resource allocations at each coalition for the optimal coalition structure is then offered by the mediator. Upon receiving such proposal, the players will provide their degrees of satisfactions in each coalition according to their inherent satisfaction functions. Hereafter, we call this as "investment satisfaction" as the fraction of resources of a player to be allocated to a fuzzy coalition refers to her investments in the coalition. On the basis of this information the mediator will update her belief and propose the next resource allocations and the process continues until a stopping condition is met. Thus the mediator would offer successive proposals of resource allocations to the players judging on their reactions to the previous offers. We have developed a stopping rule and proposed the process of updating the belief of the mediator by use of a suitable function towards the possible reactions of the players upon different offers of resource allocations. We call this function the approximate investment satisfaction function. Furthermore, a variance function is defined to measure the closeness among the investment satisfaction levels of the individual players in a coalition over a single proposal. Variance would determine acceptability of a particular resource investment allocation. If the variance of the investment satisfactions associated with a resource investment allocation (possibly with an abuse of terminology, we can call this as the variance of a resource investment allocation) is below certain threshold to be determined by all the players collectively, then it would be considered as a possible trade-off resource investment allocation to the problem. When the variance becomes static at some stage and is still greater than the threshold, the corresponding resource investment allocation will be accepted as an optimal resource investment allocation. Our method is based on the assumption that the mediator proposes successive resource investment allocations with variance getting smaller at every successive stage while keeping the following two conditions in mind throughout the negotiation process:

(a) there should exist at least one coalition where investment satisfaction of a player is greater than the aggregated investment satisfaction of all players in that coalition

(b) there should exist at least one coalition where investment satisfaction of a player is less than the aggregated investment satisfaction of all players in that coalition.

A mathematical expression of the above two conditions is provided in the beginning of subsection 3.1.2.

An exact resource investment allocation is one for which variance is zero. Thus, for an exact resource investment allocation, all the investment satisfactions are equal. We will show that under conditions (a) and (b), the negotiation process converges to an exact resource investment allocation. In general, every player keeps her investment satisfaction function secret from the others. If a player discloses it before negotiation, we say that she facilitates “arbitration” by providing incentives to the others. The negotiation strategy is so designed that the mediator would propose only offers for which the variance would be minimum at each stage of the negotiation process. What we have also kept in mind is that, in the negotiation process, each of the players has a single motive: maximizing her individual payoffs by investing maximum of her resources. This is well represented by some monotonic increasing functions characterizing the fuzzy sets of their satisfactions. However, negotiation appeals a player to accommodate the desires and views of all the other players. This suggests that an appropriate negotiation process should discourage the players from insisting on illogical and abnormal coalition structures while it should reward those who are more open in forming coalitions.

The remaining part of the paper is organized as follows. Section 2 introduces the preliminary ideas required to formulate our model. In section 3, we develop the theoretical background of our model and prove the existence of a better efficient offer. An iterative method for obtaining a coalition structure for the n-person cooperative game with fuzzy coalitions is also described here. Examples pertaining to the model and the existence of optimal resource investment allocation are presented in section 4. Section 5 includes the concluding remarks.

2 Preliminaries

This section reviews the concept of a cooperative game with fuzzy coalitions and the related properties. A fuzzy set is characterized by a membership function from the universal set to $[0, 1]$. Thus, without loss of generality, we denote the fuzzy sets here by their membership functions. We consider the class of fuzzy games defined by Azrieli and Lehrer [2]. This class seems to be more general than the other existing classes and includes the class of crisp games as a subclass. Its interpretation, however, is rather different. A fuzzy subset of a crisp set X is a function from X to $[0, 1]$, assigning every element of X a membership between 0 and 1. Let N be a finite set representing the types of agents in a large population. There is a continuum of agents of each type and $Q_i \geq 0$ is the size of type i $(i = 1, 2, ..., n)$ agents. The entire population is, therefore, represented by a non negative vector $\mathbf{Q} = (Q_1, ..., Q_n)$, and possible coalitions are identified with the vectors that are (coordinate-wise) smaller than $\mathbf{Q}$. By what is called an abuse of notation we shall represent the sum $\sum_{i=1}^{n} Q_i$ by Q here.
Thus formalizing the notion, we have the following:
For every non-negative vector $\mathbf{Q} \in \mathbb{R}^n$, let $F(\mathbf{Q})$ be the box given by

$$F(\mathbf{Q}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{Q}\}.$$

The point $\mathbf{Q}$ is interpreted as the "grand coalition" in fuzzy sense, and every $\mathbf{x} \in F(\mathbf{Q})$ is a possible fuzzy coalition while $\mathbf{0} \in F(Q)$ is the zero vector signifying 0-size of all types of players.
For every $\mathbf{Q} \geq \mathbf{0} : \mathbf{Q} \in \mathbb{R}^n$, a cooperative fuzzy game is a pair $(\mathbf{Q}, v)$ such that

(i) $\mathbf{Q} \in \mathbb{R}^n$ and $\mathbf{Q} \geq \mathbf{0}$.

(ii) $v : F(\mathbf{Q}) \rightarrow \mathbb{R}^+ \cup 0$ is bounded and satisfies $v(\mathbf{0}) = 0$.

Thus if x_i represents the amount of agents of type i $(i = 1, 2, ..., n)$ that participate in a coalition $\mathbf{x}$, then the total profit from $\mathbf{x} = (x_1, x_2, ..., x_n)$ is given by the real number $v(\mathbf{x})$[see [2] for more details]. Thus the worth of a fuzzy coalition is identified with the profit it incurs due to its formation.

This model has another interpretation due to Azrieli et al [2]. Assume that for every i $(i = 1, 2, 3, ..., n)$, the amount of resources available for agent i is $Q_i \geq 0$ (this can be time, money, etc.). Each agent can choose to invest any fraction of his resources $x_i \leq Q_i$ in a joint project. Note that a fuzzy coalition in Aubin's [1] sense is given by a membership function from N to $[0, 1]$, however the two approaches are equivalent in the following sense:
If for every $\mathbf{x} = (x_1, x_2, ..., x_n) \in F(\mathbf{Q})$, x_i $(0 \leq x_i \leq Q_i)$ is the amount of resources that agent i invests, then we can uniquely define a function $S_{\mathbf{x}}^{\mathbf{Q}} : N \rightarrow [0, 1]$ as follows

$$S_{\mathbf{x}}^{\mathbf{Q}}(i) = \begin{cases} \dfrac{x_i}{Q_i} & \text{if } Q_i \neq 0 \text{ and } x_i \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The function $S_{\mathbf{x}}^{\mathbf{Q}}$ can be interpreted as the membership function for a possible fuzzy coalition in Aubin's sense pertaining to $\mathbf{x}$ in $F(\mathbf{Q})$. Thus under this interpretation, every $\mathbf{x} \in F(\mathbf{Q})$ corresponds to a unique fuzzy coalition $S_{\mathbf{x}}^{\mathbf{Q}}$ in membership function form and vice versa. The support of $\mathbf{x}$ denoted by $Supp(\mathbf{x})$ is the set $\{i \in N \mid x_i > 0\}$. Note that $F(\mathbf{Q})$ is a lattice under ordinary inclusion of fuzzy sets. Let us denote respectively by $\vee$ and $\wedge$, the maximum and minimum operators in $F(\mathbf{Q})$. A fuzzy game $(\mathbf{Q}, v)$ is said to be superadditive if $v(\mathbf{x} \vee \mathbf{y}) \geq v(\mathbf{x}) + v(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in F(\mathbf{Q}) \ : \ \mathbf{x} \wedge \mathbf{y} = 0$. Let us denote by $G_F(\mathbf{Q})$, the class of superadditive cooperative fuzzy games with respect to a grand coalition $\mathbf{Q}$. Hereafter, we shall consider the members of $G_F(\mathbf{Q})$ only for our study and to simplify our notations, denote by v , any member $(\mathbf{Q}, v) \in G_F(\mathbf{Q})$. The following are few important excerpts from our previous paper [6] for a ready reference:

Definition 1. A vector of payoffs $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, one for each player is called a profit allocation. A profit allocation $\mathbf{y}$ is efficient for coalition $\mathbf{x} \in F(\mathbf{Q})$ if $\sum_{i=1}^{n} y_i = v(\mathbf{x})$.

Definition 2. The minimum deal index of a fuzzy game $v \in G_F(\mathbf{Q})$ with respect to a fuzzy coalition $\mathbf{x}$ is the vector $\mathbf{y}(i, \mathbf{x}) \in \mathbb{R}^n$ such that

$$\mathbf{y}(i, \mathbf{x}) = \begin{cases} v(x_i) + \dfrac{x_i}{\sum x_i}[v(\mathbf{x}) - \sum_i v(\mathbf{x_i})] & \text{if } i \in \text{Supp } \mathbf{x} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\mathbf{x_i} = (0, ...0, x_i, 0, ...0) \in F(\mathbf{Q})$, and $v(\mathbf{x}).x_i$ may be interpreted as the proportion of resource of the ith component in $v(\mathbf{x})$.

Remark 1. The minimum deal index is an efficient allocation.

Note that if the cooperative game $v : R^n \to \mathbb{R}^+ \cup \{0\}$ is continuous and all the resources of the players are of same kind, then v depends on the total resource Q of the coalition s, rather on different distributions, i.e. v is constant on each set $\{(x_1, x_2, ..., x_n) : \sum_{i=1}^{n} x_i = Q\}$ for each $\mathbf{Q} \in \mathbb{R}$. For example if resources are considered in monetary units, then v being symmetric in all variables, generates a unique function $F : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ such that $v = FoS$, where $S : \mathbb{R}^n \to \mathbb{R}$ defined for every $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, by $S(\mathbf{x}) = \sum_{i=1}^{n} x_i$. Therefore, F and v can be used alternatively in finding the optimal fuzzy coalition structure $\{s_1, s_2, ...s_m\}$ such that their resource allocation vector $(Q_1, Q_2, ...Q_m)$ maximizes $\sum_{i=1}^{m} v(Q_j)$. We illustrate this by means of the following example:

Example 1. Take a cooperative fuzzy game v with 3 players, where $v : \mathbb{R}^3 \to \mathbb{R}$ is defined as $v(x_1, x_2, x_3) = \frac{(x_1+x_2+x_3)^2}{9}$. Here v is symmetric with respect to x_1,x_2 and x_3. Hence we can find the function $F : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ by $F(x) = \frac{x^2}{9}$, such that $v = FoS$, where $S : R^n \to \mathbb{R}$ is the sum function given by $S(x_1, x_2, x_3) = \sum_{i=1}^{3} x_i$.

3 Our Model

Here we assume that only a finite number of fuzzy coalitions (where resource will be invested) exists for finite number of players. This may be the case when the resources are measured in finite denominations or when arbitrarily small fractions of resources have no practical utility. At first all players $i = 1, 2, ..., n$ would submit their available resources R_1 ,R_2 , ... , R_n to the mediator which, she will allocate in the fuzzy coalitions. Let m be the maximum possible size of the fuzzy coalition structure to be accrued by the players in presence of the mediator. Each player i would estimate the total budget for each of the m coalitions . We assume that for the jth coalition s_j, the total budget vector estimated by the players be $(Q_{j1}, Q_{j2}, ..., Q_{jn})$. We further assume that s_j can work with budget $a_j = \min\{Q_{j1}, Q_{j2}, ..., Q_{jn}\}$ at high risk while with $b_j = \max\{Q_{j1}, Q_{j2}, ..., Q_{jn}\}$ at low risk. This assumption is due to the fact that each coalition requires a sufficient amount of budgets to work with and we can expect that risk level will vary according to the investment. Let the set of possible coalitions be $s = \{s_1, s_2, ..., s_m\}$. The mediator will then find an optimal coalition structure $s^* = \{s_1, s_2, ..., s_{m^*}\}$ and a corresponding budget vector $(Q_1, Q_2, ..., Q_{m^*})$ such that (with regards to the cooperative game v), $\sum_{j=1}^{m^*} v(Q_j)$ is maximum and $\sum_{j=1}^{m^*} Q_j = Q$. This task would be performed in the following three steps:

Step 1: Find all the regions of the form $\{a_j \leq Q_j \leq b_j, j = 1, 2, ..., m'\}$, where $m' \leq n$ and $\sum_{j=1}^{m'} a_j \leq \sum_{i=1}^{n} R_i \leq \sum_{j=1}^{m'} b_j$.

Step 2: Solve the following problem for each region obtained in Step 1,

$$\arg_{(Q_1,Q_2,...Q_{m'})} \left(\max \left(\sum_{j=1}^{m'} v(Q_j) : a_j \leq Q_j \leq b_j \right) \right). \tag{3}$$

Step 3: Select those solutions $(Q_1, Q_2, ...Q_{m^*})$ obtained in Step 2 , for which $\sum_{j=1}^{m^*} v(Q_j)$ is maximum.

Let $(Q_1^*, Q_2^*, ...Q_{m^*}^*)$ be one such solution and $s* = (s_1, s_2, ...s_{m^*})$, the corresponding coalition structure. To illustrate this process, let us take the same cooperative fuzzy game v considered in example 1.

Example 2. Let the budgets of the players submitted be $R_1 = 5, R_2 = 10, R_3 = 15$ units respectively. The players accept that there must be at the most five possible coalitions s_1, s_2, s_3, s_4 and s_5. Let their announced expected total budgets for those coalitions be given by the matrix:

$$Q = [Q_{ij}] = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 4 & 5 & 7 \\ 6 & 9 & 10 \\ 15 & 17 & 20 \end{pmatrix}$$

Therefore, $a_1 = 3, b_1 = 5, a_2 = 2, b_2 = 4, a_3 = 4, b_3 = 7, a_4 = 6, b_4 = 10, a_5 = 15, b_5 = 20$.

Step 1: Here the mediator finds the regions

$\{a_j \leq Q_j \leq b_j, j = 1, 2, ..., m'\}$, where $m' \leq 3$ and $\sum_{j=1}^{m'} a_j \leq \sum_{i=1}^{n} R_i \leq \sum_{j=1}^{m'} b_j$.

Accordingly, she obtains the following :

region 1: $\{3 \leq Q_1 \leq 5, 6 \leq Q_4 \leq 10, 15 \leq Q_5 \leq 20\}$

region 2: $\{2 \leq Q_2 \leq 5, 4 \leq Q_3 \leq 7, 15 \leq Q_5 \leq 20\}$

region 3: $\{3 \leq Q_1 \leq 5, 4 \leq Q_3 \leq 7, 15 \leq Q_5 \leq 20\}$

region 4: $\{2 \leq Q_2 \leq 4, 6 \leq Q_4 \leq 10, 15 \leq Q_5 \leq 20\}$

region 5: $\{4 \leq Q_3 \leq 7, 6 \leq Q_4 \leq 10, 15 \leq Q_5 \leq 20\}$

region 6: $\{6 \leq Q_4 \leq 10, 15 \leq Q_5 \leq 20\}$

Step 2: The mediator solves the optimization problem

$$\arg_{(Q_1, Q_2, \ldots Q_{m'})} \left(\max \left(\sum_{j=1}^{m'} v(Q_j) : a_j \leq Q_j \leq b_j \right) \right)$$

for each region obtained in **Step 1** as follows,

(a) for *region 1*, she obtains one of the solutions as $(Q_1 = 3, Q_4 = 7, Q_5 = 20)$ for which $v(Q_1) + v(Q_4) + v(Q_5) = 50.8889$.

(b) for *region 2*, she obtains one of the solutions as $(Q_2 = 3, Q_3 = 7, Q_5 = 20)$ for which $v(Q_2) + v(Q_3) + v(Q_5) = 50.8889$.

(c) for *region 3*, she obtains one of the solutions as $(Q_1 = 3, Q_3 = 7, Q_5 = 20)$ for which $v(Q_1) + v(Q_3) + v(Q_5) = 50.8889$.

(d) for *region 4*, one of the solutions would be $(Q_2 = 2, Q_4 = 8, Q_5 = 20)$ for which $v(Q_2)+v(Q_4)+v(Q_5)) = 52$.

(e) for *region 5*, one of the solutions would be $(Q_3 = 4, Q_4 = 6, Q_5 = 20)$ for which $v(Q_3)+v(Q_4)+v(Q_5) = 50.2222$.

(f) for *region 6*, one of the solutions would be $(Q_4 = 10, Q_5 = 20)$ for which $v(Q_4) + v(Q_5) = 55.5556$.

Step 3: Thus, in this step, from the above solutions, the mediator picks up those solutions for which $\sum v(Q_j)$ is maximum. Here $(Q_4 = 10, Q_5 = 20)$ gives the maximum value as $v(Q_4) + v(Q_5)) = 55.5556$. So, the optimal coalition structure is $\{s_4, s_5\}$ and the corresponding budget vector is $(Q_4 = 10, Q_5 = 20)$.

Once an optimal coalition structure $s* = (s_1, s_2, ..., s_{m*})$ is evolved, the mediator would initiate for the resource allocation process to these coalitions. A resource investment allocation matrix

$\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ is one whose rows and columns signify respectively, the number of players and the number of fuzzy coalitions (investment avenues) as decided by the mediator and the (i,j)th entry x_{ij} represents the fraction of the ith player's resource, allocated for investment in the jth coalition. Moreover, a resource investment allocation matrix $\mathbf{x}$ can be expressed as an array $(\mathbf{x}_1, \cdots, \mathbf{x}_{m^*})$ of column vectors, where each $\mathbf{x}_j$ is a fuzzy coalition (i.e $\mathbf{x}_j \in F(\mathbf{Q}),\ 1 \leq j \leq m^*$). Here after we call a "resource investment allocation matrix" as "resource allocation" in short. Let x_{ij} be the fraction of resource offered by player i for v in the jth coalition from her total resource R_i. Hence, we must have $R_i = \sum_{j=1}^{m^*} x_{ij}$.
In our model, we associate a satisfaction function of player i for its resource allocation in the jth coalition and call it the "Investment satisfaction function" denoted by $IS_j^i : \mathbb{R} \to [0,1]$. We accept the following assumptions with regards to the investment satisfaction function:

Assumption (1) $IS_j^i(v)(x) = 0$, when $x \leq 0$.

Assumption (2) $IS_j^i(v)(x) = 1$, when $x \geq R_i$.

Assumption (3) $IS_j^i(v)$ is continuously differentiable and also strictly monotonic increasing in $[0, R_i]$.

A possible physical significance of the above assumptions is that each player i is keen to invest her whole resource (R_i). Her degree of satisfaction is therefore zero if she has no investment at all and one if she has full investment there. Moreover, it is natural to expect that satisfaction of any player increases continuously with respect to her investment. Furthermore, every player tries to increase her resource investment in a coalition. So, the derivatives of investment satisfaction functions are monotonic increasing.
Thus to summarize, a "resource allocation" $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ to a cooperative fuzzy game v is a distribution of resources of the players among all fuzzy coalitions such that $\sum_{ji=1}^{m^*} x_{ij} = R_i$ and $\sum_{i=1}^{n} x_{ij} = Q_j$. Following definitions are important.

Definition 3. A resource allocation $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ is said to be an exact resource allocation if all the players in a fuzzy coalition are equally satisfied with their resource investments i.e. $\exists \lambda_j \in \mathbb{R}$ with $j = 1, 2, ..., m^*$ such that $IS_j^i(v)(x_{ij}) = \lambda_j,\ \ i = 1, 2, ...n$.

Definition 4. A resource allocation $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ is said to be an approximate resource allocation if there exist at least two players $i, k (i \neq k)$, $IS_j^i(v)(x_{ij}) \neq IS_j^k(v)(x_{kj})$ for some $j : 1 \leq j \leq m^*$.

In order to obtain a better resource allocation from the previous one, we define the variance function as follows:

Definition 5. The function var : $\mathbb{R}^n \times \mathbb{R}^{m^*} \to \mathbb{R}$, given by

$$\mathrm{var}(\mathbf{x}) = \sqrt{\sum_{j=1}^{m^*}\sum_{i=1}^{n}\left(IS_j^i(v)(x_{ij}) - \overline{IS_v(\mathbf{x}_j)}\right)^2} \tag{4}$$

is called the variance function of the resource allocation $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$, where, $\overline{IS_v(\mathbf{x}_j)} = \frac{\sum_{i=1}^{n} IS_j^i(v)(x_{ij})}{n}$ is the mean satisfaction of the players over the jth coalition.

Definition 6. A resource allocation $\mathbf{x}' = (x'_{ij})_{i=1,j=1}^{n,m^*}$ is said to be better approximate resource allocation than the resource allocation $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ if $\mathrm{var}(\mathbf{x}') < \mathrm{var}(\mathbf{x})$.

Note that every resource allocation is Pareto Optimal in the sense that a betterment to a single player in the next stage is not possible without decreasing the amount of resource to at least one of the remaining players in each coalition.

3.1 Allocation Strategies

The process of negotiation is governed by the negotiation strategies adopted by the mediator for an equitable benefit of each of the players. These strategies determine how the mediator generates and consequently how the players evaluate resource investment to reach an agreement that is most in their self interest [15]. The mediator would propose resource allocation throughout the negotiation process. Whenever a resource allocation is not accepted by the players with equal satisfactions within a coalition, she would either adopt a trade-off strategy or make further offers using an exact resource allocation searching strategy.

3.1.1 Trade-off strategy

A trade-off strategy is an approach by which the mediator generates a resource allocation without reducing the corresponding aggregated satisfaction value. Here, if the variance is below or equal to certain collectively accepted threshold, the players would agree to the current resource allocation instead of continuing the process further. Thus by this approach, one can search for an agreement that benefits all the players at an acceptable threshold.
Suppose that, $S_i(\mathbf{x}^t) = \{\mathbf{x}_j^t \in F(Q) : IS_j^i(v)(x_{ij}^t) > \overline{IS_v(\mathbf{x}_i^t)}\}$, $S'_i(\mathbf{x}^t) = \{\mathbf{x}_j^t \in F(Q) : IS_j^i(v)(x_{ij}^t) < \overline{IS_v(\mathbf{x}_j^t)}\}$ and $S''_i(\mathbf{x}^t) = \{\mathbf{x}_j^t \in F(Q) : IS_j^i(v)(x_{ij}^t) = \overline{IS_v(\mathbf{x}_j^t)}\}$ for every i, represent respectively, the sets of all coalitions in which the investment satisfaction degrees of i exceed the aggregated investment satisfaction value, the investment satisfaction degrees precede the aggregated investment satisfaction value and finally the set of all coalitions in which the investment satisfaction degrees of i are equal to the aggregated value at stage t. If either S_i^t or $S_i^{'t}$ is empty, the mediator adopts a trade-off strategy. If both S_i^t and $S_i^{'t}$ are non-empty, at each stage t, an exact resource allocation searching strategy will be adopted as described in the following.

3.1.2 Exact allocation searching strategy

Let at stage $t+1$, the proposed allocation be $\mathbf{x}^t = (x_{ij}^t)$ and $S_i(\mathbf{x}^t) \neq \emptyset$ and $S_i'(\mathbf{x}^t) \neq \emptyset$. As already mentioned that the mediator is unaware of the investment satisfaction functions of the players, she would update her belief on each player at each stage by assessing the lower and upper bounds of the satisfactions. At stage t of the negotiation process, we obtain these bounds for player $i \in N$ in the coalition $\mathbf{x}_j^t$ by defining an approximate function namely μ_{ij}^t . This function is indeed a linear approximation (from below) of player i's actual investment satisfaction function for the next offer at stage $t+1$ and is obtained by joining the pair $\left[(0,0)\ ,\ (x_{ij}^t, IS_j^i(v)(x_{ij}^t))\right]$ of points. Thus we have,

$$\mu_{ij}^t\,(x) = \frac{IS_j^i(v)\left(x_{ij}^t\right)}{x_{ij}^t}x,\ \forall i \in N,\ 1 \leq j \leq m^* \tag{5}$$

The idea of searching for the exact resource allocation depends on how every investment satisfaction $IS_j^i(v)(x_{ij}^t)$ for the player i in the coalition $\mathbf{x}_j^t$ at stage t converges to the mean investment satisfaction $\overline{IS_v(\mathbf{x}_j^t)}$. So, at stage t, for obtaining the bounds of the actual investment satisfaction degree of player i, towards a resource allocation, to be proposed in the next stage $t+1$, we need to find those values of $z \in \mathbb{R}$ for which $\mu_{ij}^t(z) = \overline{IS_v(\mathbf{x}_j^t)}$. We shall denote these values by either a_{ij}^t or b_{ij}^t keeping in mind that if z is denoted by a_{ij}^t then $x_{ij}^t = b_{ij}^t$ and if z is denoted by b_{ij}^t then $x_{ij}^t = a_{ij}^t$. A geometric representation of the idea is shown in figure 1 and 2.
Thus a_{ij}^t and b_{ij}^t for $\mathbf{x}_j^t$ in $S_i(\mathbf{x}^t)$, $S_i'(\mathbf{x}^t)$ and $S_i''(\mathbf{x}^t)$ are given separately by,

$$a_{ij}^t = \overline{IS_v(\mathbf{x}_j^t)} \times \frac{x_{ij}^t}{IS_j^i(v)(x_{ij}^t)} \ \text{ and } \ b_{ij}^t = x_{ij}^t. \tag{6}$$

where $\mathbf{x}_j^t \in S_i(\mathbf{x}^t)$. Similarly for $\mathbf{x}_j^t \in S_i'(\mathbf{x}^t)$, we have,

$$a_{ij}^t = x_{ij}^t \ \text{ and } \ b_{ij}^t = \overline{IS_v(\mathbf{x}_j^t)} \times \frac{x_{ij}^t}{IS_j^i(v)(x_{ij}^t)}. \tag{7}$$

Finally, we have, for $\mathbf{x}_j^t \in S_i''(\mathbf{x}^t)$,

$$a_{ij}^t = x_{ij}^t \ \text{ and } \ b_{ij}^t = x_{ij}^t.$$

Thus the expected proposal $\mathbf{x}^*$, is defined as follows:

Definition 7. Assuming that the mediator puts a resource allocation $\mathbf{x}^t$ to the players at stage t and that all the players subsequently announce their investment satisfaction degrees, the expected better resource allocation $\mathbf{x}^*$ for the proposal at stage $t+1$ is defined as :

$$\mathbf{x}^* = \arg_{\mathbf{x}} \left(\min \sqrt{\sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ i<k \\ k\in N}}^{n} \left(\mu_{ij}^t\,(x_{ij}) - \mu_{kj}^t\,(x_{kj}) \right)^2} : A \right) \tag{8}$$

where

$$A := \left\{ \sum_{i=1}^{n} x_{ij} = Q_j,\ \sum_{j=1}^{m^*} x_{ij} = R_i,\ a_{ij}^t \leq x_{ij} \leq b_{ij}^t, i = 1, 2, ...n,\ j = 1, 2, ..., m^* \right\}.$$

Note that, if at stage $t+1$, the proposed allocation $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ be such that for some player i either $S_i(\mathbf{x}^t) = \phi$ or $S_i'(\mathbf{x}^t) = \phi$, then the mediator would seek for a resource allocation $\mathbf{x}^{t+1}$ which minimizes

$$\text{var}(\mathbf{x}) = \sqrt{\sum_{j=1}^{m^*} \sum_{i=1}^{n} \left(IS_j^i(v)(x_{ij}) - \overline{IS_v(\mathbf{x}_j)} \right)^2} \tag{9}$$

Let D denote the mutually accepted trade-off threshold to a proposal $\mathbf{x}$, at some stage. Then if $\text{var}(\mathbf{x}) \leq D$, the mediator would resort to a trade-off strategy and we call the corresponding allocation $\mathbf{x}$ an optimal resource allocation.

3.2 The Negotiation Process

In our model, we take a combination of the two allocation strategies to carry out the process of negotiations. Thus, what follows is:

The mediator offers resource allocation at stage $t = 0$ and expects that it is the exact resource allocation. If it is so, the process terminates and we get the required resource allocation. If not, she would check for non-emptiness of the sets $S_i(\mathbf{x}^t)$ and $S_i'(\mathbf{x}^t)$ for every $i \in N$. If there exist at least one player for which either of them is empty, then the mediator will pickup the trade-off strategy. If there is no trade-off among players, the process would terminate with conflicts. Otherwise, the mediator would adopt the exact resource allocation searching strategy and obtain the upper and lower bounds of the next investment satisfaction degrees of all the players. Subsequently, she approximates the investment satisfaction function possessed by player i, at stage t for stage $t+1$ by μ_{ij}^t which is a linear approximation of the investment satisfaction function $IS_j^i(v)$ for the next offer from below the aggregated investment satisfaction degree. Thus a better approximate resource allocation (due to the approximate investment satisfaction) would be obtained and announced as the next offer.

The following lemma is important:

Lemma 1. *Let the resource allocation at the stage t be $\mathbf{x}^t = \left(x_{ij}^t\right)$, such that for every player i, $S_i(\mathbf{x}^t) \neq \phi$ and $S_i'(\mathbf{x}^t) \neq \phi$. If the corresponding degrees of investment satisfactions given by the* *players are $IS_j^i(v)(x_{ij}^t)$, then we must have*

$$\sum_{x_{ij}^t \in S_i(\mathbf{x}^t)} x_{ij}^t > \sum_{x_{ij}^t \in S_i(\mathbf{x}^t)} \overline{IS_v\left(\mathbf{x}_j^t\right)} \times \frac{x_{ij}^t}{IS_j^i(v)\left(x_{ij}^t\right)} \tag{10}$$

$$\sum_{x_{ij}^t \in S_i'(\mathbf{x}^t)} \overline{IS_v\left(\mathbf{x}_j^t\right)} \times \frac{x_{ij}^t}{IS_j^i(v)\left(x_{ij}^t\right)} > \sum_{x_{ij}^t \in S_i'(\mathbf{x}^t)} x_{ij}^t \tag{11}$$

$$\sum_{i \in S_j(\mathbf{x}^t)} x_{ij}^t > \sum_{i \in S_j(\mathbf{x^t}} \overline{IS_v\left(\mathbf{x}_j^t\right)} \times \frac{x_{ij}^t}{IS_j^i(v)\left(x_{ij}^t\right)} \tag{12}$$

$$\text{and} \quad \sum_{i \in S_j'(\mathbf{x}^t)} \overline{IS_v\left(\mathbf{x}_j^t\right)} \times \frac{x_{ij}^t}{IS_j^i(v)\left(x_{ij}^t\right)} > \sum_{i \in S_j'(\mathbf{x^t})} x_{ij}^t \tag{13}$$

where $S_j(\mathbf{x}^t) = \{i \in N : IS_j^i(v)(x_{ij}^t) > \overline{IS_v(\mathbf{x}_j^t)}\}$ and $S_j'(\mathbf{x}^t) = \{i \in N : IS_j^i(v)(x_{ij}^t) < \overline{IS_v(\mathbf{x}_j^t)}\}$ represent respectively, the sets of players having satisfaction degrees above and below the aggregated satisfaction for the resource allocation $\mathbf{x}^t$ at stage t.

Remark 2. A similar lemma was proved in our previous paper (Lemma 11, [6]) concerning profit allocation. The proof of lemma 1 goes exactly in the same way and thus we omit the same here.

Theorem 1. *Given a superadditive cooperative fuzzy game $v : F(Q) \to \mathbb{R}^+ \cup \{0\}$ and a rational efficient resource allocation $\mathbf{x}^t$ at stage t which is not exact, let us assume that $S_i(\mathbf{x}^t) \neq \phi$ and $S_i'(\mathbf{x}^t) \neq \phi$ for all player i. Then for stage $t+1$, there exists at least one better rational efficient resource allocation $\mathbf{x}^{t+1}$, such that*

$$a_{ij}^t \leq x_{ij}^{t+1} \leq b_{ij}^t.$$

for $\mathbf{x}_j^t \in S_i(\mathbf{x}^t)$ where a_{ij}^t and b_{ij}^t are given by equation (6) and

$$a_{ij}^t \leq x_{ij}^{t+1} \leq b_{ij}^t.$$

for $\mathbf{x}_j^t \in S_i'(\mathbf{x}^t)$ where a_{ij}^t and b_{ij}^t are given by equation (7)

Proof. Here, we have to prove that there exists a rational efficient resource allocation which is better than the given resource allocation.

Part I: (Existence of a Rational efficient resource allocation)

For the existence of a rational efficient resource allocation $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ at stage $t+1$, we need to show that,

$\sum_{j=1}^{m^*} a_{ij}^t \leq \sum_{j=1}^{m^*} x_{ij} = R_i \leq \sum_{j=1}^{m^*} b_{ij}^t$ for all $i = 1, 2, ..., n$.

and $\sum_{i=1}^{n} a_{ij}^t < Q_j = \sum_{i=1}^{n} x_{ij} < \sum_{i=1}^{n} b_{ij}^t$ for all $j = 1, 2, ..., m^*$

From first two inequalities of lemma 1, we have,

$$\sum_{x_{ij}^t \in S_i(\mathbf{x}^t)} x_{ij}^t > \sum_{x_{ij}^t \in S_i(\mathbf{x}^t)} \overline{IS_v\left(\mathbf{x}_j^t\right)} \times \frac{x_{ij}^t}{IS_j^i(v)\left(x_{ij}^t\right)} \tag{14}$$

$$\sum_{x_{ij}^t \in S_i'(\mathbf{x}^t)} \overline{IS_v\left(\mathbf{x}_j^t\right)} \times \frac{x_{ij}^t}{IS_j^i(v)\left(x_{ij}^t\right)} > \sum_{x_{ij}^t \in S_i'(\mathbf{x}^t)} x_{ij}^t \tag{15}$$

Adding inequalities (14) and (15) with $\sum_{\mathbf{x}_j^t \in S_i''(\mathbf{x}^t)} x_{ij}^t$ on both sides, we get

$$\sum_{\mathbf{x}_j^t \in S_i(\mathbf{x}^t)} a_{ij}^t + \sum_{\mathbf{x}_j^t \in S_i'(\mathbf{x}^t)} x_{ij}^t + \sum_{\mathbf{x}_j^t \in S_i''(\mathbf{x}^t)} x_{ij}^t < \sum_{\mathbf{x}_j^t \in S_i(\mathbf{x}^t)} x_{ij}^t + \sum_{\mathbf{x}_j^t \in S_i'(\mathbf{x}^t)} b_{ij}^t + \sum_{\mathbf{x}_j^t \in S_i''(\mathbf{x}^t)} x_{ij}^t$$

Since, $x_{ij}^t = a_{ij}^t$ for $\mathbf{x}_j^t \in S_i'(\mathbf{x}^t) \cup S_i''(\mathbf{x}^t)$ and $x_{ij}^t = b_{ij}^t$ for $\mathbf{x}_j^t \in S_i(\mathbf{x}^t) \cup S_i''(\mathbf{x}^t)$, we have

$$\sum_{j=1}^{m^*} a_{ij}^t < \sum_{j=1}^{m^*} x_{ij}^t = R_i < \sum_{j=1}^{m^*} b_{ij}^t$$

for all i. Similarly, since, $x_{ij}^t = a_{ij}^t$ for $i \in S_j(\mathbf{x}^t) \cup S_j''(\mathbf{x}^t)$ and $x_{ij}^t = b_{ij}^t$ for $i \in S_j(\mathbf{x}^t) \cup S_j''(\mathbf{x}^t)$, from the remaining two inequalities of lemma 1, we have,

$$\sum_{j=1}^{m^*} a_{ij}^t < \sum_{i=1}^{n} x_{ij}^t = Q_j < \sum_{j=1}^{m^*} b_{ij}^t$$

for all $\mathbf{x}_j^t$. Now, each investment satisfaction function $IS_j^i(v)$ being continuous in the closed interval $\left[a_{ij}^t, b_{ij}^t\right]$, and $\sum_{i=1}^n R_i = \sum_{j=1}^{m^*} Q_j$ we have, by the "Intermediate Value Theorem" that there exists $\mathbf{x} = (x_{ij})_{i=1,j=1}^{n,m^*}$ such that $a_{ij}^t \leq x_{ij} \leq b_{ij}^t$ for all i and j, along with $\sum_{j=1}^{m^*} a_{ij}^t \leq \sum_{j=1}^{m^*} x_{ij} = R_i \leq \sum_{j=1}^{m^*} b_{ij}^t$ for all i and $\sum_{i=1}^n a_{ij}^t \leq \sum_{i=1}^n x_{ij} = Q_j \leq \sum_{i=1}^n b_{ij}^t$ for each $\mathbf{x}_j^t$.

Part II :(There is a new resource allocation better than the previous resource allocation)

Since for $\mathbf{x}_j^t \in S_i(\mathbf{x}^t)$,

$$IS_j^i(v)(x_{ij}^t) > \overline{IS_v(\mathbf{x}_j^t)},$$

for $\mathbf{x}_j^t \in S_i'(\mathbf{x}^t)$,

$$IS_j^i(v)(x_{ij}^t) < \overline{IS_v(\mathbf{x}_j^t)},$$

for $i \in S_j(\mathbf{x}^t)$,

$$IS_j^i(v)(x_{ij}^t) > \overline{IS_v(\mathbf{x}_j^t)}$$

and for $i \in S_j'(\mathbf{x}^t)$,

$$IS_j^i(v)(x_{ij}^t) < \overline{IS_v(\mathbf{x}_j^t)},$$

by part I and using the following facts:

(i) $\sum_{i=1}^n R_i = \sum_{j=1}^{m^*} Q_j$,

(ii) for some player i, $IS_j^i(v)(x_{ij}) < IS_j^i(v)(x_{ij}^t)$ where $\mathbf{x}_j^t \in S_i(\mathbf{x}^t)$, and finally,

(iii) for some player i, $IS^i_j(v)(x_{ij}) > IS^i_j(v)(x^t_{ij})$ where $\mathbf{x}^t_j \in S'_i(\mathbf{x}^t)$,

we can select a resource allocation $\mathbf{x} = (x_{ij})^{n,m^*}_{i=1,j=1}$ such that,

(a) $a^t_{ij} \leq x_{ij} \leq b^t_{ij}$ for all i and j,

(b) $\sum_{j=1}^{m^*} a^t_{ij} \leq \sum_{j=1}^{m^*} x_{ij} = R_i \leq \sum_{j=1}^{m^*} b^t_{ij}$ for all i and,

(c) $\sum_{i=1}^{n} a^t_{ij} \leq \sum_{i=1}^{n} x_{ij} = Q_j \leq \sum_{i=1}^{n} b^t_{ij}$ for all j,

Since, for all player i, $S_i(\mathbf{x}^t) \neq \phi$ and $S'_i(\mathbf{x}^t) \neq \phi$ and since each investment satisfaction function $IS^i_j(v)$ is continuous in the closed interval $\left[a^t_{ij}, b^t_{ij}\right]$, thus,

$$\begin{aligned}
& \sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ k\in N \\ i<k}}^{n} \left(IS^i_j(v)\left(x_{ij}\right) - IS^k_j(v)\left(x_{kj}\right)\right)^2 \quad < \quad \sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ k\in N \\ i<k}}^{n} \left(IS^i_j(v)\left(x^t_{ij}\right) - IS^k_j(v)\left(x^t_{kj}\right)\right)^2 \\
\Rightarrow & \sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ k\in N \\ i<k}}^{n} \left(IS^i_j(v)\left(x_{ij}\right) - \overline{IS_v\left(\mathbf{x}_j\right)}\right)^2 \quad < \quad \sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ k\in N \\ i<k}}^{n} \left(IS^i_j(v)\left(x^t_{ij}\right) - \overline{IS_v\left(\mathbf{x}^t_j\right)}\right)^2 \\
\Rightarrow & \operatorname{var}\left(\mathbf{x}\right) < \operatorname{var}\left(\mathbf{x}^t\right).
\end{aligned}$$

So, $\mathbf{x} = (x_{ij})^{n,m^*}_{i=1,j=1}$ is a better efficient resource allocation than $\mathbf{x}^t$. This completes the proof. □

3.2.1 The Negotiation Protocol

We now describe the negotiation protocol of the resource allocation process, accepting that an optimal coalition structure $s* = (s_1, s_2, ..., s_{m*})$ has already evolved in the first phase.

Stage1:

The mediator will propose the initial resource allocation $\mathbf{x}^0 = (x^0_{ij})^{n,m^*}_{i=1,j=1}$ where $x^0_{ij} > 0 \;\; \forall i, j$ and the players would react to this proposal by announcing their satisfaction degrees $IS^i_j(v)(x^0_{ij})$. Based on these information, the mediator would approximate her beliefs by defining the function μ^0_{ij} as follows:(refer to equation (5))

$$\mu^0_{ij}(x) = \frac{IS^i_j(v)(x^0_{ij})}{x^0_{ij}} x, \; \forall i \in N, j = 1, 2, ..., m^* \tag{16}$$

If the investment satisfactions $IS^i_j\left(x^0_{ij}\right)$ are equal in each coalition $\mathbf{x}^0_j = \left(x^0_{1j}, x^0_{2j}, ..., x^0_{nj}\right)$, $1 \leq j \leq m^*$, then the proposal $\mathbf{x}^0$ will be the exact resource allocation and the process would terminate there itself. Otherwise, either for all players i, $S_i(\mathbf{x}^0) \neq \phi$ and $S'_i(\mathbf{x}^0) \neq \phi$ or for some i one of $S_i(\mathbf{x}^0)$ or $S'_i(\mathbf{x}^0)$ is empty. In the first situation the mediator would adopt "Exact resource allocation Searching Strategy" while the second situation will be dealt with a "Trade-off strategy". Existence of a better rational efficient resource allocation in the former case is ensured by theorem 1. However, as the investment satisfaction functions of the players are not known *a* *priori*, the mediator computes the next resource allocation using only the approximate investment satisfaction function μ^0_{ij} similar to that given in equation (8):

$$\mathbf{x}^1 = \arg_{\mathbf{x}} \left(\min \sqrt{\sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ k\in N \\ i<k}}^{n} \left(\mu^0_{ij}(x_{ij}) - \mu^0_{kj}(x_{kj}) \right)^2} : A \right)$$

where,

$$A = \left\{ \sum_{i=1}^{n} x_{ij} = Q_j, \sum_{j=1}^{m^*} x_{ij} = R_i, a^t_{ij} \leq x_{ij} \leq b^t_{ij}, j = 1, 2, ..., m^*,\ i = 1, 2, ...n \right\}.$$

such that,
$a^0_{ij} = \overline{IS_v(\mathbf{x}^0_j)} \times \frac{x^0_{ij}}{IS^i_j(v)(x^0_{ij})}$, $b^0_{ij} = x^0_{ij}$, for $\mathbf{x}^0_j \in S_i(\mathbf{x}^0)$; $a^0_{ij} = x^0_{ij}$, $b^0_{ij} = \overline{IS_v(\mathbf{x}^0_j)} \times \frac{x^0_{ij}}{IS^i_j(v)(x^0_{ij})}$, for $\mathbf{x}^0_j \in S'_i(\mathbf{x}^0)$; and finally for $\mathbf{x}^0_j \in S''_i(\mathbf{x}^0)$, $a^0_{ij} = x^0_{ij}$ and $b^0_{ij} = x^0_{ij}$.
Note that, here,
$S_i(\mathbf{x}^0) = \left\{ \mathbf{x}^0_j \in F(Q) : IS^i_j(v)(x^0_{ij}) > \overline{IS_v(\mathbf{x}^0_j)} \right\}$, $S'_i(\mathbf{x}^0) = \left\{ \mathbf{x}^0_j \in F(Q) : IS^i_j(v)(x^0_{ij}) < \overline{IS_v(\mathbf{x}^0_j)} \right\}$ and $S''_i(\mathbf{x}^0) = \left\{ \mathbf{x}^0_j \in F(Q) : IS^i_j(v)(x^0_{ij}) = \overline{IS_v(\mathbf{x}^0_j)} \right\}$.

Stage t+1:

In the case of $\mathbf{x}^t$ not being exact, whereas, for each i, $S_i(\mathbf{x}^t) \neq \phi$ and $S'_i(\mathbf{x}^t) \neq \phi$, then by theorem 1 a better rational efficient resource allocation can be obtained as follows:

$$\mathbf{x}^{t+1} = \arg_{\mathbf{x}} \left(\min \sqrt{\sum_{j=1}^{m^*} \sum_{\substack{i=1 \\ k\in N \\ i<k}}^{n} \left(\mu^t_{ij}(x_{ij}) - \mu^t_{kj}(x_{kj}) \right)^2} : A \right)$$

where

$$A = \left\{ \sum_{i=1}^{n} x_{ij} = Q_j, \sum_{j=1}^{m^*} x_{ij} = R_i, a^t_{ij} \leq x_{ij} \leq b^t_{ij}, j = 1, 2, ..., m^*, i = 1, 2, ...n \right\}.$$

and $a^t_{ij} = \overline{IS_v(\mathbf{x}^t_j)} \times \frac{x^t_{ij}}{IS^i_j(v)(x^t_{ij})}$, $b^t_{ij} = x^t_{ij}$, for $\mathbf{x}^t_j \in S_i(\mathbf{x}^t)$ and for $\mathbf{x}^t_j \in S'_i(\mathbf{x}^t)$, we have, $a^t_{ij} = x^t_{ij}$,$b^t_{ij} = \overline{IS_v(\mathbf{x}^t_j)} \times \frac{x^t_{ij}}{IS^i_j(v)(x^t_{ij})}$ for $\mathbf{x}^t_j \in S''_i(\mathbf{x}^t)$, $a^t_{ij} = x^t_{ij}$ and $b^t_{ij} = x^t_{ij}$.
We now prove that the process of searching for a better resource allocation described here, leads to an exact resource allocation.

Theorem 2. *The process of obtaining a better efficient resource allocation converges to the exact resource allocation if at each stage t, it holds that $S_i(\mathbf{x}^t) \neq \phi$ and $S'_i(\mathbf{x}^t) \neq \phi$, $\forall i \in N$.*

Proof. If at each stage t, $S_i(\mathbf{x}^t) \neq \phi$ and $S_i(\mathbf{x}^t) \neq \phi$, $\forall i \in N$, we have $\text{var}\left(\mathbf{x}^{t+1}\right) < \text{var}\left(\mathbf{x}^t\right)\ \forall t \in \mathbb{H}$. Thus, $\{\text{var}\left(\mathbf{x}^t\right),\ t \in \mathbb{H}\}$ is a strictly decreasing sequence of positive real numbers, $\mathbb{H}$ being the history.

So ,

$$\lim_{t \to \infty} \operatorname{var}\left(\mathbf{x}^t\right) = 0 = \operatorname{var}(\mathbf{x}) \tag{17}$$

for some $\mathbf{x} \in \mathbb{R}^{n \times m^*}$ such that $\sum_{i=1}^{n} x_{ij} = Q_j, \sum_{j=1}^{m^*} x_{ij} = R_i$. This resource allocation $\mathbf{x}$ is then the exact resource allocation. The existence of this exact resource allocation is ensured by theorem 1. □

4 Examples

We illustrate here, our model by means of the same example we mentioned in Introduction. We first deal with a situation where the mediator searches for solutions using exact resource allocation strategy, followed by a second one which will have bearings of trade-off strategy.

Example 3. In continuation to what we have discussed in the Introduction, let players P_1, P_2 and P_3 have resources $R_1 = 10$,$R_2 = 15$ and $R_3 = 20$ respectively (in multiples of hundred thousand rupees). In phase 1, let the mediator find that only coalitions s_1 , s_2 and s_3 with their budgets as $Q_1 = 10$, $Q_2 = 15$ and $Q_3 = 20$ respectively can maximize $\sum v(Q_j)$.

Let us assume hypothetically that players' satisfaction functions are given by,

$IS_j^1(v)(x) = \frac{x^2}{100}$

$IS_j^2(v)(x) = \frac{x^2}{225}$

$IS_j^3(v)(x) = \frac{x^2}{400}$,

for every j such that $1 \leq j \leq 3$.

The negotiation process goes on as follows:

Stage 1: Let,

$$\mathbf{x}^0 = \begin{pmatrix} 2.22874 & 3.37124 & 4.40002 \\ 3.45151 & 4.93382 & 6.61467 \\ 4.31975 & 6.69494 & 8.98531 \end{pmatrix}$$

be the first resource allocation matrix proposed by the mediator. The players will announce their satisfaction degrees using their investment satisfaction functions as:

$$[IS_j^i(v)(x_{ij}^0)] = \begin{pmatrix} 0.0496728 & 0.0505123 & 0.0484004 \\ 0.119129 & 0.108189 & 0.109385 \\ 0.186602 & 0.19921 & 0.201839 \end{pmatrix}$$

so that $\operatorname{var}\left(\mathbf{x}^0\right) = 0.356931$.

The following table illustrates the iterative steps in obtaining a better rational efficient solution:

Table 1:

stage t	$\mathbf{x}^t = \{(x_{11}^t, x_{21}^t, x_{31}^t), (x_{12}^t, x_{22}^t, x_{32}^t), (x_{13}^t, x_{23}^t, x_{33}^t)\}$	var$(\mathbf{S}^t)$
$t=1$	$\{(2.22874, 3.37124, 4.40002), (3.45151, 4.93382, 6.61467), (4.31975, 6.69494, 8.98531)\}$	0.0143861
$t=2$	$\{(2.22227, 3.30558, 4.47215), (3.25174, 5.04728, 6.70098), (4.52599, 6.64714, 8.82687)\}$	.0092807
$t=3$	$\{(2.21919, 3.3568, 4.42401), (3.39132, 4.96382, 6.64486), (4.38949, 6.67938, 8.93113)\}$	0.0067498
$t=4$	$\{(2.22502, 3.31249, 4.46249), (3.29177, 5.02927, 6.67896), (4.48321, 6.65824, 8.85855)\}$	0.0047768
$t=5$	$\{(2.21809, 3.35228, 4.42963), (3.36388, 4.97582, 6.6603), (4.41803, 6.6719, 8.91007)\}$	0.00342241
$t=6$	$\{(2.22659, 3.31563, 4.45778), (3.31061, 5.02082, 6.66857), (4.4628, 6.66355, 8.87365)\}$	0.00301652
$t=7$	$\{(2.21738, 3.3504, 4.43222), (3.35068, 4.98145, 6.66787), (4.43194, 6.66815, 8.89991)\}$	0.00193378
$t=8$	$\{(2.22719, 3.32024, 4.45257), (3.31982, 5.01231, 6.66787), (4.45299, 6.66745, 8.87956)\}$	0.00141783
$t=9$	$\{(2.21748, 3.3441, 4.43842), (3.34368, 4.98845, 6.66787), (4.43884, 6.66745, 8.89371)\}$	0.00108052
$t=10$	$\{(2.22652, 3.32583, 4.44765), (3.3252, 5.00693, 6.66787), (4.44824, 6.66724, 8.88448)\}$	0.000782514

If all the players agree to a trade-off at var$(\mathbf{x}^t) < 0.0008$ then, the corresponding resource allocation matrix is given by,

$$\mathbf{S}^{10} = \begin{pmatrix} 2.22652 & 3.32583 & 4.44765 \\ 3.3252 & 5.00693 & 6.66787 \\ 4.44824 & 6.66724 & 8.88448 \end{pmatrix}$$

We take another example but similar to the former, to show that there are situations, where after certain stage, the exact solution searching strategy would no more be applicable. In such cases, a trade-off strategy will be adopted. Thus the negotiation protocol becomes a combination of both the strategies.

Example 4. Consider the same three players P_1,$P_2$and P_3 who have resources $R_1 = 10$,$R_2 = 15$ and $R_3 = 20$ respectively (in multiples of hundred thousand rupees). Suppose the mediator desires that the total number of coalitions be 2 (i.e. $n = 2$). Let these coalitions be s_1 and s_2 with their sufficient resources as $Q_1 = 20$ and $Q_2 = 25$ respectively (These may be any two industries we have proposed in the previous example).

Let us assume hypothetically that players' satisfaction functions are given by,

$IS_j^1(v)(x) = \frac{x}{10}$

$IS_j^2(v)(x) = \frac{x^2}{225}$

$IS_j^3(v)(x) = \frac{x^2}{400}$ for $j = 1, 2$.

The negotiation process goes on as follows:

Stage 1: Let,

$$\mathbf{x}^0 = \begin{pmatrix} 4 & 7 & 4.9 \\ 6 & 8 & 11 \end{pmatrix}$$

be the first resource allocation matrix proposed by the mediator. The players would announce their satisfaction degrees using their investment satisfaction functions as follows:

$$[IS_j^i(v)(x_{ij}^0)] = \begin{pmatrix} 0.4 & 0.217778 & 0.2025 \\ 0.6 & 0.284444 & 0.3025 \end{pmatrix}$$

where, $\overline{IS_v\,(\mathbf{x}_1^0)} = 0.273426$ and $\overline{IS_v\,(\mathbf{x}_2^0)} = 0.395648$. Thus $S_1(\mathbf{x}^0) = \{\mathbf{x}_1^0, \mathbf{x}_2^0\}$ and $S_1'(\mathbf{x}^0) = \phi$. Therefore, we cannot apply exact resource allocation searching strategy here. So, we can take the above $\mathbf{x}^0$ as a trade-off allocation.

5 conclusion

This paper has presented a dynamic approach to solve the problem of resource allocation among rational players involved in a joint venture. The problem is treated with fuzzy game theoretic approach. This work is in sequel to our earlier work on profit allocation [6]. We comment that a dynamic approach would encourage the players for better performance in anticipation of a satisfactory resource allocation in a coalition structure. Moreover, if the process of coalition formation is repeated as the case may be until the players' resources are exhausted, then intuitively, they would have sufficient time to learn about each other. Consequently, more synergic gains would outweigh the one shot expectations which justifies our assumption of rationality regarding the players' actual satisfaction functions. This justification is similar to our previous work [6]. Keeping that in mind, here also, we have taken two strategies namely, exact resource allocation searching and trade-off strategies. Nevertheless, our previous and current works together will completely describe the dynamics in a cooperative fuzzy game where we have chosen satisfaction of the players a key ingredient in allocating resource and payoff to a set of players.

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