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7. July 2012

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This version: July 7, 2012

Abstract: This paper presents a dynamic small open economy version of the standard neoclassical exogenous growth model with international migration. It considers both the case of perfect world capital markets and the case of imperfect capital markets and shows that local indeterminacy always arises independently of the capital market regime. To study the dynamic implications of migration on domestic consumption, current account and capital accumulation, we simulate the model numerically by distinguishing three different scenarios depending on whether the initial immigration ratio is larger, equal or smaller than its steady-state value. In the case of perfect world capital markets, we find that migration has only a temporary impact on capital accumulation, but a permanent impact on domestic consumption and foreign debt. Instead, in the case of imperfect world capital markets, we find that migration has only temporary impacts on all the main macroeconomic variables.

JEL classification: C61, C62, F22, F46, O41

Keywords: Small Open Economy, Indeterminacy, International Migration, Capital Adjustment Costs, First Best

Acknowledgments: Financial support from the Italian Ministry for the University and Scientific Research (PRIN 2008YYBH7L) is gratefully acknowledged. Any remaining errors are mine.

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1 Introduction

International migration is one of the most debated phenomena of the globalization era. According to World Bank (2011), in 2010 about 3 percent of the world's population lives outside their countries of origin, many of whom moved from low-developed countries. In the developed world international migration is perceived both as a great opportunity for long-run growth as well as a serious threat for social cohesion because of the competition in the labor market. The trade-off between costs and benefits of migration is one of the most hotly debated issue in industrialized countries, especially in those where the fact of being a host country rather than a source country is a novelty not well understood by public opinion.

In spite of the importance of the issue, few papers have addressed this issue at a general equilibrium level, many of which focusing on illegal migration. These studies include Rodriguez (1975), Braun (1993), Meier and Wenig (1997), Kemnitz and Wigger (2000), Hazari and Sgro (2003), Barro and Sala-i-Martin (2004), Moy and Yip (2006), Palivos and Yip (2007), Palivos (2009), Liu (2010), Chassamboulli and Pavilos (2010), Palivos et al. (2011). The main limitation of these papers is that they address migration issues by using a closed economy framework.[1]

This paper analyzes the dynamic behavior of a small open economy with international migration. In doing so, we present a small open economy with capital adjustment costs, where two groups of workers - domestic workers and foreign (or immigrant) workers - enter as complement inputs in the production system. Since the domestic economy is small in comparison with the rest of the world, its economic dynamics cannot alter the world interest rates. This causes capital accumulation to react to external rather internal stimuli, with the result that changes in the composition of the domestic workforce can affect the macroeconomic equilibria in different ways from those predicted by the current literature.

The paper can be ideally split into two parts. The first part of the paper focuses on perfect world capital markets, in which natives can lend and borrow as much as they want without restrictions. We conduct our analysis at both an analytical and numerical level and show that the model displays a continuum of converging paths, each depending on the initial level of the immigration ratio - defined as the number of immigrants per native resident. Since the steady state is indeterminate, to analyze the dynamic properties of the model we perform three different numerical exercises by parametrizing three different scenarios depending on whether the initial immigration ratio is equal, larger or lower than its long-run value. We find that changes in the initial level of the immigration ratio can permanently affect the long-run level of both domestic consumption and national debt per native, while they have only temporary effects on the remaining macroeconomic variables of the model. In particular, we find that the higher the immigration ratio, the higher the level of domestic consumption and the lower the level of the stock of foreign bonds per native worker.

To check whether indeterminacy depends on the absence of borrowing constraints, in the

[1] The only exceptions are the papers by Galor (1986) and Chen (2006), in which international migration is plugged into a two-country framework with overlapping generations, and Palivos and Yip (2007), who provide an extension of their Solow-like model to the case of a small open economy.

second part of the paper we extend the basic model to the case in which the domestic economy faces an upward-sloping curve of debt. Due to the complexity of the model, we resort to numerical simulations in order to study the transitional dynamics of the domestic economy. We find that for very plausible parameters the indeterminacy result is robust to changes in the world capital markets regime. However, in contrast to the previous case of perfect world capital markets, we find that changes in the initial level of the immigration ratio have only temporary effects on the main macroeconomic variable of the model. Specifically, we find that changes in the initial value of the immigration ratio have only temporary affect the long-run level of domestic consumption, national debt per native and capital accumulation, because of the dynamic adjustment of the interest rate.

Finally, in order to check whether the competitive equilibrium is also socially efficient, the paper presents a welfare analysis in which we contrast the steady-state results of the basic/decentralized economy with those of a centrally-planned one in which a benevolent social planner maximizes the welfare of natives. We find that in competitive equilibrium the domestic economy overaccumulates capital and overattracts immigrants. This result occurs because natives do not take into account the impact that their investment decisions have on the wage bill of immigrants. Consequently, in order to establish the first best, the government can pursue two alternative policies: taxing capital earnings or introducing an immigration tax. However, we find the only policy that is effective in establishing the first best is tax capital earnings.

With respect to the closed-economy literature of international migration and macroeconomic dynamics, which predicts that the steady-state equilibrium is always saddle-path stable, our results represent a novelty. In contrast with this literature, in the present model the existence of a continuum of stable arms has several dynamic implications for the model. In particular, the presence of local indeterminacy implies that: (*i*) while the transitional growth rates of the domestic economy are indeterminate, the long-run growth is not and is equal to zero in the steady state; (*ii*) structurally identical economies can exhibit convergence patterns towards the unique steady-state equilibrium if their initial immigration ratio is not the same.

The paper is organized as follows. Section 2 sets up the model economy in the case of perfect world capital markets and analyzes its steady-state properties as well as its dynamic properties. Section 3 extends the basic model to the case of an upward-sloping curve of debt and checks whether indeterminacy is likely to disappear for a set of plausible parameters representing a small open economy. Section 4 investigates the effects of migration on the welfare of natives. Finally, Section 5 summarizes the results and draws conclusions.

2 The model economy without borrowing restrictions

2.1 Overview of the model

We begin by describing the generic structure of a small open economy that consumes and produces a single traded commodity (henceforth *domestic output*). At all points in time, population L consists of N natives and M immigrants, each of whom has an infinite planning horizon and possesses perfect foresight. Regardless of their nationality, each individual is endowed with a unit of labor which is supplied inelastically in exchange for a wage. We abstract from unemployment issues, so that at any point in time total labor supply is equal to the resident population, L.

To simplify the analysis, we assume that the native population is constant over time, whereas the foreign population might change according to a migration function whose characteristics will be described later in the paper. Once foreign workers have emigrated, they enter the labor market of the host country and try to find a job in competition with natives. Following the literature on illegal migration and growth, we assume that only natives are allowed to save, while immigrants are assumed to spend all their wages in consumption.

The domestic capital market is fully integrated into the world capital markets, which channel the national saving towards financial assets. There are two financial assets available to native savers: foreign bonds and capital assets. When accumulating capital assets though, we assume that savers bear installation costs à la Hayashi (1982). This is to prevent the domestic capital stock from instantaneously reaching its long-run level at an infinite speed of convergence.

Production is carried out by a fringe of competitive firms. The financing of domestic production can occur through both domestic and foreign saving. To start with, we suppose that the domestic economy has no borrowing restrictions, meaning that it can lend or borrow as much as it likes at a given interest rate. In Section 3 we remove this simplifying assumption by extending the basic model to the case of a small open economy with borrowing constraints.

2.2 Consumption

Aggregate consumption is denoted by $C = C_n + C_m$, so that the consumption expenditure per native is given by $c \equiv C_n/N$ and consumption expenditure per immigrant is given by $c_m \equiv C_m/M$. Immigrants are supposed to consume all their income without saving or borrowing, such that at all points in time the overall wage bill paid to immigrants, w_m, is equal to their consumption, C_m; in formulas: $C_m = w_m M$. This means that immigrants do not make any financial effort to help the domestic economy to accumulate capital.[2]

Natives are endowed by a felicity function, $u(c)$, satisfying the usual neoclassical properties: $u(0) = 0$, $u'(c) > 0$, $u''(c) < 0$ and Inada (1963) conditions. The lifetime utility of the representative native is therefore given by:

$$U(t) = \int_0^\infty e^{-\rho t} u(c)\, \mathrm{d}t, \tag{1}$$

where $\rho > 0$ is the subjective discount rate.

In contrast with immigrants, natives accumulate two types of assets: capital and foreign bonds. To accumulate capital, natives invest a positive amount of output, denoted by i, in

[2] We make this assumption for simplicity. An alternative assumption would be that of allowing immigrants to save a share of their current wage, $s_m w_m$ (with $s_m \in (0,1)$) and then send it back to their own country of origin through workers' remittances. However, this assumption, although it modifies the level of the current account, does not affect the capital accumulation pattern of the domestic economy and adds no further insight to the main results of the model.

capital investment and bear installation (or adjustment) costs given by:

$$\phi\left(i,k\right)\equiv i\left[1+\frac{h}{2}\left(\frac{i}{k}\right)\right]. \tag{2}$$

Equation (2) is an application of the familiar Hayashi (1982) installation costs framework, where $h>0$ measures the sensitivity of the adjustment costs to changes in the total amount invested. It turns out to have at least two important roles in the present model. First, (2) precludes the existence of an infinite speed of convergence of the capital stock. Second, it introduces a new endogenous variable, the so-called Tobin's q after Brainard and Tobin (1968), which equilibrates the rates of return on the two financial assets available to the domestic economy. Given (2), the net rate of capital accumulation is given by:

$$\dot{k}=i-\delta k, \tag{3}$$

where $\delta\in(0,1)$ captures the depreciation rate of the installed capital stock.

Natives also accumulate foreign bonds, B, which pay a fixed rate of return, $\bar{r}$. Since the domestic economy has unrestricted access to perfect world capital markets, the differential equation governing the accumulation of foreign bonds is given by:

$$\dot{b}=\bar{r}b+w_n+r_kk-c-\phi\left(i,k\right). \tag{4}$$

where $b\equiv B/N$ and $k\equiv K/N$ are, respectively, the stock of foreign bonds and the stock of capital assets per native, w_n is the wage bill paid to natives and r_k is the rate of return on capital assets.

Natives' decisions are to choose their rates of consumption, c, per capita investment, i, and asset accumulation, b and k, to maximize the lifetime utility (1), subject to the two accumulation equations (3) and (4), and capital adjustment costs (2).

2.2.1 Optimality conditions and Tobin's q

Let's define the shadow value of wealth in the form of internationally traded bonds by λ and the shadow value of wealth in the form of installed capital by q'. The discounted Hamiltonian for this dynamic optimization is:

$$H=e^{-\rho t}u\left(c\right)+\lambda\left[\bar{r}b+w_n+r_kk-c-i\left(1+\frac{h}{2}\frac{i}{k}\right)\right]+q'\left(i-\delta k\right).$$

where we use (2) to substitute for $\phi\left(i,k\right)$.

Consumption per native, c, and investment, i, are control variables, while the capital per domestic worker, k, and foreign bonds, b, are state variables. By using λ as *numéraire*, it is possible to write the problem in terms of a new co-state variable, $q\equiv q'/\lambda$, which can be interpreted as the relative shadow price of the installed capital in terms of the (unitary) price of foreign bonds. The first order conditions with respect to c, i, b and k are given by:

$$e^{-\rho t}u'\left(c\right)-\lambda=0 \tag{5}$$

$$-\left(1+h\frac{i}{k}\right)+q=0 \tag{6}$$

$$\bar{r}=-\frac{\dot{\lambda}}{\lambda} \tag{7}$$

$$\frac{r_k}{q}+\frac{h}{2q}\left(\frac{i}{k}\right)^2-(\bar{r}+\delta)=-\frac{\dot{q}}{q}. \tag{8}$$

Equation (5) equates the marginal utility of consumption to the shadow value of wealth in the form of foreign bonds, λ. Equation (6) equates the marginal cost of an additional unit of investment in foreign bonds to the relative market value of capital, q. It may be re-arranged to yield:

$$i=k\left(\frac{q-1}{h}\right). \tag{9}$$

According to (9), the critical determinant of capital investment is the relative shadow price of the installed capital, q, the path of which is yet to be determined. Equation (7) is the standard Keynes–Ramsey consumption rule, equating the marginal return on consumption to the growth rate of the marginal utility of consumption. With $\bar{r}$ being constant, (7) implies a constant growth rate of marginal utility, λ. Equation (8) is the arbitrage condition between foreign bonds and domestic capital assets. Using (9) to substitute for the i/k ratio, it can be rewritten as follows:

$$\frac{r_k}{q}+\frac{\dot{q}}{q}+\frac{(q-1)^2}{2hq}=\bar{r}+\delta. \tag{10}$$

According to (10), at any point in time natives equate the rate of return on capital assets - left-hand side of (10) - to the rate of return on foreign bonds plus the rate of depreciation - right-hand side of (10). The rate of return on capital assets has three components. The first component is the rate of return of installed capital valued at the relative price q. The second, is the rate of capital gain. Finally, the third component measures the marginal reduction in adjustment costs valued at the relative price q. This component reflects the fact that an additional source of benefit from higher capital stock is a reduction in the installation costs associated with new investment.

Finally, the following pair of transversality conditions ensure that natives cannot increase their stock of capital or their stock of foreign bonds indefinitely:

$$\lim_{t\to\infty} q'k=0 \tag{11}$$

$$\lim_{t\to\infty} \lambda b=0. \tag{12}$$

This completes the description of the necessary and sufficient conditions for this utility maximization problem.

2.2.2 The domestic consumption function

At all points in time, domestic consumption is given by $C_n = cN$. Taking time derivative and using (5) and (7) yields:

$$\frac{\dot{C}_n}{C_n} = \frac{\dot{c}}{c} = \sigma^{-1}(\bar{r} - \rho) \equiv g_c. \tag{13}$$

where $\sigma \equiv u'(c)\, c/u''(c)$ is the inverse of the intertemporal elasticity of substitution.

According to (13), domestic consumption grows at a constant rate, g_c, which is determined by the taste parameters, σ and ρ, and the exogenous rate of return on foreign bonds $\bar{r}$. As the level of the interest rate is completely unrelated to the level of capitalization in the domestic economy, access to perfect world capital markets dichotomizes the dynamics of the model, implying that the time evolution of domestic consumption is totally independent of that of production. Indeed, by solving (13) forwardly, it is easy to verify that the time path of consumption per native does not show links with the production side of the economy and is given by:

$$c(t) = c(0)\, e^{g_c t}, \tag{14}$$

where the initial level of consumption, $c(0)$, will be determined later in the paper in accordance with the national solvency condition.

2.3 Production

The production side of the domestic economy consists of a continuum of perfectly competitive firms, whose total size is normalized to unity. At the aggregate level, the domestic output, Y, is determined by the aggregate capital stock, K, domestic workers, N, and foreign workers M. In order to accommodate growth under more general assumptions, we assume that the production technology available to all firms is of the Cobb–Douglas type and reads:

$$Y = A \cdot K^{\alpha} M^{\beta} N^{1-\alpha-\beta} \qquad \text{with} \;\; 0 < \alpha + \beta < 1,$$

where $A > 0$ is a scale parameter measuring the level of technology.

Let $m \equiv M/N$ be the immigration ratio. The linear homogeneity of the production function allows us to write:

$$y = A \cdot k^{\alpha} m^{\beta}, \tag{15}$$

where $y \equiv Y/N$ denotes the output per native. Perfect competition in the output market implies that all the production inputs are paid at their marginal cost. Consequently, profit maximization conditions read:

$$r_k = \frac{\partial Y}{\partial K} = \alpha A \cdot k^{\alpha-1} m^{\beta} \tag{16}$$

$$w_n = \frac{\partial Y}{\partial N} = (1 - \alpha - \beta)\, A \cdot k^{\alpha} m^{\beta} \tag{17}$$

$$w_m = \frac{\partial Y}{\partial M} = \beta A \cdot k^{\alpha} m^{\beta-1}. \tag{18}$$

It is worth noticing that, according to (17), natives and immigrants are (Edgeworth) complements in production. Indeed, by differentiating (17) with respect to m, it is easy to verify that an increase in the immigration ratio raises the wage rate of natives.[3] This result stems from the fact that both domestic and foreign labor are essential inputs in production, which is a standard assumption for neoclassical growth theory.[4]

2.4 The current account dynamics

As mentioned above, only natives are allowed to trade in foreign bonds. Consequently, by aggregating (4) over the N natives and using (16), (17) and (9) to substitute for r_k, w_n and i, the time evolution of the current account of the domestic economy is described by:

$$\dot{B} = \bar{r}B + (1-\beta)\, AN{\cdot}k^{\alpha}m^{\beta} - C_n - K\left(\frac{q^2-1}{2h}\right). \tag{19}$$

Equation (19) states that the rate at which the domestic economy accumulates foreign bonds equals the interest it is earning on their capital account $\bar{r}B$, plus its net export, $NX = Y - C - I\,(1 + hI/2K)$.[5] As is usual for small open economy models, in the remainder of the paper we focus on the special case of $\bar{r} = \rho$, according to which the internationally given interest rate is stationary and equal to the subjective discount rate of the native residents.

2.5 The migration function

In this section we introduce the migration function. In doing so, we assume that the choice of emigrating or not emigrating is assumed to depend on only the existence of a positive differential

[3] As Borjas (2000) points out, the impact on the wage rate of native residents of a permanent increase in the size of the immigrant workforce depends on whether immigrants and natives are complements or substitutes in production. More specifically, if immigrants and natives are complements, then a more intensive use of immigrant workers leads to a permanent increase in the wage bill paid to natives. On the contrary, if the two groups are substitutes, then a more intensive use of immigrant workers leads to a permanent decrease in the wage bill paid to natives.

[4] The bulk of literature on illegal migration and neoclassical growth is prone to consider domestic and foreign workers as perfect substitutes in production rather than complements. This paper, instead, focuses on the case in which domestic and foreign labor are essential inputs in production, with the result that natives and immigrants are "Edgeworth" complements in production. Such a property of the production function makes it hard to consider illegal something that is necessary, even crucial, for the existence of the whole economy. As a result, in the remainder of the paper immigration is always a legal phenomenon, leaving the analysis of the alternative case in which domestic and foreign workers are substitutes to future research.

[5] The national resource constraint requires $Y = C + I\,(1 + hI/2K) + NX$. Using (15) and (9) to get rid of Y and I yields: $NX = AN{\cdot}k^{\alpha}m^{\beta} - C - K{\cdot}\left(q^2-1\right)/2h$. At each instant t, aggregate consumption equals $C = cN + c_M M = N{\cdot}[c + w_m m]$. Therefore, using (18) to substitute for w_m obtains:

$$NX = (1-\beta)\, AN{\cdot}\beta A{\cdot}k^{\alpha}m^{\beta} - cN - kN{\cdot}\left(q^2-1\right)/2h.$$

between the wage rate currently paid in the domestic economy, w_m, and a reference wage currently paid in the country of origin, $\bar{w}$. For any given value of the reference wage, $\bar{w}$, migration towards the domestic country occurs if and only if the current wage bill paid to immigrants is larger than $\bar{w}$ - i.e. $w_m - \bar{w} > 0$. Likewise, migration towards the domestic country ceases if the current wage rate paid to immigrants is equal to the reference wage - i.e. $w_m - \bar{w} = 0$ -, and eventually inverts direction if the wage rate offered in the home country is larger than that paid in the domestic economy - i.e. $w_m - \bar{w} < 0$.

By using (18) to substitute for w_m, the flow of people in and out of the domestic economy is governed by the following differential equation:

$$\dot{M} = \eta \cdot \left(\beta A \cdot k^{\alpha} m^{\beta-1} - \bar{w}\right) M, \tag{20}$$

where $\eta > 0$ is an exogenous parameter capturing the sensitivity of migration to changes in wage differentials.[6]

Note that the presence of the immigration ratio, m, into the right-hand side of (20) stops people from abandoning the country of origin and catastrophically flow towards the domestic country. Obviously, this is not the only way to prevent people from emigrating catastrophically to the domestic country. Braun (1993), for instance, introduces diminishing returns to scale into the domestic economy by assuming that an increase in population congests a natural resource. Since all residents of the domestic economy - both natives and immigrants - have free access to this natural resource, in Braun's model the world economy does not depopulate in the steady state because, from a given point in time on, some foreign individuals will not find it convenient to move to the domestic economy because of congestion.[7] Similarly, in the present model smoothly-decreasing marginal products play a similar role and prevent people from completely depopulating their native soil.

2.6 The dynamic equilibrium

2.6.1 Characterization of the equilibrium and steady-state analysis.

A competitive equilibrium for this small open economy can be characterized as follows:

Definition 1: *Suppose the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. For any pair of initial conditions, $\{k(0), b(0)\}$, a competitive equilibrium for this dynamic small open economy is given by a set of time paths for the endogenous variables, $\{c(t), k(t), q(t), m(t), b(t)\}_{t\in(0,\infty)}$, a set of time paths for wages, $\{w_n(t), w_m(t)\}_{t\in(0,\infty)}$, and a time path for the rate of return on the installed capital $\{r_k(t)\}_{t\in(0,\infty)}$, such that:*

1. *natives maximize their discounted utility function (1) subject to the two accumulation constraints (3) and (4), the installation cost function (2) and the two transversality conditions (11) and (12);*

[6] Note that in this paper $\eta = 0$ is not allowed because of the essentiality property of the production function.
[7] See Barro and Sala-i-Martin (2004, Ch.9) for technical details.

2. *firms maximize profits subject to the technology constraint (15);*

3. *people migrate either in or out of the domestic economy according to the migration function (20).*

Based on (14), if $\bar{r} = \rho$ holds, then consumption per native is always constant over time and equal to $c(0)$. Thus, to completely describe the dynamics of the production side of the domestic economy it is necessary to consider the evolution of the capital stock, k, together with those of the relative shadow price of the installed capital, q, and the immigration ratio, m. To get these paths, we resort to equations (3), (10), and (20), which comprise, together with the investment function (9) and the consumption path (14), a closed-form solution for the model.

Using (9), (16), and (18) to get rid of i, r_k and w_m, the dynamics of the model are fully described by the following system of non-linear differential equations:

$$\dot{k} = k\left(\frac{q-1}{h}\right) - \delta k \tag{21}$$

$$\dot{q} = (\rho+\delta)\,q - \frac{(q-1)^2}{2h} - \alpha A \cdot k^{\alpha-1} m^{\beta} \tag{22}$$

$$\dot{m} = \eta \cdot \left[\beta A \cdot k^{\alpha} m^{\beta-1} - \bar{w}\right] m \tag{23}$$

where, in order to obtain the dynamic equation (23), we used $\dot{m}/m = \dot{M}/M$.[8]

System (21)-(23) comprises a triple of dynamic equations in k, q, and m, that evolve independently of the consumption expenditure of natives and the current account. In the system, k is a predetermined variable, while both q and m are jump (or non-predetermined) variables. The reason why we consider m as a jump variable is simple and draws on the definition of a non-predetermined variable.[9] As pointed out earlier, foreign workers decide to migrate or not migrate depending on whether the wage rate offered by domestic firms, w_m, is higher than the reference wage, $\bar{w}_0$. This makes the flow of immigrants a forward-looking variable, whose time behavior depends on whether individuals are able to correctly anticipate the time path of w_m. This implies that the immigration ratio, m, can instantaneously respond to changes in expectations due to changes in wage differentials between countries.

For both the capital stock and the immigration ratio to be constant over time, system (21)-(23) must have at least one rest point, attained when $\dot{k} = \dot{q} = \dot{m} = 0$. In the rest point, the domestic economy is said to be in a steady-state equilibrium, whose main properties are summarized as follows

Definition 2: *Suppose the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. A steady state for the domestic economy consists of a vector $\{k^*, q^*, m^*\}$ satisfying condition $\dot{k} = \dot{q} = \dot{m} = 0$.*

[8] Indeed, log-differentiating $m \equiv M/N$ with respect to t gives $\dot{m} = \left(\dot{M}/M - \dot{N}/N\right) m$. As the native population is stationary and $\dot{N} = 0$, this equation becomes: $\dot{m}/m = \dot{M}/M$. Finally, using (20) to substitute for $\dot{M}$ yields (23).

[9] For a better description of the distinction between predetermined and non-predetermined variables see, among others, Buiter (1982).

System (21)-(23) is block recursive. From (21) the steady-state relative shadow price of the installed capital is easily obtained and equal to:

$$q^* = 1 + h\delta. \tag{24}$$

Equation (24) shows that q^* exceeds 1 because adjustment costs are borne in the steady state for the gross investment to replace the capital worn out because of depreciation. Having determined q^* from this equation, the steady-state stock of capital per native worker, k^*, and the steady-state immigration ratio, m^*, are determined simultaneously from (22)-(23) to obtain:

$$k^* = \left(\frac{A \cdot \alpha^{1-\beta} \beta^{\beta}}{\vartheta^{1-\beta} \bar{w}^{\beta}} \right)^{1/(1-\alpha-\beta)} \tag{25}$$

$$m^* = \left(\frac{A \cdot \alpha^{\alpha} \beta^{1-\alpha}}{\vartheta^{\alpha} \bar{w}^{1-\alpha}} \right)^{1/(1-\alpha-\beta)} \tag{26}$$

where $\vartheta \equiv \rho (1 + h\delta) + \delta + h\delta^2/2 > 0$ is a collection of exogenous parameters.

In order to be viable, the steady state must satisfy the transversality condition (11). Straightforward computations show that (11) holds for all $\rho > 0$.[10]

2.6.2 Transitional dynamics

The dynamic system (21)-(23) is nonlinear. Consequently, to analyze the dynamic properties of the steady state we need to linearize the system around the steady-state vector, $\{k^*, q^*, m^*\}$. By using (24)-(26) to substitute for k^*, q^* and m^*, the linearized system can be written as:

$$\begin{pmatrix} \dot{k} \\ \dot{q} \\ \dot{m} \end{pmatrix} = J^* \cdot \begin{pmatrix} k - k^* \\ q - q^* \\ m - m^* \end{pmatrix} \tag{27}$$

where

$$J^* \equiv \begin{pmatrix} 0 & \frac{1}{h} \left(\frac{A \cdot \alpha^{1-\beta} \beta^{\beta}}{\vartheta^{1-\beta} \bar{w}^{\beta}} \right)^{\frac{1}{1-\alpha-\beta}} & 0 \\ (1-\alpha) \left[\frac{\vartheta^{2(1-\beta)-\alpha} \bar{w}^{1-\alpha}}{A \cdot \alpha^{1-\beta} \beta^{\beta}} \right]^{\frac{1}{1-\alpha-\beta}} & \rho & -\left(\frac{\vartheta^{1-\beta} \bar{w}^{1-\alpha}}{A \cdot \alpha^{\alpha} \beta^{\beta}} \right)^{\frac{1}{1-\alpha-\beta}} \\ \eta \beta \vartheta & 0 & -\eta (1-\beta) \bar{w} \end{pmatrix}$$

is the Jacobian matrix evaluated at the rest point, $\{k^*, q^*, m^*\}$.[11]

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[10] Solving (7) and (21) forwardly and then substituting the resulting equations into (11) yields:

$$\lim_{t \to \infty} q^* \lambda (0) e^{-\rho t} k (0) e^{[(q^*-1)/h - \delta]t} = 0.$$

Next, by using (24) to get rid of q^*, the previous expression boils down to:

$$\lim_{t \to \infty} (1 + h\delta) \lambda (0) k (0) e^{-\rho t} = 0.$$

As is easy to verify, as $t \to 0$ the limit approaches zero if and only if $\rho > 0$.

[11] See Appendix A.1 for the computational details.

For the competitive equilibrium to be saddle-path stable, the Jacobian matrix has to have two eigenvalues with positive real parts and one eigenvalue with negative real part. This is so because the initial condition of the predetermined variable, $k(0)$, is given, while the remaining two initial conditions for the two non-predetermined variables, $q(0)$ and $m(0)$, are free. In Appendix A.1 we show that the determinant of J^* is positive, while the trace of J^* can be either negative or positive depending on the size of the subjective discount rate of natives, ρ. This implies that the number of eigenvalues with positive real parts can be either one or three depending on the sign of the trace of J^*. Yet, by using the Descartes' Rule of Sign, it is possible to demonstrate that the number of eigenvalues with positive real parts is always one, and then that the steady state of the model is always locally indeterminate.

The following proposition summarizes the dynamic properties of the model:

Proposition 1 *Suppose that the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. Then, there exists an indeterminate steady state with a continuum of equilibrium trajectories indexed by initial immigration ratio, $m(0)$, each of which converges to the steady state given by (24)-(26).*

Proof. See Appendix A.1. ■

According to Proposition 1, the presence of perfect world capital markets makes the model exhibit local indeterminacy. This result is in sharp contrast to the closed-economy literature on international migration and macroeconomic dynamics, which predicts that the steady state is always determinate with a unique stable arm converging to the steady state. In contrast, our model predicts the existence of a continuum of stable arms, indexed by $m(0)$, each converging to the same steady-state equilibrium. This implies that, (*i*) while the transitional growth rates of the domestic economy are indeterminate, the long-run growth rate is always determinate and equates zero in the steady state; (*ii*) structurally identical economies can exhibit different speeds of convergence if their initial immigration ratio, $m(0)$, is not the same.

Having demonstrated that the model exhibits local indeterminacy, in the next section we will focus on the dynamic properties of domestic consumption and the current account.

2.6.3 Domestic consumption and the current account

Equation (19) describes the dynamics of the current account of the domestic economy. To study how they evolve over time, we write (19) in per native worker terms and linearize it around the steady state. The resulting differential equation reads:[12]

$$\dot{b} = \rho b + \theta(k^*, m^*) - c(0) + \Psi(0) \cdot e^{\mu_2 t} + \Upsilon(0) \cdot e^{\mu_3 t}, \tag{28}$$

where $\Psi(0)$ and $\Upsilon(0)$ are two collections of constant parameters - including $m(0)$-, while μ_2 and μ_3 are the two negative eigenvalues of the Jacobian matrix (27). The $\theta(k^*, m^*)$ term on

[12]See Appendix B for both the analytical details of the derivation of (28) as well as the compositions and the definitions of the two collections of exogenous parameters $\Psi(0)$ and $\Upsilon(0)$.

the right-hand side of (28) is the *steady-state net current income per native*. It is given by the difference between the *steady-state gross current income per native*, $(1-\beta) A \cdot (k^*)^{\alpha} (m^*)^{\beta}$, and capital adjustment costs, $\left[(q^*)^2 - 1\right] k^*/2h$.

By solving (28) forwardly and imposing national solvency through the transversality condition (12), the complete solution of (28) is given by:

$$b(t) = \left(b(0) + \frac{\Psi(0)}{\mu_2 - \bar{r}} + \frac{\Upsilon(0)}{\mu_3 - \bar{r}}\right) - \frac{\Psi(0)}{\mu_2 - \bar{r}} e^{\mu_2 t} - \frac{\Upsilon(0)}{\mu_3 - \bar{r}} e^{\mu_3 t}, \tag{29}$$

where, in order to avoid explosive paths for $b(t)$, the following initial condition for domestic consumption has been imposed:

$$c(0) = \rho \left[b(0) + \frac{\theta(k^*, m^*)}{\bar{r}} + \frac{\Psi(0)}{\mu_2 - \rho} + \frac{\Upsilon(0)}{\mu_3 - \rho}\right]. \tag{30}$$

According to (29), the current account is subject to transitional dynamics. To see this more succinctly, suppose that the domestic country is initially a creditor in world capital markets, such that $b(0) > 0$. Suppose also that at $t = 0$ the domestic economy is not in its steady state. As time goes by, the two exponential terms on the right-hand side of (29) approach zero and the net stock of foreign bonds, $b(t)$, asymptotically approaches the final term $b(0)$. Yet, even though the initial stock of foreign bonds, $b(0)$, is known at $t = 0$, the time path of $b(t)$ is indeterminate because of the presence of $\Psi(0)$ and $\Upsilon(0)$ on the right-hand side of (29). In fact, since both $\Psi(0)$ and $\Upsilon(0)$ depend on the initial level of the immigration ratio, $m(0)$, there exists a continuum of transitional paths for $b(t)$, each for any possible initial value of the immigration ratio, $m(0)$.

A similar consideration applies to the initial value of domestic consumption, $c(0)$. A glance at (30), indeed, shows that $c(0)$ crucially depends on the two parameter collections $\Psi(0)$ and $\Upsilon(0)$. We can therefore conclude that the initial value of domestic consumption is also indeterminate because of the presence of $m(0)$ in both $\Psi(0)$ and $\Upsilon(0)$.

The following proposition summarizes the results.

Proposition 2 *Suppose the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. Then, both the transitional path of the current account, $b(t)$, and the level of consumption per native, $c(0)$, are indeterminate and depend on the initial immigration ratio, $m(0)$.*

Proof. See Appendix A.3. ■

The results summarized by Proposition 2 constitute another novelty for the literature on international migration and macroeconomic dynamics. Nevertheless, since the steady state is locally indeterminate, the model does not lend itself to a clear-cut prediction about the macroeconomic implications of migration. For this reason, in the next section we will tackle the issue of the dynamic aspects of international migration numerically, through a series of calibration exercises.

2.6.4 Quantitative Analysis

In the previous sections we have studied the dynamic properties of the model without wondering what impact migration will have on the time paths of the main macroeconomic variables of the model. Unfortunately, indeterminacy does not allow us to perform a standard comparative dynamics analysis, so in this section we will study the adjustment dynamics of the model through a series of numerical simulations.

We begin by calibrating a benchmark economy using the set of parameters representative of a small open economy reported in Table 1. We then calibrate a baseline scenario, which occurs when all the endogenous variables are at their steady-state levels, including $m(0)$. Finally, we contrast the baseline scenario with two further alternative scenarios in which the initial immigration ratio, $m(0)$, is assumed to be, respectively, larger and smaller than its steady-state value, m^*.

Initial conditions	$k(0) = k^* = 2.39$
Preference parameters	$\rho = 0.04$, $\sigma = 1$
Technology parameters	$A = 1$, $\alpha = 0.3$, $\beta = 0.1$
External sector	$\bar{r} = \rho = 0.04$
Migration	$\eta = 0.05$, $\bar{w} = 0.4$, $m(0) = m^* = 0.2$
Capital accumulation	$h = 15$, $\delta = 0.05$

Table 1: Perfect world capital markets: Baseline values

In the simulation, we assume that $\bar{r} = \rho$ and set $\rho = 0.04$ so that the annual interest rate is roughly 4 percent (see Table 1). The Arrow-Pratt relative-aversion risk index $\sigma = 1$ is chosen in order to resemble logarithmic preferences. We choose the technology parameters $A = 1$, $\alpha = 0.3$ and $\beta = 0.1$ to match the empirical evidence of Gollin (2002), and the rate of depreciation $\delta = 0.05$ consistent with Cooley (1995). The choice of adjustment costs $h = 15$ is consistent with Ortigueira and Santos (1997), who prove that values of h in the range of 10 to 16 are in line with a plausible speed of convergence. Finally, we set the elasticity parameter $\eta = 0.05$ and the reference wage $\bar{w} = 0.5528$, so that, based on (26), the steady-state immigration ratio, m^*, is roughly 20 percent.

The adjustment paths are depicted in Figure 1. The continuous line represents the benchmark case $m(0) = m^*$, while the discontinuous lines represent the two alternative scenarios: (a) $m(0) > m^*$, depicted by the dotted line; (b) $m(0) < m^*$, depicted by the dashed line.

Consider first scenario (a), in which the immigration ratio, m, is initially larger than its steady-state value. In such a scenario, foreign labor is relatively abundant, implying that the wage bill paid to natives is higher than its steady-state value and the wage bill paid to

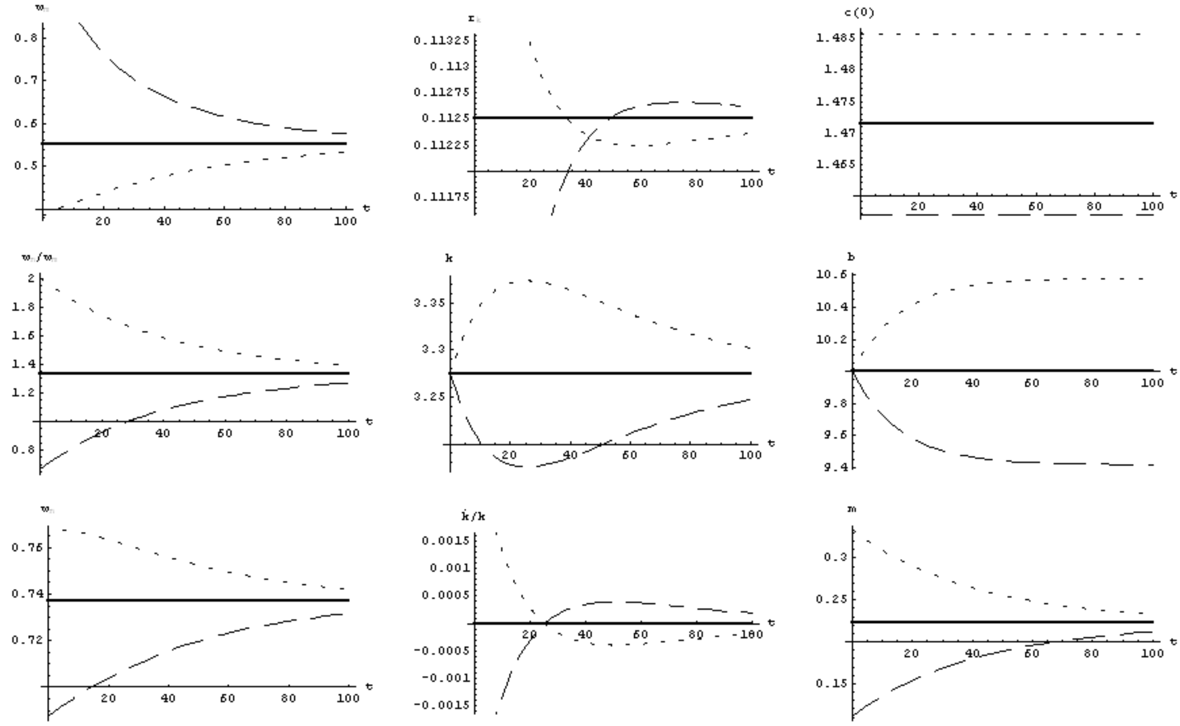

Figure 1: The basic model without borrowing constraints: Impulse response functions. Scenario (a): dashed lines; Scenario (b): dotted lines.

immigrants is lower than both its steady-state value and the reference wage, $\bar{w}$. In this case, the immigration ratio converges to the steady state from above, implying that during the transition the immigration rate is always negative, $\dot{m} < 0$, and that the wage rates move in the opposite directions so that wage inequality, measured by the index w_n/w_m, shrinks.

The abundance of foreign labor positively affects consumption per native, which turns out to be always larger than its baseline value. Not surprisingly, this result stems from the fact that the perfect capital markets assumption dichotomizes the model, by making domestic consumption totally independent of domestic production.

As regards capital accumulation, the gradual decrease in the immigration ratio has a double effect on the transitional path of capital per domestic worker. Firstly, since initially $m(0) > m^*$ and the marginal product of capital, r_k, is higher than its long-run value, the rate of return on capital assets - left-hand side of (10) - turns out to be higher than the rate of return on foreign bonds plus the rate of depreciation - right-hand side of (10). This spurs investment and provides an incentive for natives to increase the size of domestic capital per domestic worker. We refer to this effect as the *capitalization effect*, which leads to an increase in the stock of capital per domestic worker over time. Secondly, the gradual reduction in the migration ratio, m, makes the marginal product of capital, r_k, fall over time. This affects the rate of return on capital assets - left-hand side of (10) - negatively and it induces natives to reduce the domestic investment in capital accumulation. This effect is a *migration effect*, which causes the stock of capital per domestic worker to shrink over time.

The *capitalization effect* goes in the opposite direction of the *migration effect.* Consequently, the net effect on the dynamics of capital per domestic worker depends on whether the former is able to prevail over the latter. As the time path of k is hamp-shaped (see Figure 1), the *capitalization effect* initially dominates the *migration effect*, thereby leading to a temporary increase in the stock of capital per worker. However, as the immigration ratio approaches its long-run equilibrium value, the decreasing returns to capital and labor diminish the *capitalization effect* and increase the *migration effect.* Thus, after a few periods, the *migration effect* dominates the *capitalization effect* and k begins to fall monotonically to its steady-state value, k^*.

Notice that the initial increase in capital assets investment goes hand in hand with an increase in foreign bonds investment. This is so because the initial abundance of foreign labor sets both domestic output and national saving higher than their steady-state values. This induces natives to increase their stocks of foreign bonds and capital assets initially. Consequently, during the transition a part of saving flows out of the domestic country, so that, in the steady state, the domestic economy experiences a permanent increase in the stock of foreign bonds.

Consider now scenario (b), which occurs when the immigration ratio, m, is initially smaller than its steady-state value. In such a scenario, foreign labor is relatively scarce, implying that the wage bill paid to natives is lower than its equilibrium value and the wage bill paid to immigrants is higher than both its equilibrium value and the reference wage, $\bar{w}$. In this case, the immigration ratio converges to the steady state from below, implying that along the transition path the immigration rate is always positive, $\dot{m} > 0$, and that the wage rates of natives and immigrants adjust dynamically, so wage inequality diminishes over time.

In contrast with scenario (a), the shortage of foreign labor negatively affects both domestic consumption, which turns out to be always smaller than its baseline value, and capital per worker. Once again, the fall in domestic consumption is permanent, while that in capital per worker is only temporary. In particular, if the economy experiences an initial shortage of foreign labor, the adjustment dynamics of the capital stock, k, are reversed, meaning that the time path of k is initially downward sloping and then upward sloping.

The economic explanation for this U-shaped behavior of the variable k is similar to that of the previous scenario and is mainly due to the different combination of the *capitalization effect* and the *migration effect.* In contrast to scenario (a), in fact, *the migration effect* is initially positive, but it is dominated by the *capitalization effect*, which is instead negative. This causes the stock of capital per domestic worker to fall initially. In the long-run, though, the inflow of new immigrants leads to the *migration effect* prevailing over the *capitalization effect.* When this occurs, the time path of capital reaches its bottom and then starts growing over time towards its equilibrium value.

As regards the stock of foreign bonds, in this case the dynamics of the variable b are reversed. Initially, the fall in capital assets investment is accompanied by a fall in foreign bonds investment. This is so because the initial shortage of foreign labor causes both the domestic output and the national saving to be lower than their steady-state values. This induces natives to sell both foreign bonds and capital assets initially. However, in contrast with the domestic capital stock, whose initial reduction is only temporary, the fall in the stock of foreign bonds is permanent.

The following proposition summarizes the dynamic properties of the model

Proposition 3 *(Numerical) Suppose the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. Suppose also that at $t = 0$ all endogenous variables are at their steady-state values except the immigration ratio, m. Then, the adjustment dynamics of the economy depend on whether the immigration ratio, m, is initially larger or smaller than its long-run equilibrium value (26). In particular, it is possible to distinguish the following dynamics:*

(a) $m(0) > m^*$; if the immigration ratio is initially larger than its steady-state value (26), the adjustment dynamics of the economy imply: (i) a permanent increase in domestic consumption, c; (ii) a temporary increase in the stock of capital per domestic worker, k; (iii) a permanent fall in wage inequality, $\frac{w_n}{w_m}$; (iv) a permanent increase in the stock of foreign bonds per native, b.

(b) $m(0) < m^*$; if the immigration ratio is initially lower than its steady-state value (26), the adjustment dynamics of the economy imply: (i) a temporary fall in domestic consumption. c; (ii) a permanent fall in the stock of capital per domestic worker, k; (iii) a permanent increase in wage inequality, $\frac{w_n}{w_m}$; (iv) a permanent fall in the stock of foreign bonds per native, b.

The dynamic properties described by Proposition 3 rely on the assumption that natives can borrow as much as they want at a fixed interest rate. This assumption is obviously too strong, especially if one thinks of the common practice in capital markets of linking the interest rate to the amount of debt. Consequently, in the next section we provide a natural extension of the basic model to the case in which the domestic economy faces an upward-sloping curve of debt.

3 The model economy with borrowing restrictions

This section extends the model of Section 2 to the case in which the interest rate is no longer exogenously fixed but depends on the amount of debt. In doing so, we assume that the domestic economy is a net-debtor economy - i.e. $B < 0$ - and that the cost of borrowing is endogenously given and equal to:

$$r_b \equiv \bar{r} + e^{\xi\left(\frac{B}{K}\right)} - 1, \tag{31}$$

where $\xi \geq 0$ is the borrowing premium. Based on (31), the interest rate, r_b, increases with the nation's debt-capital ratio, B/K, provided that the borrowing premium, ξ, is strictly larger than zero.[13]

The representative native takes r_b as predetermined and seeks to maximize the utility function (1), subject to the capital accumulation equation (3), and the new foreign bonds accumulation equation:

[13] Observe that the previous case of perfect world capital markets can be viewed as a particular case of this model occurring when the borrowing premium disappears, $\xi = 0$, so that $r \equiv \bar{r}$ always holds.

$$\dot{b} = r_b b + c + i\left[1 + \frac{h}{2}\left(\frac{i}{k}\right)\right] - w_n - r_k k, \tag{32}$$

where, once again, (2) has been used to substitute for $\phi(i,k)$. In contrast to (4), (32) is written from the standpoint of a debtor and asserts that the stock of debt of the representative native, b, increases over time if and only if domestic consumption, c, investment expenses, $i\left[1+\frac{h}{2}\left(\frac{i}{k}\right)\right]$, plus the outstanding interest payments, $r_b b$, exceed his net current revenue, $w_n + r_k k$.

While the optimality conditions with respect to c and i remain unchanged and given by (5) and (6) - with the latter implying (9) -, the optimality conditions with respect to b and k change to incorporate the borrowing premium (31) and become:

$$\frac{\dot{c}}{c} = \sigma^{-1}\left[\bar{r} + e^{\xi\left(\frac{b}{k}\right)} - 1 - \rho\right] \tag{33}$$

$$\frac{\alpha A \cdot k^{\alpha-1} m^{\beta}}{q} + \frac{(q-1)^2}{2hq} + \frac{\dot{q}}{q} = \bar{r} + e^{\xi\left(\frac{b}{k}\right)} - 1 + \delta. \tag{34}$$

Equation (33) is the Euler equation in the case of an upward-sloping supply curve of debt. In contrast to (13), consumption is no longer independent of the production side of the economy since the interest rate is now positively related to the debt to capital ratio, b/k. As in the case of (8), equation (34) can be viewed as the new arbitrage equation between foreign bonds and domestic capital assets, in which the depreciation-adjusted rate of returns on foreign bonds, given by the right-hand side of (34), change to take into account the dependence of the interest rate on the debt to capital ratio, b/k.

By plugging (16), (17), (9) and (31) into (32), the dynamics of the net foreign debt is governed by the following differential equation:

$$\dot{b} = \left[\bar{r} + e^{\xi\left(\frac{b}{k}\right)} - 1\right] b + c + \left(\frac{q^2-1}{2h}\right) k - (1-\beta) A \cdot k^{\alpha} m^{\beta}. \tag{35}$$

The competitive equilibrium can be characterized as follows:

Definition 3: *Suppose the world capital markets are imperfect, such as that the domestic economy faces an upward-sloping supply curve of debt as that described by (31). Then, for any pair of initial conditions,* $\{k(0), b(0)\}$, *a competitive equilibrium for this dynamic small open economy is given by a set of time paths* $\{c(t), k(t), q(t), m(t), b(t)\}_{t\in(0,\infty)}$, *a pair of input prices* $\{w_n(t), w_m(t)\}_{t\in(0,\infty)}$ *and a pair of rates of return on capital assets and foreign bonds* $\{r_k(t), r_b(t)\}_{t\in(0,\infty)}$, *such that:*

1. *natives maximize their discounted utility function (1) subject to the two accumulation constraints (3) and (32), the installation cost function (2), and the two transversality conditions (12) and (11);*

2. *firms maximize profits according to production technology (15);*

3. *people migrate either in or out of the domestic economy according to the migration function (20).*

To analyze the dynamics of the domestic economy, we need to consider how consumption per native, c, capital per domestic worker, k, and external debt per native b, evolve over time, together with the relative shadow price of the installed capital, q, and the immigration ratio, m. To derive these paths, we resort to equations (3), (20), (33), (34) and (35), which we hereby summarize in the following algebraic system:

$$\dot{c} = \sigma^{-1}\left[\bar{r} + e^{\xi\left(\frac{b}{k}\right)} - 1 - \rho\right]c \tag{36}$$

$$\dot{k} = k\left(\frac{q-1}{h}\right) - \delta k \tag{37}$$

$$\dot{q} = \left[\bar{r} + e^{\xi\left(\frac{b}{k}\right)} - 1 + \delta\right]q - \frac{(q-1)^2}{2h} - \alpha A{\cdot}k^{\alpha-1}m^{\beta} \tag{38}$$

$$\dot{m} = \eta\left[\beta A{\cdot}k^{\alpha}m^{\beta-1} - \bar{w}\right]m \tag{39}$$

$$\dot{b} = \left[\bar{r} + e^{\xi\left(\frac{b}{k}\right)} - 1\right]b + c + \left(\frac{q^2-1}{2h}\right)k - (1-\beta)\,A{\cdot}k^{\alpha}m^{\beta}. \tag{40}$$

In the system, k and b are the predetermined variables, while c, q and m are the jump variables. In order for consumption, c, capital per domestic worker, k, net foreign debt b, to be stable over time, and for the relative price of installed capital, q, and the immigration ratio, m, to be constant over time, the dynamic system (36)-(40) must present at least one rest point, attained when $\dot{c} = \dot{k} = \dot{b} = \dot{q} = \dot{m} = 0$. In the rest point, the domestic economy is said to be in its steady state, which we define as follows

Definition 4: *Suppose the world capital markets are imperfect, such that the domestic economy faces an upward-sloping supply curve of debt such as that described by (31). A steady state for the domestic economy consists of a vector* $\{c^*, k^*, b^*, q^*, m^*\}$ *satisfying condition* $\dot{c} = \dot{k} = \dot{b} = \dot{q} = \dot{m} = 0$.

By imposing the steady-state condition $\dot{c} = \dot{k} = \dot{b} = \dot{q} = \dot{m} = 0$ on the dynamic system (36)-(40), the corresponding steady-state vector $\{c^*, k^*, b^*, q^*, m^*\}$ is the solution of the following steady-state system:

$$\bar{r} + e^{\xi\frac{b^*}{k^*}} - 1 = \rho \tag{41}$$

$$\frac{q^*-1}{h} = \delta \tag{42}$$

$$\alpha A{\cdot}\left(k^*\right)^{\alpha-1}\left(m^*\right)^{\beta} = \left(\bar{r} + e^{\xi\frac{b^*}{k^*}} - 1 + \delta\right)q^* - \frac{\left(q^*-1\right)^2}{2h} \tag{43}$$

$$\beta A{\cdot}\left(k^*\right)^{\alpha}\left(m^*\right)^{\beta-1} = \bar{w} \tag{44}$$

$$c^* = A{\cdot}\left(k^*\right)^{\alpha}\left(m^*\right)^{\beta} - k^*\delta\left(1 + \frac{h\delta}{2}\right) - \left(\bar{r} + e^{\xi\frac{b^*}{k^*}} - 1\right)b^* - m^*\bar{w}. \tag{45}$$

where ϑ is the same collection of exogenous parameters as in Section 3.1.

System (41)-(45) has a simple recursive structure. First, the steady-state shadow price of capital, q^*, is determined by (42). As is easy to verify, its value is the same as (24), meaning that the regime of the world capital market does not affect the shadow price of capital. Next, by plugging (24) and (41) into (43) and then solving the resulting equation simultaneously with (44), we obtain a closed-form solution for both the steady-state stock of capital per domestic worker, k^*, and for the steady-state immigration ratio, m^*. Once again, their values are the same as in (24) and (26), so we can conclude that the regime of the world capital market affects neither the steady-state level of the stock of capital per domestic worker, nor the steady-state level of the immigration ratio.

Finally, with k^*, q^*, m^* and b^* in hand, the steady-state debt to capital ratio, b^*, and the steady-state consumption per native, c^*, are determined by (41) and (45) respectively.

3.1 Transitional dynamics

The dynamic system (36)-(40) is too complex to be studied analytically. Consequently, to analyze its local stability properties we resort to numerical simulations. Our main interest is to study the sign of the real parts of the complex eigenvalues of the Jacobian matrix and check whether the introduction of capital market imperfections affects the indeterminacy result outlined in Proposition 1.

Since indeterminacy arises when $\xi = 0$, our objective is to verify whether the sign of the real parts of the eigenvalues changes when the borrowing premium, ξ, is allowed to vary.

Initial conditions	$k(0) = k^* = 2.39$, $b(0) = b^* = -0.16$
Preference parameters	$\rho = 0.04$, $\sigma = 1$
Technology parameters	$A = 1$, $\alpha = 0.3$, $\beta = 0.1$
External sector	$\bar{r} = \rho = 0.04$, $\xi = 0.15$
Migration	$\eta = 0.05$, $\bar{w} = 0.4$, $m(0) = m^* = 0.2$
Capital accumulation	$h = 15$, $\delta = 0.05$

Table 2: Imperfect capital markets: Baseline values

To set all the exogenous parameters, we use the same parametrization shown in Table 1 and allow ξ to vary within a range of values (see Table 2). For the range of variation of ξ, we set the lower bounding of the range equal to zero - which is the case of perfect capital markets -, and the upper bounding equal to $\xi = 0.3$ - which is twice the reference value used by the literature for small open economy in their simulation exercises (see Turnovsky, 2005).

ξ	μ_1	μ_2	μ_3	μ_4	μ_5
0	$+$	$-$	$-$	$n.a.$	$n.a$
0.05	$+$	$-$	$-$	$-$	$+$
0.10	$+$	$-$	$-$	$-$	$+$
0.15	$+$	$-$	$-$	$-$	$+$
0.20	$+$	$-$	$-$	$-$	$+$
0.25	$+$	$-$	$-$	$-$	$+$
0.30	$+$	$-$	$-$	$-$	$+$

Table 3: Simulation results

The simulation results are reported in Table 3. The first column of the table reports the values of the borrowing premium, ξ, while the remaining five columns report the values of the real parts of the complex eigenvalues. In the table, all five eigenvalues are real-valued, with three of them negative and the remaining two positive. Since the number of stable roots is larger than the number of predetermined variables, Table 3 confirms the indeterminacy result of the previous section. Thus, even in the presence of a borrowing constraint, there exists a continuum of converging paths in the neighborhood of the steady state depending on the initial level of the immigration ratio, $m(0)$. This implies that the dynamics of the model do not change in the presence of an endogenously determined interest rate and is robust to changes in the regime of the world capital markets.

The following proposition summarizes these results

Proposition 4 *(Numerical) Suppose the world capital markets are imperfect, such as that the domestic economy faces an upward-sloping supply curve of debt as that described by (31). Then, the Jacobian matrix has three eigenvalues with negative real parts and two eigenvalues with positive real parts. The steady state is locally indeterminate.*

Proof. See Appendix B ■

3.2 Quantitative analysis

This section extends the quantitative analysis of Section 2.8 to the case in which the domestic economy faces an upward-sloping supply curve of debt. In particular, in this section we are interested in testing whether the dynamic properties of the basic model - mostly those regarding domestic consumption, capital accumulation and net foreign debt - are robust to changes in the regime of the world capital markets. In doing so, we adopt the same two-step simulation strategy of Section 2.8, which consists in first calibrating a baseline scenario in which the immigration ratio is initially at its steady-state value, m^*, then in contrasting the baseline scenario with two further alternative scenarios in which the initial immigration ratio, $m(0)$, is assumed to be, respectively, larger and smaller than its steady-state value. In all simulations we set $\xi = 0.15$, so

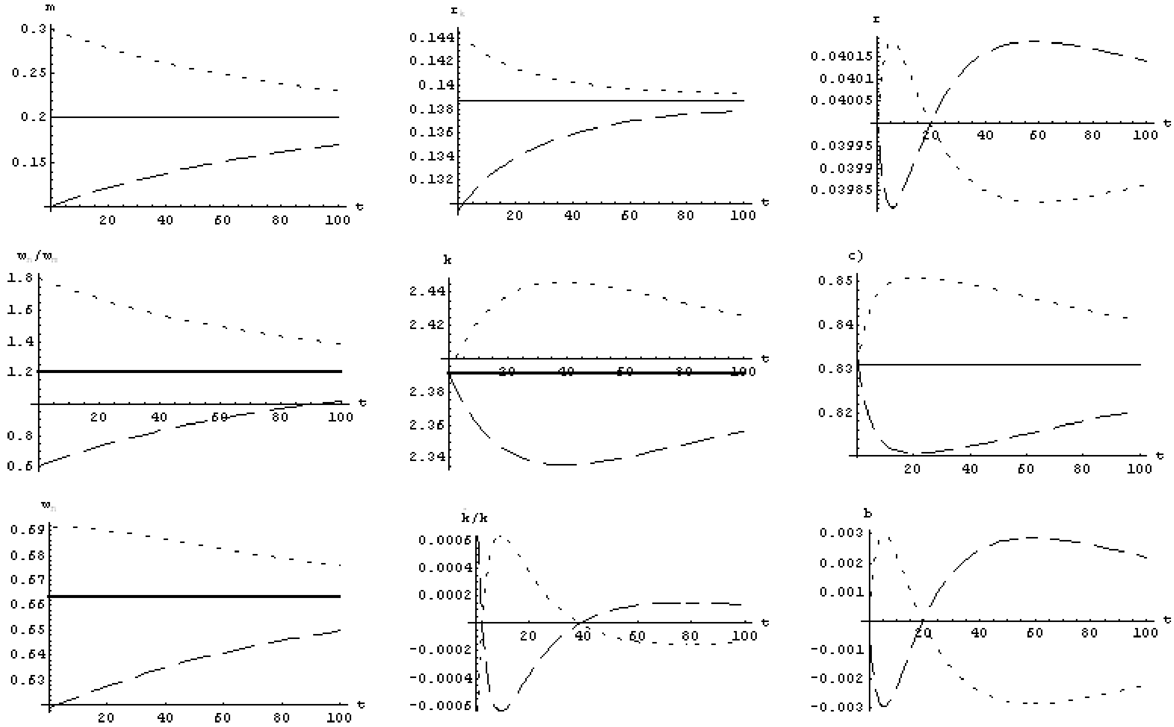

Figure 2: The basic model with borrowing constraints: Impulse response functions. Scenario (a): dotted lines; Scenario (b): dashed lines.

that the speed of convergence is in line with the small open economy literature (see Turnovsky, 2005), and $\bar{r} = \rho = 0.04$, so that the steady-state stock of foreign bonds in the domestic economy is always equal to zero.

The adjustment paths are depicted in Figure 2. For many variables, the adjustment dynamics depicted in Figure 2 are similar to those depicted in Figure 1, except for domestic consumption, c, and the net national debt, b, whose adjustment paths are now only temporarily affected by the initial level of the immigration ratio, m.

We start by considering scenario (a), which occurs when the level of the immigration ratio is initially higher that its long-run equilibrium value. In contrast to the case of perfect world capital markets, in the presence of an upward-sloping supply curve of debt the impact of changes in the initial immigration ratio, $m(0)$, on domestic consumption is only temporary. In fact, if the immigration ratio is initially larger than its long-run value, the time path of domestic consumption is hamp-shaped, meaning that it initially increases and then decreases towards its steady-state value.

The non-linear behavior of c can be explained as follows. Initially, the wage rate of domestic workers, w_n, is lower than its steady-state value, while the marginal product of capital, r_k, is higher than its steady-state value. However, as m approaches m^*, both w_n and r_k begin to approach their steady-state values. This causes the debt to capital ratio to fall and the interest rate to decrease initially, thereby causing domestic consumption to rise further because of the lower cost of borrowing. The strong increase in domestic consumption causes the net financial position of the domestic economy to deteriorate at first; i.e. the net national debt per native, b, increases initially. Thus, as the pace of capital accumulation starts falling and the cost of debt rising, natives adjust their intertemporal consumption pattern by gradually reducing c. In the long-run, both domestic consumption and capital per domestic worker approach their steady-state values, c^* and k^*, from above, while the stock of the net national debt converges towards its starting level, b^*, from below.[14]

We now turn to scenario (b), which occurs when the initial level of the immigration ratio is lower than its long-run equilibrium value. If the immigration ratio is initially lower than its long-run value, the time path of domestic consumption is U-shaped, meaning that c initially decreases and then increases to approach its long-run equilibrium value, c^*. In this case, the low immigration ratio causes the wage rate of domestic workers, w_n, to be higher than their steady-state value and the marginal product of capital, r_k, to be lower than their steady-state value. As w_n and r_k adjust to approach their steady states, both domestic consumption and capital per domestic worker fall initially, thereby making the debt to capital ratio rise and the interest rate increase. This, in turn, causes domestic consumption to decrease further because of the higher cost of borrowing and the net financial position of the domestic economy to improve initially; i.e. the net national debt per native, b, decreases initially. However, as the pace of capital accumulation starts to increase and the cost of debt to fall, natives adjust their intertemporal consumption pattern by gradually increasing c. As domestic consumption and capital per worker approach their long-run value, c^* and k^*, from below, the net national debt increases over time and converges towards its starting level, b^*, from above.

The following proposition summarizes all the results

Proposition 5 *(Numerical) Suppose the world capital markets are imperfect, such that the domestic economy faces an upward-sloping supply curve of debt such as that described by (31). Suppose further that at* $t = 0$ *all endogenous variables are at their steady-state values, except the immigration ratio, m. Then, the adjustment dynamics of the economy depend on whether the immigration ratio, m, is initially larger or smaller than its long-run equilibrium value. In particular, it is possible to distinguish the following dynamics:*

(a) $m(0) > m^*$*; if the immigration ratio is initially larger than its steady state value (26), the adjustment dynamics of the economy imply:* (i) *a temporary increase in domestic consumption, c;* (ii) *a temporary increase in the stock of capital per domestic worker, k;* (iii) *a permanent fall in wage inequality,* $\frac{w_n}{w_m}$*;* (iv) *a temporary increase in the stock of foreign bonds per native, b.*

(b) $m(0) < m^*$*; if the immigration ratio is initially lower than its steady state value (26), the adjustment dynamics of the economy imply:* (i) *a temporary fall in domestic consumption.*

[14] Being a high-order dynamic system, the model allows for nonlinear adjustment paths, meaning that some endogenous variables - such as the net national debt in Figure 2 - may overshoot their respective steady-state values.

c; (ii) a temporary fall in the stock of capital per domestic worker, k; (iii) a permanent increase in wage inequality, $\frac{w_n}{w_m}$*; (iv) a temporary fall in the stock of foreign bonds per native, b.*

4 Centrally-planned economy

4.1 Characterization of the equilibrium

As a final exercise, this section addresses the question of whether the competitive equilibrium of Section 2 is Pareto optimal or not. In doing so, we assume that the domestic economy is governed by a benevolent social planner, who chooses consumption and investment, c and i, and the rates of accumulation of capital assets and foreign bonds, $\dot{k}$ and $\dot{b}$, directly so as to maximize the utility function (1), subject to the accumulation equation (3) and the aggregate resource constraint of the economy:

$$\dot{b} = \bar{r}b + (1-\beta)\, A{\cdot}k^{\alpha}m^{\beta} - c - \phi\,(i,k) \tag{46}$$

where the installation cost function, $\phi\,(i,k)$, is still given by (2).

The discounted Hamiltonian for this problem reads:

$$H = e^{-\rho t}u\,(c) + \lambda\left[\bar{r}b + (1-\beta)\, A{\cdot}k^{\alpha}m^{\beta} - c - i\left(1+\frac{h}{2}\frac{i}{k}\right)\right] + q'\,(i-\delta k)$$

where λ is the shadow value of wealth in the form of internationally traded bonds and q' is the shadow value of wealth in the form of installed capital.

The first order conditions with respect to c, i, b and k are given by:

$$e^{-\rho t}u'\,(c) - \lambda = 0 \tag{47}$$

$$-\left(1+h\frac{i}{k}\right) + q = 0 \tag{48}$$

$$\bar{r} = -\frac{\dot{\lambda}}{\lambda} \tag{49}$$

$$\frac{(1-\beta)\,\alpha A{\cdot}k^{\alpha-1}m^{\beta}}{q} + \frac{h}{2q}\left(\frac{i}{k}\right)^2 - (\bar{r}+\delta) = -\frac{\dot{q}}{q}. \tag{50}$$

where, once again, we used λ as *numéraire* so as to rewrite the whole problem in terms of the new co-state variable, $q \equiv q'/\lambda$.

Notice that the optimality conditions with respect to consumption, private investment and foreign bonds, (47), (48) and (49) are identical to the corresponding conditions (5), (6) and (7) of the basic model. As a result, we can solve (48) for i and get the same investment function (9), and combine (47) and (49) to get the same domestic consumption function (14). The growth rate of domestic consumption of the centrally-planned economy therefore coincides with that of the market economy, though its level will be different due to the fact that the determinants of $c\,(0)$ are no longer the same.

The optimality condition with respect to the installed capital per worker, (50), is now modified and can be rearranged to get:

$$(1-\beta)\frac{\alpha A{\cdot}k^{\alpha-1}m^{\beta}}{q}+\frac{\dot{q}}{q}+\frac{q^2-1}{2hq}=\bar{r}+\delta. \tag{51}$$

Equation (51) is the arbitrage condition between the foreign bonds and domestic capital assets of the social planner. Mathematically, (51) is identical to (10) except for the presence of the term $1-\beta$ on the rate of return of installed capital (first term on the left-hand side). Since $\beta<1$, the rate of return of installed capital is lower in the case of a centrally-planned economy. This is so because the social planner internalizes the fact that at each point in time a share of national resources are consumed by immigrants.

Equations (47)-(50), along with two transversality conditions identical to (11) and (12), complete the description of the necessary and sufficient conditions of the social planner's utility maximization problem. By letting "O" denote optimal paths, a dynamic equilibrium for this centrally-planned small open economy can be defined as follows:

Definition 5: *Suppose the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. For any pair of initial conditions, $\{k_O(0), b_O(0)\}$, a dynamic equilibrium for this centrally-planned small open economy is given by a set of time paths for the endogenous variables, $\{c_O(t), k_O(t), q_O(t), m_O(t), b_O(t)\}_{t\in(0,\infty)}$, such that:*

1. *the social planner maximizes the discounted flow of utility (1) subject to the aggregate resource constraint (46), the accumulation constraints (3), the installation cost function (2), and the two transversality conditions (11) and (12);*

2. *people migrate either in or out of the domestic economy according to the migration function (20).*

Using (9) to get rid of i, the dynamics of this small open economy are summarized by the following system of three non-linear differential equations:

$$\dot{k}_O=k_O\left(\frac{q_O-1}{h}\right)-\delta k_O \tag{52}$$

$$\dot{q}_O=(\rho+\delta)\,q_O-\frac{(q_O-1)^2}{2h}-\alpha(1-\beta)\,A{\cdot}k_O^{\alpha-1}m_O^{\beta} \tag{53}$$

$$\dot{m}_O=\eta{\cdot}\left[\beta A{\cdot}k_O^{\alpha}m_O^{\beta-1}-\bar{w}\right]m_O. \tag{54}$$

A steady state is constituted by a rest point of system (52)-(54). Thus, imposing the steady-state condition $\dot{k}_O=\dot{q}_O=\dot{m}_O=0$ yields:

$$q_O=1+h\delta \tag{55}$$

$$k_O=\left[\frac{A{\cdot}(1-\beta)^{1-\beta}\alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\bar{w}^{\beta}}\right]^{1/(1-\alpha-\beta)} \tag{56}$$

$$m_O = \left[\frac{A \cdot (1-\beta)^{\alpha} \alpha^{\alpha} \beta^{1-\alpha}}{\vartheta^{\alpha} \bar{w}^{1-\alpha}} \right]^{1/(1-\alpha-\beta)} . \tag{57}$$

The steady-state solutions (55)-(57) represent the first best resource allocation for the domestic economy. Notice that the solution for the optimal shadow price of capital, (55), is identical to that of the competitive economy, (5). This result is not surprising. Indeed, since - via (6) and (48) - the investment function did not change because of the presence of the social planner, the pace of capital accumulation of the two economies - i.e. the centrally-planned economy and the competitive economy - has to be the same. This means that the equilibrium shadow price of the social planner and of the decentralized economy must coincide, implying that the private incentive of natives to invest in capital assets is equal to that of the social planner.

As regards the two remaining endogenous variables, k_O and m_O, the difference between the first best values, (56) and (57), and the competitive equilibrium values, (25) and (26), is due to the presence of the term $1-\beta$ in the numerator of both (56) and (57). Since $\beta < 1$, long-run equilibrium value of the immigration ratio of the decentralized economy, m^*, is larger than the first best solution, m_O. This means that the market economy tends to have more immigrants in the equilibrium. By the same token, long-run equilibrium value of capital per domestic worker of the decentralized economy, k^*, is larger than its optimal value, k_O, meaning that in the competitive economy there will be an over-accumulation of capital in the steady state.

The economic explanation for this result is not difficult to grasp. When deciding about capital investment, natives do not take into account that a share of the national resources is currently used to pay immigrants. On the one hand, this causes both the level of investment, i^*, and the stock of capital per domestic worker, k^*, to be larger than the first-best values. On the other, over-accumulation of capital increases the wage bill paid to immigrants with respect to the first best level, thereby leading to an overflow of foreign workers into the domestic economy.

Summing up

Proposition 6 *Suppose the internationally given interest rate, $\bar{r}$, is stationary and equal to the subjective discount rate of natives, ρ. Then, the competitive equilibrium is not Pareto optimal and presents the following characteristics:*

1. *the steady-state level of the capital per domestic worker of the competitive economy is higher than the first best, $k^* > k_O$;*

2. *the steady-state immigration ratio of the competitive economy is higher than the first best, $m^* > m_O$.*

4.2 Policy implications

In the previous section we saw that the market economy is incapable of reaching the first best. The government can pursue two different policies to fix the market's inefficiencies. The first policy consists in correcting private incentive to accumulate capital assets through the introduction of a tax on capital earnings equal to $\tau_k \in (0,1)$. The second policy consists in adjusting the immigration ratio through the introduction of an immigration fee, $\tau_m > 0$. To avoid distorcionary effects, the proceeding of taxation are supposed to be rebated to natives as lump-sum transfers.

Let us consider the first policy. The presence of a tax on capital earnings modifies the natives' arbitrage condition between capital assets and foreign bonds, which becomes:

$$\frac{(1-\tau_k)\, r_k}{q} + \frac{\dot{q}}{q} + \frac{(q-1)^2}{2hq} = \bar{r} + \delta.$$

Though the steady-state shadow price of capital does not change because of capital taxation, the two remaining steady-state solutions, (25) and (26), change accordingly and read:

$$\tilde{k} = \left[\frac{A\cdot (1-\tau_k)^{1-\beta}\, \alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\bar{w}^{\beta}}\right]^{1/(1-\alpha-\beta)} \tag{58}$$

$$\tilde{m} = \left[\frac{A\cdot (1-\tau_k)^{\alpha}\, \alpha^{\alpha}\beta^{1-\alpha}}{\vartheta^{\alpha}\bar{w}^{1-\alpha}}\right]^{1/(1-\alpha-\beta)}, \tag{59}$$

where the "tilda" denotes steady-state values in the presence of capital taxation.

The first best is established if the government sets τ_k such that $k^* = \tilde{k}$ and $m^* = \tilde{m}$. By using (25) and (26) to replace k^* and $\tilde{k}$ and (58) and (59) to substitute for $\tilde{k}$ and $\tilde{m}$, it is easy to verify that the first best allocation can be established if and only if the optimal tax rate on capital earnings is equal to:

$$\tau_k = \beta.$$

Consider now the second policy. As said, it consists in introducing an immigration fee, τ_m, acting as a deterrent for migration. It can also be thought of as kind of "ticket" that each immigrant needs to pay at each moment in time in order to be considered a legal immigrant.

The introduction of the immigration fee modifies the migration function, which becomes

$$\dot{M} = \eta\cdot \left[\beta A\cdot k^{\alpha} m^{\beta-1} - (1+\tau_m)\, \bar{w}\right] M.$$

Once again, the steady-state shadow price of capital does not change because of the immigration fee, while the two remaining steady-state solutions, (25) and (26), change accordingly and read:

$$\hat{k} = \left[\frac{A\cdot\alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\,(1+\tau_m)^{\beta}\, \bar{w}^{\beta}}\right]^{1/(1-\alpha-\beta)} \tag{60}$$

$$\hat{m} = \left[\frac{A\cdot\alpha^{\alpha}\beta^{1-\alpha}}{\vartheta^{\alpha}\,(1+\tau_m)^{1-\alpha}\, \bar{w}^{1-\alpha}}\right]^{1/(1-\alpha-\beta)}, \tag{61}$$

where the "hat" denotes steady-state values in the presence of the immigration fee.

The first best is established if the government sets τ_m such that $k^* = \hat{k}$ and $m^* = \hat{m}$. However, using (25) and (26) to substitute for k^* and m^*, and (60) and (61) to substitute for $\hat{k}$ and $\hat{m}$, it is easy to verify that there does not exist a unique optimal fee capable of ensuring

$k^* = \hat{k}$ and $m^* = \hat{m}$ simultaneously. Indeed, by solving $k^* = \hat{k}$ and $m^* = \hat{m}$ separately, it turns out that there exists two possible fees (respectively $\tau_m^1 = (1-\beta)^{\beta/(\beta-1)} - 1$ and $\tau_m^2 = (1-\beta)^{\alpha/(\alpha-1)} - 1$), both incapable of getting the first best level of both k and m, except in the particular case in which the parameter restriction, $\alpha = \beta$, holds. This result leads us to the conclusion that taxing immigration cannot be viewed as effective as taxing capital earnings to establish the first best allocation.

5 Concluding remarks

In this paper we have developed a dynamic small open economy model to study the macroeconomic effects of international migration. In the model, individuals are supposed to be identical in all but their nationality and move from one country to another because of the existence of international differences in wages. The paper focuses on the standpoint of a host country and can be ideally split into two parts according to whether the world capital markets are assumed to be perfect or imperfect.

In the first part of the paper, we focus on the case of perfect world capital markets, in which the cost of borrowing does not depend on the level of the external debt of the domestic country. We find that the steady-state equilibrium is indeterminate, so there exists a continuum of converging transitional paths depending on the initial value of the immigration ratio. To study the dynamic implication of migration on domestic consumption, foreign debt and capital accumulation, we simulate the model by distinguishing three different scenarios depending on whether the immigration ratio is larger, equal or smaller than its long-run equilibrium value. We find that changes in the initial level of the immigration ratio have only a temporary effect on capital accumulation, whereas they have permanent effects on domestic consumption, net national debt and wage inequality.

In the second part of the paper, we extend the basic model to the case in which the cost of borrowing increases with the level of the external debt. Due to the complexity of the dynamic system, the stability analysis is carried out numerically and considers the borrowing premium as the key parameter differentiating the two capital market regimes. We find that the indeterminacy result is robust to changes in the world capital markets regime, but also that the presence of an upward-sloping curve of debt modifies the adjustment dynamics of the model. More specifically, we find that changes in the initial level of the immigration ratio have only a temporary impact on domestic consumption, net national debt and capital accumulation, while they have permanent impacts on wage inequality.

Finally, we conclude the paper by investigating the normative properties of the basic model. In doing so, we adopt the Social planner's approach and use the steady-state solutions of the centrally-planned economy as a benchmark to check whether the competitive equilibrium is also Pareto optimal. We find that in the decentralized equilibrium natives overaccumulate capital and the domestic economy overattracts immigrants. This result occurs because natives do not take into account the impact that their investment decisions have on the wage bill of immigrants. Thus, to establish the first best the government can pursue two alternative policies:

taxing capital earnings or introducing an immigration tax. Surprisingly, we find that the only policy that is effective in establishing the first best is taxing capital earnings.

Appendix A

A.1 *Linearization of the dynamic system and stability analysis*

This appendix provides the analytical details of the stability analysis as well as the formal proof of Proposition 1. First-order Taylor expansion around the steady-state vector, $\{k^*, q^*, m^*\}$, gives:

$$\begin{pmatrix} \dot{k} \\ \dot{q} \\ \dot{m} \end{pmatrix} = J\cdot \begin{pmatrix} k-k^* \\ q-q^* \\ m-m^* \end{pmatrix}$$

where the matrix of coefficients is given by:

$$J = \begin{pmatrix} \frac{q^*-1}{h}-\delta & \frac{k^*}{h} & 0 \\ \alpha(1-\alpha)A\cdot(k^*)^{\alpha-2}(m^*)^{\beta} & \bar{r}+\delta-\frac{q^*-1}{h} & \alpha\beta A\cdot(k^*)^{\alpha-1}(m^*)^{\beta-1} \\ \alpha\beta\eta A\cdot(k^*)^{\alpha-1}(m^*)^{\beta} & 0 & \beta\eta A\cdot(k^*)^{\alpha}(m^*)^{\beta-1}-\eta\bar{w}_o \end{pmatrix}$$

where $\vartheta \equiv \rho(1+h\delta)+\delta+h\delta^2/2$. By using the steady-state conditions (24)-(26), the matrix of coefficients can be simplified to:

$$J^* = \begin{pmatrix} 0 & \frac{1}{h}\left(\frac{A\cdot\alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\bar{w}^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} & 0 \\ (1-\alpha)\left[\frac{\vartheta^{2(1-\beta)-\alpha}\bar{w}^{1-\alpha}}{A\cdot\alpha^{1-\beta}\beta^{\beta}}\right]^{\frac{1}{1-\alpha-\beta}} & \rho & -\left(\frac{\vartheta^{1-\beta}\bar{w}^{1-\alpha}}{A\cdot\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} \\ \eta\beta\vartheta & 0 & -\eta(1-\beta)\bar{w} \end{pmatrix}$$

The eigenvalues of J^* are the solution of its characteristic equation:

$$-\mu^3+\mathrm{Tr}J^*\cdot\mu^2-B\cdot\mu+\mathrm{Det}J^*=0, \tag{A.1}$$

where:

$$\mathrm{Tr}J^* = \rho-\eta(1-\beta)\bar{w} \tag{A.2}$$

$$B = |J^*_{11}|+|J^*_{22}|+|J^*_{33}| = -\frac{(1-\alpha)\vartheta+\rho h\eta(1-\beta)\bar{w}}{h} < 0 \tag{A.3}$$

$$\mathrm{Det}J^* = \frac{(1-\alpha-\beta)\vartheta\eta\bar{w}_o}{h} > 0 \tag{A.4}$$

and $\left|J^*_{j'j}\right|$ is the minor of J^* formed by removing from A its j'th row and jth column.

According to Routh-Horwitz theorem, the sign of the real parts of the roots of (A.1) depends on the following Theorem:

Theorem 1 : *The number of roots of (A.1) with positive real parts is equal to the number of variations of sign in the scheme:*

$$-1, \qquad \mathrm{Tr}J^*, \qquad -B+\frac{\mathrm{Det}J^*}{\mathrm{Tr}J^*}, \qquad \mathrm{Det}J^*. \tag{S}$$

Proof. According to the Routh-Hurwitz criterion, the table of coefficients associated to the characteristic polynomial *(A.1)* reads:

$$\begin{bmatrix} -1 & -B & 0 \\ \mathrm{Tr}J^* & \mathrm{Det}J^* & 0 \\ -B+\frac{\mathrm{Det}J^*}{\mathrm{Tr}J^*} & 0 & \\ \mathrm{Det}J^* & 0 & \end{bmatrix}$$

Thus, the number of eigenvalues with positive real parts is given by the number of sign changes in the first column. While the sign of the determinant of the matrix of coefficients is positive, the sign of the trace is ambiguous and depends on exogenous parameters. To apply the criterion, we must know the sign of (A.2) as well as the sign of $-B+\frac{\mathrm{Tr}J^*}{\mathrm{Det}J^*}$. Using (A.3), (A.2) and (A.4) to substitute for B, $\mathrm{Tr}J^*$ and $\mathrm{Det}J^*$ yields:

$$-B+\frac{\mathrm{Det}J^*}{\mathrm{Tr}J^*}=\frac{(1-\alpha)\,\vartheta+\rho h\eta\,(1-\beta)\,\bar{w}}{h}+\frac{(1-\alpha-\beta)\,\vartheta\eta\bar{w}_o}{h\left[\rho-(1-\beta)\,\eta\bar{w}_o\right]}. \tag{A.6}$$

The sign of $\mathrm{Tr}J^*$ depends on the size of interest rate, ρ. If $\rho > (1-\beta)\,\eta\bar{w}_o$, the trace of the matrix of coefficients is positive and the sign of (A.6) is unambiguously positive. If $\rho < (1-\beta)\,\eta\bar{w}_o$, the trace of the matrix of coefficients is negative while the sign of (A.6) is ambiguous and depends on the size of ρ. More specifically, the following table summarizes the number of sign changes in scheme (S):

Restrictions	-1	$\mathrm{Tr}J^*$	$-B+\frac{\mathrm{Tr}J^*}{\mathrm{Det}J^*}$	$\mathrm{Det}J^*$
$\rho > (1-\beta)\,\eta\bar{w}_o$	$-$	$+$	$+$	$+$
$\rho < (1-\beta)\,\eta\bar{w}_o$ and $(1-\beta)\,\eta\bar{w}_o > \rho > \tilde{\rho}$	$-$	$-$	$+$	$+$
$\rho < (1-\beta)\,\eta\bar{w}_o$ and $(1-\beta)\,\eta\bar{w}_o > \tilde{\rho} > \rho$	$-$	$-$	$-$	$+$

As is easy to check, the number of sign changes is always one, implying that the number of eigenvalues with positive real parts is one. ■

Proof of Proposition 1: From (A.4), we know that the determinant of J^* is positive, which is equal to saying that the product of the three eigenvalues is positive. This means that the number of eigenvalues with positive real parts is either three or one. To have three eigenvalues with positive real terms, there must be three changes of sign in scheme (S). However, Theorem 1 proves that the number of changes of sign in scheme (S) is always one, implying that J^* has only one eigenvalue with positive real part. Since both the relative price of the installed capital, q, and the immigration ratio, m, can jump instantaneously, while the stock of installed capital, k, is constrained to adjust continuously, we conclude that the number of unstable roots is less than the number of jump variables and that there exists a continuum of converging path to the steady-state equilibrium (24)-(26), one for each possible initial condition for the jump variable m.■

The equilibrium paths: Once demonstrated that J^* has one unstable and two stable eigenvalues, we are now ready to write down the time paths of the three endogenous variables, k, q and m. Let's $\mu_1 > 0$ denote the eigenvalue with positive root and $\mu_3 < \mu_2 < 0$ denote the two eigenvalues with negative roots. The complete solution of system (21)-(23) can be written as:

$$\begin{bmatrix} k(t) \\ q(t) \\ m(t) \end{bmatrix} = \begin{bmatrix} k^* \\ q^* \\ m^* \end{bmatrix} + B_1 \begin{bmatrix} 1 \\ v_{21} \\ v_{31} \end{bmatrix} e^{\mu_1 t} + B_2 \begin{bmatrix} 1 \\ v_{22} \\ v_{32} \end{bmatrix} e^{\mu_2 t} + B_3 \begin{bmatrix} 1 \\ v_{23} \\ v_{33} \end{bmatrix} e^{\mu_3 t},$$

where B_1, B_2 and B_3 are arbitrary constants and the vector $(1, v_{2\omega}, v_{3\omega})'$ (where the prime denotes vector transpose) is the normalized eigenvector associated with the eigenvalue ω.

Transversality condition (*11*) implies $B_1 = 0$. Thus, by taking $t \to 0$ yields:

$$\begin{bmatrix} k(0) - k^* \\ q(0) - q^* \\ m(0) - m^* \end{bmatrix} = B_2 \begin{bmatrix} 1 \\ v_{22} \\ v_{32} \end{bmatrix} + B_3 \begin{bmatrix} 1 \\ v_{23} \\ v_{33} \end{bmatrix}. \tag{A.7}$$

The arbitrary constants B_2, and B_3, appearing in (A.7), are obtained from initial conditions. Suppose the domestic economy starts out with a given initial stock of installed capital, $k(0)$, and with a given initial relative supply of immigrant labor, $m(0)$. By solving for the two arbitrary constant B_2 and B_3, and the initial condition, $q(0)$, yields:

$$B_2 = \frac{(k(0) - k^*) v_{33} - (m(0) - m^*)}{v_{33} - v_{32}}$$

$$B_3 = \frac{(m(0) - m^*) - (k(0) - k^*) v_{32}}{v_{33} - v_{32}}$$

$$q(0) = q^* - \left(\frac{v_{33} v_{22} - v_{32}^2}{v_{33} - v_{32}} \right) (k(0) - k^*) - \left(\frac{v_{32} - v_{22}}{v_{33} - v_{32}} \right) (m(0) - m^*).$$

Both the two arbitrary constants B_2, and B_3, and the initial relative price, $q(0)$, depend upon the specific shocks $(k(0) - k^*)$, and $(m(0) - m^*)$, and the elements of the two eigenvectors $v_{2\omega}$, and $v_{3\omega}$. Both $v_{2\omega}$ and $v_{3\omega}$ satisfy:

$$\begin{pmatrix} -\mu_\omega & \frac{1}{h} \left(\frac{A \cdot \alpha^{1-\beta} \beta^\beta}{\vartheta^{1-\beta} \bar{w}^\beta} \right)^{\frac{1}{1-\alpha-\beta}} & 0 \\ (1-\alpha) \left[\frac{\vartheta^{2(1-\beta)-\alpha} \bar{w}^{1-\alpha}}{A \cdot \alpha^{1-\beta} \beta^\beta} \right]^{\frac{1}{1-\alpha-\beta}} & \rho - \mu_\omega & -\left(\frac{\vartheta^{1-\beta} \bar{w}^{1-\alpha}}{A \cdot \alpha^\alpha \beta^\beta} \right)^{\frac{1}{1-\alpha-\beta}} \\ \eta\beta\vartheta & 0 & -\eta(1-\beta)\bar{w} - \mu_\omega \end{pmatrix} \begin{bmatrix} 1 \\ v_{2\omega} \\ v_{3\omega} \end{bmatrix} = 0,$$

with $\omega = \{2, 3\}$.

By solving for $v_{2\omega}$ and $v_{3\omega}$ yields:

$$v_{2\omega} = \mu_\omega h \left(\frac{A \cdot \alpha^{1-\beta} \beta^\beta}{\vartheta^{1-\beta} \bar{w}^\beta} \right)^{-\frac{1}{1-\alpha-\beta}} > 0 \qquad \wedge \qquad v_{3\omega} = \frac{\eta\beta\vartheta}{\eta(1-\beta)\bar{w} + \mu_\omega} > 0.$$

Consequently, the equilibrium paths of the three endogenous variables crucially depends on $m(0)$ and read:

$$k(t) = k^* + \left[\tfrac{(k(0)-k^*)v_{33}-(m(0)-m^*)}{v_{33}-v_{32}}\right] e^{\mu_2 t} - \left[\tfrac{(m(0)-m^*)-(k(0)-k^*)v_{32}}{v_{33}-v_{32}}\right] e^{\mu_3 t} \tag{A.8}$$

$$q(t) = q^* + \left[\tfrac{(k(0)-k^*)v_{33}-(m(0)-m^*)}{v_{33}-v_{32}}\right] v_{22} e^{\mu_2 t} - \left[\tfrac{(m(0)-m^*)-(k(0)-k^*)v_{32}}{v_{33}-v_{32}}\right] v_{23} e^{\mu_3 t} \tag{A.9}$$

$$m(t) = m^* + \left[\tfrac{(k(0)-k^*)v_{33}-(m(0)-m^*)}{v_{33}-v_{32}}\right] v_{32} e^{\mu_2 t} - \left[\tfrac{(m(0)-m^*)-(k(0)-k^*)v_{32}}{v_{33}-v_{32}}\right] v_{33} e^{\mu_3 t} \tag{A.10}$$

A.3 *The steady-state dynamics of the current account*

In this appendix we determine the equilibrium path of the current account. In doing so, we follow Eicher and Turnovsky (1999) and restrict attention to the special case in which the interest rate is stationary and equal to the subjective discount rate of natives, $\bar{r} = \rho$. By dividing both sides of (19) by N yields:

$$\dot{b}(t) = b(k(t), q(t), m(t)), \tag{B.1}$$

where:

$$b(k(t), q(t), m(t)) \equiv \rho b(t) + (1-\beta) A \cdot k(t)^{\alpha} m(t)^{\beta} - c(0) - \frac{q(t)^2 - 1}{2h} k(t).$$

First-order Taylor expansion around the steady-state vector, $\{k^*, q^*, m^*\}$, gives:

$$\dot{b}(t) = b(k^*, q^*, m^*) + \tfrac{\partial b(k^*,q^*,m^*)}{\partial k}(k(t)-k^*) + \tfrac{\partial b(k^*,q^*,m^*)}{\partial q}(q(t)-q^*) + \tfrac{\partial b(k^*,q^*,m^*)}{\partial m}(m(t)-m^*)$$

which can be rewritten as:

$$\begin{aligned} \dot{b}(t) \quad = \quad & \rho b(t) + (1-\beta) A \cdot (k^*)^{\alpha} (m^*)^{\beta} - c(0) - \tfrac{(q^*)^2-1}{2h} k^* \\ & \left[\alpha(1-\beta) A \cdot (k^*)^{\alpha-1} (m^*)^{\beta} - \tfrac{(q^*)^2-1}{2h}\right](k(t)-k^*) - \\ & -\tfrac{q^* k^*}{h}(q(t)-q^*) - \beta(\beta-1) A \cdot (k^*)^{\alpha} (m^*)^{\beta-1}(m(t)-m^*). \end{aligned} \tag{B.2}$$

By using time path (A.8), (A.9) and (A.10) to substitute for $k(t)$, $q(t)$ and $m(t)$, equation (B.2) becomes:

$$\dot{b}(t) = \rho b(t) + \theta(k^*, m^*) - c(0) + \Psi(0) \cdot e^{\mu_2 t} + \Upsilon(0) \cdot e^{\mu_3 t}.$$

where

$$\theta(k^*, m^*) \equiv (1-\beta) A \cdot (k^*)^{\alpha} (m^*)^{\beta} - \tfrac{(q^*)^2-1}{2h} k^*$$

$$\Psi(0) \equiv \left[(1-\beta) A \cdot (k^*)^{\alpha-1} (m^*)^{\beta}\left(\alpha + \tfrac{\beta v_{32} k^*}{m^*}\right) - \tfrac{(q^*)^2-1}{2h} - \tfrac{q^* k^* v_{22}}{h}\right]\left[\tfrac{(k(0)-k^*)v_{32}-(m(0)-m^*)}{v_{33}-v_{32}}\right]$$

$$\Upsilon(0) \equiv \left[(1-\beta) A \cdot (k^*)^{\alpha-1} (m^*)^{\beta}\left(\alpha + \tfrac{\beta v_{33} k^*}{m^*}\right) - \tfrac{(q^*)^2-1}{2h} + \tfrac{q^* k^* v_{23}}{h}\right]\left[\tfrac{(m(0)-m^*)-(k(0)-k^*)v_{32}}{v_{33}-v_{32}}\right].$$

Starting from a given initial stock, $b(0)$, the solution of (B.2) is:

$$
\begin{aligned}
b(t) &= e^{\rho t}\left(b(0)+\frac{\theta(k^*,m^*)}{\rho}+\frac{\Psi(0)}{\mu_2-\rho}+\frac{\Upsilon(0)}{\mu_3-\rho}-\frac{c(0)}{\rho}\right)- \\
&\quad -\frac{\theta(k^*,m^*)}{\rho}+\frac{c(0)}{\rho}+\frac{\Psi(0)}{\mu_2-\rho}e^{\mu_2 t}+\frac{\Upsilon(0)}{\mu_3-\rho}e^{\mu_3 t}.
\end{aligned}
$$

As $\rho>0$, No-Ponzi scheme requires the following initial value for consumption:

$$
c(0)=\rho\left(b(0)+\frac{\theta(k^*,m^*)}{\rho}+\frac{\Psi(0)}{\mu_2-\rho}+\frac{\Upsilon(0)}{\mu_3-\rho}\right),
$$

which in turn leads the following complete solution:

$$
b(t)=\left(b(0)+\frac{\Psi(0)}{\mu_2-\bar{r}}+\frac{\Upsilon(0)}{\mu_3-\bar{r}}\right)-\frac{\Psi(0)}{\mu_2-\bar{r}}e^{\mu_2 t}-\frac{\Upsilon(0)}{\mu_3-\bar{r}}e^{\mu_3 t}.
$$

Appendix B

B.1 *Linearization of the dynamic system*

This appendix provides the analytical details of the stability analysis as well as the formal proof of Proposition 3. First-order Taylor expansion around the steady-state vector, $\{k^*,q^*,m^*,b^*,c^*\}$, gives:

$$
\begin{pmatrix} \dot{c} \\ \dot{k} \\ \dot{q} \\ \dot{m} \\ \dot{b} \end{pmatrix} = J\cdot \begin{pmatrix} c-c^* \\ k-k^* \\ q-q^* \\ m-m^* \\ b-b^* \end{pmatrix}.
$$

The matrix of coefficients, J, is given by:

$$
\begin{bmatrix}
\frac{\bar{r}+e^{\xi\left(\frac{b^*}{k^*}\right)}-1-\rho}{\sigma} & -\frac{\xi e^{\xi\left(\frac{b^*}{k^*}\right)}c^*b^*}{\sigma(k^*)^2} & 0 & 0 & \frac{e^{\xi\left(\frac{b^*}{k^*}\right)}c^*}{\sigma k^*} \\
0 & \frac{q^*-1}{h}-\delta & \frac{k^*}{h} & 0 & 0 \\
0 & -\frac{\xi e^{\xi\left(\frac{b^*}{k^*}\right)}q^*z^*-\alpha(1-\alpha)y^*}{(k^*)^2} & a_{33} & -\alpha\beta\frac{y^*}{k^*m^*} & \frac{e^{\xi\left(\frac{b^*}{k^*}\right)}q^*}{k^*} \\
0 & \eta\alpha\beta\frac{y^*}{k^*} & 0 & \eta\beta^2\frac{y^*}{m^*}-\eta\bar{w}_o & 0 \\
1 & -\frac{2h\xi e^{\xi\left(\frac{b^*}{k^*}\right)}+(k^*)^2\left[(q^*)^2-1\right]-2h\alpha(1-\beta)\frac{y^*}{k^*}}{2h(k^*)^2} & \frac{k^*q^*}{h} & -\beta(1-\beta)\frac{y^*}{m^*} & a_{55}
\end{bmatrix},
$$

where:

$$
y^* \equiv A\cdot(k^*)^{\alpha}(m^*)^{\beta}
$$

$$
a_{33} \equiv \bar{r}+e^{\xi\left(\frac{b^*}{k^*}\right)}-1-\frac{q^*-1}{h}+\delta
$$

$$a_{55} \equiv \bar{r} + e^{\xi\left(\frac{b^*}{k^*}\right)}\left(1 + \xi\frac{b^*}{k^*}\right) - 1.$$

By using the steady-state conditions (41)-(45), the matrix of coefficients can be rewritten as follows:

$$\begin{bmatrix} 0 & \zeta\frac{\phi}{\xi} & 0 & 0 & (1-\bar{r}+\rho)\phi \\ 0 & 0 & \frac{1}{h}\left(\frac{A\cdot\alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\bar{w}^{\beta}}\right)^{\frac{1}{1-\alpha-\beta}} & 0 & 0 \\ 0 & \frac{[(1-\alpha)\vartheta-(1+h\delta)\zeta]}{\left(\frac{A\cdot\alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\bar{w}^{\beta}}\right)^{-\frac{1}{1-\alpha-\beta}}} & \rho & -\left(\frac{\bar{w}^{1-\alpha}\vartheta^{1-\beta}}{A\alpha^{\alpha}\beta^{\beta}}\right)^{\frac{\beta}{1-\alpha-\beta}} & \frac{(1+h\delta)(1-\bar{r}+\rho)}{\left(\frac{\bar{w}^{\beta}\vartheta^{1-\beta}}{A\alpha^{1-\beta}\beta^{\beta}}\right)^{-\frac{\beta}{1-\alpha-\beta}}} \\ 0 & \beta\eta\vartheta & 0 & -\eta(1-\beta)\bar{w} & 0 \\ 1 & -\frac{\left[\delta\left(1+\frac{h\delta}{2}\right)-(1-\beta)\vartheta\right]\xi+\zeta\log(1-\bar{r}+\rho)}{\xi} & \frac{\left(\frac{1+h\delta}{h}\right)}{\left(\frac{A\cdot\alpha^{1-\beta}\beta^{\beta}}{\vartheta^{1-\beta}\bar{w}^{\beta}}\right)^{-\frac{1}{1-\alpha-\beta}}} & -(1-\beta)\bar{w} & \rho+\zeta \end{bmatrix}$$

where

$$\zeta \equiv (1-\bar{r}+\rho)\log(1-\bar{r}+\rho)$$

$$\phi \equiv (\alpha\sigma)^{-1}\left\{\left[\alpha\delta\left(1+\frac{h\delta}{2}\right) - (1-\beta)\vartheta\right]\xi + \alpha\rho\log(1-\bar{r}+\rho)\right\}.$$

Given the complexity of the Jacobian, the local stability analysis is not feasible analytically. Consequently, in we examine the local stability properties of the model numerically using the parametrization provided by Table 2. Simulations are conducted by using Mathematica 5. The results reported in Table B.1:

ξ	μ_1	μ_2	μ_3	μ_3	μ_3
0.00	0.1025	−0.0641	−0.0233	n.a.	n.a.
0.05	0.3103	−0.2771	−0.0493	−0.0225	0.0837
0.10	0.3535	−0.3203	−0.0496	−0.0225	0.0840
0.15	0.3918	−0.3587	−0.0498	−0.0225	0.0843
0.20	0.4265	−0.3934	−0.0499	−0.0225	0.0844
0.25	0.1958	−0.1619	−0.0470	−0.0224	0.0807
0.30	0.2597	−0.2263	−0.0487	−0.0225	0.0829

Table B.1: Simulation results

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