Stability, efficiency and monotonicity in two-sided matching

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ABSTRACT. we propose a fairness property called P-monotonicity that we would like a matching mechanism to satisfy. We show that it is impossible to have a mechanism which is both stable and P-monotonic. Moreover, we show that it is impossible to have a mechanism which is both efficient and P-monotonic.

KEYWORDS: Stability, Efficiency, Monotonicity, Two-Sided Matching.

JEL CLASSIFICATION: C78.

1. INTRODUCTION

Since the initiation of the theory of two-sided matching by Gale and Shapley (1962), it has been successfully used in designing real world markets like labor and education markets[1]. A central solution concept used in the literature is stability. A matching mechanism is stable if no agent from any side of the market is matched to unacceptable allocation and if there exist no pair of agents who would prefer to be matched to each other than to their current assigned allocations. Stability is also empirically important since Roth (2002) finds that markets which adopted stable matching mechanisms have mostly succeeded while those who adopted not stable matching mechanisms have mostly failed. Nevertheless, there are some impossibility results about the existence of a matching mechanism that satisfies stability and other required properties like strategy proofness (Roth 1982), Non bossiness (Kojima 2010)

[1]For student assignment systems reforms, see Abdulkadiroglu and Sönmez (2003), Abdulkadiroglu, Pathak, and Roth (2005) for the US, Biró (2008) for Hungary, and Selim and Salem (2010) for Egypt.

and non damaging bossiness (Matsubae 2010). In this note, we propose another fairness property that we would like matching mechanisms to satisfy, which is P-monotonicity. It simply requires that if a student s becomes more *popular*[2] for at least one college c, then s and c should not end up with a worse allocation[3]. Our proposed notion is closely related to Balinski and Sönmez (1999) notion of *respecting improvement.* A mechanism respects improvement if a student s is ranked higher by any college c, then s becomes weakly better off[4]. The incentive for proposing P-monotonicity is that if we take into account the possibility that c might end with a worse allocation than this may invite c to strategically manipulate its preference list[5], but if we require that c becomes weakly better off then it has no incentive to misreport its preference list. Unfortunately, it turns out that it is impossible to have a mechanism satisfying stability and P-monotonicity. Moreover, it is impossible to have a mechanism satisfying efficiency and P-monotonicity.

2. Model

We consider a one-to-one matching problem between students and colleges. Let S and C be two finite disjoint sets of students and colleges, respectively. Each student $s \in S$ has a strict preference relation $\succ_s$ over $C \cup \{s\}$, similarly Each college $c \in C$ has a strict preference relation $\succ_c$ over $S \cup \{c\}$. Let $\succ = (\succ_i)_{i \in S \cup C}$ denotes the set of all possible preferences for i. A *matching* is a mapping μ from $S \cup C$ to $S \cup C$ such that: (i) for each $c \in C$, $\mu(c) = s \cup \{c\}$ and for each $s \in S$, $\mu(s) = c \cup \{s\}$. (ii) for each $(s, c) \in S \times C$, $\mu(c) = s$ if and only if $\mu(s) = c$. A matching μ is said to be *individually rational* if for all $i \in S \cup C$ either $\mu(i) \succ_i i$ or $\mu(i) = i$. A matching μ is *blocked* by a pair $(s, c) \in S \times C$ if $c \succ_s \mu(s)$ and $s \succ_c \mu(c)$. If μ is not blocked and individually rational, then it is *stable*. Also, a matching is efficient[6] if there exists no matching μ' such that $\mu(i)' \succsim_i \mu(i)$ for all $i \in S \cup C$, and there exists $\mu(i)' \succ_i \mu(i)$ for at least one $i \in S \cup C$. We denote the set of all possible matchings over $S \cup C$ by $\mathcal{M}$. A *matching mechanism* is a function ϕ mapping from $\succ$ to $\mathcal{M}$. A matching mechanism ϕ is *stable* if for any preference profile it produces a stable matching $\phi(\succ)$. Gale and Shapley (1962) show that there will always exist a stable matching mechanism and they proposed the Deferred acceptance algorithm to find it[7]. Similarly, A matching mechanism ϕ is *efficient* if for any preference profile it produces

[2]Becoming more popular means that she moves at least one rank in the preference list of at least one college.

[3]Note that the definition is symmetric between students and colleges.

[4]Balinski and Sönmez (1999) show that the student optimal stable mechanism respects improvement.

[5]See Roth (1982) for the manipulability of stable mechanisms and Alcalde and Barberà (1994) and Sönmez (1994) for the manipulability of efficient mechanisms.

[6]In a Pareto sense.

[7]See Roth (2008) for a survey on the Deferred acceptance algorithm .

an efficient matching $\phi(\succ)$. An example of an efficient mechanism is the Top trading cycle mechanism proposed by Abdulkadiroglu and Sönmez (2003).

3. Results

We propose a fairness criterion called *P-monotonicity* which we would like matching mechanisms to satisfy. For all $i, j \in S \cup C$, Fix a $j^*$and let P_i^{j*} denotes a specific preference list of i over j^* (i.e. the order that j^* takes in i's preference list). Similarly, let $\tilde{P}_i^{j^*}$ denotes a difference preference list of i over the same j^*. Now let $\boldsymbol{P_i^{j^*}}$be the set of all possible preference lists of i over j^* such that $P_i^{j^*}, \tilde{P}_i^{j^*} \in \boldsymbol{P_i^{j^*}}$. Define $R_{j^*}$to be a weak preference relation of j^*over $\boldsymbol{P_i^{j^*}}$where $(\tilde{P}_i^{j^*})R_{j^*}(P_i^{j^*})$ means that j^* weakly prefers her order in i's preference list $\tilde{P}_i^{j^*}$ than to her order in i's preference list $P_i^{j^*}$.

Definition 3.1. A matching is *P-monotonic* if $(\tilde{P}_i^{j^*})R_{j^*}(P_i^{j^*})$, then $\tilde{\mu}(j^*) \succsim_{j^*} \mu(j^*)$, and $\tilde{\mu}(i) \succsim_i \mu(i)$.

It means that if j^* moves at least one rank in the preference list of any i, then j^* and i should not end up with a worse allocation according to j^*'s and i's preference lists, respectively. A mechanism ϕ is P-monotonic if for every preference profile in $\succ$ and every preference profile in $\boldsymbol{P_i^{j^*}}$ the matching resulting $\phi(\succ, \boldsymbol{P_i^{j^*}})$ is P-monotonic.

Theorem 3.2. *There does not exist a matching mechanism that is stable and P-monotonic.*

Proof. Consider a 2×2 market with the following preferences for colleges and students, $\succ_{c_1}$: $s_1, s_2, \varnothing$;$\succ_{c_2}$: $s_2, s_1, \varnothing$;$\succ_{s_1}$: $c_2, c_1, \varnothing$;$\succ_{s_2}$: $c_2, c_1, \varnothing$. The preference list for s_2 simply means that she prefers to be matched to college c_2, then to be matched to college c_1, then to be unmatched[8]. There exists a unique stable matching:

$$\mu : \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix}$$

Where s_2 and c_1 are getting their top choices. Now assume that s_2 changes her preference list to $\succ_{s_2}^{'}$: $c_1, c_2, \varnothing$[9], then we have two stable matchings:

$$\mu_1^{'} : \begin{pmatrix} c_1 & c_2 \\ s_2 & s_1 \end{pmatrix}$$

$$\mu_2^{'} : \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix}$$

[8]$\varnothing$ denotes being unmatched.

[9]Note that $\succ_{s_2}^{'}$ is not a false preference list, but it is s_2 new preference list where c_1 becomes more popular for s_2.

Under $\mu_1^{'}$, c_1 is allocated to its second preference and, under $\mu_2^{'}$, s_2 is allocated to her second preference. Hence the mechanism is stable but not P-monotonic.

□

Our second result checks whether we can have an efficient and P-monotonic matching.

Theorem 3.3. *There does not exist a matching mechanism that is efficient and a P-monotonic.*

Proof. Consider a 2×2 market with the following preferences for colleges and students, $\succ_{c_1}$: $s_1, s_2, \varnothing$; $\succ_{c_2}$: $s_2, s_1, \varnothing$;$\succ_{s_1}$: $c_1, \varnothing$; $\succ_{s_2}$: $c_2, \varnothing$. Then the efficient matching we have is unique:

$$\pi : \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix}$$

Note that s_1 and c_2 are getting their top choices. Now let s_1 changes his preference list to $\succ_{s_1}^{'}$: $c_2, c_1, \varnothing$. Then we have two efficient matchings:

$$\pi_1^{'} : \begin{pmatrix} c_1 & c_2 \\ s_1 & s_2 \end{pmatrix}$$

$$\pi_2^{'} : \begin{pmatrix} c_1 & c_2 & \varnothing \\ \varnothing & s_1 & s_2 \end{pmatrix}$$

Under $\pi_1^{'}$, s_1 is allocated to his second preference, and under $\pi_2^{'}$, c_2 is allocated to its second preference. Hence the mechanism is efficient but not P-monotonic.

□

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