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Tacit collusion and capacity withholding in repeated uniform price auctions

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This article analyzes tacit collusion in infinitely repeated multiunit uniform price auctions in a symmetric oligopoly with capacity-constrained firms. Under two popular definitions of the uniform price, when each firm sets a price-quantity pair, perfect collusion with equal sharing of profit is easier to sustain in the uniform price auction than in the corresponding discriminatory auction. Moreover, capacity withholding may be necessary to sustain this outcome. Even when firms may set bids that are arbitrary finite step functions of price-quantity pairs, in repeated uniform price auctions maximal collusion is attained with simple price-quantity strategies exhibiting capacity withholding.

1. Introduction

■ This article contributes to the study of tacit collusion by analyzing infinitely repeated multiunit uniform price auctions with capacity-constrained firms. As in our earlier work on discriminatory auctions, we modify the Bertrand-Edgeworth approach by allowing each firm to simultaneously set a price-quantity pair specifying the firm's minimum acceptable price and the maximum quantity the firm is willing to sell at this price.[1] Using this game, we analyze the feasibility of perfect collusion using two different rules for determining the uniform price. Under

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[1] See Dechenaux and Kovenock (2003). Fabra (2003) provides a comparison of infinitely repeated uniform price and discriminatory auctions based on the Bertrand-Edgeworth approach.

the first rule, which we call the Market Clearing Price rule, the uniform price is equal to the minimum price at which the quantity offered by the firms is greater than or equal to demand. Under the second rule, called the Maximum Accepted Price rule, the uniform price is equal to the highest submitted price at which the residual demand left over from supply provided at strictly lower prices is strictly greater than zero. Both definitions have been used extensively in the literature (see, for example, Green and Newbery, 1992; von der Fehr and Harbord, 1993).

When each firm sets a price-quantity pair, there exists a range of discount factors for which the monopoly outcome with equal sharing is sustainable in either of the uniform price auctions, but not in the corresponding discriminatory auction. Moreover, capacity withholding may be necessary to sustain this outcome.

We extend these results to the case where firms may set bids that are arbitrary step functions of price-quantity pairs with any finite number of price steps. Surprisingly, under the Maximum Accepted Price rule, firms need employ no more than two price steps to minimize the value of the discount factor above which the perfectly collusive outcome with equal sharing is sustainable on a stationary path. Under the Market Clearing Price rule, only one step is required. That is, within the class of step bidding functions with a finite number of steps, maximal collusion is attained with simple price-quantity strategies exhibiting capacity withholding.

These results are particularly relevant for markets such as electricity markets in which uniform price and discriminatory auctions govern exchange. Our simple model captures some of the basic features of operating electricity markets, such as the UK spot market, the Spanish wholesale market, or the Victoria Power Exchange. In these markets, capacity-constrained firms compete by offering step bidding functions that vary in their complexity depending on the market.

The theoretical literature on capacity-constrained uniform price auctions applied to electricity markets can be traced back to Green and Newbery (1992) and von der Fehr and Harbord (1993).[2] The former assumes that capacity-constrained firms offer continuous supply functions, whereas the latter assumes that firms submit discrete step functions similar to those in this article. In both papers the analysis is static, and thus ignores the strategic implications of repeated interaction. Although, as Borenstein, Bushnell, and Wolak (2002) note, most electricity markets provide favorable conditions for firms to collude, surprisingly, little attention has been paid to the theoretical modelling of collusion in electricity markets. An exception is Fabra's (2003) comparison of the uniform price and discriminatory auctions in Bertrand-Edgeworth duopoly supergames.

Fabra (2003) has shown that under Bertrand-Edgeworth (B-E) duopoly, divisions of the monopoly profit can be supported in the infinitely repeated uniform price auction for strictly lower discount factors than in the infinitely repeated discriminatory auction. However, this result is only valid for a subset of symmetric capacities for which nonstationary paths with bid rotation can be sustained as perfect equilibria of the uniform price auction. For example, in the duopoly, if each firm's capacity is large enough to supply the monopoly output, incentives to deviate from perfectly collusive paths in the uniform price auction are no less than in the discriminatory auction. Furthermore, on the nonstationary paths with bid rotation that minimize incentives to deviate in the uniform price auction, firms do not equally share monopoly profit. Expanding the strategy space to price-quantity pairs, thereby allowing for physical withholding, has important implications for the sustainability of perfect collusion in the uniform price auction. A direct implication of capacity withholding is that, in contrast to B-E competition, when capacity is such that $n - 1$ firms can supply the monopoly output, the monopoly outcome can be supported for a strictly wider range of discount factors in the uniform price auction than in the discriminatory

[2] More recent theoretical work related to this study includes Baldick and Hogan (2001), Boom (2003), Borenstein, Bushnell, and Stoft (2000), Ciarreta and Espinosa (2005), Crampes and Créti (2005), Crawford, Crespo, and Tauchen (2005), Fabra, von der Fehr, and Harbord (2006), García-Díaz and Marín (2003), Gutiérrez-Hita and Ciarreta (2003), Lave and Perekhodstev (2001), Le Coq (2002), and Ubéda (2004).

auction. Moreover, this result holds even if we restrict attention to stationary paths on which each firm obtains an equal share of the monopoly profit.

In the discriminatory auction, the incentive to deviate from perfect collusion is minimized on a stationary path on which each firm sets the monopoly price and offers its whole capacity. On the other hand, in the uniform price auction, if the uniform price is given by the Market Clearing Price rule, the stationary path on which each firm withholds capacity to offer its share of monopoly output at a price below some critical level (strictly lower than the monopoly price) minimizes firms' incentives to deviate in the class of stationary paths with equal sharing of the monopoly profit. If the uniform price is given by the Maximum Accepted Price rule, then incentives to deviate from perfect collusion are minimized when $n - 1$ firms withhold capacity to offer their share of the monopoly output. The remaining firm acts as the price setter and offers capacity at the monopoly price. Together, these two results provide a conclusive theoretical link between equilibrium capacity withholding and the ability to support tacitly collusive outcomes.

The remainder of the article is organized as follows. In Section 2, we describe the model and the simultaneous move price-quantity uniform price auction under two alternative definitions of the uniform price and characterize the Nash equilibria of the game. In Section 3, we introduce notation and definitions used in analyzing the price-quantity supergame. In Section 4, we show that under both formulations of the uniform price, capacity withholding relaxes incentives to deviate on perfectly collusive stationary perfect equilibrium paths with equal sharing. On such paths, incentives to deviate are minimized when n firms withhold capacity under the Market Clearing Price rule and when $n - 1$ firms withhold capacity under the Maximum Accepted Price rule. Section 5 extends the results in Section 4 to L-step bidding functions, $L \geq 1$, and shows that bidding functions with at most two steps are sufficient in order to minimize firms' incentives to deviate from a perfectly collusive path. One step is required under the Market Clearing Price rule and two steps under the Maximum Accepted Price rule. Section 6 concludes.

2. The simultaneous move price-quantity game

■ **The model.** Consider a market for a homogeneous good. There are n firms in the industry. Let $N = \{1, \ldots, n\}$ denote the set of firms. Firm i's cost function is such that unit cost c_i is constant up to capacity k_i. Firms are symmetric: $k_i = k$ and $c_i = c = 0$ for all i. Let $d(p)$ be market demand and assume that it satisfies the following assumptions.

Assumption 1. $d(p)$ is continuous on $[0, \infty)$. $\exists \bar{p} > 0$ such that $d(p) = 0$ if $p \geq \bar{p}$ and $d(p) > 0$ if $p < \bar{p}$. $d(p)$ is twice continuously differentiable and $d'(p) < 0$ on $(0, \bar{p})$. Finally, $pd(p)$ is strictly concave on $[0, \bar{p}]$ with maximizer p^m.

These assumptions guarantee that there exists a unique unconstrained monopoly price, p^m. Inverse demand exists and is denoted by $P(y)$, where y is output. To ensure that there exists a unique Cournot equilibrium with a strictly positive price in the quantity-setting game with n symmetric firms (without capacity constraints), demand given by $d(p)$, and zero marginal cost, we further assume[3]

Assumption 2. $d'(p) + pd''(p) < 0$ on $(0, \bar{p})$.

Under assumptions analogous to Assumption 1 for $P(y)$, this is equivalent to assuming that $\log P(y)$ is strictly concave over the relevant range and implies that Cournot quantity

[3] See Deneckere and Kovenock (1999), who also compare and contrast these conditions to inverse-demand-based conditions guaranteeing the existence and uniqueness of Cournot equilibrium. Note also that in the absence of capacity constraints, if $c_i = 0$ for every i, bootstrap Cournot equilibria exist in which equilibrium price is zero and every group of $n - 1$ firms sets their aggregate quantity $q > d(0)$.

best-response functions are downward sloping (see Deneckere and Kovenock, 1999). Denote by $r(z)$ a firm's Cournot best response to an aggregate quantity z set by other firms. That is, $r(z)$ maximizes $P(x + z)x$ with respect to x. Let y^c be the quantity set by each firm in the Cournot equilibrium with strictly positive price.

In the one-shot simultaneous move price-quantity game, firms simultaneously set price-quantity pairs, (p, q), where $p \in \mathbb{R}_+$ and $q \in [0, k]$. Firm i's strategy space is thus $S_i = \mathbb{R}_+ \times [0, k]$. A strategy profile $(\mathbf{p}, \mathbf{q}) = ((p_1, q_1), \dots, (p_n, q_n))$ is an element of $\times_{i=1}^n S_i$. In this article, we restrict the analysis to pure strategies.

Define $\hat{q}_i = \min\{q_i, d(0)\}$ to be the effective quantity offered by firm i. Given a strategy profile $(\mathbf{p}, \mathbf{q})$ and a coordinate $p \in \mathbb{R}_+$ of the price vector $\mathbf{p}$, define the set $L(p \mid \mathbf{p}, \mathbf{q}) \equiv \{i \in N \mid p_i = p\}$. $L(p \mid \mathbf{p}, \mathbf{q})$ is the set of firms setting price p. We have $L(p \mid \mathbf{p}, \mathbf{q}) = \emptyset$ if for all i, $p_i \neq p$. Let $L^-(p \mid \mathbf{p}, \mathbf{q}) \equiv \cup_{z<p} L(z \mid \mathbf{p}, \mathbf{q})$ be the set of all firms charging a price strictly less than p. To simplify notation, we often drop the argument $(\mathbf{p}, \mathbf{q})$.

We assume efficient rationing. Hence, given a strategy profile $(\mathbf{p}, \mathbf{q})$, the residual demand faced by firms in $L(p)$ is

$$R(p \mid \mathbf{p}, \mathbf{q}) = \max\left\{d(p) - \sum_{j \in L^-(p)} q_j, 0\right\}.$$

If $L^-(p)$ is empty, then we define $R(p \mid \mathbf{p}, \mathbf{q}) = d(p)$. Note that here the residual demand is the demand left over from supply provided at strictly lower prices.

If, in case of a tie in price at p, we assume that firms share residual demand in proportion to their effective quantities offered, then for $i \in L(p \mid \mathbf{p}, \mathbf{q})$, sales are

$$s_i(p \mid \mathbf{p}, \mathbf{q}) = \min\left\{\hat{q}_i, \frac{\hat{q}_i}{\sum_{l \in L(p)} \hat{q}_l} R(p \mid \mathbf{p}, \mathbf{q})\right\}.$$

In this context, the literature has defined a uniform price auction in two distinct ways. We will examine each in turn. In the first definition, we follow Green and Newbery (1992), who use a specification in which the uniform price is the price at which the quantity demanded is equal to the quantity supplied (see also Boom, 2003; Ubéda, 2004; Fabra, von der Fehr, and Harbord, 2006). This formulation leaves open the possibility that the uniform price will not be one of the submitted bids. See Figure 1 for an illustration.

Definition 1 (market clearing price). Given a strategy profile $(\mathbf{p}, \mathbf{q})$ in the uniform price auction, the uniform price $\mathbf{P}^e(\mathbf{p}, \mathbf{q})$ is the unique price that solves

$$\min\left\{p \,\middle|\, \sum_{i \in \mathcal{L}(p \mid \mathbf{p}, \mathbf{q})} \hat{q}_i \geq d(p)\right\},$$

where $\mathcal{L}(p \mid \mathbf{p}, \mathbf{q}) = L^-(p \mid \mathbf{p}, \mathbf{q}) \cup L(p \mid \mathbf{p}, \mathbf{q})$.

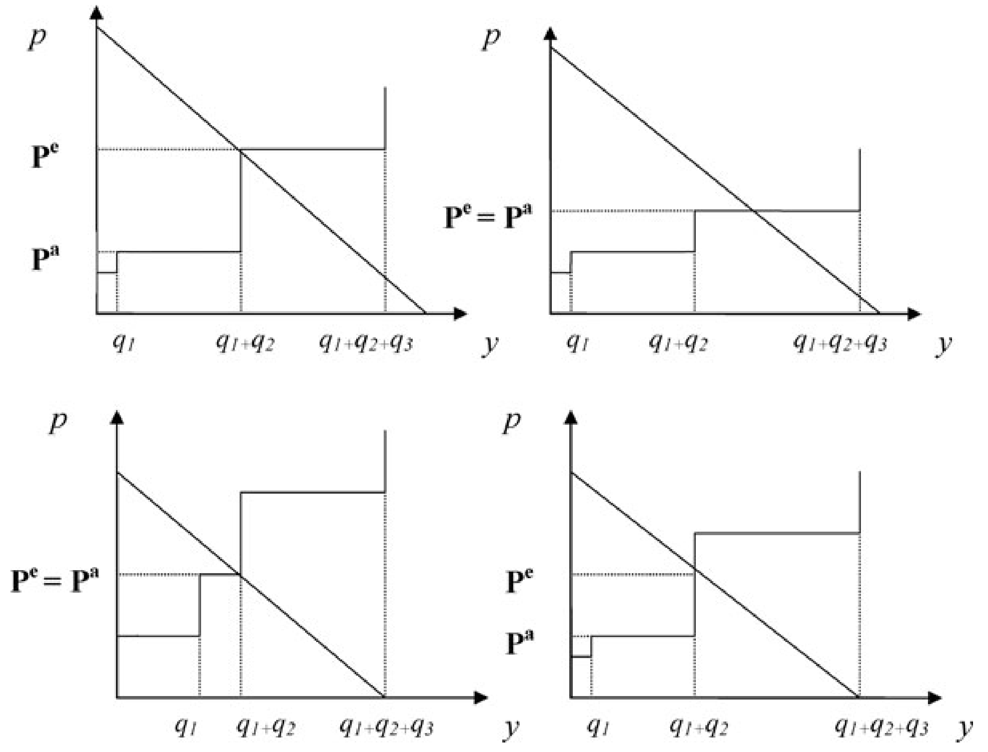
Definition 2 is the approach used by von der Fehr and Harbord (1993) (see also Crampes and Créti, 2005; Fabra, 2003; Le Coq, 2002). The price each firm receives in the uniform price auction is equal to the maximum accepted price, where the maximum accepted price is the highest submitted price at which the residual demand left over from supply provided at strictly lower prices is strictly positive. Note that in this definition, the uniform price must be one of the submitted prices, and thus may not clear the market. See Figure 1 for an illustration. For Definition 2, we require slightly more notation. Let $\mathbf{p} = (p_1, \dots, p_n)$ and define

$$\mathcal{P}(\mathbf{p}, \mathbf{q}) = \{p \in \{p_1, \dots, p_n\} \mid R(p \mid \mathbf{p}, \mathbf{q}) > 0\}.$$

$\mathcal{P}(\mathbf{p}, \mathbf{q})$ is the set of submitted prices with $R(p \mid \mathbf{p}, \mathbf{q}) > 0$.

FIGURE 1

DETERMINING THE UNIFORM PRICE UNDER DEFINITION 1 AND DEFINITION 2



Definition 2 (maximum accepted price). Given a strategy profile $(\mathbf{p}, \mathbf{q})$ in the uniform price auction, the uniform price $\mathbf{P}^a(\mathbf{p}, \mathbf{q})$ is equal to the maximum accepted price, that is

$$\mathbf{P}^a(\mathbf{p}, \mathbf{q}) = \max \mathcal{P}(\mathbf{p}, \mathbf{q}) \quad \text{if} \quad \mathcal{P}(\mathbf{p}, \mathbf{q}) \neq \emptyset,$$

and

$$\mathbf{P}^a(\mathbf{p}, \mathbf{q}) = \bar{p} \quad \text{if} \quad \mathcal{P}(\mathbf{p}, \mathbf{q}) = \emptyset.$$

For $u \in \{e, a\}$, firm i's payoff, $i = 1, \ldots, n$, under the two alternative definitions is simply

$$\pi_i(\mathbf{p}, \mathbf{q}) = \mathbf{P}^u(\mathbf{p}, \mathbf{q}) s_i(p_i \mid \mathbf{p}, \mathbf{q}).$$

Before proceeding with the characterization of equilibria, we first justify our assumption of efficient rationing. Note that our choice of the efficient rationing rule does not play a role in Definition 1. The market clearing price does not depend on the specific rationing rule used, but only on the vector of price-quantity pairs submitted by the firms. Moreover, every consumer that obtains a unit of the good pays the same price per unit no matter which firm supplies it. It follows that without cross-subsidization between consumers, any consumer who obtains a unit must be willing to pay at least the uniform price for that unit.

When the uniform price is defined as in Definition 2, the firms' prices are ranked in increasing order, with the lowest-price firms selling first. If demand were not rationed efficiently, at some strategy profiles, there would exist consumers who would be required to pay more than their reservation value at the uniform price. Implementation of other rationing rules would therefore require cross-subsidization of consumers.[4]

□ **Pure strategy equilibria.** We now define critical prices that are useful in characterizing a firm's profit from deviating from a given profile. We also characterize a firm's minmax payoff.

First, for $q < d(0)$, define the residual demand monopoly price for a firm with capacity k, $p^r(k, q)$:

$$p^r(k, q) \equiv \max\left\{\arg\max_p\{p[d(p) - q]\},\ P(k + q)\right\}.$$

$p^r(k, q)$ is unique for every pair (k, q) given our assumptions on demand. From the strict concavity of $pd(p)$, it is clear that whenever $p^r(k, q)$ is strictly positive, it is strictly decreasing in q. A firm's profit from setting $p^r(k, q)$ after lower-price firms have sold a quantity q is $\underline{\pi}(k, q) \equiv p^r(q)[d(p^r(q)) - q]$. For $q \geq d(0)$, a firm's residual demand after other firms have sold a quantity q is zero for all p. In this case, we define $p^r(k, q) \equiv 0$ and it follows that $\underline{\pi}(k, q) = 0$ for all $q \geq d(0)$.

Defining $p^r \equiv p^r(k, (n-1)k)$, it is straightforward to show that if $(n-1)k < d(0)$, then a firm's minmax payoff, $\underline{\pi}$, is $\underline{\pi} = p^r[d(p^r) - (n-1)k] > 0$. If $(n-1)k \geq d(0)$, then by definition, $p^r((n-1)k) = 0$ and each firm's minmax payoff is $\underline{\pi} = 0$.

Following Deneckere and Kovenock (1992), let $\underline{p}(k, q)$ be the unique price less than or equal to $p^r(k, q)$ at which a firm is indifferent between being the low-price firm at $\underline{p}(k, q)$ and being a monopolist on residual demand left after q is sold and earning $\underline{\pi}(k, q)$. $\underline{p}(k, q)$ is equal to the smallest solution to

$$p \times \min\{d(p), k\} = \underline{\pi}(k, q).$$

If $q < d(0)$, then $\underline{p}(k, q) > 0$. If $q \geq d(0)$, by definition $\underline{\pi}(k, q) = 0$, and thus $\underline{p}(k, q) = 0$. In the continuation, we will use the notation $\underline{p}$ to denote $\underline{p}(k, (n-1)k)$. Moreover, because $k_i = k$ for every i, when there is no ambiguity we use $p^r(q)$ to denote $p^r(k, q)$, $\underline{\pi}(q)$ to denote $\underline{\pi}(k, q)$, and $\underline{p}(q)$ to denote $\underline{p}(k, q)$.

We can now state the following proposition describing equilibrium in the one-shot price-quantity uniform price auction with common capacities $k_i = k$ and common unit costs $c_i = 0$, for every i, which we denote by $\Gamma^u(k, 0)$, where $u \in \{e, a\}$ indicates the definition of the uniform price that is employed.

Proposition 1. The sets of pure strategy equilibria, $E^u(k, 0)$, of the one-shot uniform price auctions $\Gamma^u(k, 0)$, $u = e, a$, are completely characterized as follows.

(i) Suppose $k \leq y^c$. Then $E^a(k, 0) = \{(\mathbf{p}^*, \mathbf{q}^*) \mid p_i^* \leq P(nk) \text{ and } q_i^* = k,\ \forall i \in N, \text{ with } p_j^* = P(nk) \text{ for at least one } j \in N\}$ and $E^e(k, 0) = \{(\mathbf{p}^*, \mathbf{q}^*) \mid p_i^* \leq P(nk) \text{ and } q_i^* = k,\ \forall i \in N\}$.

(ii) Suppose $k \geq \frac{d(0)}{n-1}$. Then $E^a(k, 0) = \{(\mathbf{p}^*, \mathbf{q}^*) \mid \exists\, i, j,\ i \neq j, \text{ such that } p_i^* = p_j^* = 0$ and $\forall h \in L(0 \mid \mathbf{p}^*, \mathbf{q}^*), \sum_{l \in L(0 \mid \mathbf{p}^*, \mathbf{q}^*) \setminus \{h\}} \hat{q}_l^* \geq d(0)\}$. Define $C(0) \equiv \{(\mathbf{p}^*, \mathbf{q}^*) \mid p_i^* \leq \underline{p}((n-1)y^c) \text{ and } q_i^* = y^c,\ \forall i \in N\}$. Then $E^e(k, 0) = E^a(k, 0) \cup C(0)$.

(iii) Suppose $k \in (y^c, \frac{d(0)}{n-1})$. Define $\underline{y}$ to be the unique $y \in (d(p^r) - (n-1)k, k)$ such that $\underline{\pi}((n-2)k + y) = p^r k$. Then $E^a(k, 0) = \{(\mathbf{p}^*, \mathbf{q}^*) \mid \exists\, j \in N \text{ such that } p_j^* = p^r \text{ and } q_j^* \in [\underline{y}, k] \text{ and } \forall i \neq j,\ p_i^* \leq \underline{p}, \text{ and } q_i^* = k\}$ and $E^e(k, 0) = E^a(k, 0) \cup C(0)$, where $C(0)$ is as defined in (ii).

Moreover, for every $k \in \mathbb{R}_+$, $u \in \{e, a\}$, and $i \in N$, there exists a pure strategy equilibrium of $\Gamma^u(k, 0)$ in which $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = \underline{\pi}$.

[4] As Davidson and Deneckere (1986) have shown, in discriminatory auctions, the firms' payoffs depend on the particular rationing scheme employed. One prominent alternative to efficient rationing is proportional rationing. The implications of assuming efficient rationing rather than proportional rationing are explored in Section 4, where we compare uniform price and discriminatory auctions.

Proof. See the Appendix.

Proposition 1 demonstrates that, under both rules for determining the uniform price, if each firm's capacity k is less than or equal to its n-firm Cournot output, each firm sets price at or below the capacity clearing price $P(nk)$ and sells its capacity. If, for this range of capacities, the Maximum Accepted Price rule is used in determining the market price, at least one of the firms must set price equal to $P(nk)$.

For k in the classical Bertrand region where any $(n-1)$ firms have sufficient capacity to satisfy the whole market demand $d(0)$ at unit cost $c = 0$, the uniform price is always zero in the auction with the Maximum Accepted Price rule. This requires that at least two firms price at zero and set quantities sufficiently large that any unilateral deviation to a higher price yields zero sales. These classical Bertrand equilibria are also contained in the set of equilibria under the Market Clearing Price rule. However, under this rule, the equilibrium set also contains Cournot-like equilibria in which all firms set prices at or below $\underline{p}((n-1)y^c)$ and sell their Cournot quantity y^c. The uniform price in these equilibria is the Cournot price $P(ny^c)$.

For $k \in (y^c, \frac{d(0)}{(n-1)})$, the intermediate range between the Cournot output and the classical Bertrand region, for each $j \in N$, there exists a continuum of equilibria in which firm j sets $p_j^* = p^r$ and all other firms price at or below $\underline{p}$. The $(n-1)$ low-price firms all have sales equal to capacity k and firm j sells to residual demand. In these equilibria, the quantity that firm j places on the market must be sufficient to deter a unilateral deviation by a low-price firm to a price above p^r. The critical supply that achieves this is the quantity $\underline{y}$ defined in Proposition 1, so j must supply at least $\underline{y}$, which we show is greater than residual demand $d(p^r) - (n-1)k$. Under the Maximum Accepted Price rule, these equilibria define the complete set of equilibria. For the Market Clearing Price rule, the set must again be augmented by the set of Cournot-like equilibria described in the previous paragraph.

In our analysis of the repeated uniform price auctions that follows, the most important aspect of the characterization in Proposition 1 is the fact that, under both uniform price rules, for any common capacity k and any firm i there exists a one-shot Nash equilibrium in which i receives its minmax profit $\underline{\pi}$. This allows the direct construction of credible punishments in the repeated uniform price auctions that force any unilaterally deviating firm down to its minmax per-period continuation payoff.

3. The price-quantity supergame

■ In this section, we examine the supergame $\Gamma^u(k, 0, \delta)$ obtained by infinitely repeating the one-shot game $\Gamma^u(k, 0)$ and discounting payoffs with discount factor $\delta < 1$. In the supergame, a path τ is an infinite sequence of action profiles $\{(\mathbf{p}^t, \mathbf{q}^t)\}_{t=0}^{\infty}$. A pure strategy σ_i for firm i is a sequence of functions, $\{\sigma_i(t)\}_{t=0}^{\infty}$, such that for every t, $\sigma_i(t): H_t \to S_i$, where H_t is the set of possible histories $h_t = (\mathbf{p}^0, \mathbf{q}^0, \ldots, \mathbf{p}^{t-1}, \mathbf{q}^{t-1})$ up to time t and h_0 is the null history. A strategy profile is a vector $\sigma = (\sigma_1, \ldots, \sigma_n)$. Each strategy profile generates an infinite path $\tau(\sigma)$. Firm i's normalized discounted value from period s along a given path $\tau = \{(\mathbf{p}^t, \mathbf{q}^t)\}_{t=0}^{\infty}$ is given by

$$V_i(\tau, s) = (1-\delta)\sum_{t=s}^{\infty} \delta^{t-s}\pi_i(\mathbf{p}^t, \mathbf{q}^t).$$

We refer to $V_i(\tau, t)$ for $t = 0, 1, 2, 3, \ldots,$ as firm i's continuation value at t. We let $V_i(\tau) \equiv V_i(\tau, 0)$ denote the payoff associated with the entire path. A security-level punishment for firm i is a path on which firm i obtains the discounted sum of its minmax profit, equal to $\underline{\pi}$ in normalized terms. The result below establishes that a perfect equilibrium security-level punishment in pure strategies exists under both definitions of the uniform price. After any unilateral deviation by firm i, firm i's punishment consists of reverting to a static equilibrium in every period.

Proposition 2. For every $k \in \mathbb{R}_+$, $\delta \in (0, 1)$, $u \in \{e, a\}$, and $i \in N$, there exists a perfect equilibrium of $\Gamma^u(0, k, \delta)$ which serves as a security-level punishment for firm i.

Proof. From Proposition 1, the simultaneous move game has a pure strategy equilibrium in which firm i obtains its minmax payoff, $\forall i$ and k. Because repeating a minmax one-shot Nash equilibrium forever is a perfect equilibrium security-level punishment, the result follows directly. *Q.E.D.*

In the continuation, we assume that each firm's capacity is larger than its share of the monopoly output, $k > \frac{d(p^m)}{n}$. If, on the other hand, $k \leq \frac{d(p^m)}{n}$, all firms are capacity constrained in the one-shot Nash equilibrium and equilibrium payoffs are Pareto optimal. Thus, the simultaneous move equilibrium is a collusive outcome immune to deviations for any discount factor.

Consider a stationary path $\tau = \{(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)\}$. Let $\pi_i^*(p_{-i}, q_{-i})$ be firm i's optimal deviation profit as a function of the prices and quantities set by the remaining $n - 1$ firms. Formally, $\pi_i^*(p_{-i}, q_{-i}) = \sup_{\{p_i, q_i\}} \pi_i(p_i, q_i, p_{-i}, q_{-i})$. Because from Proposition 2 a perfect equilibrium security-level punishment exists, the incentive constraints that provide the perfect equilibrium conditions for the stationary path τ are simply

$$(1 - \delta)\big[\pi_i^*(p_{-i}, q_{-i}) - \mathbf{P}^u(\mathbf{p}, \mathbf{q})s_i(p_i \mid \mathbf{p}, \mathbf{q})\big] \leq \delta[\mathbf{P}^u(\mathbf{p}, \mathbf{q})s_i(p_i \mid \mathbf{p}, \mathbf{q}) - \underline{\pi}] \tag{1}$$

for every $i \in N$ and $u \in \{e, a\}$. In the continuation, we refer to the difference between firm i's one-period profit obtained from deviating optimally in period t and its one-period profit from conforming to the prescribed path as firm i's *incentive to deviate*. In equation (1), the incentive to deviate is given by the term in the square brackets on the left-hand side.

In the next section, we characterize all stationary paths that achieve the monopoly outcome and on which firms share monopoly profits equally. We show that under both uniform pricing rules, there is a range of discount factors for which capacity withholding is necessary for such paths to be supported as perfect equilibrium paths.

4. Capacity withholding and market sharing

■ In this section, we focus attention on a specific class of paths. We consider paths τ that are stationary and on which the normalized payoffs satisfy $\sum_{i\in N} V_i(\tau) = p^m d(p^m) \equiv \Pi^m$. We say that such paths are *perfectly collusive*. In the remainder of this section, we also impose the condition that $\pi_i(\mathbf{p}, \mathbf{q}) = \frac{\Pi^m}{n}$; that is, firms share monopoly profits equally. We call a path satisfying the two conditions above a perfectly collusive stationary path with equal sharing. Note that on such a path, sales are symmetric, although price-quantity pairs need not be.

Lemma 1 below is useful in characterizing the perfectly collusive stationary paths with equal sharing on which firms' incentives to deviate are minimized. Consider two profiles of price-quantity pairs for firm i's rivals, (p'_{-i}, q_{-i}) and (p_{-i}, q_{-i}), such that for $j \neq i$, firm j's price is no lower in (p'_{-i}, q_{-i}) than it is in (p_{-i}, q_{-i}), but the ordering of the prices across firm i's rivals, as well as their quantity ceilings, are the same in both profiles. Lemma 1 states that firm i's profit from an optimal deviation when its rivals set (p'_{-i}, q_{-i}) cannot be lower than when they set (p_{-i}, q_{-i}). Note that this result follows from the conditions imposed on the profiles $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q})$ in the statement of the lemma, which are sufficient to guarantee that for every p, residual demand at price p is at least as large when i's rivals set (p'_{-i}, q_i) as when they set (p_{-i}, q_i).

Lemma 1. Suppose $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{p}' = (p'_1, \ldots, p'_n)$ satisfy the two following conditions: for some $i \in N$, (i) $\forall j \neq i$, $p'_j \geq p_j$ and (ii) $\forall j, h \in N\backslash\{i\}$, $p_j \geq p_h$ implies $p'_j \geq p'_h$. Then for any vector of quantities $\mathbf{q} = (q_1, \ldots, q_n)$, $\pi_i^*(p'_{-i}, q_{-i}) \geq \pi_i^*(p_{-i}, q_{-i})$.

Proof. Suppose that when the $n - 1$ remaining firms set (p_{-i}, q_{-i}), firm i's optimal deviation is to undercut all $n - 1$ firms. Because for every $j \neq i$, $p'_j \geq p_j$, firm i's optimal deviation profit under (p_{-i}, q_{-i}) can also be obtained by undercutting all other firms when they set (p'_{-i}, q_{-i}). Thus, its deviation profit cannot be lower in this case. Now suppose that if i's rivals set (p_{-i}, q_{-i}), firm i's optimal deviation consists of setting the residual demand monopoly price after a group of $l \geq 1$ firms have sold a quantity q, $p^r(q)$, to earn $\underline{\pi}(q)$. We show that i cannot obtain less than

$\underline{\pi}(q)$ by deviating when its rivals set (p'_{-i}, q_{-i}). First, consider the difference in firm i's residual demand at a given price p when firms set $(\mathbf{p}', \mathbf{q})$ and $(\mathbf{p}, \mathbf{q})$:

$$\max\left\{d(p) - \sum_{j \in L^-(p \mid \mathbf{p}', \mathbf{q}) \setminus \{i\}} \hat{q}_j, 0\right\} - \max\left\{d(p) - \sum_{j \in L^-(p \mid \mathbf{p}, \mathbf{q}) \setminus \{i\}} \hat{q}_j, 0\right\}.$$

From conditions (i) and (ii) in the statement of the lemma, it follows that $L^-(p \mid \mathbf{p}', \mathbf{q}) \subseteq L^-(p \mid \mathbf{p}, \mathbf{q})$ and, thus, $\sum_{j \in L^-(p \mid \mathbf{p}', \mathbf{q}) \setminus \{i\}} \hat{q}_j \leq \sum_{j \in L^-(p \mid \mathbf{p}, \mathbf{q}) \setminus \{i\}} \hat{q}_j$. Consequently, residual demand at p is weakly larger when other firms set (p'_{-i}, q_i) than when they set (p_{-i}, q_{-i}). It follows that if $\nexists j$ for which $p'_j = p^r(q)$, then firm i obtains a deviation profit $\pi^*_i \geq \underline{\pi}(q)$ from setting exactly $p^r(q)$. If $\exists j$ such that $p'_j = p^r(q)$, then i can obtain a payoff arbitrarily close to $\underline{\pi}(q)$ by infinitesimally undercutting $p'_j = p^r(q)$. Therefore, we have shown that any deviation profit level firm i can guarantee itself under (p_{-i}, q_i) can also be obtained when the other firms set (p'_{-i}, q_{-i}). *Q.E.D.*

Lemma 2 characterizes all perfectly collusive stationary paths with equal sharing. Note that the characterization is independent of the definition of the uniform price except for statement (ii).

Lemma 2. Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^u(k, 0, \delta)$, on every perfectly collusive stationary path with equal sharing $\tau = ((p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n))$, the following must hold: (i) $\mathbf{P}^u(\mathbf{p}, \mathbf{q}) = p^m$, (ii) $p_i \leq p^m$ with equality for at least one firm if $u = a$, and (iii) for every firm $i \in N$, $q_i \geq \frac{d(p^m)}{n}$, with equality if firm i sets $p_i < p^m$.

Proof. (i) and (ii) follow from the fact that there exists a unique maximizer to $pd(p)$. Thus, if industry profit is Π^m in every period, p^m must be the uniform price in every period as well. It is straightforward to see that if there exists a firm i for which $p_i > p^m$, then, either the uniform price is strictly greater than p^m, or firm i does not have any sales, a contradiction to $V_i(\tau) = \frac{\Pi^m}{n}$. Furthermore, if the uniform price is given by Definition 2, p^m must also be one of the accepted bids, thus at least one firm $i \in N$ must set $p_i = p^m$. To prove (iii), note that it is clear that if $q_i < \frac{d(p^m)}{n}$, then $s_i < \frac{d(p^m)}{n}$, and thus $V_i(\tau) < \frac{\Pi^m}{n}$. Hence, $q_i \geq \frac{d(p^m)}{n}$. To complete the proof of (iii), consider first the Maximum Accepted Price rule. Suppose to the contrary that there exists a firm i for which $p_i < p^m$ and $q_i > \frac{d(p^m)}{n}$. From the definition of s_i, it is clear that the only time a firm's sales are strictly below its quantity ceiling occurs when it sells to all or a fraction of residual demand. If this is the case, however, for firm i, then $R(p \mid \mathbf{p}, \mathbf{q}) = 0$ for every $p > p_i$, which implies that p^m cannot be the uniform price, thus contradicting (i). Hence, it follows immediately that if $p_i < p^m$, $q_i = \frac{d(p^m)}{n}$. Consider now the Market Clearing Price rule and suppose that there exists a firm i with $p_i < p^m$ and $q_i > \frac{d(p^m)}{n}$. First, note that from Definition 1 and (i), if $p_j < p^m$, $\forall j$, then it must be the case that $\sum_{h \in N} \hat{q}_h = d(p^m)$. Otherwise, at least one firm must be setting p^m. Because we have established above that for every j, $q_j \geq \frac{d(p^m)}{n}$, it follows directly that if, additionally, for every j, $p_j < p^m$, then $\sum_{j \in N} \hat{q}_j > d(p^m)$, so that the uniform price is not equal to p^m, a contradiction to (i). Suppose now that l firms, $l \geq 1$, are setting p^m. Because for every firm j setting $p_j < p^m$, $q_j \geq \frac{d(p^m)}{n}$ and there exists i for which $q_i > \frac{d(p^m)}{n}$, it follows that residual demand at p^m is strictly less than $\frac{l}{n} d(p^m)$, so that $s_h < \frac{d(p^m)}{n}$ for at least one firm h setting $p_h = p^m$, a contradiction to the fact that $V_h(\tau) = p^m s_h = \frac{\Pi^m}{n}$. *Q.E.D.*

Lemma 2 shows that there are essentially two ways in which firms can achieve a perfectly collusive outcome with equal sharing on a stationary path. All firms may set the monopoly price and offer the same quantity ceiling. In this case, the sales function prescribes that each firm will obtain an equal share of demand at the monopoly price. Alternatively, a group of firms (possibly empty under the Market Clearing Price rule) may set the monopoly price and share residual demand after another group of (strictly lower-price) firms offer their share of the monopoly output and sell their quantity. We show in the following sections that the crucial difference between the Market Clearing Price and Maximum Accepted Price approaches is that if the former is used, in sustaining collusion there can be as many as n low-price firms, whereas there must be at least one firm setting p^m under the latter.

□ **The Market Clearing Price rule.** Building on the insight from Lemma 2, we construct a perfect equilibrium path τ^{me} and show that given the imposed stationarity and equal division, such a path minimizes incentives to deviate for all firms among perfectly collusive stationary paths achieving the same payoff per firm. To this effect, let $q_-^m = (n-1)\frac{d(p^m)}{n}$. τ^{me} is characterized as follows.

$$p_i = \underline{p}(q_-^m) \quad \text{and} \quad q_i = \frac{d(p^m)}{n}, \forall i \in N.$$

Lemma 3. Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^e(k, 0, \delta)$, the path τ^{me} minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing.

Proof. See the Appendix.

The path constructed in Lemma 3 is symmetric. Each firm withholds capacity and sets a quantity equal to its share of the monopoly output and a price equal to the maximum price that no firm would want to unilaterally undercut. From Definition 1, it is straightforward to check that the uniform market price is then equal to the monopoly price. The proposition below shows that for some discount factors, withholding is necessary to sustain perfect collusion with equal sharing on a stationary path. The crucial effect of capacity withholding on collusion is that in markets with a large aggregate capacity, it allows firms to set lower prices without affecting the uniform price, thereby substantially reducing incentives to deviate for all firms.

It is important to note that setting $p_i = \underline{p}(q_-^m)$ is not necessary to support this outcome. Every path on which each firm offers exactly its share of monopoly output at a price less than or equal to $\underline{p}(q_-^m)$, for example at the marginal cost of zero, yields the same outcome and exactly the same incentives to deviate as τ^{me}.

Proposition 3. Suppose $k > \frac{d(p^m)}{n}$. There exists a $\bar{\delta} < 1$ such that in $\Gamma^e(k, 0, \delta)$, perfect collusion with equal sharing is sustainable on a stationary path on which no firm withholds capacity if and only if $\delta \geq \bar{\delta}$. Furthermore, there exists a $\underline{\delta}^e < \bar{\delta}$ such that if $\delta \in [\underline{\delta}^e, \bar{\delta})$, perfect collusion with equal sharing is sustainable on a stationary path and such a path requires that some subset of the firms withholds capacity. If $\delta = \underline{\delta}^e$, perfect collusion with equal sharing is sustainable on a stationary path and such a path requires that all firms withhold capacity.

Proof. See the Appendix.

Note that from Lemma 2, part (iii), $\bar{\delta}$ is the lowest discount factor such that for all $\delta \geq \bar{\delta}$, the path on which all firms set the monopoly price p^m and offer k can be supported in a perfect equilibrium. Clearly, both $\bar{\delta}$ and $\underline{\delta}^e$ depend on the common capacity level k and the number of firms n. In the continuation, as in Proposition 3, we will abuse notation by dropping the arguments k and n.

□ **The Maximum Accepted Price rule.** If the uniform price is defined as the maximum accepted price, then the result is slightly different. The fact that the uniform price has to be one of the submitted prices adds the extra constraint that one firm must set p^m on the path. This clearly increases incentives to deviate for low-price firms when compared to the path τ^{me} characterized above.

Let τ_i^{ma} be characterized as follows, $\forall i \in N$.

$$p_i = p^m \quad \text{and} \quad q_i = k,$$

$$p_j = \underline{p}(q_-^m) \quad \text{and} \quad q_j = \frac{d(p^m)}{n}, \ \forall j \neq i.$$

There exist n such paths corresponding to each of the n players with price p^m. Note that on any such path, τ_i^{ma}, the maximum incentive to deviate across firms is the incentive to deviate of a low-price firm.

Lemma 4. Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^a(k, 0, \delta)$, paths of the form of τ_i^{ma} minimize the maximum incentive to deviate in the class of perfectly collusive stationary paths with equal sharing.

Proof. See the Appendix.

Lemma 4 shows that paths on which $n - 1$ firms withhold capacity to offer an equal share of the monopoly output at a low price and one firm sets the monopoly price and offers its whole capacity minimize the maximum incentive to deviate on paths in that particular class. The next result shows that if $n > 2$ and capacity is large enough, by withholding output, firms can sustain an equal division of the monopoly outcome for a strictly wider range of discount factors than by offering full capacity.

Again, as for the Market Clearing Price approach, it is important to note that it is not necessary that low-price firms set $p_j = \underline{p}(q^m_-)$. Every path on which each firm j, $j \neq i$, offers exactly its share of monopoly output at a price less than or equal to $\underline{p}(q^m_-)$ yields the same outcome and exactly the same incentives to deviate as τ_i^{ma}.

Proposition 4. Suppose $k > \frac{d(p^m)}{n}$. In $\Gamma^a(k, 0, \delta)$, perfect collusion with equal sharing is sustainable on a stationary path on which no firm withholds capacity if and only if $\delta \geq \bar{\delta}$. Furthermore, suppose $n > 2$ and $k > \frac{2}{n}d(p^m)$. Then there exists a $\underline{\delta}^a < \bar{\delta}$ such that if $\delta \in [\underline{\delta}^a, \bar{\delta})$, perfect collusion with equal sharing is sustainable on a stationary path and such a path requires that some subset of the firms withholds capacity. If $\delta = \underline{\delta}^a$, perfect collusion with equal sharing is sustainable on a stationary path and such a path requires that at least $n - 1$ firms withhold capacity.

Proof. See the Appendix.

Proposition 4 shows that for $n > 2$ and for sufficiently large capacity, withholding capacity facilitates collusion. The intuition behind the dependence of the result in Proposition 4 on the number of firms is as follows. For every n, on a perfectly collusive path with equal sharing that does not feature withholding, an optimal deviation always consists of undercutting the monopoly price to obtain $p^m \min\{k, d(p^m)\}$. On τ_i^{ma}, if there are only two firms in the industry, the single low-price firm can always undercut the high-price firm, and offer and sell the minimum of demand and its capacity. On the other hand, when there are more than two firms in the industry, an optimal deviation takes one of two forms depending on k. For a range of relatively low capacity values, a low-price firm offers and sells its capacity at a price below p^m. In this case, because k is low, the deviating firm does not affect the uniform price and receives p^m for its capacity independently of the price it sets as long as this price is below p^m. However, this statement is only valid if undercutting the low-price firms and offering capacity does not lower the market price. If k is large enough that a deviating firm would have to withhold in order not to affect the uniform price by expanding its quantity up to its capacity, the optimal deviation consists of pricing above the low-price firms but below the high-price firm. For such values of k, a low-price firm's deviation profit is strictly below $p^m \min\{k, d(p^m)\}$. Thus, k relatively large is required for withholding to strictly relax incentive constraints as compared to paths on which no firm withholds.

Fabra (2003) analyzes the feasibility of collusion in a Bertrand-Edgeworth duopoly supergame with the Maximum Accepted Price rule. In Fabra's model, the two firms must offer capacity at the price they set. She shows that to support a given level of industry profit, the paths that minimize firms' incentives to deviate are nonstationary. On such paths, firms alternate between being high-price and low-price, so that only one firm, the high-price firm, has an incentive to deviate in any given period. To support perfect collusion, the high-price firm's incentive to deviate on the nonstationary path is lower than that of the low-price firm on the stationary path τ_i^{ma} constructed in Lemma 4 in this article. Thus, perfect collusion can be supported for lower discount factors on the nonstationary path with firms switching roles every period than on the stationary path with equal sharing. However, on the nonstationary paths, firms do not share industry profit equally, so that the question of role selection in the first period arises. Moreover, to support perfect collusion, such paths are feasible only if $k < d(p^m)$, because if $k \geq d(p^m)$, the firms' inability to withhold capacity implies that to obtain industry profit equal to Π^m, each firm must set p^m.[5] The latter observations are all the more relevant in the case $n \geq 3$.[6] Indeed, the range of capacities for which alternating nonstationary paths with one high-price firm and no withholding are feasible, $k < \frac{d(p^m)}{n-1}$, shrinks as n increases. If $k \geq \frac{d(p^m)}{n-1}$, we conjecture that nonstationary perfect equilibrium paths with more than one high-price firm setting p^m can be constructed for a range of discount factors, but it is not clear that they minimize incentives to deviate.

We have focused on one possible division of monopoly profits, namely the symmetric allocation, in which each firm receives an equal share of industry profit. Other allocations can also be sustained as perfect equilibria. With symmetric firms and the Market Clearing Price rule, equal sharing is optimal because on the path τ^{me}, all firms have the same incentive to deviate. However, when the Maximum Accepted Price rule is used, equal sharing may not minimize the critical value of the discount factor above which perfect collusion is sustainable. This is best illustrated by considering the duopoly case. For $d(p) = \max\{0, 1 - p\}$ and $k = \frac{1}{2}$, Figure 2 shows the set of sustainable allocations for the low-price firm as a function of the discount factor on perfectly collusive stationary perfect equilibrium paths similar to τ_i^{ma}. On such paths, the low-price firm obtains $p^m s^m$ and the high-price firm obtains $p^m(d(p^m) - s^m)$. $\underline{s}^m$ is the smallest allocation and $\bar{s}^m$ the largest allocation the low-price firm can obtain on such paths. Along the curve $\underline{s}^m$, the low-price firm's incentive constraint is binding, whereas along the curve $\bar{s}^m$, the high-price firm's constraint binds. At every allocation between $\underline{s}^m$ and $\bar{s}^m$, both constraints are slack. It is clear that the critical discount factor obtains at an allocation at which the low-price firm receives a greater share of monopoly profits.

To summarize, there are two main insights to be gleaned from the analysis of the price-quantity supergames in this section. First, capacity withholding facilitates collusion on stationary paths no matter which uniform pricing rule is used. This is because it allows firms to set a price that minimizes the other firms' deviation profit without preventing the market price from remaining at p^m. Second, under the Market Clearing Price rule, the reduction in deviation profit achieved on paths with withholding is larger than under the Maximum Accepted Price Rule because under the former, no single firm is required to price at p^m. Using these results, we now compare repeated uniform price auctions to repeated discriminatory auctions.

□ **Uniform price and discriminatory auctions.** Another commonly employed auction mechanism is the discriminatory auction, in which each firm receives the price it bids for its quantity. Propositions 3 and 4 provide a simple way to compare the two institutions. Recall that $\bar{\delta}$ is the critical value of the discount factor above which a path on which each firm sets the monopoly price p^m and offers its capacity in every period can be supported as a perfect equilibrium in the repeated uniform price auction. In both the uniform price and the discriminatory auctions, an optimal deviation from such a path consists of undercutting the monopoly price and offering capacity. Therefore, $\bar{\delta}$ also represents the critical value of the discount factor above which a perfectly collusive path with equal sharing is sustainable in the repeated discriminatory auction.

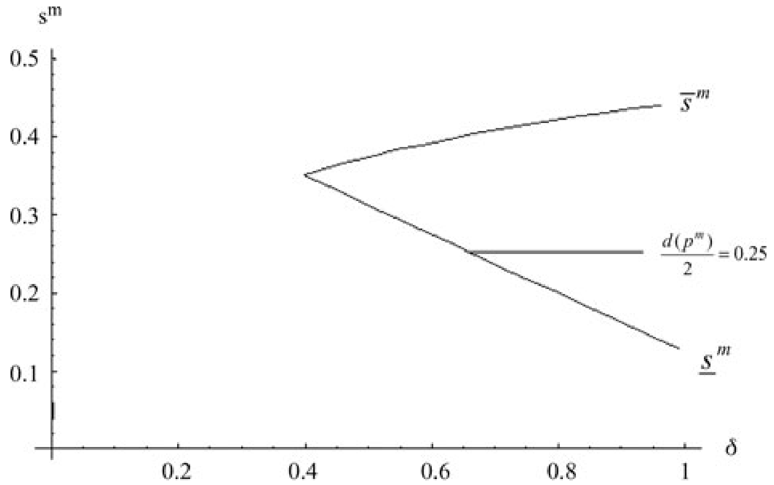
Theorem 1. Suppose $k > \frac{d(p^m)}{n}$. Under both definitions of the uniform price, perfect collusion with equal sharing is sustainable on a stationary path in the repeated uniform price auction whenever it is sustainable on a stationary path in the repeated discriminatory auction. Moreover, if either (i) $u = e$ or (ii) $u = a$, $n > 2$, and $k > \frac{2}{n}d(p^m)$ hold, there exists a nondegenerate interval of discount factors $[\underline{\delta}^u, \bar{\delta})$ for which perfect collusion with equal sharing is sustainable on a stationary path in the repeated uniform price auction, but not in the repeated discriminatory auction.

[5] This is because if one firm sets p^m and the other firm $p < p^m$, then the price is equal to p.

[6] Fabra (2003) only analyzes the duopoly.

FIGURE 2

RANGE OF POSSIBLE DIVISIONS OF MONOPOLY OUTPUT FOR THE LOW-PRICE FIRM ON A STATIONARY PERFECT EQUILIBRIUM PATH FOR $d(p) = \max\{0, 1 - p\}$ AND $k = \frac{1}{2}$

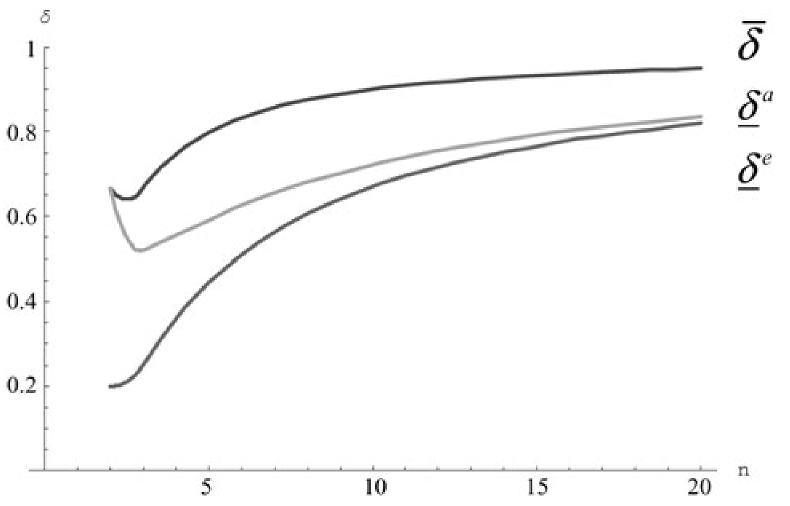


Proof. In the discriminatory auction, it follows from a simple extension of Fabra's (2003) Proposition 3 to our price-quantity supergame with n symmetric firms that incentives to deviate are minimized on symmetric paths on which firms set the same price, p^m. Thus, the minimum value of the discount factor for which a perfectly collusive stationary perfect equilibrium path with equal sharing exists is obtained from a firm's incentive constraint on a path on which all firms offer capacity k at a price equal to p^m and is given by $\bar{\delta}$. Hence the result follows from Propositions 3 and 4. *Q.E.D.*

The results in Theorem 1 are illustrated for an example with linear demand in Figure 3. Let $d(p) = \max\{0, 1 - p\}$. In this case, $p^m = \frac{1}{2}$, $d(p^m) = \frac{1}{2}$, and $y^c = \frac{1}{n+1}$. Suppose that $k > y^c$. Using the Market Clearing Price approach, the optimal deviation on τ^{me} yields profit equal to

FIGURE 3

CRITICAL VALUES OF THE DISCOUNT FACTOR AS A FUNCTION OF THE NUMBER OF FIRMS FOR $d(p) = \max\{0, 1 - p\}$ and $k = \frac{1}{2}$



$\frac{(n+1)^2}{16n^2}$. Using the Maximum Accepted Price rule, it is simple to show that if $n > 2$, for $j \neq i$, the optimal deviation on τ_i^{ma} is for j to undercut i but not $h \neq i, j$. j's deviation profit is then equal to $\frac{(n+2)^2}{16n^2}$. Assuming $k = \frac{1}{2}$ so that each firm's capacity is sufficient to supply the monopoly output, we obtain[7]

$$\bar{\delta} = \begin{cases} \frac{2}{3} & \text{if } n = 2, \\ \frac{n-1}{n} & \text{if } n = 3, 4, \ldots, \end{cases}$$

$$\underline{\delta}^e = \begin{cases} \frac{1}{5} & \text{if } n = 2, \\ \left(\frac{n-1}{n+1}\right)^2 & \text{if } n = 3, 4, \ldots, \end{cases}$$

$$\underline{\delta}^a = \begin{cases} \frac{2}{3} & \text{if } n = 2, \\ \frac{4+n^2}{(n+2)^2} & \text{if } n = 3, 4, \ldots. \end{cases}$$

Under the Maximum Accepted Price rule, as in Brock and Scheinkman (1985),[8] the relationship between the lower bound on the discount factor for which perfect collusion is sustainable and the number of firms is always nonmonotonic.[9] Therefore, when demand is linear, the result that for sufficiently low capacity (such that minmax payoffs are strictly positive for some n), decreasing the number of firms in the industry makes collusion more difficult continues to hold in the repeated price-quantity uniform price auction. Furthermore, independent of k, when the Maximum Accepted Price rule is used in the uniform price auction, the number of firms at which the critical value of the discount factor is minimized is greater in the uniform price auction than in the discriminatory auction. To see this, recall that in the linear demand case, Brock and Scheinkman (1985) show that $\bar{\delta}$ is initially decreasing in n. This is because a decrease in punishment payoffs more than offsets an increase in incentives to deviate for such values of n. The same holds in the uniform price auction. However, note that $\frac{d}{dn}[\frac{(n+2)^2}{16n^2}] < 0$, that is, deviation profit decreases with n in the repeated uniform price auction, whereas deviation profit is independent of n in the discriminatory auction. Therefore, $\underline{\delta}^a$ must decrease with n for a range of values of n above the minimizer of $\bar{\delta}$.[10] In the uniform price auction with the Market Clearing Price rule, in our example, even at low values of n an increase in incentives to deviate is not offset by a decrease in punishment value. Therefore, $\underline{\delta}^e$ increases in n for all $n \geq 2$. However, this is not generally true. The relationship between $\underline{\delta}^e$ and n depends on capacity, k. If k is sufficiently low, then $\underline{\delta}^e$ decreases with n initially and the relationship between the critical discount factor and the number of firms is again nonmonotonic.

In ending this section, we should note that, in the discriminatory auction, perfect collusion with equal sharing cannot arise for a larger set of discount factors under proportional rationing than under efficient rationing. Under proportional rationing, a firm's incentive to deviate is still minimized on a path on which each firm offers capacity at the monopoly price p^m and, at this minimum, the relevant payoffs are equal to those under efficient rationing. It follows that if incentive constraints are to be relaxed under proportional rationing, it must be because firms can be held to a payoff in the punishment phase that is lower than their security level under the efficient rationing rule. However, it is straightforward to show that in the context of our model,

[7] All formulas are derived from equations (A1), (A3), (A4), and (A8), which can be found in the Appendix. Figure 3 is drawn treating n as a continuous variable. For $n \in [2, 3)$, the specific functional form for each critical value of δ can be obtained from straightforward calculations.

[8] See also Davidson and Deneckere (1984) and Lambson (1987) for analyses of discriminatory auctions in a Bertrand-Edgeworth oligopoly supergame.

[9] Brock and Scheinkman's (1985) Figure 2 is drawn for $\frac{1-\delta}{\delta}$.

[10] In our example, the optimal number of firms is between two and three in both the discriminatory and the uniform price auction. However, the same intuition as that outlined above also works at values of k for which the optimal number of firms is greater than three.

security-level payoffs are always at least as large as those under efficient rationing.[11] Thus, collusion under the proportional rationing rule can be no easier to sustain than under efficient rationing. We argued in Section 2 that proportional rationing in a uniform price auction does not affect the analysis under the Market Clearing Price rule and does not make sense under the Maximum Accepted Price rule. Hence, to the extent that the rationing rule may be applied, proportional rationing only reinforces the result of Theorem 1.[12]

5. Bid functions with an arbitrary number of steps

■ In this section, we generalize the results obtained in the previous section to a setting in which firms can submit bid functions with an arbitrary finite number of steps. In the one-shot simultaneous move game $\Gamma^u(k, 0, L)$, assume that a firm's strategy is a vector of price-quantity pairs defining an incremental bid function $((p_i^{l_i}, q_i^{l_i}))_{l_i=1}^{L_i}$, $L_i \leq L$ and $p_i^1 < \cdots < p_i^{L_i}$. L is a finite number representing the maximum number of admissible steps in the (nondecreasing) bid function of each firm. For each pair, the price $p_i^{l_i}$ represents the minimum price at which firm i is willing to sell the quantity increment $q_i^{l_i}$. We assume $p_i^{l_i} \in \mathbb{R}_+$ and $\sum_{l_i} q_i^{l_i} \leq k$. A strategy profile is a vector of bid functions $(\mathbf{p}, \mathbf{q}) = ((p_1^{l_1}, q_1^{l_1})_{l_1=1}^{L_1}, \ldots, (p_n^{l_n}, q_n^{l_n})_{l_n=1}^{L_n})$. We denote by $(\mathbf{p}_{-i}, \mathbf{q}_{-i})$ the vector composed of firm i's rivals' bid functions.

Below we define residual demand and a firm's sales at a given price in this setting. To this effect, let $l_i(p)$ be the index of the step associated with price p in firm i's strategy, that is, if there exists l_i such that $p_i^{l_i} = p$, then $l_i(p) = l_i$. Otherwise, define $l_i(p) \equiv \emptyset$. Let p_i^s be the set of prices submitted by firm i for each of its quantity increments, $p_i^s = \{p \in \mathbb{R}_+ \mid l_i(p) \neq \emptyset\}$, and let $p_i^s(p)$ be the set of prices submitted by i that are less than or equal to p. That is, $p_i^s(p) = [0, p] \cap p_i^s$. Define $q_i(p)$ to be the quantity increment associated with price p in firm i's strategy. Formally,

$$q_i(p) = \begin{cases} q_i^{l_i} & \text{if } l_i(p) = l_i, \\ 0 & \text{if } l_i(p) = \emptyset, \end{cases}$$

so that firm i's quantity supplied at p is $\sum_{z \in p_i^s(p)} q_i(z)$.

Define $\hat{q}_i(p) = \min\{q_i(p), d(0)\}$. Assuming efficient rationing and given a strategy profile $(\mathbf{p}, \mathbf{q})$, residual demand at p is then easily defined as follows.

$$R(p \mid \mathbf{p}, \mathbf{q}) = \max\left\{ d(p) - \sum_{i \in N} \sum_{z \in p_i^s(p) \setminus \{p\}} \hat{q}_i(z), 0 \right\}.$$

Note that if the minimum price submitted by any firm is greater than or equal to p, then $R(p \mid \mathbf{p}, \mathbf{q}) = d(p)$. Given a strategy profile $(\mathbf{p}, \mathbf{q})$, firm i's sales at a price $p \in p_i^s$ are equal to

$$s_i(p \mid \mathbf{p}, \mathbf{q}) = \min\left\{ \hat{q}_i(p), \frac{\hat{q}_i(p)}{\sum_{l \in N} \hat{q}_l(p)} R(p \mid \mathbf{p}, \mathbf{q}) \right\},$$

[11] Note that for any value of the capacity parameter k, a firm can always set p^r, its residual demand monopoly price after its $n-1$ rivals have sold their capacity under efficient rationing, and be certain to obtain at least $p^r\,[d(p^r) - (n-1)k]$, which is its minmax payoff under efficient rationing. It is possible to show that for a wide range of values of k, by setting p^r, a firm obtains a strictly higher payoff under proportional rationing than it does under efficient rationing.

[12] The literature on electricity markets that has used the Bertrand-Edgeworth model to analyze both uniform price and discriminatory auctions has often assumed box demand (see, for instance, Fabra, von der Fehr, and Harbord, 2006). In this case, every commonly used rationing rule defines the same residual demand. Fabra (2003), Ubéda (2004), and, in part of their analysis, Fabra, von der Fehr, and Harbord (2006) allow for a decreasing demand. In her analysis, Fabra assumes efficient rationing, whereas Ubéda and Fabra, von der Fehr, and Harbord assume that even in the discriminatory auction, every consumer pays the same price given by the market clearing price. Hence, in these studies, residual demand again coincides with efficient rationing. Although we could find no evidence that efficient rationing has been used in existing electricity markets, such as the UK pool, that were once organized as discriminatory auctions, as argued above, the results in this article continue to be relevant even with random rationing. Finally, note that efficient rationing would arise as the optimal choice of a price-taking system operator who maximizes efficiency with complete information of consumers' willingness to pay for each unit.

so that firm i's total sales amount to

$$\mathbf{s}_i(\mathbf{p}, \mathbf{q}) = \sum_{p \in p_i^s} s_i(p \mid \mathbf{p}, \mathbf{q})$$

and firm i's profit is $\pi_i(\mathbf{p}, \mathbf{q}) = \mathbf{P}^u(\mathbf{p}, \mathbf{q})\, \mathbf{s}_i(\mathbf{p}, \mathbf{q})$, $\forall u \in \{e, a\}$. Finally, we let $\pi_i^*(\mathbf{p}_{-i}, \mathbf{q}_{-i})$ denote firm i's profit from an optimal deviation when its rivals play $(\mathbf{p}_{-i}, \mathbf{q}_{-i})$.

The supergames obtained by repeating the component game above are defined in a similar manner as the supergame with $L = 1$. We denote the supergame in which the number of admissible steps is L by $\Gamma^u(k, 0, \delta, L)$, for $u \in \{e, a\}$. Lemma 5 below simplifies the analysis of the games with L-step bidding functions by showing that, when compared to the price-quantity approach, expanding the strategy space does not allow firms to obtain greater one-period deviation payoffs.

Lemma 5. For every $k \in \mathbb{R}_+$, $L \geq 1$, $u \in \{e, a\}$, and $i \in N$ in the game $\Gamma^u(k, 0, L)$, $\pi_i^*(\mathbf{p}_{-i}, \mathbf{q}_{-i})$, firm i's optimal deviation payoff when its rivals set $(\mathbf{p}_{-i}, \mathbf{q}_{-i})$ can be obtained by restricting its response to bidding functions using a single step ($L_i = 1$).

Proof. See the Appendix.

The extension of Lemmas 3 and 4 to the more general setting introduced above is relatively straightforward. Statements similar to those in Proposition 2 and Lemmas 1 and 2 are valid. Beginning with Proposition 1 and making use of Lemma 5, note that the pure strategy equilibria for the simultaneous move game characterized for $L = 1$ are clearly equilibria of the simultaneous move game with $L \geq 2$ admissible steps. We hence have the following result.

Proposition 5. For every $k \in \mathbb{R}_+$, $L \geq 1$, $u \in \{e, a\}$, and $i \in N$, there exists a pure strategy equilibrium of the game $\Gamma^u(k, 0, L)$ in which $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = \underline{\pi}$.

Allowing for L steps does not change a firm's minmax profit $\underline{\pi}$. This is because the worst that can be imposed on a given firm i is obtained by maximizing i's profit after its rivals have sold their capacity at a price of 0. Therefore, Proposition 2 continues to hold when $L \geq 2$ because perfect equilibrium security-level punishments are attained with strategies using only one step. The analog of Lemma 1 for the case $L \geq 2$ states that if for some $i \in N$, $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$ satisfy the two following conditions: first, the total quantity put on the market by i's rivals does not change, and second, the residual demand faced by firm i at every p is no less when firms set $(\mathbf{p}', \mathbf{q}')$ than when they set $(\mathbf{p}, \mathbf{q})$, then firm i's deviation profit under $(\mathbf{p}, \mathbf{q})$ is no greater than under $(\mathbf{p}', \mathbf{q}')$. Formally, these two conditions are

$$\sum_{j \in N \setminus \{i\}} \sum_{p \in p_j^s} q_j(p) = \sum_{j \in N \setminus \{i\}} \sum_{p \in p_j^{s\prime}} q_j'(p)$$

and, for every $p \in [0, \bar{p}]$,

$$\sum_{j \in N \setminus \{i\}} \sum_{z \in p_j^s(p) \setminus \{p\}} q_j(z) \geq \sum_{j \in N \setminus \{i\}} \sum_{z \in p_j^{s\prime}(p) \setminus \{p\}} q_j'(z).$$

To generalize Lemma 2, note that the requirements are that the uniform price be p^m and that $s_i(\mathbf{p}, \mathbf{q}) = \frac{d(p^m)}{n}$. Moreover, supply of lower-priced units should sum up to a quantity less than $d(p^m)$,[13] while still allowing firms that do not have p^m in the support of their strategy to sell $\frac{d(p^m)}{n}$. Thus, on a perfectly collusive stationary path with equal sharing,

$$\sum_{j \in N} \sum_{p \in p_j^s(p^m) \setminus \{p^m\}} \hat{q}_j(p) \leq d(p^m) \tag{2}$$

with a strict inequality if the uniform price is given by the Maximum Accepted Price rule. Furthermore, if $p^m \notin p_h^s$, then

[13] This quantity must be strictly less than $d(p^m)$ if the uniform price is given by the Maximum Accepted Price rule.

$$\sum_{p \in p_h^s(p^m)} \hat{q}_h(p) = \frac{d(p^m)}{n}.$$

As before, when the uniform price is given by the Market Clearing Price rule, on the perfectly collusive stationary path that minimizes incentives to deviate, the price will be set at p^m by having firms withhold their capacity to offer only $\frac{d(p^m)}{n}$ each and set a price that minimizes deviation profits. When the uniform price is given by the Maximum Accepted Price rule, the highest accepted bid has to be p^m so that at least one firm has to offer a positive quantity at p^m.

□ **The Market Clearing Price rule.** For every $L \geq 1$, it follows from the above discussion that a perfectly collusive stationary path that minimizes incentives to deviate is such that all firms set $p = \underline{p}(q_-^m)$ and $q = \frac{d(p^m)}{n}$. Under the Market Clearing Price definition, one step is sufficient to minimize incentives to deviate on perfectly collusive stationary paths with equal sharing. Thus, we have Proposition 6.

Proposition 6. Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 1$, in $\Gamma^e(k, 0, \delta, L)$, the path τ^{me} minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing. That is, the incentive to deviate is minimized over all possible finite step functions by employing a simple (one-step) price-quantity strategy.

Theorem 2 below then follows directly from Propositions 3 and 6. It shows that a path on which firms use a bidding function with only one step minimizes the critical value of the discount factor above which a stationary perfect equilibrium path that achieves the monopoly outcome with equal sharing exists.

Theorem 2. Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 1$, in $\Gamma^e(k, 0, \delta, L)$, perfect collusion with equal sharing is sustainable on a stationary path if and only if $\delta \geq \underline{\delta}^e$. That is, if the strategy space is extended to allow for arbitrary finite step functions, whenever perfect collusion with equal sharing is sustainable using strategies involving more than one step, it is sustainable employing simple one-step price-quantity strategies.

□ **The Maximum Accepted Price rule.** The difference between the Market Clearing Price rule and the Maximum Accepted Price rule is that under the latter, on a perfectly collusive path with equal sharing, at least one firm must set $p = p^m$, whereas this is not required in the former. However, in contrast to the price-quantity supergame, a firm setting p^m may also offer part of its capacity at prices below p^m. The main insight from Lemma 4, namely that incentives to deviate are minimized when as much capacity as possible is offered at a sufficiently low price, applies here as well. However, when $L \geq 2$, we show that all firms offer some quantity at p^m. Because in our model nothing prevents firms from offering infinitesimally small quantities at a given price, we conduct the analysis for a given aggregate quantity ϵ sold at p^m and show that as this quantity goes to zero, the lowest discount factor for which perfect collusion is sustainable on a stationary path with the Maximum Accepted Price approach converges to that in the Market Clearing Price approach, $\underline{\delta}^e$. We may interpret ϵ as being part of the tacitly collusive agreement between the firms.

Define $q_-^\epsilon \equiv \frac{(n-1)(d(p^m)-\epsilon)}{n}$ and for a given minimum quantity agreement $\epsilon > 0$, consider the two-step bidding function $(p_i^\epsilon, q_i^\epsilon)$ for firm $i \in N$ where

$$\left(p_i^\epsilon, q_i^\epsilon\right) = \left(\left(\underline{p}(q_-^\epsilon), \frac{d(p^m)-\epsilon}{n}\right), \left(p^m, k - \frac{d(p^m)-\epsilon}{n}\right)\right).$$

Let the path τ^ϵ be such that firm i sets $(p_i^\epsilon, q_i^\epsilon)$, $\forall i \in N$. Proposition 7 shows that for a given quantity agreement $\epsilon > 0$, τ^ϵ minimizes incentives to deviate in the class of perfectly collusive paths with equal sharing. Therefore, in characterizing paths that minimize incentives to deviate in the class of perfectly collusive stationary paths with equal sharing, two steps are sufficient.

Proposition 7. Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 2$ and quantity agreement $\epsilon \in (0, d(p^m)]$, in $\Gamma^a(k, 0, \delta, L)$, the path τ^ϵ minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing. That is, the incentive to deviate is minimized over all possible finite step functions by employing a two-step strategy.

Proof. See the Appendix.

The intuition behind Proposition 7 is as follows. Suppose that the aggregate quantity offered at a common price $p < p^m$ is $d(p^m) - \epsilon$, for some quantity agreement $\epsilon > 0$, where p is sufficiently low that no firm would ever undercut it. Then, if l firms each offer $\frac{d(p^m)}{n} - \frac{\epsilon}{l}$ at p and $k - [\frac{d(p^m)}{n} - \frac{\epsilon}{l}]$ at p^m, while the remaining firms simply offer $\frac{d(p^m)}{n}$ at p, the uniform price is indeed p^m and each firm earns $p^m \frac{d(p^m)}{n}$. Furthermore, it is clear that the $n - l$ low-price firms have greater incentives to deviate than the high-price firms, because they face a strictly higher residual demand at every price between p and p^m. If k is sufficiently small that the price would remain at p^m if a low-price firm offered its capacity at p, then an optimal deviation consists of offering capacity at p to earn $p^m k$. Otherwise, the optimal deviation is to set the residual monopoly price after $n - 1$ firms sell their quantity offered at p, where this quantity is equal to

$$(n - l - 1)\frac{d(p^m)}{n} + l\left(\frac{d(p^m)}{n} - \frac{\epsilon}{l}\right) = (n - 1)\frac{d(p^m)}{n} - \epsilon. \tag{3}$$

Now suppose that $l = n$, so that all firms set $k - [\frac{d(p^m)-\epsilon}{n}]$ at p^m. Then each firm's optimal deviation is to either offer capacity at p or set the residual monopoly price after $n - 1$ firms sell their quantity offered at p, where this quantity is equal to

$$(n - 1)\left(\frac{d(p^m)}{n} - \frac{\epsilon}{n}\right) = q_-^\epsilon. \tag{4}$$

Equation (4) is clearly greater than (3), so that a firm's deviation profit is lowest when all firms offer the same quantity both at the low price and the monopoly price. It follows that on a path that minimizes incentives to deviate, all n firms will offer some quantity at p^m. Thus all firms will have two steps in their bidding function. One of the steps must be at a low price in order to prevent rivals from undercutting and the quantity offered at that price must be the highest quantity consistent with p^m being the uniform price. The second step effectively sets the price at p^m. To sustain the perfectly collusive outcome, there is nothing to gain from being allowed to include additional steps in the bidding functions. Theorem 3 below follows directly from Proposition 7.

Theorem 3. Suppose $k > \frac{d(p^m)}{n}$. For every number of admissible steps $L \geq 2$ and quantity agreement $\epsilon \in (0, d(p^m)]$, perfect collusion with equal sharing is sustainable on a stationary path in $\Gamma^a(k, 0, \delta, L)$, if and only if τ^ϵ is sustainable. Moreover, there exists $\underline{\delta}^a(\epsilon)$ such that τ^ϵ is sustainable if and only if $\delta \geq \underline{\delta}^a(\epsilon)$ and $\underline{\delta}^a(\epsilon) \downarrow \underline{\delta}^e$ as $\epsilon \to 0$.

Apart from directly relaxing firms' incentive constraints on perfectly collusive paths, allowing for more than one step in a firm's bidding function generates another interesting difference as compared to the simple price-quantity approach. If $L \geq 2$, on a perfectly collusive stationary path that minimizes incentives to deviate, firms set identical two-step bidding functions. This symmetry in firms' actions is not a property of the most collusive path in the price-quantity supergame, as a single firm must play the role of the high-price firm. This further implies that under the Maximum Accepted Price rule, capacity withholding is effective in relaxing incentive constraints even in a duopoly, while as we have shown in Proposition 4, this is not the case in the price-quantity approach.

6. Conclusion

■ We have examined the nature of collusive stationary perfect equilibrium paths in an infinitely repeated multiunit uniform price auction with capacity-constrained firms. Under two different definitions of the market price in a uniform price auction, each appearing prominently in the literature, we characterize the set of paths that minimize the incentive to deviate while supporting the monopoly price with equal sharing of output. We then show that these paths can be supported as stationary perfect equilibria for a wider range of discount factors than under a repeated discriminatory price auction.

Using the Market Clearing Price rule to determine the uniform price, we show that extending firms' strategy spaces to allow them to place bids involving any finite number of price-quantity pairs neither enhances nor hinders the firms' ability to collude. Surprisingly, L-step bidding functions, $L \geq 2$, cannot improve upon the ability to collude with one-step bidding functions, which involve the simple choice of a price-quantity pair, nor can they make collusion more difficult to sustain. Because such step functions are quite common in electricity markets, our analysis extends the analysis of collusion to these more complicated markets with step supply functions.

Using the Maximum Accepted Price rule, the capacity withholding properties stated above continue to hold, although with this rule, bidding functions that involve two steps ($L \geq 2$) strictly lower the incentive to deviate from the most collusive outcome when compared to the price-quantity game ($L = 1$). Further increases in the number of steps in the bidding function provide no advantage: collusive outcomes are no easier or harder to support when the strategy space is extended to $L = 3$ than with $L = 2$. Hence, under the Maximum Accepted Price rule as well, optimal collusion can be attained with a drastically restricted set of available step supply functions requiring only two steps.

Appendix

■ Proofs of Propositions 1, 3, 4, and 7 and Lemmas 3–5 follow.

Proof of Proposition 1. We first show that all strategy profiles described in (i)–(iii) of Proposition 1 form Nash equilibria of the respective games $\Gamma^u(k, 0)$, $u \in \{e, a\}$. For $k \leq y^c$, we show in (a) that $E^u(k, 0)$ is a set of Nash equilibria for $u \in \{e, a\}$. For $k \geq \frac{d(0)}{n-1}$ and $k \in (y^c, \frac{d(0)}{n-1})$, we show in (b) and (c), respectively, that $E^a(k, 0)$ is a set of Nash equilibria under both the Market Clearing Price and the Maximum Accepted Price rules. Finally, in (d), we show that for $k > y^c$, $C(0)$ is an additional set of Nash equilibria under the Market Clearing Price rule. We complete the proof of the proposition by demonstrating that for each $u \in \{e, a\}$, the sets of strategy profiles described in (i)–(iii) characterize the complete set of Nash equilibria of $\Gamma^u(k, 0)$.

(a) Suppose $k \leq y^c$. Note that in this case, all firms are capacity constrained at $p^r = \underline{p} = P(nk)$. Furthermore, for each $u \in \{e, a\}$, all strategy profiles in the statement of the proposition yield $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = P(nk)k$, $\forall i \in N$. To prove (i), we examine each definition of the uniform price separately. Consider first the Market Clearing Price rule or $u = e$. Suppose that there exists a firm i such that $\forall j \neq i$, $p_j^* \leq P(nk)$, and $q_j^* = k$. We show that firm i's best response to (p_{-i}^*, q_{-i}^*) is the set of price-quantity pairs (p_i^*, q_i^*) such that $p_i^* \leq P(nk)$ and $q_i^* = k$. Let (p_i', q_i') be firm i's strategy and $\mathbf{P}^{e\prime}$ the resulting uniform price. Then we either have $p_i' > \mathbf{P}^{e\prime}$, in which case firm i's profit from (p_i', q_i') is equal to zero, or $p_i' \leq \mathbf{P}^{e\prime}$. In the latter case, one can easily check that $\mathbf{P}^{e\prime} = \max\{p_i', P((n-1)k + q_i')\}$ and that firm i's sales are equal to $\min\{q_i', R(p_i' \mid p_i', q_i', p_{-i}^*, q_{-i}^*)\}$. Furthermore, firm i's payoff is equal to $\max\{p_i', P((n-1)k + q_i')\} \min\{q_i', R(p_i' \mid p_i', q_i', p_{-i}^*, q_{-i}^*)\}$, which is maximized by setting $p_i' \leq P(nk)$ and $q_i' = k$, at which firm i obtains a payoff equal to $P(nk)k > 0$.

Consider now the Maximum Accepted Price rule or $u = a$. Suppose that there exists a firm i such that $\forall j \neq i$, $p_j^* \leq P(nk)$, and $q_j^* = k$. We show that firm i's best response to (p_{-i}^*, q_{-i}^*) is the set of price-quantity pairs (p_i^*, q_i^*) such that $p_i^* \leq P(nk)$ and $q_i^* = k$, unless $p_j^* < P(nk)$, $\forall j \neq i$, in which case, $p_i^* = P(nk)$ and $q_i^* = k$ is firm i's best response. Let (p_i', q_i') be firm i's strategy and $\mathbf{P}^{a\prime}$ the resulting uniform price. Again, if $p_i' > \mathbf{P}^{a\prime}$, firm i obtains a profit of zero. On the other hand, if (p_i', q_i') is such that $p_i' \leq \mathbf{P}^{a\prime}$, then firm i's payoff from (p_i', q_i') is given by $\mathbf{P}^{a\prime} \min\{q_i', R(p_i' \mid p_i', q_i', p_{-i}^*, q_{-i}^*)\}$. Because $p_j^* \leq P(nk)$, $\forall j \neq i$, if $p_i' > P(nk)$, $P^{a\prime} = p_i'$ and thus firm i's payoff is equal to $p_i' \min\{q_i', R(p_i' \mid p_i', q_i', p_{-i}^*, q_{-i}^*)\} < P(nk)k$. However, firm i could obtain $P(nk)k$ by decreasing the price to $P(nk)$ and offering $q_i' = k$ instead. Furthermore, given any q_i', firm i's payoff from setting $p_i' < P(nk)$ can be no greater than $P(nk)k$ because $k < y^c$ implies $\underline{p} = P(nk)$ and $\underline{p}k = P(nk)k$. So suppose $p_j^* < P(nk)$, $\forall j \neq i$. If $p_i' < P(nk)$, it is clear that $P^{a\prime} < P(nk)$, in which case firm i obtains strictly less than $P(nk)k$, a payoff it could obtain by raising the price to $P(nk)$ and offering capacity. On the other hand, if there exists $h \neq i$ for which $p_h^* = P(nk)$, then for any $p_i' \leq P(nk)$, we have $P^{a\prime} = P(nk)$ and, thus, by setting $q_i' = k$, firm i obtains exactly $P(nk)k$.

(b) Suppose $k \geq \frac{d(0)}{n-1}$. In this case, $n - 1$ firms can serve demand at a price equal to marginal cost. To prove that all strategy profiles $(\mathbf{p}^*, \mathbf{q}^*) \in E^a(k, 0)$ are equilibria, note that $\mathbf{P}^u(\mathbf{p}^*, \mathbf{q}^*) = 0$ for every $(\mathbf{p}^*, \mathbf{q}^*) \in E^a(k, 0)$ and

$u \in \{e, a\}$. Hence $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = \underline{\pi} = 0, \forall i \in N$. Because $\forall h \in L(0 \mid \mathbf{p}^*, \mathbf{q}^*)$, $\sum_{l \in L(0 \mid \mathbf{p}^*, \mathbf{q}^*) \setminus \{h\}} \hat{q}_l^* \geq d(0)$, residual demand is zero at every $p \in (0, \bar{p}]$. It thus follows that no firm has a profitable deviation in this case.

(c) Suppose $k \in (y^c, \frac{d(0)}{n-1})$. We show that strategy profiles in $E^a(k, 0)$ are Nash equilibria under both rules. For $u \in \{e, a\}$, note that if $\underline{y} > d(p^r) - (n-1)k$ (which we show below), all strategy profiles in $E^a(k, 0)$ yield $P^u(\mathbf{p}^*, \mathbf{q}^*) = p^r$, $\pi_j(\mathbf{p}^*, \mathbf{q}^*) = \underline{\pi}$ and for every firm $i \neq j$, $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = p^r k$. By definition of $\underline{p}$, firm j has no incentive to deviate from (p^r, q_j^*). Moreover, a low-price firm, say i, $i \neq j$, can neither increase its payoff by slightly undercutting other low-price firms nor by setting a price in the interval $(\underline{p}, p^r)$, because in both cases it would obtain exactly $p^r k$. We now show that such a firm i has no incentive to deviate to the residual demand monopoly price after $n - 2$ firms have sold k and firm j has sold q_j^*. To this effect, define $\underline{y}$ as the unique solution in y to

$$\underline{\pi}((n-2)k + y) = p^r k.$$

The uniqueness of $\underline{y}$ in the interval $(-(n-2)k, k)$ follows from $\underline{\pi}(0) = \max\{p^m, P(k)\}\min\{d(p^m), k\} \geq P(k)k > p^r k$, $\underline{\pi}((n-1)k) < p^r k$, and the fact that $\underline{\pi}((n-2)k + y)$ is continuous and strictly decreasing as a function of y on the closed interval $[-(n-2)k, k]$. Because $\underline{\pi}((n-2)k + q_j^*)$ is strictly decreasing in q_j^*, if $q_j^* > \underline{y}$, then $p^r k > \underline{\pi}((n-2)k + q_j^*)$. Therefore, if $q_j^* \geq \underline{y}$, firm i prefers to obtain $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = p^r k$ rather than raising its price to $p^r((n-2)k + q_j^*)$ and serving residual demand.

It remains to show that $\underline{y} > d(p^r) - (n-1)k$. By way of contradiction, suppose first that $\underline{y} < d(p^r) - (n-1)k$. Let $q_j^* = d(p^r) - (n-1)k$. Then $\underline{\pi}((n-2)k + q_j^*) = \max_{q \leq k}\{P(q + (n-2)k + q_j^*)q\} \geq P(k + (n-2)k + q_j^*)k = P(d(p^r))k = p^r k$, a contradiction to $q_j^* > \underline{y}$.

Now suppose, again by way of contradiction, that $\underline{y} = d(p^r) - (n-1)k$. Then $(n-2)k + \underline{y} = d(p^r) - k$, so that $\underline{\pi}((n-2)k + \underline{y}) = \underline{\pi}(d(p^r) - k)$. We now show that for $k > y^c$, $\underline{\pi}(d(p^r) - k) > p^r k$ holds, implying that to obtain the equality $\underline{\pi}((n-2)k + \underline{y}) = p^r k$, we must have $\underline{y} > d(p^r) - (n-1)k$. To show that $\underline{\pi}(d(p^r) - k) > p^r k$ holds, suppose first that $k > r(d(p^r) - k)$. In this case, we have $\underline{\pi}(d(p^r) - k) = P(d(p^r) - k + r(d(p^r) - k))r(d(p^r) - k) > P(d(p^r) - k + k)k = p^r k$. If $k \leq r(d(p^r) - k)$ instead, then $\underline{\pi}(d(p^r) - k) = P(d(p^r) - k + k)k = p^r k$. However, we show that $k \leq r(d(p^r) - k)$ cannot arise for $k \in (y^c, \frac{d(0)}{n-1})$. Indeed, $k \leq r(d(p^r) - k)$ implies $d(p^r) \leq d(p^r) - k + r(d(p^r) - k)$. Because $d'(p) < 0$, the last inequality implies $p^r \geq P(d(p^r) - k + r(d(p^r) - k))$ or $p^r((n-1)k) \geq p^r(d(p^r) - k)$. It thus follows from the fact that $p^r(q)$ is decreasing in q that $(n-1)k \leq d(p^r) - k$, which in turn implies $nk \leq d(p^r)$. Because $nk \leq d(p^r)$ is equivalent to $k \leq y^c$, however, we have a contradiction to $k \in (y^c, \frac{d(0)}{n-1})$. Therefore, $\underline{y} > d(p^r) - (n-1)k$. This completes our proof that for $k \in (y^c, \frac{d(0)}{n-1})$, all strategy profiles in $E^a(k, 0)$ are Nash equilibria under both the Market Clearing Price and the Maximum Accepted Price rules.

(d) Consider the game $\Gamma^e(k, 0)$. Define $q_-^c \equiv (n-1)y^c$. For $k > y^c$, we now show that any $(\mathbf{p}^*, \mathbf{q}^*)$ where $p_i^* \leq \underline{p}(q_-^c)$ and $q_i^* = y^c, \forall i$, is a Nash equilibrium of $\Gamma^e(k, 0)$. For such strategy profiles, the uniform price is equal to $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(ny^c)$ and each firm i's payoff $\pi_i(\mathbf{p}^*, \mathbf{q}^*)$ is equal to $P(ny^c)y^c$. By definition of $\underline{p}(q_-^c)$, no firm has an incentive to undercut any of its rivals' prices and expand output above y^c. Moreover, because y^c is the unique Cournot equilibrium output, $p^r(q_-^c) = P(ny^c)$ and $\underline{\pi}(q_-^c) = P(ny^c)y^c$, so that no firm has an incentive to set its price above $P(ny^c)$ (to sell strictly less than y^c) either. It follows that $(\mathbf{p}^*, \mathbf{q}^*)$ is an equilibrium of $\Gamma^e(k, 0)$.

In the remainder of the proof, we show that for $u \in \{e, a\}$, the equilibria characterized above are the only equilibria of $\Gamma^u(k, 0)$. In the analysis below, let $(\mathbf{p}^*, \mathbf{q}^*)$ denote a pure strategy equilibrium and, for $i \in N$, let $\bar{\pi}_i$ be firm i's profit in the equilibrium.

□ Maximum Accepted Price rule

Lemma A1. Suppose $k < \frac{d(0)}{n-1}$; then, for every $(\mathbf{p}^*, \mathbf{q}^*)$, $\bar{p} > \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$ and $p_i^* \leq \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*), \forall i \in N$.

Proof. It is clear that $\bar{p} > \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$ because both $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = \bar{p}$ and $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = 0$ imply $\bar{\pi}_j = 0, \forall j$. Because $k < \frac{d(0)}{n-1}$, however, we have $\underline{\pi} > 0$, a contradiction to $\bar{\pi}_j \geq \underline{\pi}$ in equilibrium. Finally, suppose there exists a firm i setting $p_i^* > \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. Then firm i's profit is equal to zero because its sales are zero. Hence, an argument similar to the above applies. *Q.E.D.*

Lemma A2. If $(\mathbf{p}^*, \mathbf{q}^*)$ is such that there exists a firm $i \in N$ for which $0 \leq p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$, then $q_i^* = s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) = k$.

Proof. The proof is in two parts. First we show that each firm i setting $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ offers q_i^* such that $q_i^* = k$. Then, we show that $s_i^* = k$ follows. Suppose contrary to the statement of the lemma that there exists a firm i setting $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ and $q_i^* < k$. It is straightforward to show that for small enough $\epsilon > 0$, $\mathbf{P}^a(\mathbf{p}^*, q_i^* + \epsilon, q_{-i}^*) = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. However, then $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)(q_i^* + \epsilon) > \bar{\pi}_i = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)q_i^*$, a contradiction to (p_i^*, q_i^*) being part of an equilibrium. Suppose now that $q_i^* = k > s_i^*$. This can only be the case if firm i sells to residual demand at $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. Then, however, by Definition 2, p_i^* must be the uniform price, which contradicts $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. Therefore, $p_i^* < \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) \Rightarrow q_i^* = s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) = k$. *Q.E.D.*

Lemma A3. If $(\mathbf{p}^*, \mathbf{q}^*)$ is such that $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$, then either (i) exactly one firm sets $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$, or (ii) $s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) = k, \forall i \in N$.

Proof. Suppose contrary to the statement of the lemma that a group of l firms, $l \geq 2$, tie at $p = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$. It follows immediately that if firm h sets $p_h^* = p$, but $s_h(p \mid \mathbf{p}^*, \mathbf{q}^*) < k$, then firm h can strictly increase its profit by slightly undercutting firms setting p and offering capacity to earn:

$$p \min\{\hat{k}, R(p \mid \mathbf{p}^*, \mathbf{q}^*)\} > ps_h(p \mid \mathbf{p}^*, \mathbf{q}^*) = p \min\left\{\hat{q}_h, \frac{\hat{q}_h}{\sum_{j \in L(p)} \hat{q}_j} R(p \mid \mathbf{p}^*, \mathbf{q}^*)\right\}.$$

This inequality and Lemma A2 imply that in equilibrium, either there is only one firm setting $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ or $s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) = k, \forall i \in N$. *Q.E.D.*

Lemma A4. Suppose $k \leq y^c$; then in every equilibrium, $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = P(nk)$.

Proof. Suppose $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) < P(nk)$. Then the (possibly multiple) firm(s) setting $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$ could increase profit strictly by offering and selling capacity at $P(nk)$ instead. Suppose $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > P(nk) > 0$. In this case, from Lemma A3, there can only be one firm, say i, setting the uniform price. As from Lemma A2, $s_j(p_j^* \mid \mathbf{p}^*, \mathbf{q}^*) = k$, $\forall j \neq i$, it follows that $\pi_i(\mathbf{p}^*, \mathbf{q}^*) = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) < P(nk)k = \underline{\pi}$, a contradiction to equilibrium behavior. *Q.E.D.*

Lemma A5. Suppose $k \geq \frac{d(0)}{n-1}$; then in every equilibrium, $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = 0$.

Proof. Suppose to the contrary that $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$. Then from Lemma A3 and $k \geq \frac{d(0)}{n-1}$, it must be the case that there is a unique h setting $p_h^* = \mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*)$. However, from Lemma A2 and $k \geq \frac{d(0)}{n-1}$, $\sum_{j \in L^-(\mathbf{P}^a)} \hat{q}_j = (n-1)\frac{d(0)}{n-1} = d(0)$, contradicting the fact that residual demand is strictly positive at $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) > 0$ and, thus, the definition of the uniform price. Therefore, $\mathbf{P}^a(\mathbf{p}^*, \mathbf{q}^*) = 0$. *Q.E.D.*

The fact that the complete set of equilibria of $\Gamma^a(k, 0)$ is given by (i)–(iii) follows from combining Lemmas A1–A5 and constructing strategy profiles that satisfy the properties in the lemmas. It is straightforward to show that the only such strategy profiles from which no firm has an incentive to deviate are those characterized in Proposition 1. It is then clear that all Nash equilibria of $\Gamma^a(k, 0)$ are given by (i)–(iii).

Market Clearing Price rule. Lemma A6 below identifies the major difference between the two definitions of the uniform price. When the Market Clearing Price rule is used, the uniform price may be a price not set by any firm. Lemma A6 identifies the properties that an equilibrium in which no firm sets the resulting uniform price must satisfy. It follows from Lemma A6 that all such equilibria are given by strategy profiles we characterized and are contained in $E^e(k, 0)$ if $k \leq y^c$ and in $C(0)$ if $k > y^c$. In all other equilibria, at least one firm must set the uniform price and it is straightforward to show that in this case, Lemmas A1–A5 derived for the Maximum Accepted Price rule apply to the Market Clearing Price rule as well. Thus, all equilibria of $\Gamma^e(k, 0)$ are characterized by (i)–(iii) in the statement of Proposition 1.

Lemma A6. If $(\mathbf{p}^*, \mathbf{q}^*)$ is such that $p_i^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$, $\forall i \in N$, then $p_i^* < \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(n \min\{k, y^c\})$ and $q_i^* = s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) = \min\{k, y^c\}$.

Proof. First, it is clear that $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) > 0$, because otherwise, it must be the case that there exists a firm j setting the uniform price $p_j^* = 0$. Second, $p_i^* < \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ must hold for every i. If this were not the case, a firm setting its price strictly above $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ would obtain a payoff of zero, which is strictly less than what it would obtain by offering a positive quantity at exactly $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$. It follows from the above arguments that $p_i^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$, $\forall i$ implies $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(\sum_{i \in N} q_i^*)$. Now suppose that $k \leq y^c$. Then it is clear that in equilibrium, $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = P(nk)$. Otherwise it must be the case that some firm, say j, sets $q_j^* < k$, to earn $P(\sum_{i \in N} q_i^*) q_j^*$. However, $k \leq y^c$, such a firm could clearly increase its profit by offering k at a price at or below the resulting uniform price to earn $P(k + \sum_{i \in N \setminus \{j\}} q_i^*)k$ instead. Hence, we must have $p_i^* < P(nk)$ and $q_i^* = s_i(p_i^* \mid \mathbf{p}^*, \mathbf{q}^*) = k$.

Suppose now that $k > y^c$. If an equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$ in which no firm sets the uniform price and at least one firm does not offer the Cournot output exists, then, for $i \in N$, either $q_i^* = k$ or $q_i^* = r(\sum_{h \neq i} q_h^*)$. Equilibria with $q_i^* = k$ and $p_i^* < \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ $\forall i \in N$ are not possible because each firm earns $P(nk)k < \underline{\pi}$ at such profiles. Moreover, we can easily rule out more than one firm having $q_i^* \neq k$. Now consider strategy profiles such that there exists j for which $q_j^* \neq k$ and $\forall i \neq j$, $q_i^* = k$. Then if $p_h^* \neq \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ $\forall h$ is to hold at such profiles, it must be the case that $q_j^* = r((n-1)k) = d(p^r) - (n-1)k$, $p_i^* \leq \underline{p}$ $\forall i \neq j$, and $\mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*) = p^r$. If $k \in (y^c, \frac{d(0)}{n-1})$, it follows immediately from $\underline{y} > d(p^r) - (n-1)k$ that a firm i for which $q_i^* = k$ has an incentive to deviate from such profiles. If $k \geq \frac{d(0)}{n-1}$, then $p^r = 0$. Hence, $p_i^* < p^r = \mathbf{P}^e(\mathbf{p}^*, \mathbf{q}^*)$ is not possible. This completes the proof of Lemma A6. *Q.E.D.*

Finally, for both uniform price rules, simple computations show that for all values of capacity k and each firm $i \in N$, there exists a pure strategy equilibrium in which firm i obtains its minmax payoff $\underline{\pi}$. *Q.E.D.*

Proof of Lemma 3. To prove Lemma 3, we first show that τ^{me} satisfies the properties of a perfectly collusive stationary path with equal sharing. From Definition 1, on τ^{me}, the assumption $k > \frac{d(p^m)}{n}$ implies that $p_i = \underline{p}(\underline{q}^m) < p^m$. Moreover, $\sum_{i=1}^n q_i = d(p^m)$ implies that the uniform price is equal to p^m. Because each firm sells its quantity ceiling of $\frac{d(p^m)}{n}$, its one-period profit is equal to $p^m \frac{d(p^m)}{n}$, as required on a perfectly collusive stationary path with equal sharing. We now show that there does not exist a perfectly collusive stationary path with equal sharing τ, $\tau \neq \tau^{me}$, on which some firm i's incentive to deviate is strictly lower than it is on τ^{me}, $i = 1, \ldots, n$.

Let τ be some perfectly collusive stationary path with equal sharing. Then τ must satisfy the properties stated in Lemma 2.

(i) Suppose that on τ, some firms set their price equal to the monopoly price. We first show that the incentive to deviate cannot increase for any of the firms if every firm i setting $p_i = p^m$ also sets $q_i = k$. To this effect, let $l \geq 1$ be the number of elements of $L(p^m)$. By (iii) in Lemma 2, for firms in $L(p^m)$, the residual demand is $R(p^m) = d(p^m) - (n-l)\frac{d(p^m)}{n} = l\frac{d(p^m)}{n}$. Furthermore, equal sharing requires that if $i \in L(p^m)$, $s_i = \min\{\hat{q}_i, \frac{\hat{q}_i}{\sum_{j \in L(p^m)} \hat{q}_j} l \frac{d(p^m)}{n}\} = \frac{d(p^m)}{n}$. However, then it is easy to see that setting $q_i = k$, $\forall i \in L(p^m)$ implies $s_i = \frac{d(p^m)}{n}$. Hence every firm is selling its share of monopoly output at a uniform price of p^m. Moreover, the incentive to deviate of a firm in $L(p^m)$ cannot be larger than if $q_i < k$, whereas the incentive to deviate of a firm j setting $p_j < p^m$ is unchanged.

(ii) We now show that if on some perfectly collusive stationary path with equal sharing τ, there exist i and j, $i \neq j$, for which $p_i < p_j < p^m$, then, if the path τ' differs from τ only insofar as p_j is reduced to equal p_i on τ', the incentive to deviate of any firm cannot be greater on τ' than it is on τ. First, from Lemma 1, firm h's deviation profit is weakly lower on τ' than it is on τ, $h = 1, \ldots, n$. Second, note that each firm's one-period profit is the same on τ' as it is on τ. Because firm h's incentive to deviate on τ is given by the difference $\pi_h^*(p_{-h}, q_{-h}) - p^m \frac{d(p^m)}{n}$, $\pi_h^*(p'_{-h}, q_{-h}) \leq \pi_h^*(p_{-h}, q_{-h})$ implies that, for every $h = 1, \ldots, n$, the incentive to deviate on the perfectly collusive stationary path with equal sharing τ' is weakly lower than it is on τ. It follows from the above arguments that in the remainder, we can restrict attention to paths on which any firm setting its price equal to p^m offers its capacity at that price and all firms setting price strictly below p^m all set the same price.

(iii) Next we show that on a perfectly collusive stationary path with equal sharing τ, a firm i's incentive to deviate is minimized when the remaining $n-1$ firms set their price equal to $\underline{p}(q^m_-)$ (and thus offer exactly $\frac{d(p^m)}{n}$). In general, by the definition of $p^r(q)$, the worst possible deviation profit firm i can guarantee itself when lower-price firms offer $q < d(0)$ is equal to

$$p^r(q)[d(p^r(q)) - q],$$

its residual demand monopoly profit after lower-price firms have sold their aggregate quantity. The above expression is decreasing in q. Because on τ, every firm setting $p < p^m$ must offer $\frac{d(p^m)}{n}$, firm i's worst possible deviation profit is minimized when the number of low-price firms is the largest, that is, when the remaining firms' aggregate quantity offered is q^m_-. In this case, firm i's deviation profit is given by

$$p^r(q^m_-)[d(p^r(q^m_-)) - q^m_-] = \underline{\pi}(q^m_-). \tag{A1}$$

Note that it follows from $k > \frac{d(p^m)}{n}$ that $\underline{p}(q^m_-) \leq p^r(q^m_-) < p^m$. Furthermore, if on τ, each firm setting $p_h < p^m$ sets $p_h = \underline{p}(q^m_-)$, then firm i's profit from an optimal deviation is indeed equal to its worst possible deviation profit given by (A1). It follows that if on the path τ, every firm sets its price equal to $\underline{p}(q^m_-)$ and offers $\frac{d(p^m)}{n}$, all firms have the same incentive to deviate. Furthermore, each firm's deviation profit is the lowest that can be obtained from a one-period deviation on a perfectly collusive stationary path with equal sharing. Because $\tau = \tau^{me}$, we have shown that the path τ^{me} minimizes all firms' incentives to deviate in the class of perfectly collusive stationary paths with equal sharing. *Q.E.D.*

Proof of Proposition 3. We first characterize $\bar{\delta}$. From Lemma 2, it follows that the only perfectly collusive stationary path with equal sharing on which every firm offers its capacity is τ^{sm}, where on τ^{sm} the price-quantity pairs for each firm are as follows.

$$p_i = p^m \quad \text{and} \quad q_i = k, \forall i.$$

From the definition of s_i, sales on τ^{sm} are equal to $\frac{k}{nk}d(p^m) = \frac{d(p^m)}{n}$ and the uniform price is p^m. Hence τ^{sm} is a perfectly collusive stationary path with equal sharing. From (1), on τ^{sm}, the symmetric incentive constraints are

$$(1-\delta)p^m\left(\min\{k, d(p^m)\} - \frac{d(p^m)}{n}\right) \leq \delta\left(p^m\frac{d(p^m)}{n} - \underline{\pi}\right). \tag{A2}$$

Thus τ^{sm} is sustainable if and only if (A2) holds. Solving for δ from (A2) satisfied with equality, we obtain

$$\bar{\delta} = \frac{p^m \min\{k, d(p^m)\} - p^m \frac{d(p^m)}{n}}{p^m \min\{k, d(p^m)\} - \underline{\pi}}. \tag{A3}$$

It follows from the above arguments that if perfect collusion with equal sharing is sustainable for $\delta < \bar{\delta}$, it must be the case that some firm withholds. Hence, for $\delta < \bar{\delta}$, if perfect collusion with equal sharing is sustainable on a stationary path, it must be the case that there exists $N(\delta) \in \{1, \ldots, n\}$ such that on that path, $N(\delta)$ firms withhold capacity. We now show that there exists $\underline{\delta}^e < \bar{\delta}$ such that if $\delta \geq \underline{\delta}^e$ and n firms withhold capacity, then there exists a perfect equilibrium stationary path that supports perfect collusion with equal sharing. To this effect, consider the path τ^{me} characterized in Lemma 3. From Lemma 3, τ^{me} minimizes incentives to deviate. Furthermore, on τ^{me}, each firm's deviation profit is given by (A1). It thus follows that $\underline{\delta}^e$ is the value of δ that solves

$$(1-\delta)\left(\underline{\pi}(q_-^m) - p^m \frac{d(p^m)}{n}\right) = \delta\left(p^m \frac{d(p^m)}{n} - \underline{\pi}\right).$$

After rearranging, we obtain

$$\underline{\delta}^e = \frac{\underline{\pi}(q_-^m) - p^m \frac{d(p^m)}{n}}{\underline{\pi}(q_-^m) - \underline{\pi}}. \tag{A4}$$

Because $\underline{\pi}(q_-^m) < p^m \min\{k, d(p^m)\}$ holds for all values of $k > \frac{d(p^m)}{n}$, $\underline{\delta}^e < \bar{\delta}$ follows. We now show that the maximum incentive to deviate on τ^{me} is strictly lower than on a perfectly collusive stationary path with equal sharing on which no more than $n-1$ firms withhold. Suppose $\tau'' \neq \tau^{me}$ is such a path. Then, on τ'', there must exist a firm i whose deviation profit is greater than or equal to $\underline{\pi}(q)$, where $q < \frac{(n-1)d(p^m)}{n}$ is the quantity offered by firm i's rivals at prices strictly below p^m. Because $q_-^m = \frac{(n-1)d(p^m)}{n}$ and $\underline{\pi}(y)$ is strictly decreasing in y, $q_-^m > q$ implies $\underline{\pi}(q_-^m) < \underline{\pi}(q)$. Hence, on τ'', the maximum incentive to deviate is strictly greater than on τ^{me}. Hence, it follows that for $\delta = \underline{\delta}^e$, withholding by all firms is required to sustain perfect collusion with equal sharing on a stationary path. *Q.E.D.*

Proof of Lemma 4. To prove Lemma 4, we first show that, for each $i \in N$, τ_i^{ma} satisfies the properties of a perfectly collusive stationary path with equal sharing. On τ_i^{ma}, the assumption $k > \frac{d(p^m)}{n}$ implies that for $j \neq i$, $p_j = \underline{p}(q_-^m) < p^m$. From Definition 2, $p_i = p^m$ and $\sum_{j \neq i} q_j = (\frac{n-1}{n})d(p^m)$ implies that the uniform price is equal to p^m. Because each low-price firm sells its quantity ceiling of $\frac{d(p^m)}{n}$, its one-period profit is equal to $p^m \frac{d(p^m)}{n}$. Moreover, firm i's profit is equal to $p^m(d(p^m) - (n-1)d(p^m)/n) = p^m \frac{d(p^m)}{n}$ as well. Hence, τ_i^{ma} satisfies all of the requirements of a perfectly collusive stationary path with equal sharing. We now show that there does not exist a perfectly collusive stationary path with equal sharing τ on which the maximum incentive to deviate across firms is strictly less than on τ_i^{ma}.

First, note that from Lemma 2, at least one firm has to set its price equal to p^m. It follows from an argument similar to (i) in the proof of Lemma 3 that if firm i sets $p_i = p^m$, then the incentive to deviate cannot increase for any of the firms if $q_i = k$. Arguments similar to (iii) in Lemma 3 establish that to minimize the incentive to deviate of a firm setting its price equal to p^m on the path, the remaining $n-1$ firms should offer a total quantity equal to q_-^m at prices no greater than $\underline{p}(q_-^m)$.

Suppose $l \geq 1$ firms set $p < p^m$ and offer $\frac{d(p^m)}{n}$ each, as required by Lemma 2. Arguments similar to (ii) in the proof of Lemma 3 imply that profits will not change for any of the firms, and incentives to deviate will be no higher, if each of these firms sets its price equal to the minimum of the submitted prices. Furthermore, incentives to deviate cannot increase for any of the firms, if the l low-price firms set $p = \underline{p}(q_-^m)$ rather than some $p' \in (\underline{p}(q_-^m), p^m)$. Suppose then that the l low-price firms set $p = \underline{p}(q_-^m)$. It is straightforward to check that for such a firm, deviating from the path by undercutting $\underline{p}(q_-^m)$ cannot be optimal if it results in the uniform price being reduced to $\underline{p}(q_-^m)$. If $l = 1$, the optimal deviation clearly consists of undercutting p^m and offering k, for every k. If $l > 1$ and firm j is a low-price firm, an optimal deviation takes one of two possible forms depending on the value of k. Either firm j offers the minimum of k and a quantity infinitesimally less than $\frac{(n-l+1)d(p^m)}{n}$ at any price at or below $\underline{p}(q_-^m)$ to obtain

$$p^m \min\left\{k, \frac{(n-l+1)d(p^m)}{n}\right\}; \tag{A5}$$

or firm j maximizes profit on residual demand after $l-1$ low-price firms have sold their quantity $\frac{d(p^m)}{n}$ to obtain

$$p^r\left(q_l^H\right)\left[d\left(p^r\left(q_l^H\right)\right) - q_l^H\right], \text{ where } q_l^H \equiv \frac{(l-1)d(p^m)}{n}. \tag{A6}$$

Upon inspecting (A5) and (A6), it is clear that optimal deviation profits do not depend on p, the common price of the low-price firms, as long as $p \leq \underline{p}(q_-^m)$. Therefore, if each of the l firms sets $p < \underline{p}(q_-^m)$, incentives to deviate would not be lower for any of the firms. If $k \leq \frac{(n-l+1)d(p^m)}{n}$, it is clear that $p^r(q_l^H) \geq p^m$, and thus the deviation that yields (A6) is not a possible deviation (as then the deviating firm would be setting a higher price than all $n-1$ remaining firms). In this case, firm j's optimal deviation profit is given by (A5) and equals $p^m k$. If $\frac{(n-l+1)d(p^m)}{n} < k \leq r(q_l^H)$, then $p^r(q_l^H) = P(k + q_l^H) < p^m$. In this case, firm j's optimal deviation consists of selling its capacity at $p^r(q_l^H) < p^m$. Finally, if $k > r(q_l^H)$, $p^r(q_l^H) = P(r(q_l^H) + q_l^H) < p^m$ and firm j's optimal deviation consists of setting the residual demand monopoly price after $l-1$ firms have sold their quantity, and offering $q_j \in [r(q_l^H), k]$ for sale. It is clear that deviation profits in (A5) and (A6) are nonincreasing in l. Therefore, they are minimized subject to the constraint that $l \leq n-1$, by setting $l = n-1$. It follows that on the path τ_i^{ma}, all low-price firms have the same incentive to deviate. Moreover, it is straightforward to check that this incentive to deviate is strictly higher than firm i's. As we have argued that it is not possible to hold a firm for which $p_j < p^m$ to a lower incentive to deviate on a perfectly collusive stationary path with equal sharing, the proof of Lemma 4 is complete. *Q.E.D.*

Proof of Proposition 4. For every $k > \frac{d(p^m)}{n}$, it is clear that if no firm withholds output, perfect collusion with equal sharing is sustainable on the stationary path if and only if the path τ^{sm} defined in the proof of Proposition 3 is sustainable. Hence, without any withholding, perfect collusion with equal sharing is sustainable on the stationary path if and only if $\delta \geq \bar{\delta}$. A firm's optimal one-period deviation profit on τ^{sm} is equal to $p^m \min\{k, d(p^m)\}$. From Lemma 4, on the path that minimizes the maximum incentive to deviate, τ_i^{ma}, the maximum incentive to deviate across firms, is the incentive to deviate of any low-price firm j, for $i, j \in N$, $j \neq i$. On τ_i^{ma}, for $j \neq i$, firm j's optimal deviation profit is equal to either

(A5) or (A6) above after substituting for $l = n - 1$. (A5) is then equivalent to $p^m \min\{k, \frac{2d(p^m)}{n}\}$. It is clear that if $n = 2$ or $n > 2$ and $k \leq \frac{2d(p^m)}{n}$, for $j \neq i$, firm j's optimal deviation yields $p^m k$, in which case τ_i^{ma} does not strictly relax incentives to deviate when compared to τ^{sm}.

Now suppose $n > 2$ and $k > \frac{2d(p^m)}{n}$. An argument similar to that made in the proof of Proposition 3 implies that for $\delta < \bar{\delta}$, if perfect collusion with equal sharing is sustainable on a stationary path, it must be the case that there exists $M(\delta) \in \{1, \ldots, n\}$ such that on that path, $M(\delta)$ firms withhold capacity. We now show that there exists $\underline{\delta}^a < \bar{\delta}$ such that if $\delta \geq \underline{\delta}^a$ and $n - 1$ firms withhold capacity, then there exists a perfect equilibrium stationary path that supports perfect collusion with equal sharing. Indeed, consider the path τ_i^{ma}. Because $n > 2$ and $k > \frac{2d(p^m)}{n}$, on τ_i^{ma}, for $j \neq i$, firm j's payoff from an optimal deviation is given by (A6). Letting $q^H \equiv \frac{(n-2)}{n} d(p^m)$, (A6) is equivalent to

$$\pi^H \equiv p^r\left(q^H\right)\left[d\left(p^r\left(q^H\right)\right) - q^H\right].$$

Because $n > 2$ and $k > \frac{2d(p^m)}{n}$ hold by assumption, $P(k + q^H) < p^m$. Moreover, if k is such that $p^r(q^H) \neq P(k + q^H)$, then it is clear that because $q^H > 0$, $p^r(q^H) < p^m$ as well. Hence,

$$\pi^H < p^m \min\{k, d(p^m)\}. \tag{A7}$$

Because Lemma 4 shows that τ_i^{ma} minimizes the maximum incentive to deviate, if $n > 2$ and $k > \frac{2d(p^m)}{n}$ hold, $\underline{\delta}^a$ is the solution in δ from any low-price firm's incentive constraint satisfied with equality, which using (1) is given by

$$(1 - \delta)\left(\pi^H - p^m \frac{d(p^m)}{n}\right) = \delta\left(p^m \frac{d(p^m)}{n} - \underline{\pi}\right).$$

After solving for δ in the above, we obtain

$$\underline{\delta}^a = \frac{\pi^H - p^m \frac{d(p^m)}{n}}{\pi^H - \underline{\pi}}. \tag{A8}$$

Using (A3) and (A8), it follows from (A7) that $\underline{\delta}^a < \bar{\delta}$.

Finally, it is clear that the maximum deviation profit is strictly lower on τ_i^{ma} than on any other perfectly collusive stationary path with equal sharing on which no more than $n - 2$ firms withhold. Hence, for $\delta = \underline{\delta}^a$, withholding by at least $n - 1$ firms is required to sustain perfect collusion with equal sharing on a stationary path and $M(\underline{\delta}^a) = n - 1$. *Q.E.D.*

Proof of Lemma 5. Let $(p_i, q_i) = ((p_i^{l_i}, q_i^{l_i}))_{l_i=1}^{L_i}$ denote firm i's strategy and let $(\mathbf{p}_{-i}, \mathbf{q}_{-i})$ be a vector of firm i's rival's strategies. We show that for both uniform price auction rules, $\pi_i^* = \pi_i^*(\mathbf{p}_{-i}, \mathbf{q}_{-i}) = \sup_{(p_i, q_i)} \pi_i(p_i, q_i, \mathbf{p}_{-i}, \mathbf{q}_{-i})$ remains the supremum of deviation payoffs when restricting firm i to strategies using a single step ($L_i = 1$). Throughout the proof, we ignore price steps l_i such that $p_i^{l_i}$ is strictly greater than the uniform price. Such price steps are indeed irrelevant because the quantity sold at such steps is equal to zero, so that firm i cannot increase its profit by including them in its strategy. Let p_i^{s*} be the set of prices submitted by firm i and $\mathbf{P}^{u*} = \mathbf{P}^u(p_i, q_i, \mathbf{p}_{-i}, \mathbf{q}_{-i})$ be the uniform price. First, $\mathbf{P}^{u*} = 0$ is only possible if firm i's residual demand at strictly positive prices is equal to zero. In this case, $\pi_i^* = 0$ is independent of firm i's strategy and can therefore be obtained by using a single-step bidding function (for instance, offering k at $p_i = 0$). Suppose that $\mathbf{P}^{u*} > 0$. There are two cases: either $\mathbf{P}^{u*} \in p_i^{s*}$ or $p_i^{l_i} < \mathbf{P}^{u*}, \forall l_i$. Consider first the case $\mathbf{P}^{u*} \in p_i^{s*}$. Under both pricing rules, tying with a group of firms at $\mathbf{P}^{u*}$ cannot be optimal if $s_i(\mathbf{P}^{u*} \mid \mathbf{p}, \mathbf{q}) > 0$ but $\mathbf{s}_i^* = \mathbf{s}_i((p_i, q_i), \mathbf{p}_{-i}, \mathbf{q}_{-i}) < \min\{d(\mathbf{P}^{u*}), k\}$. Indeed, in this case, firm i could strictly increase its profit by slightly undercutting $\mathbf{P}^{u*}$. Suppose then that $\mathbf{s}_i^* = \min\{d(\mathbf{P}^{u*}), k\}$. It is clear that $\mathbf{s}_i^* = d(\mathbf{P}^{u*})$ is not possible if firm i is not the only firm setting the uniform price. Suppose then that $\mathbf{s}_i^* = k$. In this case, under both Definitions 1 and 2 for the uniform price, it is straightforward to show that firm i can obtain $\pi_i^* = \mathbf{P}^{u*}k$ by using a one-step strategy (p, k), where $p < \mathbf{P}^{u*}$. The case in which $\mathbf{P}^{u*} \in p_i^{s*}$ and $s_i(\mathbf{P}^{u*} \mid \mathbf{p}, \mathbf{q}) = 0$ is similar to the case in which $\mathbf{P}^{u*} \notin p_i^{s*}$ and is discussed below. We first address the case in which firm i is the only firm with $\mathbf{P}^{u*}$ in its strategy. If firm i is the only firm with $\mathbf{P}^{u*}$ in its strategy, then

$$\pi_i^* = \mathbf{P}^{u*}\left[\min\left\{q_i(\mathbf{P}^{u*}), d(\mathbf{P}^{u*}) - \sum_{j \in N \backslash \{i\}} \sum_{p \in p_j^s} q_j(p) - \sum_{p \in p_i^{s*} \backslash \mathbf{P}^{u*}} q_i(p)\right\} + \sum_{p \in p_i^{s*} \backslash \mathbf{P}^{u*}} q_i(p)\right].$$

Note that unless $q_i(\mathbf{P}^{u*}) = k < d(\mathbf{P}^{u*}) - \sum_{j \in N \backslash \{i\}} \sum_{p \in p_j^s} q_j(p) - \sum_{p \in p_i^{s*} \backslash \mathbf{P}^{u*}} q_i(p)$ (in which case $\mathbf{s}_i^* = \min\{d(\mathbf{P}^{u*}), k\}$), firm i would never set $q_i(\mathbf{P}^{u*}) < d(\mathbf{P}^{u*}) - \sum_{j \in N \backslash \{i\}} \sum_{p \in p_j^s} q_j(p) - \sum_{p \in p_i^{s*} \backslash \mathbf{P}^{u*}} q_i(p)$. Thus

$$\pi_i^* = \mathbf{P}^{u*}\left[d(\mathbf{P}^{u*}) - \sum_{j \in N \backslash \{i\}} \sum_{p \in p_j^s} q_j(p)\right].$$

However, this is exactly what firm i would obtain by setting the single step $(\mathbf{P}^{u*}, q_i^*)$, where $q_i^* = d(\mathbf{P}^{u*}) - \sum_{j \in N \backslash \{i\}} \sum_{p \in p_j^s} q_j(p)$, instead. Finally, consider the case $p_i^{l_i} < \mathbf{P}^{u*}, \forall l_i$. Then, from Definitions 1 and 2, it must be the case that $\pi_i^* = \mathbf{P}^{u*}\mathbf{s}_i^*$ is independent of firm i's prices $p_i^{l_i}$ because $p_i^{l_i} < \mathbf{P}^{u*}, \forall l_i$. A similar argument applies to the case in which $\mathbf{P}^{u*} \in p_i^{s*}$ and $s_i(\mathbf{P}^{u*} \mid \mathbf{p}, \mathbf{q}) = 0$, as the only price steps relevant to firm i's sales are those for which $p_i^{l_i} < \mathbf{P}^{u*}$. Therefore, in both cases, the single-step bidding function $(p, \sum_{z \in p_i^{s*} \backslash \{\mathbf{P}^{u*}\}} q_i(z))$, where $p < \mathbf{P}^{u*}$, achieves a payoff equal to π_i^*. *Q.E.D.*

Proof of Proposition 7. First, it is straightforward to show that τ^ϵ is a perfectly collusive stationary path with equal sharing. Furthermore, for $\epsilon \in (0, d(p^m)]$, on τ^ϵ, every firm has the same incentive to deviate in every period. It is simple to check that if $\frac{d(p^m)+(n-1)\epsilon}{n} \geq k$, a firm's profit from an optimal unilateral deviation is obtained by expanding output up to k at $\underline{p}(q^\epsilon_-)$ and is equal to

$$\pi_i^* = p^m k. \tag{A9}$$

If $k > \frac{d(p^m)+(n-1)\epsilon}{n}$, then profit from an optimal deviation is obtained by setting the residual demand monopoly price after firm i's rivals have sold q^ϵ_- and is equal to

$$\pi_i^* = \underline{\pi}(q^\epsilon_-). \tag{A10}$$

We now show that τ^ϵ minimizes incentives to deviate in the class of perfectly collusive stationary paths with equal sharing. First note that if $\epsilon = d(p^m)$, then the quantity offered at the price p^m must be no lower than $d(p^m)$. It follows immediately that the path that minimizes incentives to deviate is τ^ϵ, the path on which all firms offer their capacity at the monopoly price in every period. In the remainder of the proof of Proposition 7, assume $\epsilon \in (0, d(p^m))$.

Because on a perfectly collusive stationary path with equal sharing it must be the case that $\mathbf{P}^a = p^m$, it is clear that incentives to deviate cannot be lowered by moving some quantity offered at $\underline{p}(q^\epsilon_-)$ to some price strictly between $\underline{p}(q^\epsilon_-)$ and p^m. Furthermore, for every i, we can rule out steps such that $p > \mathbf{P}^a = p^m$ and $q_i(p) > 0$ because setting such steps does not affect the outcome (because a firm's sales are equal to zero at such prices), but may increase incentives to deviate. Because on a perfectly collusive stationary path with equal sharing, for a given $(p_i^{l_i}, q_i^{l_i})$ such that $p_i^{l_i} < p^m$, $p_i^{l_i}$ affects neither firm i's sales nor its profit, it follows that we can, without loss of generality, restrict attention to stationary paths on which firms use at most two steps in their bidding function. Moreover, the analysis may be further restricted to the two-step bidding functions that satisfy the following: if there exists an i and an l_i for which $p_i^{l_i} \in p_i^s$ is such that $p_i^{l_i} < p^m$, then $p_i^{l_i}$ can be reduced to $p = \min_{h,l_h}\{p_h^{l_h}\}$ with the corresponding quantity $q_i^{l_i}$ provided at that p. Indeed, this neither affects the profit obtained by any of the firms nor can it increase incentives to deviate. Hence all firms with a step at a price strictly below p^m may move all such steps to the common price $p = \min_{h,l_h}\{p_h^{l_h}\}$ without affecting firm profits or increasing the incentive of any firm to deviate. Furthermore, letting $q^{-i} = \sum_{h\neq i} q_h(p)$ be the total quantity offered by firm i's rivals at the common price p, the fact that firms have at most two steps in their strategy implies that q^{-i} is the quantity offered at every price in the interval $[p, p^m)$. It follows that, other things being equal, the maximum incentive to deviate is minimized if p is set equal to $p_l = \min_{i\in N}\{\underline{p}(q^{-i})\}$, the highest price that no firm would want to undercut. Additionally, for every firm j such that $p^m \in p_j^s$, setting $q_j(p^m) = k - q_j(p_l)$ cannot increase any firm's incentive to deviate (as then, when firm j ties with a group of firms at p^m, it cannot deviate by increasing its quantity offered at p^m without decreasing its quantity offered at p_l). Finally, to minimize firm i's deviation profit (and thus its incentive to deviate), q^{-i}, the quantity offered by firm i's rivals at a price p strictly below p^m, must be as large as possible. This quantity must also satisfy $\sum_{h\in N} q_h(p) = q^{-i} + q_i(p) \leq d(p^m) - \epsilon$. Thus for every i, this maximum quantity is obtained when $\sum_{h\in N} q_h(p) = d(p^m) - \epsilon$ or, equivalently, $q^{-i} = d(p^m) - \epsilon - q_i(p)$. It follows that the maximum incentive to deviate is minimized at $(q_1(p), \ldots, q_n(p))$, satisfying $q^{-i} = d(p^m) - \epsilon - q_i(p) = d(p^m) - \epsilon - q_j(p) = q^{-j}$, $\forall i, j, i \neq j$ and $\sum_{h\in N} q_h(p) = d(p^m) - \epsilon$. This is a system of n linear equations in n unknowns ($q_i(p)$ for $i = 1, \ldots, n$). Solving the system yields $q_1(p) = \cdots = q_i(p) = \cdots = q_n(p) = \frac{d(p^m)-\epsilon}{n}$. It then follows directly that to minimize the maximum incentive to deviate, $p_l = \underline{p}(q^\epsilon_-)$ is sufficient, and that each firm must offer a quantity equal to $k - [\frac{d(p^m)-\epsilon}{n}]$ at p^m. *Q.E.D.*

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