Inequality averse criteria for evaluating infinite utility streams: The impossibility of Weak Pareto

José Carlos R. Alcantud

Universidad de Salamanca

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José Carlos R. Alcantud [1]

Facultad de Economía y Empresa, Universidad de Salamanca, E 37008 Salamanca, Spain

Abstract

This paper investigates ethical aggregation of infinite utility streams by representable social welfare relations. We prove that the Hammond Equity postulate and other variations of it like the Pigou-Dalton transfer principle are incompatible with positive responsiveness to welfare improvements by every generation. The case of Hammond Equity for the Future is investigated too.

JEL classification: D63; D71; D90.

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Email address: `jcr@usal.es`, Tel. `+34-923-294640 ext 3180`, Fax `+34-923-294686` (José Carlos R. Alcantud).

URL: `http://web.usal.es/jcr` (José Carlos R. Alcantud).

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1 Introduction and motivation

Representability stands out among the rationality properties of a ranking that make it intuitively tractable. Nevertheless in the analysis of ranking infinite utility streams, the literature has seldom provided representable social welfare relations (SWRs) on $[0,1]^{\mathbb{N}}$ that verify some form of equity and interesting versions of the Pareto principle for efficiency. The abundance of impossibility results involving continuity or representability invites to argue that "continuity and representability can be considered rather demanding in infinite-horizon settings" (Bossert et al. [8, p. 588]). In this work we produce new evidences that lead us to concur in the position that there is an intrinsic incompatibility of representability with attractive sets of axioms on $[0,1]^{\mathbb{N}}$.

In this regard the seminal contribution is Basu and Mitra [6]. It establishes that any Strongly Paretian social welfare function (or SWF, i.e., representable SWR) on $\{0,1\}^{\mathbb{N}}$ must contradict Anonymity, i.e., the equal treatment of the generations. The literature has provided further results in that line. Relaxing the requirement of Strong Pareto to Weak Pareto also excludes Anonymity in the $[0,1]^{\mathbb{N}}$ case (v., Basu and Mitra [7, Theorem 4]). Likewise, social evaluations of $[0,1]^{\mathbb{N}}$ that verify a weaker form of the Pareto principle named Weak Dominance must contradict a fairly weak distributional principle named Hammond Equity for the Future (cf., Banerjee [5]). Although Basu and Mitra [7, Theorem 5] assures that there are Weakly Dominant and Anonymous SWFs on $[0,1]^{\mathbb{N}}$, the authors are aware that such evaluations have no practical utility because they must contradict Monotonicity. [2] It is also known that Strongly Paretian SWFs on $[0,1]^{\mathbb{N}}$ must contradict other two fundamental equity properties, namely the Pigou-Dalton transfer principle and Hammond Equity. The reason is that any evaluation with the properties of Monotonicity and either the Pigou-Dalton transfer principle or Hammond Equity must verify Hammond Equity for the Future (cf., Asheim et al. [3]) thus Banerjee's impossibility result applies. These arguments hint that Strong Pareto is a source of incompatibilities in the study of intergenerational conflicts on $[0,1]^{\mathbb{N}}$ under the assumption of representability.

But Strong Pareto is excessively demanding in e.g., the analysis of economies with an infinite population, therefore it is arguable that for certain purposes it may suffice to be equipped with weaker forms of Paretianism. A natural question arises as to whether possibility results can emerge by replacing Strong Pareto with Monotonicity plus Weak Pareto as the efficiency requirement. We believe that this specification is natural for the following reasons. As was

[2] This condition demands that if no generation is worse off at $\mathbf{x}$ than at $\mathbf{y}$ then the social evaluation does not rank $\mathbf{y}$ above $\mathbf{x}$. It is widely considered to be a necessary condition for efficiency.

mentioned, any applicable restricted version of the Pareto axiom must incorporate Monotonicity. This excludes adopting certain additional variants of Paretianity like the aforementioned Weak Dominance, because in the presence of Monotonicity it yields the full force of the Pareto axiom. However, Weak Pareto does not suffer from this handicap and its spirit rarely raises objections. Besides, it has been adopted in related studies like Basu and Mitra [7] and Hara et al. [10].

Nevertheless in this work we complement the negative conclusions that have been presented in the following sense: For social evaluations of $[0,1]^{\mathbb{N}}$ that verify Weak Pareto, not only Anonymity but also various equity axioms relating to inequality aversion are banned. To be precise, we check that there is no SWF satisfying Weak Pareto and any one of the following three consequentialist equity principles: the Pigou-Dalton transfer principle, Hammond Equity, and a variant of the latter that we call Very Weak Inequality Aversion. The case of Hammond Equity for the Future is different but also negative: Although it is trivial that there are SWFs that verify Hammond Equity for the Future, Monotonicity, and Weak Pareto, this set of axioms does not prevent a dictatorial behavior by any future generation (consider the case of the evaluation $\mathbf{W}(\mathbf{x} = (x_1, x_2, ..., x_n, ...)) = x_i$ for any fixed $i \geqslant 2$, where generation i is dictatorial). Thus unless one can guarantee the existence of SWFs with additional properties that avoid dictatorships, such compatibility is without interest as to applications. To this purpose, adding Anonymity is a natural proposal but as was mentioned, it is of no avail –because Anonymity is incompatible with Weak Pareto. We here prove that another natural alternative, namely, adding Independent Future instead, leads to impossibility as well (which bears some comparison with the recent analysis in Asheim et al. [3]).

The overall picture that we obtain confirms the position that for an increasingly large list of important equity axioms, the representability axiom is incompatible with sufficiently nice efficiency when the domain of utilities is $[0,1]^{\mathbb{N}}$. A possible positive escape to this situation that does not reject representability is the appeal to more reduced programmes spaces, e.g., of the form $Y^{\mathbb{N}}$ with $Y \subseteq \mathbb{N}$. For purposes like the analysis of infinitely repeated finite games, or benefits given by monetary endowments that therefore are multiples of the smallest indivisible unit, this is a very realistic possibility. The interested reader may find some preliminary evidences in that line in Alcantud and García-Sanz [1,2]. Alternatively, one may well wonder about the parallel analysis when both continuity and representability are dropped. Bossert et al. [8] gives positive answers in all the cases we have considered. From their contribution it follows that the Pigou-Dalton transfer principle, Hammond Equity, Very Weak Inequality Aversion, and Hammond Equity for the Future are each compatible with Anonymous and Strongly Paretian social orderings. To wit, their Theorem 1 proves that the Pigou-Dalton transfer principle is compatible with Strong Pareto and Anonymity (properly speaking, the class of orderings with these three properties is fully identified), and their Theorem 2 proves that Hammond Equity is compatible with Strong Pareto and Anonymity (the class of orderings with these three properties is identified too). The fact that under Strong Pareto, Hammond Equity is equivalent to the requirement that we call Very Weak Inequality Aversion and it implies Hammond Equity for the Future, completes the claim.

The remainder of the paper is organized as follows. In Section 2 we do two things. For one, Subsection 2.1 sets our notation and axioms. For another, Subsection 2.2 proves some relationships among the axioms. Section 3 presents our results. All proofs are relegated to Appendix A.

2 Notation and axioms

Let $\mathbf{X}$ denote a subset of $\mathbb{R}^{\mathbb{N}}$, that represents a domain of utility sequences or infinite-horizon utility streams. We adopt the usual notation for such utility streams: $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$. By $(y)_{con}$ we mean the constant sequence $(y, y, ...)$, $(x, (y)_{con})$ holds for $(x, y, y, y, ...)$, and $(x_1, ..., x_k, (y)_{con}) = (x_1, ..., x_k, y, y, ...)$ denotes an eventually constant sequence. We write $\mathbf{x} \geqslant \mathbf{y}$ if $x_i \geqslant y_i$ for each $i = 1, 2, ...$, and $\mathbf{x} \gg \mathbf{y}$ if $x_i > y_i$ for each $i = 1, 2, ...$ Also, $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

A *social welfare function* (SWF) is a function $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$. In this paper we are concerned with two sets of axioms of different nature on SWFs, that can be rephrased for social welfare relations (i.e., binary relations on $\mathbf{X}$) in a direct manner. We proceed to state and discuss them.

2.1 The axioms

Firstly we introduce some equity axioms of two different classes. Anonymity is the usual "equal treatment of all generations" postulate à-la-Sidgwik and Diamond. We then list some consequentialist equity axioms that implement preference for egalitarian allocations of utilities among generations in various senses. Afterwards we discuss about efficiency in this context.

Axiom AN (*Anonymity*). Any finite permutation of a utility stream produces a utility stream with the same social utility

Axioms HE below is another equity principle stating that in case of a conflict between two generations, every other generation being as well off, the stream where the least favoured generation is better off must be weakly preferred.

Axiom HE (*Hammond Equity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

A variant of this principle is the following postulate of aversion to inequality (or pure preference for equity against inequality).

Axiom VWIA (*Very Weak Inequality Aversion*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j = y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

VWIA captures weak preference for streams that do not produce inequality between two generations when the alternative unequal endowments are conflicting, every other generation being as well off. Although it seems a especially basic requirement, it is closely related to HE as will be shown later on in Subsection 2.2.

We now consider a notion of inequality aversion in a cardinal vein.

Axiom PDT (*Pigou-Dalton transfer principle*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that there is $\varepsilon > 0$ with $y_j = x_j - \varepsilon \geqslant y_k = x_k + \varepsilon$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

PDT has been introduced in this literature by Hara et al. [10], Bossert et al. [8] –under the name *strict transfer principle*– and Sakai [11]. [3] It states that a non-costly transfer of utility from a richer generation to a poorer one must increase intergenerational welfare if it is not so large as to reverse their relative ranking.

Further we are concerned with the following axiom that was introduced in Asheim and Tungodden [4].

Axiom HEF (*Hammond Equity for the Future*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x} = (x_1, (x)_{con})$, $\mathbf{y} = (y_1, (y)_{con})$ and $x_1 > y_1 > y > x$, then $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$.

HEF states the following ethical restriction on the ranking of streams where the level of utility is constant from the second period on and the present generation is better-off than the future: If the sacrifice by the present generation conveys a higher utility for all future generations, then such trade off is weakly preferred.

[3] The formulation in Bossert et al. [8] is different but equivalent: If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ and $x_j + x_k = y_j + y_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$. This version is parallel in structure to the HE-related axioms.

As is apparent there is a similarity in structure among HE, VWIA, and PDT that is not shared in full by HEF. The latter has been designed as a necessary condition for equity in the present context. The former three properties are not exclusive to the infinite-dimensional setting and establish which trade-offs between two generations are socially favoured. Their ethical implications are more controvesial than those of HEF, where only particular types of transfers from the present to the future generations are enforced. In this sense, Asheim and Tungodden [4] and Asheim et al. [3, Section 3] explain that HEF is a very weak equity condition that can be endorsed both from an egalitarian and utilitarian point of view.

We intend to account for some kind of efficiency too. Various axioms capture the general principle that with respect to a given infinite utility stream, adequate changes must improve the social welfare if every generation is at least as well off after the change. The *Weak Dominance* axiom captures the following spirit: Improving the welfare of *exactly one* generation suffices to improve the social welfare. In turn, the *Weak Pareto* axiom requests that *all* generations increase their utility for the social welfare to improve. The *Strong Pareto* axiom is stronger: It imposes that if *at least one* generation increases its utility then the social welfare must improve. Formally:

Axiom WD (*Weak Dominance*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and there is $j \in \mathbb{N}$ such that $x_j > y_j$, and $x_i = y_i$ for all $i \neq j$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

Axiom WP (*Weak Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \gg \mathbf{y}$, then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

Axiom SP (*Strong Pareto*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} > \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) > \mathbf{W}(\mathbf{y})$.

Another relaxed form of Strong Pareto that is unrelated to either WP or WD is Monotonicity.

Axiom M (*Monotonicity*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\mathbf{x} \geqslant \mathbf{y}$ then $\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y})$.

Finally, a condition that relates to Koopmans' [9] stationarity postulate is the following independence condition.

Axiom IF (*Independent Future*). For each $\mathbf{x} = (x_1, ..., x_n, ...) \in \mathbf{X}$, $\mathbf{y} = (y_1, ..., y_n, ...) \in \mathbf{X}$ such that $x_1 = y_1$:

$$\mathbf{W}(\mathbf{x}) \geqslant \mathbf{W}(\mathbf{y}) \text{ if and only if } \mathbf{W}(x_2, ..., x_n, ...) \geqslant \mathbf{W}(y_2, ..., y_n, ...)$$

This axiom captures the idea that decisions affecting the future (i.e., from period 2 on) generations can be made as if the present was one period forward, when the allocations to the present are the same along the alternatives. It is central in the analysis by Asheim et al. [3].

2.2 Some relationships

In this Subsection we give some direct relationships for Monotonic or Weakly Dominant SWFs when $\mathbf{X} = [0,1]^{\mathbb{N}}$. Lemma 1 below proves that axioms HE and VWIA are equivalent under either M or WD, and that in addition, in the presence of WD both egalitarian axioms are equivalent to the following strengthened form of their conjunction (and of PDT):

Axiom SEP (*Strong Equity Preference*). If $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$, then $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$.

Lemma 1 *Let* $\boldsymbol{W}$ *be an SWF on* $\boldsymbol{X}$.

(a) If $\boldsymbol{W}$ *is M then axioms HE and VWIA are equivalent.*

(b) If $\boldsymbol{W}$ *is WD then axioms HE, VWIA, and SEP are equivalent and imply PDT.* [4]

3 Implementing inequality aversion under weak Pareto: Impossibility results

In this Section we produce various impossibility results for representable social welfare relations on $\mathbf{X} = [0,1]^{\mathbb{N}}$. Our first result establishes the incompatibility of PDT and WP:

Proposition 1 *There are not SWFs on* $\boldsymbol{X}$ *that verify PDT and WP.* [5]

Because we renounced Weak Dominance in order to investigate if some possibility result emerges by relaxing Strong Pareto, our next Proposition is independent of Proposition 1.

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[4] The proof that VWIA implies HE under M does not need any specific form of the programmes space. Bossert et al. [8] prove a fact in line with (*b*): SP and HE together imply SEP which in turn, obviously implies PDT. PDT can be regarded as a cardinal version of SEP.

[5] The full force of PDT and WP is not used along the proof. Our argument ensures incompatibility of a weaker version of PDT with a weaker version of WP. Furthermore, incompatibility of SEP with WP is a trivial consequence.

Proposition 2 *There are not SWFs on $\boldsymbol{X}$ that verify HE and WP.*[6]

We now put forward a result apropos VWIA:

Proposition 3 *There are not SWFs on $\boldsymbol{X}$ that verify VWIA and WP.* [7]

Finally we check that although there are SWFs that verify HEF, M, and WP, the occurrence of dictatorial generations can not be avoided by imposing IF to the criteria. In fact we can prove something stronger:

Proposition 4 *There are not SWFs on $\boldsymbol{X}$ that verify IF, HEF, and WP.*

The latter Proposition deserves further comments. For SWRs under M and a restricted continuity assumption, Asheim et al. [3, Theorem 2] prove that IF, a weak sensitivity condition, and another postulate in Koopmans' [9] analysis named Independent Present contradict HEF. In view of Koopmans' characterization of discounted utilitarianism, this accounts for its failure to verify HEF. By refusing to impose any continuity restriction, thus giving up numerical representability, [3, Proposition 13] proves the existence of complete orderings that verify Independent Present, IF, HEF, and SP. With respect to this achievement, Proposition 4 shows that even if we are willing to deny Independent Present and relax the efficiency criteria to WP then representability can not be gained.

Remark 1 *Propositions 1 to 4 would be compelling even if M were added to WP as a requirement. We emphasize that according to our proofs, Monotonicity does not play any role in the incompatibilities we have scrutinized. The fact that we can prove the current statements brings causes of incompatibility to light with a better accuracy. Observe also that under Monotonicity, Proposition 3 reduces to Proposition 2 by Lemma 1 b).*

Appendix A

Proof of Lemma 1. *Part (a).* Let us assume that $\mathbf{W}$ is Monotonic and satisfies VWIA. In order to check for HE let us select $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $x_j > y_j > y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. Define $\mathbf{z} \in \mathbf{X}$ such that $z_j = y_k$, and $z_t = y_t$ when $t \neq j$. Then VWIA entails $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$. Now M implies $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$ and the conclusion $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$ follows.

Now let us assume that $\mathbf{W}$ is Monotonic and satisfies HE. In order to check for VWIA let us select $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $x_j > y_j = y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. Define $\mathbf{z} \in \mathbf{X}$ such that $z_j = y_j$, $y_k > z_k > x_k$,

[6] Our argument ensures incompatibility of HE with a weaker version of WP.
[7] Our argument ensures incompatibility of VWIA with a weaker version of WP.

and $z_t = y_t$ when $j \neq t \neq k$. Then HE entails $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$. Because M implies $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$, the conclusion $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{x})$ follows. SEP obviously implies HE, VWIA, and PDT. Let $\mathbf{W}$ be a Weakly Dominant SWF.

Part (b). SEP obviously implies HE, VWIA, and PDT. By mimicking the proof that Monotonic SWFs that satisfy VWIA are HE we can conclude that Weakly Dominant SWFs that satisfy VWIA verify a strengthened version of HE. Now let us assume that $\mathbf{W}$ is Weakly Dominant and satisfies HE. In order to check for SEP, we select $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $x_j > y_j \geqslant y_k > x_k$ for some $j, k \in \mathbb{N}$, and $x_t = y_t$ when $j \neq t \neq k$. We separately consider two cases:

(i) If $y_j > y_k$ then we generate $\mathbf{z} \in \mathbf{X}$ such that $z_t = x_t$ when $t \neq k$, and $y_k > z_k > x_k$. Now WD implies $\mathbf{W}(\mathbf{z}) > \mathbf{W}(\mathbf{x})$, and the conclusion follows because $\mathbf{W}(\mathbf{y}) \geqslant \mathbf{W}(\mathbf{z})$ due to HE.

(ii) If $y_j = y_k$ then we generate $\mathbf{z} \in \mathbf{X}$ such that $z_t = x_t$ when $j \neq t \neq k$, and $y_j = y_k > z_j > z_k > x_k$. Due to HE, $\mathbf{W}(\mathbf{z}) \geqslant \mathbf{W}(\mathbf{x})$. By WD, $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{z})$ and the conclusion $\mathbf{W}(\mathbf{y}) > \mathbf{W}(\mathbf{x})$ follows. $\square$

Proof of Proposition 1. If $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$ satisfies PDT and WP then we can assign an uncountable number of different rational numbers, which is impossible, in the following manner. For each $0 < x < \frac{1}{2}$ we fix $0 < \varepsilon_x$ such that $x + \varepsilon_x < \frac{1}{2}$, and then let

$$L(x) := \mathbf{W}\,(1-x, x, x, ...) \quad \text{and } R(x) := \mathbf{W}\left(1 - \frac{x}{2}, x + \varepsilon_x, x + (\varepsilon_x)^2, ...\right)$$

thus $I(x) := (L(x), R(x)) \neq \varnothing$ due to WP. We proceed to prove that $I(x) \cap I(y) = \varnothing$, i.e., $L(y) \geqslant R(x)$, for every $\frac{1}{2} > y > x > 0$.

Associated with x and y we select n_0, the minimum natural number with the property that $n > n_0$ implies $y > x + (\varepsilon_x)^{n-1}$. We also select $y - x - (\varepsilon_x)^{n_0} > \varepsilon > 0$ sufficiently small to allow for the existence of k_1 and k_2, natural numbers with the properties $1 - \frac{x}{2} < 1 - y + k_1\varepsilon \leqslant 1$ and $x + \varepsilon_x < y + k_2\varepsilon \leqslant 1$. A sequential application of (a relaxed version of) PDT proves that

$$L(y) = \mathbf{W}\,(1-y, y, y, ...) \geqslant$$

$$\geqslant \mathbf{W}\,(1 - y + \varepsilon, y, \overset{n_0-1}{\ldots\ldots}, y, y - \varepsilon, y, y, ...) \geqslant ... \geqslant$$

$$\geqslant \mathbf{W}\left(1 - y + k_1\,\varepsilon, y, \overset{n_0-1}{\ldots\ldots}, y, y - \varepsilon, \overset{k_1}{\ldots\ldots}, y - \varepsilon, y, y, ...\right) \geqslant ... \geqslant$$

$$\geqslant \mathbf{W}\left(1 - y + k_1\,\varepsilon, y + k_2\,\varepsilon, \overset{n_0-1}{\ldots\ldots}, y + k_2\,\varepsilon, y - \varepsilon, \overset{k_1 + k_2\cdot(n_0-1)}{\ldots\ldots\ldots\ldots}, y - \varepsilon, y, y, ...\right)$$

(Intuitively: we compare streams where a positive amount ε of utility is exchanged between a generation beyond the n_0 threshold and another before it, which preserves their relative ranking. And we do this $k_1 + k_2 \cdot (n_0 - 1)$ many times). Now (a relaxed version of) WP assures that

$$\mathbf{W}\left(1-y+k_1\,\varepsilon, y+k_2\,\varepsilon, \overset{n_0-1}{\ldots\ldots}, y+k_2\,\varepsilon, y-\varepsilon, \ldots\overset{k_2\cdot n_0}{\ldots\ldots}\ldots, y-\varepsilon, y, y, \ldots.\right) \geqslant$$

$$\mathbf{W}\left(1-\frac{x}{2}, x+\varepsilon_x, x+(\varepsilon_x)^2, \ldots.\right) = R(x)$$

because $1-y+k_1\varepsilon > 1-\frac{x}{2}$, $y+k_2\varepsilon > x+\varepsilon_x > x+(\varepsilon_x)^2 > \ldots.$, and $y > y-\varepsilon > x+(\varepsilon_x)^{n_0} > x+(\varepsilon_x)^{n_0+1} > \ldots.$ This completes the argument. □

Proof of Proposition 2. If $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$ verifies HE and WP then we can assign an uncountable number of different rational numbers, which is impossible, as follows. Let $\varepsilon = 0.1$, $\delta = 0.2$. For each $0 < x < \frac{1}{2}$ we let

$$L(x) := \mathbf{W}\left(x+\varepsilon^2, x+\varepsilon^3, x+\varepsilon^4, \ldots.\right), \quad R(x) := \mathbf{W}\left(x+\varepsilon, x+\varepsilon^2, x+\varepsilon^3, \ldots.\right)$$

thus $I(x) := (L(x), R(x)) \neq \varnothing$ due to WP. We proceed to prove that $I(x) \cap I(y) = \varnothing$, or $\mathbf{W}(y+\varepsilon^2, y+\varepsilon^3, y+\varepsilon^4, \ldots.) \geqslant \mathbf{W}(x+\varepsilon, x+\varepsilon^2, x+\varepsilon^3, \ldots.)$, for every $\frac{1}{2} > y > x > 0$.

Case 1: $y+\varepsilon^2 > x+\varepsilon$, i.e., $y-x > \varepsilon-\varepsilon^2$. Then $y+\varepsilon^{n+1} > x+\varepsilon^n$ follows from trivial computations for each $n = 1, 2, \ldots.$, thus WP yields the thesis.

Case 2: $y+\varepsilon^2 \leqslant x+\varepsilon$. Let $m > 1$ denote the first index for which $y+\varepsilon^{m+1} > x+\varepsilon^m$. This number is well defined because $\lim_k(y+\varepsilon^{k+1}) = y > x = \lim_k(x+\varepsilon^k)$. Observe that $y+\varepsilon^{n+1} > x+\varepsilon^n$ for each $n > m$ too, because $y-x > \varepsilon^m(1-\varepsilon) > \varepsilon^n(1-\varepsilon)$ for all $n > m$. We use the trivial consequence $y+\varepsilon^{n+1} > \frac{1}{2}(y+\varepsilon^{n+1}+x+\varepsilon^n) = \frac{x+y+\varepsilon^n(1+\varepsilon)}{2} > x+\varepsilon^n$ for each $n \geqslant m$.

A sequential application of HE proves that

$$L(y) \overset{(\dagger)}{\geqslant} \mathbf{W}\left(x+\delta, y+\varepsilon^3, y+\varepsilon^4, \ldots., y+\varepsilon^m, \frac{x+y+\varepsilon^m(1+\varepsilon)}{2}, y+\varepsilon^{m+2}, \ldots\right) \overset{(\ddagger)}{\geqslant}$$

$$\mathbf{W}\left(x+\delta, x+\varepsilon, y+\varepsilon^4, \ldots., y+\varepsilon^m, \frac{x+y+\varepsilon^m(1+\varepsilon)}{2}, \frac{x+y+\varepsilon^{m+1}(1+\varepsilon)}{2}, y+\varepsilon^{m+3}, \ldots\right)$$

$$\geqslant \ldots \geqslant$$

$$\geqslant \mathbf{W}\left(x+\delta, x+\varepsilon, \ldots., x+\varepsilon^{m-2}, \frac{x+y+\varepsilon^m(1+\varepsilon)}{2}, \ldots, \frac{x+y+\varepsilon^{2m-2}(1+\varepsilon)}{2}, y+\varepsilon^{2m}, \ldots\right)$$

To be precise, inequality (†) derives from $x+\delta > x+\varepsilon \geqslant y+\varepsilon^2 > y+\varepsilon^{m+1} > \frac{x+y+\varepsilon^m(1+\varepsilon)}{2}$, (‡) derives from $x+\varepsilon \geqslant y+\varepsilon^2 > y+\varepsilon^3 > y+\varepsilon^{m+2} > \frac{x+y+\varepsilon^{m+1}(1+\varepsilon)}{2}$, and so forth. Now WP assures that

$$\mathbf{W}\left(x+\delta, x+\varepsilon, \ldots., x+\varepsilon^{m-2}, \frac{x+y+\varepsilon^m(1+\varepsilon)}{2}, \ldots, \frac{x+y+\varepsilon^{2m-2}(1+\varepsilon)}{2}, y+\varepsilon^{2m}, \ldots\right) \geqslant$$

$$\mathbf{W}\left(x+\varepsilon, x+\varepsilon^2, x+\varepsilon^3, ...\right) = R(x)$$

because $x+\delta > x+\varepsilon$, $x+\varepsilon > x+\varepsilon^2$, ... , and $\frac{x+y+\varepsilon^m(1+\varepsilon)}{2} > x+\varepsilon^m$, ..., $\frac{x+y+\varepsilon^{2m-2}(1+\varepsilon)}{2} > x+\varepsilon^{2m-2}$, $y+\varepsilon^{2m} > x+\varepsilon^{2m-1}$, ... This ends the argument. □

Proof of Proposition 3. If $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$ verifies VWIA and WP then we can assign an uncountable number of different rational numbers, which is impossible, in the following manner. Let $\varepsilon = 0.1$, $\delta = 0.2$. For each $0 < x < \frac{1}{2}$ we let

$$L(x) := \mathbf{W}\left((x)_{con}\right) \quad \text{and } R(x) := \mathbf{W}\left(x+\varepsilon, x+\varepsilon^2, x+\varepsilon^3, ...\right)$$

thus $I(x) := (L(x), R(x)) \neq \varnothing$ due to WP. We proceed to prove that $I(x) \cap I(y) = \varnothing$ for every $\frac{1}{2} > y > x > 0$.

Case 1: $y > x+\varepsilon$. Then the conclusion follows from WP since $y > x+\varepsilon^n$ for each n thus $L(y) = \mathbf{W}\left((y)_{con}\right) \geqslant R(x)$.

Case 2: $x+\varepsilon \geqslant y$. Now we select z, a number strictly between x and y, such that $x+\varepsilon > x+\varepsilon^2 > ... > x+\varepsilon^n > z > x+\varepsilon^{n+1} > ...$ for some n (that is: a number $y > z > x$ that does not coincide with any $x+\varepsilon^n$). We use the trivial consequence $z > \frac{z+x+\varepsilon^{m+1}}{2} > x+\varepsilon^{m+1}$ for each $m = n, n+1, ...$. A sequential application of VWIA proves that

$$L(y) \geqslant \mathbf{W}\left((z)_{con}\right) \overset{(\dagger)}{\geqslant} \mathbf{W}\left(x+\delta, z, \overset{n-1}{\ldots\ldots\ldots}, z, \frac{z+x+\varepsilon^{n+1}}{2}, z, ...\right) \overset{(\ddagger)}{\geqslant}$$

$$\geqslant \mathbf{W}\left(x+\delta, x+\varepsilon, z, \overset{n-2}{\ldots\ldots\ldots}, z, \frac{z+x+\varepsilon^{n+1}}{2}, \frac{z+x+\varepsilon^{n+2}}{2}, z, ...\right) \geqslant ... \geqslant$$

$$\geqslant \mathbf{W}\left(x+\delta, x+\varepsilon, ..., x+\varepsilon^{n-1}, \frac{z+x+\varepsilon^{n+1}}{2}, ..., \frac{z+x+\varepsilon^{2n}}{2}, z, z, ...\right)$$

To be precise, inequality ($\dagger$) derives from $x+\delta > x+\varepsilon > z > \frac{z+x+\varepsilon^{n+1}}{2}$, ($\ddagger$) derives from $x+\varepsilon > z > \frac{z+x+\varepsilon^{n+2}}{2}$, and so forth. Now WP assures that

$$\mathbf{W}\left(x+\delta, x+\varepsilon, ..., x+\varepsilon^{n-1}, \frac{z+x+\varepsilon^{n+1}}{2}, ..., \frac{z+x+\varepsilon^{2n}}{2}, z, z, ...\right) \geqslant$$

$$\mathbf{W}\left(x+\varepsilon, x+\varepsilon^2, x+\varepsilon^3, ...\right) = R(x)$$

This ends the argument. □

Proof of Proposition 4. We need the following auxiliary result.

Lemma 2 *If $\boldsymbol{W}: \boldsymbol{X} \longrightarrow \mathbb{R}$ verifies IF, WP, and HEF then: For each $n \in \mathbb{N}$ and $y_1, ..., y_n, x, z \in [0,1]$ such that $y_1 \geqslant x, \; ... \;, y_n \geqslant x$, $x > z$,*

$$\boldsymbol{W}(\,(x)_{con}\,) > \boldsymbol{W}(y_1, ..., y_n, (z)_{con}\,)$$

Proof of Lemma 2. The case $n = 1$ amounts to the following statement that we use along this proof:

$$\mathbf{W}(\,(x)_{con}\,) > \mathbf{W}(y, (z)_{con}\,) \text{ for each } x, y, z \in [0,1] \text{ with } y \geqslant x > z \qquad (1)$$

To prove (1) observe that the case $x = y$ is direct from IF and WP. Assume $y > x$, and select $x > x' > z$. IF and WP entail $\mathbf{W}(\,(x)_{con}\,) > \mathbf{W}(x, (x')_{con}\,)$, and the claim follows because HEF implies $\mathbf{W}(x, (x')_{con}\,) \geqslant \mathbf{W}(y, (z)_{con}\,)$.

Let us now prove the thesis. Select $\varepsilon > 0$ such that $x > z + (n-1)\varepsilon$. Due to (1) we can assure $\mathbf{W}(\,(x)_{con}\,) > \mathbf{W}(y_1, (z + (n-1)\varepsilon)_{con}\,)$. Now we refer to $y_2 > z + (n-1)\varepsilon > z + (n-2)\varepsilon$ in order to apply (1) and deduce the inequality $\mathbf{W}((z+(n-1)\varepsilon)_{con}\,) > \mathbf{W}(y_2, (z+(n-2)\varepsilon)_{con}\,)$. Using IF we obtain $\mathbf{W}(y_1, (z + (n-1)\varepsilon)_{con}\,) > \mathbf{W}(y_1, y_2, (z + (n-2)\varepsilon)_{con}\,)$. An iterative process yields $\mathbf{W}(\,(x)_{con}\,) > \mathbf{W}(y_1, (z+(n-1)\varepsilon)_{con}\,) > ... > \mathbf{W}(y_1, .., y_n, (z)_{con}\,)$. □

If $\mathbf{W} : \mathbf{X} \longrightarrow \mathbb{R}$ verifies IF, WP, and HEF then we can assign an uncountable number of different rational numbers, which is impossible, as follows. For each $x \in (0,1)$ let $a_i(x) = x + \frac{1-x}{i+1}$, $i = 1, 2, ...$, in such way that the $\{a_i(x)\}_i$ sequence is strictly decreasing, tends to x, and $1 > a_i(x) > x$ throughout. Let

$$L(x) := \mathbf{W}\left((x)_{con}\right) \quad \text{and} \quad R(x) := \mathbf{W}\left(a_1(x), ..., a_n(x), ...\right)$$

thus $I(x) := (L(x), R(x)) \neq \varnothing$ by WP. We prove $I(x) \cap I(y) = \varnothing$, i.e., $L(y) = \mathbf{W}\left((y)_{con}\right) > R(x) = \mathbf{W}\left(a_1(x), ..., a_n(x), ...\right)$, for each $1 > y > x > 0$.

If $((y)_{con}) \gg (a_1(x), ..., a_n(x), ...)$ then WP yields the conclusion. Therefore assume that there is $m > 1$, the first index for which $y > a_n(x)$ whenever $n \geqslant m$. Select $b < 1$ such that $b > a_1(x) \geqslant y$. An application of Lemma 2 produces $\mathbf{W}\left((y)_{con}\right) > \mathbf{W}\left(b, \overset{m-1}{\cdots\cdots}, b, (\frac{y+a_m(x)}{2})_{con}\right)$, and furthermore WP yields $\mathbf{W}\left(b, \overset{m-1}{\cdots\cdots}, b, (\frac{y+a_m(x)}{2})_{con}\right) > \mathbf{W}\left(a_1(x), ..., a_m(x), ...\right)$. This ends the proof. □

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