Axiomatics for the Hirsch research output index

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18. December 2009

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18th December 2009

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Abstract
The Hirsch index is a number that synthesizes a researcher's output. It is defined as the maximum number *h* such that the researcher has *h* papers with at least *h* citations each. Two axiomatic characterizations of this index are suggested. One of them provides a simple conceptualization of the Hirsch index: after selecting those outputs deserving index 1, the Hirsch index of any other output *x* is the minimum value of a two-part decomposition of *x*.

Keywords: Hirsch index, publications, citations, research quality, scientific productivity.

JEL Classification: C43, A11, D80, D70

[†] E-mail address: aqa@urv.cat. Financial support from the Spanish *Ministerio de Educación y Ciencia* under research project SEJ2007-67580-C02-01 and from the *Departament d'Universitats, Recerca i Societat de la Informació* (*Generalitat de Catalunya*) under research project 2005SGR-00949 is gratefully acknowledged.

1. Introduction

The physicist Jorge E. Hirsch (2005) has suggested the *h*-index, also known as the Hirsch index, as a way to characterize the scientific output of a researcher. The Hirsch index of a researcher i is the maximum number h of i's papers having at least h citations each; see Wikipedia (2009) for a discussion of advantages and criticisms.

The Hirsch index, by now "famous" according to Sidiropoulos et al. (2007, p. 253), has been axiomatically characterized by Woeginger (2008a, 2008b) within the domain of scientific impact indices taking values in the set of non-negative integers. This note offers another two axiomatizations, but for a larger domain, consisting of scientific impact indices taking values in the set of non-negative real numbers.

The second characterization (Proposition 3.6) is probably the most valuable because it hinges on an axiom expressing a sort of non-manipulability condition. According to this condition, it cannot be in a researcher's interest to partition his or her output into two parts, compute the index of each part, and next claim that the index of the whole output is the sum of the index of the two parts.

2. Definitions and axioms

Following Woeginger (2008a, p. 225; 2008b, p. 299), let $\mathbb{N}$ designate the set of non-negative integers and X designate the set of all vectors $x = (x_1, x_2, \dots, x_n)$ such that: (i) $n \geq 1$; (ii) every component x_i belongs to $\mathbb{N}$; and (iii) $x_1 \geq x_2 \geq \dots \geq x_n$. A researcher with $n \geq 1$ publications can be represented by the member $x = (x_1, \dots, x_n)$ of X such that x_i is the number of citations of paper i, with citations are arranged in a non-increasing order. For $x \in X$, d_x denotes the number of components (or dimension) of vector x, $x^- = min\{x_1, \dots, x_{d_x}\}$ and $x^+ = max\{x_1, \dots, x_{d_x}\}$.

Definition 2.1. A research output index (or index, for short) is a mapping $f : X \to \mathbb{R}_+$, where $\mathbb{R}_+$ designates the set of non-negative real numbers.

For $x \in X$, $f(x)$ can be viewed as a measure of the value, relevance, impact… of output x or a measure of the productivity, quality, visibility… of the researcher who generated x.

Definition 2.2. The Hirsch index is the research output index h such that, for all $x \in X$, $h(x) = max\{k \in \{0, 1, \dots, d_x\}: x_k \geq k\}$.

BOU. *Boundedness*. For all $x \in X$, $min\{x^-, d_x\} \leq f(x) \leq min\{x^+, d_x\}$.

The condition BOU states that $f(x)$ cannot be greater than the minimum between the number of papers and the number of citations of the most cited paper. It also says that $f(x)$ cannot be smaller than the minimum between the number of papers and the number of citations of the least cited paper. BOU allows $f(x)$ to be interpreted as a measure of the number of quality papers in output x. In particular, more citations are necessary but not sufficient for the index to rise.

For $x \in X$ and $r \in \{1, \dots , d_x\}$, let x_{-r} designate the member of X obtained from x by removing x_r. For instance, if $x = (9, 7, 1)$ then $x_{-1} = (7, 1)$, $x_{-2} = (9, 1)$ and $x_{-3} = (9, 7)$.

MAX. *Maximization under minimal exclusion*. For $x \in X$, if $d_x \geq 2$ and $x^- < d_x$ then $f(x) = max\{f(x_{-r})\}_{1 \leq r \leq d_x}$.

Suppose $x^- \geq d_x$. If BOU is assumed, then all the papers are quality papers, so the researcher achieves the maximum index that BOU allows. MAX presumes that $x^- < d_x$ implies that some paper is not a quality paper and deals with this case by considering the ways in which output x can be reached from outputs containing one paper less. MAX holds that the value of x is the maximum among the values of the outputs lacking one paper. By MAX, if not all the papers are quality papers, then, in some history leading to the final output, the marginal contribution of the last paper to research quality is null: a researcher not achieving the maximum index with a given set of papers can choose to remove some paper without altering the resulting Hirsch index.

Define $X_1 = \{x \in X$: $x^+ = 1$ or there is exactly one $i \in \{1, \dots , d_x\}$ such that $x_i > 1\}$. The set X_1 is formed by two types of output: (i) outputs consisting of one citation papers; and (ii) outputs in which only one paper has more than one citation.

UNI. *Outputs with value equal to unity*. For all $x \in X_1$, $f(x) = 1$.

UNI sets the unit of measure by selecting outputs deserving index 1. By UNI, if some paper has some citation, the index is 1 when no paper obtains more than one citation or there is only one paper. For instance, UNI implies that having 10 papers each of which is cited once is equivalent to producing only one paper and having it cited 10 times.

For $x \in X$, define $x^{\Sigma} = x_1 + \dots + x_{d_x}$. Let X' designate the set of all vectors $x = (x_1, \dots , x_n)$ such that $n \geq 1$ and each x_i belongs to $\mathbb{N}$. For $x \in X'$ and $y \in X'$ with $d_x \geq d_y$, $x \oplus y$ is the member z of X' such that: (i) for all $i \in \{1, \dots, d_y\}$, $z_i = x_i + y_i$; and (ii) for all $i \in \{d_y + 1, \dots, d_x\}$, $z_i = x_i$.

Definition 2.3. A simple decomposition of $x \in X$ is a pair $(y, z) \in X \times X$ such that: (i) if $d_x \geq 2$ or $x^{\Sigma} \geq 2$, then $y \neq x \neq z$; and (ii) there are $y' \in X'$ and $z' \in X'$ such that $y' \oplus z' = x$ and, for each $\alpha \in \{y, z\}$, $d_{\alpha'} \leq d_x$ and α is the member of X obtained from α' by arranging the components of α' in a non-increasing order. The set of simple decompositions of $x \in X$ is $\delta(x)$.

A simple decomposition (y, z) of output x can be seen as a history of how x could have been reached in two periods, with the output of each period ready to be evaluated by an index and with x different from y and z if there are least two papers or at least two citations. As an illustration, let $x = (7, 5, 3, 3, 0)$ be the output in which paper 1 has 7 citations, paper 2 has 5, papers 3 and 4 have 3 citations each, and paper 5 has no citation. Then (y, z) with $y = (5, 2, 1)$ and $z = (6, 3, 1, 0)$ is a simple decomposition representing the history such that: in the first period, paper 1 receives 1 citation, paper 2 receives 5 and paper 3 receives 2; and, in the second period, paper 1 receives 6 citations more, paper 3 receives 1 citation more, paper 4 receives 3 citations, and paper 5 receives none. In decomposition (y, z), paper 2 implicitly receives 0 citations in the second period. If this is made explicit with $z^* = (6, 3, 1, 0, 0)$, then (y, z^*) is also a simple decomposition of x.

MIN. *Minimization under simple decompositions*. For all $x \in X \backslash X_1$, $f(x) = min\{f(y) + f(z)\}_{(y,z)\in\delta(x)}$.

For outputs not in X_1, MIN suggests that the index is two-period history insensitive. If the value of a history $(y, z) \in \delta(x)$ is the sum of the values of y and z, then, by MIN, the value of x is the smallest value of a history leading to x. Implicit in MIN is some form of non-manipulability: MIN does not allow a researcher i to improve his or her own index by breaking his or her career into two periods (junior and senior periods, for example) and by recalculating next the index of the total output as the sum of the index of the two partial outputs. MAX and MIN can be viewed as symmetrical requirements: whereas MAX deals with the effect of varying the number of papers, MIN considers the effect of splitting the number of citations (and, possibly, papers). Both MAX and MIN express some compromise between a “liberal” and a “conservative” output evaluation. MAX is liberal in considering maximum values and conservative in letting the value of an n paper output coincide with the value of some $n - 1$ paper output. MIN is liberal is considering the sum of outputs and conservative in choosing a minimum value.

3. Results

Remark 3.1. The Hirsch index satisfies BOU.

Let $x \in X$ and $n = d_x$. Case 1: $x^+ \geq n$. By definition, $h(x) \leq n$. Hence, $h(x) \leq n = min\{x^+, n\}$. Case 2: $x^+ < n$. Since $h(x) > x^+$ would imply that some paper has more citations than the maximum number of citations, $h(x) \leq x^+ = min\{x^+, n\}$. Case 3: $x^- \geq n$. This means that each of the n papers receives at least n citations, so $h(x) = n = min\{x^-, n\}$. Case 4: $x^- < n$. In this case, each of the n papers receives at least x^- citations, for which reason $h(x) \geq x^- = min\{x^-, n\}$.

Remark 3.2. The Hirsch index satisfies MAX.

Let $x \in X$, $x^- < d_x = n \geq 2$ and $h = h(x)$. It follows from $x^- < n$ that $h(x) < n$. This means that the last component x_n of x is irrelevant to compute $h(x)$, so $h(x_{-n}) = h(x)$ and $max\{h(x_{-1}), \dots, h(x_{-n})\} \geq h(x_{-n}) = h(x)$. To show that $max\{h(x_{-1}), \dots, h(x_{-n})\} \leq h(x)$ suppose that, for some $i \in \{1, \dots, n\}$, $h(x_{-i}) > h(x)$. Then there are at least $h + 1$ components in x_{-i} not smaller than $h + 1$. Accordingly, there are at least $h + 1$ components in x not smaller than $h + 1$. This implies $h(x) \geq h + 1$: contradiction.

Proposition 3.3. An index f satisfies BOU and MAX if and only if f is the Hirsch index.

Proof. "$\Leftarrow$" Remarks 3.1 and 3.2. "$\Rightarrow$" Let f satisfy BOU and MAX. For $n \in \mathbb{N}\backslash\{0\}$, define $D_n = \{x \in X: d_x = n\}$. The proof is by induction on the sets D_n. Step 1: $f = h$ on D_1. Choose $x \in D_1$. This makes $x = (x_1)$. By BOU, $f(x) = min\{x_1, 1\}$. Hence, $x_1 = 0$ implies $f(x) = 0 = h(x)$; and $x_1 \geq 1$ implies $f(x) = 1 = h(x)$. Step 2: for $n \geq 2$, if $f = h$ on $D_1 \cup \dots \cup D_{n-1}$, then $f = h$ on $D_1 \cup \dots \cup D_{n-1} \cup D_n$. Choose $n \geq 2$ and assume that $f = h$ on $D_1 \cup \dots \cup D_{n-1}$. To show that $f = h$ on $D_1 \cup \dots \cup D_n$, choose $x \in D_n$. Case 1: for all $i \in \{1, \dots, n\}$, $x_i \geq n$. By BOU, $min\{x^-, n\} \leq f(x) \leq min\{x^+, n\}$. Therefore, $f(x) = n = h(x)$. Case 2: for some $i \in \{1, \dots, n\}$, $x_i < n$. This yields $x^- < n$. By MAX, $f(x) = max\{f(x_{-1}), \dots, f(x_{-n})\}$. By the induction hypothesis, for all $i \in \{1, \dots, n\}$, $f(x_{-i}) = h(x_{-i})$. Thus, $f(x) = max\{h(x_{-1}), \dots, h(x_{-n})\}$. By Remark 3.2, h satisfies MAX. Hence, as $x^- < d_x \geq 2$, $max\{h(x_{-1}), \dots, h(x_{-n})\} = h(x)$.■

For $x \in X$ with $h = h(x) \geq 2$, set $r = h/2$ if h is even and $r = (h + 1)/2$ otherwise. Then define $(x^1, x^2) \in \delta(x)$ as follows: (i) $\tilde{x}^1 \in X'$ collects, for paper $i \in \{1, \dots, r\}$, x_i citations and, for paper $i \in \{h + 1, \dots, d_x\}$, $x_i/2$ citations if x_i is even and $(x_i + 1)/2$ citations otherwise; (ii) $\tilde{x}^2 \in X'$ collects, for paper $i \in \{r + 1, \dots, h\}$, x_i citations and, for paper i

$\in \{h + 1, \dots, d_x\}$, $x_i/2$ citations if x_i is even and $(x_i - 1)/2$ citations otherwise; and (iii) for $k \in \{1, 2\}$, x^k is obtained from $\tilde{x}^k$ by arranging citations in a non-increasing order. Roughly speaking, (x^1, x^2) is the simple decomposition of x in which x^i collects the citations of half of the papers contributing to the Hirsch index and half of the citations of those papers not contributing to the index.

Remark 3.4. For all $x \in X \backslash X_1$, $h(x^1) + h(x^2) = h(x)$, where $(x^1, x^2) \in \delta(x)$.

Let $x \in X \backslash X_1$ and $h = h(x)$. Case 1: h even. Then $h(x^1) \geq h/2$ because x^1 has at least $h/2$ papers with at least h citations each. That $h(x^1) \leq h/2$ follows from the fact that $h(x) = h$ implies $x_{h+1} \leq h$ and, thus, paper $i \in \{h + 1, \dots, n\}$ has at most $h/2$ citations in x^1. In sum, $h(x^1) = h/2$. The same reasoning proves that $h(x^2) = h/2$. Case 2: h odd. Let $h' = (h + 1)/2$. Then $h(x^1) \geq h'$ because x^1 has at least h' papers with at least h citations each. That $h(x^1) \leq h'$ follows from the fact that $h(x) = h$ implies $x_{h+1} \leq h$ and, consequently, paper $i \in \{h + 1, \dots, n\}$ has at most h' citations in x^1. This shows that $h(x^1) = h'$. With $h'' = (h - 1)/2$, $h(x^2) \geq h''$ because x^2 has at least h'' papers with at least h citations each. Being h odd, paper $i \in \{h + 1, \dots, n\}$ has at most $h'' - 1$ citations in x^2, for which reason $h(x^2) \leq h''$. Summarizing, $h(x^2) = h''$ and $h(x^1) + h(x^2) = h' + h'' = h$.

Remark 3.5. The Hirsch index satisfies UNI and MIN.

UNI follows immediately from the definition of the Hirsch index. As for MIN, let $x \in X \backslash X_1$ and $h = h(x)$. Step 1: $min\{h(y) + h(z)\}_{(y,z) \in \delta(x)} \leq h$. Follows from Remark 3.4. Step 2: $min\{h(y) + h(z)\}_{(y,z) \in \delta(x)} \geq h$. Choose $(y, z) \in \delta(x)$. Let $h_1 = h(y)$ and $h_2 = h(z)$. Clearly, $h_1 \leq h$ and $h_2 \leq h$: with less citations the Hirsch index cannot be higher. As a result, of the h papers that, in x, have at least h citations each, at most h_1 may have more than h_1 citations in y. This leaves at least $h - h_1$ papers in z with at least $h - h_1$ citations each. In consequence, $h_2 \geq h - h_1$ and $h_1 + h_2 \geq h$.

Proposition 3.6. An index f satisfies UNI and MIN if and only if f is the Hirsch index.

Proof. "$\Leftarrow$" Remark 3.5. "$\Rightarrow$" Let f satisfy UNI and MIN. For $n \in \mathbb{N} \backslash \{0\}$, define $D_n = \{x \in X: d_x = n\}$ and let 0_n stand for the $x \in D_n$ such that, for all $i \in \{1, \dots, n\}$, $x_i = 0$. Step 1: for all $n \in \mathbb{N} \backslash \{0\}$, $f(0_n) = 0$. Case 1: $n = 1$. Suppose not: $f(0_1) > 0$. Since $\delta(0_1) = \{(0_1, 0_1)\}$, by MIN, $f(0_1) = f(0_1) + f(0_1)$. Hence, $f(0_1) = 0$: contradiction. Case 2: $n \geq 2$. Taking case 1 as the base case of an induction argument, choose $n \geq 2$ and suppose that, for all $k \in \{1, \dots, n - 1\}$, $f(0_k) = 0$. Choose $(y, z) \in \delta(0_n)$. By definition of simple decomposition: (i) there are $r \leq n$ and $k \leq n$ such that $y = 0_r$ and $z = 0_k$; and (ii) since $n \geq$

2, $r \neq n \neq k$. Consequently, $r < n > k$. In view of this, by the induction hypothesis, $h(y) + h(z) = 0$. In sum, for all $(y, z) \in \delta(0_n)$, $h(y) + h(z) = 0$. By MIN, $h(0_n) = 0$.

Step 2: $f = h$ on D_1. Choose $x \in D_1$. If $x = 0_1$, then, by step 1, $f(x) = 0 = h(x)$. Otherwise, $x \in X_1$ and, by UNI, $f(x) = 1 = h(x)$.

Step 3: for $n \geq 2$, if $f = h$ on $D_1 \cup \ldots \cup D_{n-1}$ then $f = h$ on $D_1 \cup \ldots \cup D_{n-1} \cup D_n$. Choose $n \geq 2$ and suppose $f = h$ on $D_1 \cup \ldots \cup D_{n-1}$. To show that $f = h$ on $D_1 \cup \ldots \cup D_n$, let $x \in D_n$. If $x \in X_1$, then, by UNI, $f(x) = 1 = h(x)$. If $x \notin X_1$, then consider $(x^1, x^2) \in \delta(x)$. By MIN, $f(x) \leq f(x^1) + f(x^2)$. As $x \notin X_1$, at least two papers get at least two citations, so $h(x) \geq 2$. Because of this, the dimension of both x^1 and x^2 is smaller than n. By the induction hypothesis, $f(x^1) = h(x^1)$ and $f(x^2) = h(x^2)$. Consequently, $f(x) \leq f(x^1) + f(x^2) = h(x^1) + h(x^2) = h(x)$, the last equality by Remark 3.4. To show that $f(x) \geq h(x)$, suppose otherwise: $f(x) < h(x)$.

By MIN, there is $(y, z) \in \delta(x)$ with $f(x) = f(y) + f(z)$. Clearly, $x^\Sigma \geq 2$: if $x^\Sigma = 0$, then, by step 1, $f(x) = 0 = h(x)$; and if $x^\Sigma = 1$, then $f(x) = 1 = h(x)$. In addition, $d_x \geq 2$. Hence, by definition of simple decomposition, $y \neq x \neq z$. By Remark 3.5, h satisfies MIN, so $h(x) \leq h(y) + h(z)$. As $f(x) < h(x)$, $f(y) + f(z) < h(y) + h(z)$. This implies that, for some $v \in \{y, z\}$, $f(v) < h(v)$. Given that $v \neq x$, $v^\Sigma < x^\Sigma$. All in all, $f(x) < h(x)$ implies that, for some $v \in X$, $v^\Sigma < x^\Sigma$ and $f(v) < h(v)$. The same reasoning could be then applied to $f(v) < h(v)$ to conclude that, for some $w \in X$, $w^\Sigma < v^\Sigma$ and $f(w) < h(w)$. By replicating this reasoning successively, a sequence $(x, v, w, \ldots)$ is generated in which some member t must eventually satisfy $t \in D_1 \cup \ldots \cup D_{n-1}$ or $t \in X_1$. Both cases contradict $f(t) < h(t)$: the first one, by the induction hypothesis; the second one, by UNI.■

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