The axiomatic approach to three values in games with coalition structure

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Abstract

We study three values for transferable utility games with coalition structure, including the Owen coalitional value and two weighted versions with weights given by the size of the coalitions. We provide three axiomatic characterizations using the properties of Efficiency, Linearity, Independence of Null Coalitions, and Coordination, with two versions of Balanced Contributions inside a Coalition and Weighted Sharing in Unanimity Games, respectively.

Keywords: coalition structure, coalitional value.

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1 Introduction

Coalition structures are important in many real-world contexts, such as the formation of cartels or bidding rings, alliances or trading blocs among nation states, research joint ventures, and political parties.

These situations can be modelled through transferable utility (TU, for short) games, in which the players partition themselves into coalitions for the purpose of bargaining. All players in the same coalition agree before the play that any cooperation with other players will only by carried out collectively. That is, either all the members of the coalition take part of it or none of them (Malawski, 2004).

Given a coalition structure, bargaining occurs between coalitions and between players in the same coalition. The main idea is that the coalitions play among themselves as individual agents in a game among coalitions, and then, the profit obtained by each coalition is distributed among its members. Owen (1977) studied the allocation that arises from applying the Shapley value (Shapley, 1953b) twice: first in the game among coalitions, and then in a reduced game inside each coalition. In this latter step, the worth a subcoalition in the reduced game is defined as the Shapley value that the subcoalition would get in the game among coalitions, assuming that their partners are out.

Owen's approach assumes a symmetric treatment for each coalition. As Harsanyi (1977) points out, in unanimity games this procedure implies that players would be better off bargaining by themselves than joining forces. This is know as the join-bargaining paradox, or the Harsanyi paradox.

An alternative approach is to give a different treatment, or weight, to each coalition. Following this idea, Levy and McLean (1989) apply the weighted Shapley value (Shapley, 1953a; Kalai and Samet, 1987, 1988) in the game among coalitions, as well as in the reduced games.

A natural weight for each coalition is its own size. In fact, a motivation for the weighted Shapley value is precisely the difference in size[1]. Moreover, Kalai and Samet (1987, Corollary 2 in Section 7) show that the size of coalitions are appropriate weights for the players. The reason is that if we force the players in a coalition to work together (by destroying their resources when they are not all together), then the aggregated Shapley value of each

[1] Kalai and Samet (1987) present the example of large constituencies with many individuals, in contrast with constituencies composed by a small number of individuals.

coalition in the new game coincides with the weighted Shapley value of the game among coalitions, with weights given by the size of the coalition[2].

It is then reasonable to apply the Levy and McLean value with intracoalitional symmetry and weights given by the size of the coalition. However, in Levy and McLean's model, the weight of the subcoalitions in the reduced game remains constant, even though these subcoalitions may have different size. An alternative approach is to vary the weight of the coalitions in the reduced game. Vidal-Puga (2006) follows this approach to define a new coalitional value. This new coalitional value does not present the Harsanyi paradox.

In this paper, we characterize the above coalitional values: the coalitional Owen value (Owen, 1977), the coalitional Levy-McLean weighted value (Levy and McLean, 1989) with the weights given by the size of the coalition, and the new value presented by Vidal-Puga (2006). These three values have in common the following feature: First, the worth of the grand coalition is divided among the coalitions following either the Shapley value (Owen), or the weighted Shapley value with weights given by the size of the coalitions (Levy and McLean, Vidal-Puga), and then the profit obtained by each coalition is distributed among its members following the Shapley value.

Some of the axioms used in the characterizations (*efficiency, intracoalitional symmetry,* and *linearity*) are standard in the literature, others (*independence of null coalitions* and two intracoalitional versions of *balanced contributions*) are used in many different frameworks. Moreover, we introduce new properties in this kind of problems: *coordination* (which asserts that internal changes in a coalition which do no affect the game among coalitions, do not influence the final payment of the rest of the players) and two properties of *sharing in unanimity games* (which establish how should the payment be under the grand coalition unanimity game).

The properties of efficiency, linearity, intracoalitional symmetry and independence of null coalitions are natural extensions of the classical properties that characterize the Shapley value (efficiency, linearity, symmetry and null player, respectively) to the game among coalitions. On the other hand, the properties of balanced contributions are applied to the game inside a coalition, and each of them is a natural extension of the property of balanced

[2]Another possibility is to give the worth of any coalition to any of its nonempty subcoalitions. In this case, the aggregated Shapley value of each coalition coincides with the weigthed Shapley value of the *dual* game among coalitions (see Kalai and Samet, 1987, Section 7, for further details).

contributions that also characterizes, with efficiency, the Shapley value (Myerson, 1980). Hence, the three values proposed here can be seen as natural extensions of the Shapley value for games with coalition structure. Additionally, the property of coordination formalizes the idea presented by Owen that the players inside a coalition negotiate among them, but always assuming that the rest of the coalitions remain together (see for example the game v_1 defined by Kalai and Samet, 1987, Section 7).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we define a family that includes the three coalitional values. In Section 4 we present the properties used in the characterization and we study which properties satisfy the coalitional values. In Section 5 we present the characterization results. In Section 6 we prove that the properties are independent. In Section 7 we present some concluding remarks.

2 Notation

Let $U = \{1, 2, ...\}$ be the (infinite) set of potential players.

Given a finite subset $N \subset U$, let $\Pi(N)$ denote the set of all orders in N. Given $\pi \in \Pi(N)$, let $\text{Pre}(i, \pi)$ denote the set of the elements in N which come before i in the order given by π, *i.e.* $\text{Pre}(i, \pi) = \{j \in N : \pi(j) < \pi(i)\}$. For any $S \subset N$, π_S denotes the order induced in S by π (for all $i, j \in S$, $\pi_S(i) < \pi_S(j)$ if and only if $\pi(i) < \pi(j)$).

A *transfer utility game*, *TU game*, or simply a *game*, is a pair (N, v) where $N \subset U$ is finite and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. When N is clear, we can also denote (N, v) as v. Given a TU game (N, v) and $S \subset N$, $v(S)$ is called the *worth* of S. Given $S \subset N$, we denote the restriction of (N, v) to S as (S, v).

For simplicity, we write $S \cup i$ instead of $S \cup \{i\}$, $N \backslash i$ instead of $N \backslash \{i\}$, and $v(i)$ instead of $v(\{i\})$.

Two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subset N \backslash \{i, j\}$. A player $i \in N$ is *null* in (N, v) if $v(T \cup i) = v(T)$ for all $T \subset N \backslash i$. The set of non-null players in (N, v) is the *carrier* of (N, v), and we denote it as $Carr(N, v)$. Given two games (N, v), (N, w), the game $(N, v+w)$ is defined as $(v+w)(S) = v(S)+w(S)$ for all $S \subset N$. Given a game (N, v) and a real number α, the game $(N, \alpha v)$ is defined as $(\alpha v)(S) = \alpha v(S)$ for all $S \subset N$.

Given $N \subset U$ finite, we call *coalition structure* over N a partition of the player set N, *i.e.* $\mathcal{C} = \{C_1, C_2, ..., C_m\} \subset 2^N$ is a coalition structure if it satisfies $\bigcup_{C_q \in \mathcal{C}} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. We also assume $C_q \neq \emptyset$ for all q.

We say that $C_q \in \mathcal{C}$ is a *null coalition* if all its members are null players.

For any $S \subset N$, we denote the restriction of $\mathcal{C}$ to the players in S as $\mathcal{C}_S$, *i.e.* $\mathcal{C}_S = \{C_q \cap S : C_q \in \mathcal{C}$ and $C_q \cap S \neq \emptyset\}$.

For any $S \subset C_q \in \mathcal{C}$, we will frequently study the case in which the players in $C_q \backslash S$ leave the game. In this case, we write $\mathcal{C}^S$ instead of the more cumbersome $\mathcal{C}_{N \backslash (C_q \backslash S)}$.

Given a game (N, v) and a coalition structure $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N, the *game among coalitions* is the TU game $(M, v/\mathcal{C})$ where $M = \{1, 2, ...m\}$ and $(v/\mathcal{C})(Q) = v\left(\bigcup_{q \in Q} C_q\right)$ for all $Q \subset M$.

We denote the *game* (N, v) *with coalition structure* $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N as $(N, v, \mathcal{C})$ or $(v, \mathcal{C})$. When N and $\mathcal{C}$ are clear, we also write v instead of $(N, v, \mathcal{C})$.

Given $S \subset N$, $S \neq \emptyset$, the *unanimity game* with carrier S, (N, u_N^S) is defined as $u_N^S(T) = 1$ if $S \subset T$ and $u_N^S(T) = 0$ otherwise, for all $T \subset N$.

A *value* is a function that assigns to each game (N, v) a vector in $\mathbb{R}^N$ representing the amount that each player in N expects to get in the game. One of the most important values in TU games is the *Shapley value* (Shapley, 1953b). We denote the Shapley value of the TU game (N, v) as $Sh(N, v) \in \mathbb{R}^N$.

Similarly, a *coalitional value* is a function that assigns to each game with coalition structure $(N, v, \mathcal{C})$ a vector in $\mathbb{R}^N$. Each value can also be considered as a coalitional value by simply ignoring the coalition structure. Hence, we define the *coalitional Shapley value* of the game $(N, v, \mathcal{C})$ as $Sh(N, v, \mathcal{C}) = Sh(N, v)$. One of the most important coalitional values is the *Owen value* (Owen, 1977).

Another generalization for a value is the following: a *weighted value* ϕ^ω is a function that assigns to each TU game (N, v) and each $x \in \mathbb{R}^N_{++}$ a vector ϕ^x in $\mathbb{R}^N$. For each $i \in N$, x_i is the *weight* of player i. We will say that a weighted value ϕ^ω *extends* or *generalizes* a value ϕ if $\phi^x(N, v) = \phi(N, v)$ for any weight vector x with $x_i = x_j$ for all $i, j \in N$. The most prominent weighted generalization of the Shapley value is the *weighted Shapley value* Sh^ω (Shapley (1953a), Kalai and Samet (1987, 1988)).

3 Games with coalition structure

We now focus on games with coalition structure. Fix $\mathcal{C} = \{C_1, ..., C_m\}$ and let $M = \{1, ..., m\}$. For each pair (γ, ϕ^ω), where γ is a value and ϕ^ω is a weighted value, we define two coalitional values $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$. In both cases, the idea is to divide the worth of the grand coalition in two steps: In the first step, ϕ^ω is used to divide the worth of the grand coalition in the game among coalitions, with weights given by the size of each coalition. In the second step, γ is used to divide the worth inside each coalition.

For each coalition structure $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N, let $\sigma(\mathcal{C}) \in \mathbb{R}^M_+$ be defined as $\sigma_q(\mathcal{C}) = |C_q|$ for all[3] $q \in M$. Given $C_q \in \mathcal{C}$, the *reduced TU game with fixed weights* $\left(C_q, v_{C_q}^{[\phi^\omega]N}\right)$ is defined as

$$v_{C_q}^{[\phi^\omega]N}(S) := \phi_q^{\sigma(\mathcal{C})}\left(M, v/\mathcal{C}^S\right)$$

for all $S \subset C_q$. The *reduced TU game with relaxed weights* $\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$ is defined as

$$v_{C_q}^{\langle\phi^\omega\rangle N}(S) := \phi_q^{\sigma\left(\mathcal{C}^S\right)}\left(M, v/\mathcal{C}^S\right)$$

for all $S \subset C_q$.

Thus, both $v_{C_q}^{[\phi^\omega]N}(S)$ and $v_{C_q}^{\langle\phi^\omega\rangle N}(S)$ are interpreted as the value that ϕ^ω assigns to coalition S in the game among coalitions assuming that the members of $C_q \backslash S$ are out. In the first case, coalition S maintains the weight of the original coalition C_q. In the second case, coalition S plays with a weight proportional to its own (reduced) size.

In the particular case $\phi^x = \phi$ for all x, both reduced TU games coincide and we write $\left(C_q, v_{C_q}^{(\phi)N}\right)$ instead of $\left(C_q, v_{C_q}^{[\phi^\omega]N}\right)$ or $\left(C_q, v_{C_q}^{\langle\phi^\omega\rangle N}\right)$.

Definition 1 *Given a value γ and a weighted value ϕ^ω, we define respectively the coalitional values $\gamma[\phi^\omega]$ and $\gamma\langle\phi^\omega\rangle$ as*

$$\gamma[\phi^\omega]_i(N, v, \mathcal{C}) := \gamma_i\left(C_q, v_{C_q}^{[\phi^\omega]N}\right)$$

[3] To be precise, given $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ over N, $\sigma(\mathcal{C}) = \sigma(\mathcal{C}, M, f)$ where $f : \mathcal{C} \to M$ is a one-to-one correspondence that matches each coalition in $\mathcal{C}$ with each index in M.

and

$$\gamma \langle \phi^\omega \rangle_i (N, v, \mathcal{C}) := \gamma_i \left(C_q, v_{C_q}^{\langle \phi^\omega \rangle N} \right)$$

for all $i \in C_q \in \mathcal{C}$.

In the particular case $\phi^x = \phi$ *for all* x, *both expressions coincide and hence we write* $\gamma(\phi) := \gamma[\phi^\omega] = \gamma\langle \phi^\omega \rangle$.

We concentrate on three particular members of this family, that have been previously studied in the literature:

Example 2 $Sh(Sh)$ *is the Owen value (Owen, 1977).*

$Sh[Sh^\omega]$ *is the weighted coalitional value with intracoalitional symmetry, and weights given by the size of the coalitions (Levy and McLean, 1989).*

$Sh\langle Sh^\omega \rangle$ *has been studied by Vidal-Puga (2006).*

There exist other relevant coalitional values that belong to this family. Let Ba be the *Banzhaf value* (Banzhaf 1965, Owen 1975). Let In be the *individual value* (Owen[4], 1978) defined as $In_i(N, v) = v(\{i\})$ for all $i \in N$. Given $p \in [0, 1]$, let B^p be the p-binomial value (Puente, 2000). Let DP be the *Deegan-Packel value* (Deegan and Packel, 1979). Let LSP be the *least square prenucleolus* (Ruiz, Valenciano and Zarzuelo, 1996).

Example 3 $Sh(In)$ *is the Aumann-Drèze value (Aumann and Drèze, 1974).*

$Ba(Ba)$ *is the Banzhaf-Owen value (Owen 1975).*

$Sh(Ba)$ *is the symmetric coalitional Banzhaf value (Alonso-Meijide and Fiestras-Janeiro, 2002).*

$Ba(Sh)$ *is defined and studied by Amer, Carreras and Giménez (2002).*

$\{Sh(B^p)\}_{p\in[0,1]}$ *is the family of symmetric coalitional binomial values (Carreras and Puente, 2006).*

$DP(DP)$ *and* $LSP(LSP)$ *are defined and studied by Młodak (2003).*

4 Properties

In this section we present some properties of the values. Moreover, we provide several results.

[4] Owen uses the term *dictatorial* instead of *individual*.

4.1 Classical properties

Efficiency (Eff) For any game $(N, v, \mathcal{C})$, $\sum_{i \in N} f_i(N, v, \mathcal{C}) = v(N)$.

That is, the worth of the grand coalition is distributed.

Linearity (Lin) Given $(N, v, \mathcal{C})$, $(N, w, \mathcal{C})$ and real numbers α and β,

$$f(N, \alpha v + \beta w, \mathcal{C}) = \alpha f(N, v, \mathcal{C}) + \beta f(N, w, \mathcal{C}).$$

That is, if a game is a linear combination of two games, the value assigns the linear combination of the values of the games.

Symmetry (Sym) Given two symmetric players $i, j \in N$ in a game $(N, v, \mathcal{C})$, $f_i(N, v, \mathcal{C}) = f_j(N, v, \mathcal{C})$.

That is, two symmetric players in (N, v) receive the same.

Null Player (NP) Given a null player $i \in N$ in a game $(N, v, \mathcal{C})$, $f_i(N, v, \mathcal{C}) = 0$.

That is, any null player receives zero.

Independence of Null Players (INP) Given a null player $i \in N$ in a game $(N, v, \mathcal{C})$,

$$f_j(N, v, \mathcal{C}) = f_j\left(N \backslash i, v, \mathcal{C}_{N \backslash i}\right)$$

for all $j \in N \backslash i$.

That is, no agent gets a different value if a null player is removed from the game.

We say that a weighted value ϕ^ω satisfies some property if ϕ^x satisfies this property for each x.

Proposition 4 *a) The Shapley value Sh is the only value that satisfies Eff, Lin, Sym and INP.*

b) The weighted Shapley value Sh^ω satisfies INP.

Proof. a) It is well-known that Sh satisfies Eff, Lin and Sym. It is also clear that Sh satisfies INP. On the other hand, it is straightforward to check that Eff and INP imply NP. Since Sh is the only value that satisfies Eff, Lin, Sym and NP (Shapley, 1953b), we deduce the result.

b) From Kalai and Samet (1987, Theorem 1) and a classical induction hypothesis on the number of players, it is straightforward to check that Sh^{ω} satisfies INP. ■

Lin and Eff can be adapted to games with coalition structure without changes. For Sym and INP, we will apply them inside the coalitions and to null coalitions, respectively:

Intracoalitional Symmetry (IS) Given two symmetric players in the same coalition $i, j \in C_q \in \mathcal{C}$, $f_i(N, v, \mathcal{C}) = f_j(N, v, \mathcal{C})$.

Independence of Null Coalitions (INC) Given a game $(N, v, \mathcal{C})$ and a null coalition $C_q \in \mathcal{C}$, $f_i(N, v, \mathcal{C}) = f_i\left(N \backslash C_q, v, \mathcal{C}_{N \backslash C_q}\right)$ for all $i \in N \backslash C_q$.

INC asserts that if a coalition is *null*, it does not influence the allocation within the rest of the players. It is a weaker property than INP. Notice that INC and Eff imply that the aggregated payment of the agents in a null coalition is zero.

Proposition 5 *a) If both γ and ϕ^{ω} satisfy Eff, then both $\gamma[\phi^{\omega}]$ and $\gamma\langle\phi^{\omega}\rangle$ satisfy Eff.*

b) If both γ and ϕ^{ω} satisfy Lin, then both $\gamma[\phi^{\omega}]$ and $\gamma\langle\phi^{\omega}\rangle$ satisfy Lin.

c) If γ satisfies Sym, then both $\gamma[\phi^{\omega}]$ and $\gamma\langle\phi^{\omega}\rangle$ satisfy IS.

d) If ϕ^{ω} satisfies INP, then both $\gamma[\phi^{\omega}]$ and $\gamma\langle\phi^{\omega}\rangle$ satisfy INC.

Proof. *Parts a), b) and c) are straightforward from the definition.*

d) We prove the result for $\gamma[\phi^{\omega}]$. The result for $\gamma\langle\phi^{\omega}\rangle$ is analogous. Let $\mathcal{C} = \{C_1, ..., C_m\}$ and let $C_q \in \mathcal{C}$ be a null coalition. Denote $M = \{1, 2, ..., m\}$. To prove that $\gamma[\phi^{\omega}]_i(N, v, \mathcal{C}) = \gamma[\phi^{\omega}]_i(N \backslash C_q, v, \mathcal{C}_{N \backslash C_q})$ for all $i \in N \backslash C_q$ it is enough to prove that $v_{C_r}^{[\phi^{\omega}]N}(S) = v_{C_r}^{[\phi^{\omega}]N \backslash C_q}(S)$ for all $S \subset C_r \in \mathcal{C} \backslash C_q$.

Take $S \subset C_r \in \mathcal{C} \backslash C_q$. By definition,

$$v_{C_r}^{[\phi^{\omega}]N}(S) = \phi_r^{\sigma(\mathcal{C})}(M, v/\mathcal{C}^S).$$

Since $\phi^{\sigma(\mathcal{C})}$ satisfies INP, we have

$$\phi_r^{\sigma(\mathcal{C})}\left(M, v/\mathcal{C}^S\right) = \phi_r^{\sigma(\mathcal{C})}\left(M \backslash q, v/\mathcal{C}_{N \backslash C_q}^S\right).$$

Notice that there is no ambiguities in the notation $v/\mathcal{C}^S_{N\backslash C_q}$ *because* $\left(\mathcal{C}^S\right)_{N\backslash C_q} = \left(\mathcal{C}_{N\backslash C_q}\right)^S$*. By definition,*

$$\phi_r^{\sigma(\mathcal{C})}\left(M\backslash q, v/\mathcal{C}^S_{N\backslash C_q}\right) = v_{C_r}^{[\phi^\omega]N\backslash C_q}(S).$$

Combining the three last expressions we obtain the result. ■

Corollary 6 *$Sh\,(Sh)$, $Sh\,[Sh^\omega]$ and $Sh\,\langle Sh^\omega\rangle$ satisfy Eff, Lin, IS and INC.*

4.2 Properties of Balanced Contributions

The principle of Balanced Contributions is used in different contexts. Myerson (1977) was the first to use it for *games with graphs*. He called it *Fairness*. Later, Myerson (1980) characterized the Shapley value with balanced contributions and efficiency. The principle of balanced contributions has also been used in other contexts: Amer and Carreras (1995) and Calvo, Lasaga and Winter (1996) characterized the Owen value; Calvo and Santos (2000) characterized a value for multi-choice games; Bergantiños and Vidal-Puga (2005) characterized an extension of the Owen value for non-transferable utility games; Calvo and Santos (2006) characterized the subsidy-free serial cost sharing method (Moulin, 1995) in discrete cost allocation problems; and Alonso-Meijide, Carreras and Puente (2007) characterized a parametric family of coalitional values.

Balanced Contributions (BC) Given a game (N, v), for all $i, j \in N$,

$$f_i\,(N, v) - f_i\,(N\backslash j, v) = f_j\,(N, v) - f_j\,(N\backslash i, v)\,.$$

This property states that for any two players, the amount that each player would gain or lose by the other's withdrawal from the game should be equal.

A remarkable property of this principle is that it completely characterizes the Shapley value with the only help of efficiency.

Proposition 7 *(Myerson, 1980) Sh is the only value that satisfies Eff and BC.*

A similar, yet different version of BC arises when we make the players to become null, instead of leaving the game: Given (N, v) and $i \in N$, we define (N, v^{-i}) as $v^{-i}(S) = v(S \cap (N \backslash i))$ for all $S \subset N$. Hence, in (N, v^{-i}) player i becomes a null player.

Null Balanced Contributions (NBC) Given a game (N, v), for all $i, j \in N$,

$$f_i(N, v) - f_i\left(N, v^{-j}\right) = f_j(N, v) - f_j\left(N, v^{-i}\right).$$

Under Eff and Sym, NBC and BC are equivalent:

Proposition 8 *Sh is the only value that satisfies Eff, NBC and Sym.*

Proof. It is well-known that Sh satisfies Eff, Sym and INP. Since Sh satisfies Eff and INP, we have $Sh_i(N, v^{-j}) = Sh_i(N \backslash j, v)$ for any null player j and any $i \in N \backslash j$. Hence, BC and NBC are equivalent for Sh. Since Sh satisfies BC (Proposition 7), Sh also satisfies NBC.

To see the uniqueness, let f be a value satisfying these properties. Fix (N, v). We proceed by induction on $|Carr(N, v)|$. If $|Carr(N, v)| = 0$, the result holds from Eff and Sym. Assume the result holds for less than $|Carr(N, v)|$ non-null players, with $|Carr(N, v)| > 0$. Let $i \in N$.

Assume first that player i is a null player. Obviously, $(N, v) = (N, v^{-i})$. For any $j \in Carr(N, v)$, under NBC,

$$f_i(N, v) - f_i\left(N, v^{-j}\right) = f_j(N, v) - f_j\left(N, v^{-i}\right) = 0$$

and hence $f_i(N, v) = f_i(N, v^{-j})$. By induction hypothesis, $f_i(N, v) = Sh_i(N, v^{-j}) = 0$ because i is also a null player in (N, v^{-j}).

Assume now $i \in Carrier(N, v)$. Under NBC, $f_i(N, v) - f_i(N, v^{-j}) = f_j(N, v) - f_j(N, v^{-i})$ for all $j \in N \backslash i$, and hence

$$(n-1) f_i(N, v) - \sum_{j \in N \backslash Carr(N,v)} f_i\left(N, v^{-j}\right) - \sum_{j \in Carr(N,v) \backslash i} f_i\left(N, v^{-j}\right)$$
$$= \sum_{j \in N \backslash i} f_j(N, v) - \sum_{j \in N \backslash i} f_j\left(N, v^{-i}\right).$$

Obviously, $f_i(N, v) = f_i(N, v^{-j})$ for all $j \in N \backslash Carr(N, v)$. Hence,

$$(|Carr(N, v)| - 1) f_i(N, v) - \sum_{j \in Carr(N,v) \backslash i} f_i\left(N, v^{-j}\right) = \sum_{j \in N \backslash i} f_j(N, v) - \sum_{j \in N \backslash i} f_j\left(N, v^{-i}\right).$$

Under Eff, $\sum_{j\in N\backslash i} f_i(N,v) = v(N) - f_i(N,v)$ and hence,

$$f_i(N,v) = \frac{1}{|Carr(N,v)|}\left[v(N) + \sum_{j\in Carr(N,v)\backslash i} f_i\left(N,v^{-j}\right) - \sum_{j\in N\backslash i} f_j\left(N,v^{-i}\right)\right].$$

Under the induction hypothesis, $f(N,v^{-j}) = Sh(N,v^{-j})$ for all $j \in Carr(N,v)$ and hence

$$f_i(N,v) = \frac{1}{|Carr(N,v)|}\left[v(N) + \sum_{j\in Carr(N,v)\backslash i} Sh_i\left(N,v^{-j}\right) - \sum_{j\in N\backslash i} Sh_j\left(N,v^{-i}\right)\right]$$

from where we deduce that $f_i(N,v)$ is unique for all $i \in Carr(N,v)$. ■

Remark 9 *Sym is needed in the previous characterization. Let $f^{\{1,2\}}$ be defined as follows: If $\{1,2\} \subseteq N$, then $f_1^{\{1,2\}}(N,v) = Sh_1(N,v)+1$, $f_2^{\{1,2\}}(N,v) = Sh_2(N,v) - 1$, and $f_i^{\{1,2\}}(N,v) = Sh_i(N,v)$ otherwise. If $\{1,2\} \nsubseteq N$, then $f^{\{1,2\}}(N,v) = Sh(N,v)$. This value satisfies Eff and NBC, but $f^{\{1,2\}} \neq Sh$.*

Remark 10 *Young (1985) characterized Sh as the only value that satisfies Eff, Sym and Strong Monotonicity (SM). This last property says that $f_i(N,v) \geq f_i(N,v')$ whereas $v(S\cup i) - v(S) \geq v'(S\cup i) - v'(S)$ for all $S \subset N\backslash i$. Hence, Proposition 8 implies that NBC and SM are equivalent under Eff and Sym.*

In order to keep the essence of the Shapley value at the intracoalitional level, we force (null) balanced contributions inside a coalition:

Balanced Intracoalitional Contributions (BIC) Given a game $(N,v,\mathcal{C})$, for all $i,j \in C_q \in \mathcal{C}$,

$$f_i(N,v,\mathcal{C}) - f_i\left(N\backslash j, v, \mathcal{C}_{N\backslash j}\right) = f_j(N,v,\mathcal{C}) - f_j\left(N\backslash i, v, \mathcal{C}_{N\backslash i}\right).$$

This property states that for any two agents that belong to the same coalition in $\mathcal{C}$, the amount that each agent would gain or lose by the other's withdrawal from the game should be equal.

Null Balanced Intracoalitional Contributions ($NBIC$) Given a game $(N, v, \mathcal{C})$, for all $i, j \in C_q \in \mathcal{C}$,

$$f_i(N, v, \mathcal{C}) - f_i\left(N, v^{-j}, \mathcal{C}\right) = f_j(N, v, \mathcal{C}) - f_j\left(N, v^{-i}, \mathcal{C}\right).$$

This property states that for any two agents that belong to the same coalition in $\mathcal{C}$, the amount that each agent would gain or lose if the other becomes null should be equal.

Proposition 11 *a) If γ satisfies NBC, then $\gamma[\phi^\omega]$ satisfies $NBIC$.*

b) If γ satisfies BC, then $\gamma\langle\phi^\omega\rangle$ satisfies BIC.

Proof. *Fix $C_q \in \mathcal{C}$ and $i, j \in C_q$.*

a) By definition,

$$\gamma[\phi^\omega]_i(N, v, \mathcal{C}) - \gamma[\phi^\omega]_i(N, v^{-j}, \mathcal{C}) = \gamma_i\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_i\left(C_q, (v^{-j})_{C_q}^{[\phi^\omega]N}\right).$$

By definition of (N, v^{-j}), we have $v_{C_q}^{[\phi^\omega]N}(S) = (v^{-j})_{C_q}^{[\phi^\omega]N}(S)$ for all $S \subset C_q \backslash j$. Hence, $\left(v_{C_q}^{[\phi^\omega]N}\right)^{-j}(S) = (v^{-j})_{C_q}^{[\phi^\omega]N}(S)$ for all $S \subset C_q$, which implies that $\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-j}\right)$ coincides with $\left(C_q, (v^{-j})_{C_q}^{[\phi^\omega]N}\right)$ and so, expression above can be restated as

$$\gamma[\phi^\omega]_i(N, v, \mathcal{C}) - \gamma[\phi^\omega]_i(N, v^{-j}, \mathcal{C}) = \gamma_i\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_i\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-j}\right).$$

Since γ satisfies NBC, we have

$$\gamma[\phi^\omega]_i(N, v, \mathcal{C}) - \gamma[\phi^\omega]_i(N, v^{-j}, \mathcal{C}) = \gamma_j\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_j\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-i}\right).$$

Reasoning as before, it is straightforward to check that

$$\gamma[\phi^\omega]_j(N, v, \mathcal{C}) - \gamma[\phi^\omega]_j(N, v^{-i}, \mathcal{C}) = \gamma_j\left(C_q, v_{C_q}^{[\phi^\omega]N}\right) - \gamma_j\left(C_q, \left(v_{C_q}^{[\phi^\omega]N}\right)^{-i}\right)$$

and hence the result.

b) By definition,

$$\gamma\left\langle\phi^{\omega}\right\rangle_i(N,v,\mathcal{C})-\gamma\left\langle\phi^{\omega}\right\rangle_i(N\backslash j,v,\mathcal{C}_{N\backslash j})=\gamma_i\left(C_q,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right)-\gamma_i\left(C_q\backslash j,v_{C_q\backslash j}^{\langle\phi^{\omega}\rangle N\backslash j}\right).$$

By definition of the reduced game, $v_{C_q}^{\langle\phi^{\omega}\rangle N}(S)=v_{C_q\backslash j}^{\langle\phi^{\omega}\rangle N\backslash j}(S)$ *for all* $S\subset C_q\backslash j$. *Thus,* $\left(C_q\backslash j,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right)$ *coincides with* $\left(C_q\backslash j,v_{C_q\backslash j}^{\langle\phi^{\omega}\rangle N\backslash j}\right)$ *and so, expression above can be restated as*

$$\gamma\left\langle\phi^{\omega}\right\rangle_i(N,v,\mathcal{C})-\gamma\left\langle\phi^{\omega}\right\rangle_i(N\backslash j,v,\mathcal{C}_{N\backslash j})=\gamma_i\left(C_q,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right)-\gamma_i\left(C_q\backslash j,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right).$$

Since γ *satisfies* BC, *we have*

$$\gamma\left\langle\phi^{\omega}\right\rangle_i(N,v,\mathcal{C})-\gamma\left\langle\phi^{\omega}\right\rangle_i(N\backslash j,v,\mathcal{C}_{N\backslash j})=\gamma_j\left(C_q,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right)-\gamma_j\left(C_q\backslash i,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right).$$

Reasoning as before, it is straightforward to check that

$$\gamma\left\langle\phi^{\omega}\right\rangle_j(N,v,\mathcal{C})-\gamma\left\langle\phi^{\omega}\right\rangle_j(N\backslash i,v,\mathcal{C}_{N\backslash j})=\gamma_j\left(C_q,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right)-\gamma_j\left(C_q\backslash i,v_{C_q}^{\langle\phi^{\omega}\rangle N}\right)$$

and hence the result. ∎

Corollary 12 *The Owen value* $Sh\,(Sh)$ *satisfies both* BIC *and* $NBIC$; $Sh\,[Sh^{\omega}]$ *satisfies* $NBIC$; $Sh\,\langle Sh^{\omega}\rangle$ *satisfies* BIC.

Even though Proposition 7 and Proposition 8 show that BC and NBC are equivalent under Eff and Sym, this is not the case for their intracoalitional versions:

Remark 13 *a)* $Sh\,[Sh^{\omega}]$ *does not satisfy* BIC. *Let* $N=\{1,2,3\}$ *and* v *defined as* $v(S)=1$ *if* $\{1,2\}\subset S$ *or* $\{1,3\}\subset S$, *and* $v(S)=0$ *otherwise. Let* $\mathcal{C}=\{\{1,2\},\{3\}\}$. *Then,*

$$\begin{aligned}Sh\,[Sh^{\omega}]_1(N,v,\mathcal{C})-Sh\,[Sh^{\omega}]_1\left(N\backslash 2,v,\mathcal{C}_{N\backslash 2}\right)&=\frac{5}{6}-\frac{1}{2}=\frac{1}{3}\\ Sh\,[Sh^{\omega}]_2(N,v,\mathcal{C})-Sh\,[Sh^{\omega}]_2\left(N\backslash 1,v,\mathcal{C}_{N\backslash 1}\right)&=\frac{1}{6}-0=\frac{1}{6}.\end{aligned}$$

b) $Sh\langle Sh^{\omega}\rangle$ does not satisfy NBIC. Let $(N, v, \mathcal{C})$ be defined as in the previous section. Then,

$$\begin{aligned} Sh\langle Sh^{\omega}\rangle_1 (N, v, \mathcal{C}) - Sh\langle Sh^{\omega}\rangle_1 \left(N, v^{-2}, \mathcal{C}\right) &= \frac{3}{4} - \frac{7}{12} = \frac{-1}{6} \\ Sh\langle Sh^{\omega}\rangle_2 (N, v, \mathcal{C}) - Sh\langle Sh^{\omega}\rangle_2 \left(N, v^{-1}, \mathcal{C}\right) &= \frac{1}{4} - 0 = \frac{1}{4}. \end{aligned}$$

4.3 Other properties

Coordination (Co) For all v, v' and $C_q \in \mathcal{C}$, if

$$v\left(T \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = v'\left(T \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$$

for all $T \subset C_q$ and all $\mathcal{R} \subset \mathcal{C} \backslash C_q$, then,

$$f_i(N, v, \mathcal{C}) = f_i(N, v', \mathcal{C}) \text{ for all } i \in C_q.$$

This property says that, given a coalition C_q, if there are changes inside other coalitions, but these changes do not affect to the worth of any subset of C_q with the rest of coalitions, then these internal changes in the other coalitions do not affect the final payment of each agent in C_q.

Proposition 14 *$\gamma(\phi)$, $\gamma[\phi^{\omega}]$ and $\gamma\langle\phi^{\omega}\rangle$ satisfy Co.*

Proof. *Let $\mathcal{C}$, v and v' such that $v\left(T \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = v'\left(T \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$ for all $T \subset C_q$ and all $\mathcal{R} \subset \mathcal{C} \backslash \{C_q\}$. It is enough to prove that $v_{C_q}^{[\phi^{\omega}]N}(S) = v_{C_q}^{\prime[\phi^{\omega}]N}(S)$ and $v_{C_q}^{\langle\phi^{\omega}\rangle N}(T) = v_{C_q}^{\prime\langle\phi^{\omega}\rangle N}(T)$ for all $S \subset C_q$. By the condition satisfied by v and v' we have that $\left(M, v/\mathcal{C}^S\right) = \left(M, v'/\mathcal{C}^S\right)$ for all $S \subset C_q$. Hence, $\phi_q^{\sigma(\mathcal{C})}\left(M, v/\mathcal{C}^S\right) = \phi_q^{\sigma(\mathcal{C})}\left(M, v'/\mathcal{C}^S\right)$ and $\phi_q^{\sigma\left(\mathcal{C}^S\right)}\left(M, v/\mathcal{C}^S\right) = \phi_q^{\sigma\left(\mathcal{C}^S\right)}\left(M, v'/\mathcal{C}^S\right)$ for all $S \subset C_q$. By the definition of the reduced games, we have the result.* ■

Frequently, is interpreted that players form coalitions in order to improve their bargaining strength (Hart and Kurz, 1983). However, as Harsanyi (1977) points out, the bargaining strength does not improve in general. An individual can be worse off bargaining as a member of a coalition than bargaining alone. This is what is known as the "Harsanyi paradox".

The following property avoids the "Harsanyi paradox" in the case where all the agents are symmetric. In the unanimity game with carrier N all the agents are necessary to obtain a positive payment. Hence it seems reasonable that their assignment should be independent of the coalitional structure:

Equal Sharing in Unanimity Games (ESUG) For any $\mathcal{C}$,

$$f_i\left(N, u_N^N, \mathcal{C}\right) = f_j\left(N, u_N^N, \mathcal{C}\right)$$

for all $i, j \in N$.

This property asserts that under the unanimity game with carrier N, each agent should receive the same payment, regardless of $\mathcal{C}$.

The Owen value does not satisfy $ESUG$ but a weighted version:

Inverse Proportional Sharing in Unanimity Games ($IPSUG$) For any game $\left(N, u_N^N, \mathcal{C}\right)$, and any coalitions $C_q, C_r \in \mathcal{C}$,

$$|C_q|\, f_i\left(N, u_N^N, \mathcal{C}\right) = |C_r|\, f_j\left(N, u_N^N, \mathcal{C}\right)$$

for all $i \in C_q$ and $j \in C_r$.

This property asserts that under the unanimity game with carrier N, each agent should receive a payment inversely proportional to the size of the coalition he belongs to. A similar property is the following:

Coalitional Symmetry in Unanimity Games ($CSUG$) For any game $\left(N, u_N^N, \mathcal{C}\right)$, and any coalitions $C_q, C_r \in \mathcal{C}$,

$$\sum_{i \in C_q} f_i\left(N, u_N^N, \mathcal{C}\right) = \sum_{i \in C_r} f_j\left(N, u_N^N, \mathcal{C}\right).$$

It is straightforward to check that, under IS, $CSUG$ is equivalent to $IPSUG$. We use $IPSUG$ because it follows the same formulation as $ESUG$.

In addition to Eff, either $ESUG$ or $IPSUG$ would determine the coalitional value for $\left(N, u_N^N, \mathcal{C}\right)$:

Proposition 15 *a) If a coalitional value f satisfies Eff and $ESUG$, then $f_i\left(N, u_N^N, C\right) = \frac{1}{|N|}$ for all $i \in N$.*

b) If a coalitional value f satisfies Eff and $IPSUG$, then $f_i\left(N, u_N^N, \mathcal{C}\right) = \frac{1}{|C_q||\mathcal{C}|}$ for all $i \in C_q \in \mathcal{C}$.

Proof. Part a) is trivial. As for part b), notice that $IPSUG$ implies that all the coalitions should receive the same aggregate value, and hence, under Eff, this value is $\frac{1}{|\mathcal{C}|}$. Moreover, $IPSUG$ also implies that all the players in the same coalition should receive the same value. Hence the result. ■

However, these properties are still very weak, since they only apply to a very specific unanimity game u_N^N. The following result gives us sufficient conditions to have these properties for the family of coalitional values defined before:

Proposition 16 *a) If both γ and ϕ^ω satisfy Eff and Sym, then $\gamma\left[\phi^\omega\right]$ and $\gamma\left\langle\phi^\omega\right\rangle$ satisfy $IPSUG$.*

b) If γ satisfies Eff and Sym, ϕ^ω satisfies Eff, and $\phi_i^x\left(N, u_N^N\right)/x_i = \phi_j^x\left(N, u_N^N\right)/x_j$ for all $i, j \in N$ and all $x \in \mathbb{R}_+^N$, then $\gamma\left[\phi^\omega\right]$ and $\gamma\left\langle\phi^\omega\right\rangle$ satisfy $ESUG$.

Proof. *Clearly, $\left(M, u_N^N/\mathcal{C}\right) = \left(M, u_M^M\right)$ and $\left(M, u_N^N/\mathcal{C}^S\right) = (M, null)$ for all $S \subsetneq C_q \in \mathcal{C}$, where $null\,(Q) = 0$ for all $Q \subset M$.*

a) Under Eff and Sym of ϕ^ω, we have $\left(u_N^N\right)_{C_q}^{[\phi^\omega]N} = \left(u_N^N\right)_{C_q}^{\langle\phi^\omega\rangle N} = \frac{1}{|\mathcal{C}|}u_{C_q}^{C_q}$ for all $C_q \in \mathcal{C}$. Under Eff and Sym of γ, we conclude that $\gamma\left[\phi^\omega\right]_i(N, v, \mathcal{C}) = \gamma\left\langle\phi^\omega\right\rangle_i(N, v, \mathcal{C}) = \frac{1}{|C_q||\mathcal{C}|}$ for all $i \in C_q \in \mathcal{C}$ and hence the result.

b) Under our hypothesis over ϕ^ω, we have $\left(u_N^N\right)_{C_q}^{[\phi^\omega]N} = \left(u_N^N\right)_{C_q}^{\langle\phi^\omega\rangle N} = \frac{|C_q|}{|N|}u_{C_q}^{C_q}$ for all $C_q \in \mathcal{C}$. Under Eff and Sym of γ, we conclude that $\gamma\left[\phi^\omega\right]_i(N, v, \mathcal{C}) = \gamma\left\langle\phi^\omega\right\rangle_i(N, v, \mathcal{C}) = \frac{1}{|C_q|}\frac{|C_q|}{|N|} = \frac{1}{|N|}$ for all $i \in C_q \in \mathcal{C}$ and hence the result. ■

Corollary 17 *a) The Owen value $Sh\,(Sh)$ satisfies $IPSUG$.*

b) $Sh\left[Sh^\omega\right]$ and $Sh\left\langle Sh^\omega\right\rangle$ satisfy $ESUG$.

5 Characterization

In this section, we present our main result:

Theorem 18 *Among all the coalitional values that satisfy Eff, Lin, INC and Co,*

a) the Owen value $Sh\,(Sh)$ is the only one that satisfies $NBIC$, $IPSUG$ and IS;

b) the Owen value $Sh\,(Sh)$ is the only one that satisfies BIC and $IPSUG$;

c) $Sh\,[Sh^{\omega}]$ is the only one that satisfies $NBIC$, $ESUG$ and IS; and

d) $Sh\,\langle Sh^{\omega}\rangle$ is the only one that satisfies BIC and $ESUG$.

Proof. We know by Corollary 6, Corollary 12, Proposition 14 and Corollary 17 that these rules satisfy the corresponding properties. Let $\mathcal{C} = \{C_1, ...C_m\}$ be a coalition structure. Let $M = \{1, ..., m\}$.

Let f^1 and f^2 be two coalitional values satisfying Eff, Lin, INC, Co, and the properties stated in one of the four sections. We prove $f^1 = f^2$ by induction over the number of players n. If $n = 1$, under Eff, $f^1(N, v, \mathcal{C}) = f^2(N, v, \mathcal{C})$ and hence the result holds.

Assume the result holds for less than n players. Now we prove that the result holds for n players.

It is well-know that every TU game can be expressed as a linear combination of unanimity games. Since f^1 and f^2 satisfy Lin, we can restrict our proof to unanimity games.

Let $S \subset N$, $S \neq \emptyset$. Consider the game u_N^S. First, we will show that it is enough to restrict the proof to the case where all the coalitions intersect the carrier S. To prove that, suppose that there exists some coalition, say $C_m \in \mathcal{C}$, that does not intersect the carrier; that is, $S \cap C_m = \emptyset$.

Clearly, C_m is a null coalition. Under INC, $f_i^x(N, u_N^S, \mathcal{C}) = f_i^x(N \backslash C_m, u_{N\backslash C_m}^S, \mathcal{C}_{N\backslash C_m})$ for all $i \in N\backslash C_m$ and $x = 1, 2$. By induction hypothesis,

$$f_i^1\left(N\backslash C_m, u_{N\backslash C_m}^S, \mathcal{C}_{N\backslash C_m}\right) = f_i^2\left(N\backslash C_m, u_{N\backslash C_m}^S, \mathcal{C}_{N\backslash C_m}\right)$$

for all $i \in N\backslash C_m$. Moreover, as an implication of INC and Eff, $\sum\limits_{i \in C_m} f_i^x(N, u_N^S, \mathcal{C}) = 0$ for $x = 1, 2$. We still need to prove that every agent in C_m receives the same under both coalitional values. In particular, we will prove that each of them receives zero. We have two possibilities:

Cases a and c (the coalitional values satisfy IS): Under IS, it is clear that $f_i^x(N, u_N^S, \mathcal{C}) = 0$ for all $i \in C_m$, $x = 1, 2$, because all the players in C_m are symmetric and their values sum up zero.

Cases b and d (the coalitional values satisfy BIC): If $|C_m| = 1$, it is clear that $f_i^x\left(N, u_N^S, \mathcal{C}\right) = 0$ for all $i \in C_m$, $i = 1, 2$. Assume $f_i^x\left(N, u_N^S, \mathcal{C}\right) =$

0 for all null coalitions with less than l players. If $|C_m| = l$, $l > 1$, from BIC,

$$f_i^x(N, u_N^S, \mathcal{C}) - f_i^x(N\backslash j, u_{N\backslash j}^S, \mathcal{C}_{N\backslash j}) = f_j^x(N, u_N^S, \mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^S, \mathcal{C}_{N\backslash i})$$

for all $i, j \in C_m$, $x = 1, 2$. By induction hypothesis on $|C_m|$, $f_i^x(N\backslash j, u_{N\backslash j}^S, \mathcal{C}_{N\backslash j}) = f_j^x(N\backslash i, u_{N\backslash i}^S, \mathcal{C}_{N\backslash i}) = 0$, for all $i, j \in C_m$, $x = 1, 2$. Hence, we have that $f_i^x(N, u_N^S, \mathcal{C}) = f_j^x(N, u_N^S, \mathcal{C})$ for all $i, j \in C_m$ and $x = 1, 2$. Moreover, since $\sum_{i \in C_m} f_i^x(N, u_N^S, \mathcal{C}) = 0$, we obtain that $f_i^x(N, u_N^S, \mathcal{C}) = 0$ for all $i \in C_m$ and $x = 1, 2$.

From now on, we assume that $S \cap C_q \neq \emptyset$ for all $C_q \in \mathcal{C}$.
Fix $i \in C_q \in \mathcal{C}$. We should prove that $f_i^1(N, u_N^S, \mathcal{C}) = f_i^2(N, u_N^S, \mathcal{C})$.
Let $S_q := C_q \cap S$ and $T := S_q \cup (N\backslash C_q)$.

Claim 19

$$u_N^S\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = u_N^T\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$$

for all $T' \subset C_q$ and all $\mathcal{R} \subset \mathcal{C}\backslash C_q$.
Proof. *Fix $T' \subset C_q$. We distinguish three cases:*

Case 1: *$S_q \subset T'$ and $\mathcal{R} = \mathcal{C}\backslash C_q$. In this case, $S \subset \left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$ and $T \subset \left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$. Thus by definition of u_N^S and u_N^T, we have that $u_N^S\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = u_N^T\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = 1$.*

Case 2: *$S_q \not\subset T'$. In this case, there exists some $i \in S_q$ such that $i \notin T'$, and so, $S \not\subset \left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$ and $T \not\subset \left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$. Hence, $u_N^S\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = u_N^T\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = 0$.*

Case 3: *$\mathcal{R} \neq \mathcal{C}\backslash C_q$. In this case, there exists some $C_k \in \mathcal{C}\backslash C_q$ such that $C_k \notin \mathcal{R}$. Since by hypothesis, $C_r \cap S \neq \emptyset$ for all $C_r \in \mathcal{C}$, we have that $S \not\subset \left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$ and $T \not\subset \left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right)$. Hence, $u_N^S\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = u_N^T\left(T' \cup \bigcup_{C_r \in \mathcal{R}} C_r\right) = 0$.* ■

Since we are under the assumptions of Co (Claim 19), we have $f_i^x(N, u_N^S, \mathcal{C}) = f_i^x(N, u_N^T, \mathcal{C})$ for $x = 1, 2$. Hence, it is enough to prove that $f_i^1(N, u_N^T, \mathcal{C}) = f_i^2(N, u_N^T, \mathcal{C})$. As a previous step, consider the unanimity game $\left(N, u_N^N\right)$. By an analogous argument as in the proof of Claim 19, we have

$$u_N^T\left(T' \cup \bigcup_{C_l \in \mathcal{Q}} C_l\right) = u_N^N\left(T' \cup \bigcup_{C_l \in \mathcal{Q}} C_l\right)$$

for all $T' \subset C_r \in \mathcal{C}\backslash C_q$ and all $\mathcal{Q} \subset \mathcal{C}\backslash C_r$. Under Co,

$$f_j^x(N, u_N^N, \mathcal{C}) = f_j^x(N, u_N^T, \mathcal{C}) \tag{1}$$

for all $j \in N\backslash C_q$.

We have two possibilities:

Cases a and c (the coalitional values satisfy $NBIC$ and IS): Under Eff and $ESUG/IPSUG$, by Proposition 15, we have $\sum_{i \in C_q} f_i^x(N, u_N^T, \mathcal{C}) = \beta_q$ where $\beta_q = \frac{1}{|\mathcal{C}|}$ (when f^x satisfies $IPSUG$) or $\beta_q = \frac{|C_q|}{|N|}$ (when f^x satisfies $ESUG$).

Under IS, we have $f_i^x(N, u_N^T, \mathcal{C}) = f_j^x(N, u_N^T, \mathcal{C})$ for all $i, j \in S_q$ (respectively, $i, j \in C_q\backslash S_q$) and $x = 1, 2$. Hence it is enough to prove $f_i^x(N, u_N^T) = 0$ for all $i \in C_q\backslash S_q$, $x = 1, 2$. This is clear for $S_q = C_q$. Let $i \in S_q$ and $j \in C_q\backslash S_q$. Player j is a null player in (N, u_N^T) and hence $(N, u_N^T) = (N, \left(u_N^T\right)^{-j})$. Under $NBIC$,

$$0 = f_i^x(N, u_N^T) - f_i^x\left(N, \left(u_N^T\right)^{-j}\right) = f_j^x(N, u_N^T) - f_j^x\left(N, \left(u_N^T\right)^{-i}\right).$$

Obviously, $(N, \left(u_N^T\right)^{-i})$ is the null game $\left(u_N^T\right)^{-i}(S) = 0$ for all $S \subset N$ and thus Eff and IS imply $f_j^x(N, \left(u_N^T\right)^{-i}) = 0$. Thus, $f_j^x(N, u_N^T) = 0$ for $x = 1, 2$.

Cases b and d (the coalitional values satisfy BIC): Fix $x \in \{1, 2\}$. Under BIC,

$$f_i^x(N, u_N^T, \mathcal{C}) - f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}) = f_j^x(N, u_N^T, \mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i})$$

for all $j \in C_q\backslash i$. Hence,

$$\begin{aligned} &\sum_{j \in C_q\backslash i} \left(f_i^x(N, u_N^T, \mathcal{C}) - f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})\right) \\ = &\sum_{j \in C_q\backslash i} \left(f_j^x(N, u_N^T, \mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i})\right). \end{aligned}$$

Rearranging terms, $(|C_q|-1)\, f_i^x\left(N, u_N^T, \mathcal{C}\right) =$

$$= \sum_{j\in C_q\backslash i} \left(f_j^x(N,u_N^T,\mathcal{C}) - f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}) \right). \quad (2)$$

On the other hand, by Proposition 15,

$$f_j^x(N,u_N^N,\mathcal{C}) = \alpha_q \text{ for all } j\in C_q\in\mathcal{C} \quad (3)$$

where $\alpha_q = \frac{1}{|N|}$ (if f^x satisfies $ESUG$) and $\alpha_q = \frac{1}{|C_q||\mathcal{C}|}$ (if f^x satisfies $IPSUG$).

Hence,

$$\sum_{j\in N\backslash C_q} f_j^x(N,u_N^T,\mathcal{C}) \overset{(1)}{=} \sum_{j\in N\backslash C_q} f_j^x(N,u_N^N,\mathcal{C}) \overset{(3)}{=} \sum_{C_r\in\mathcal{C}\backslash C_q} |C_r|\,\alpha_r.$$

Moreover, by Eff,

$$\sum_{j\in C_q\backslash i} f_j^x(N,u_N^T,\mathcal{C}) = u_N^T(N) - f_i^x(N,u_N^T,\mathcal{C}) - \sum_{C_r\in\mathcal{C}\backslash C_q} |C_r|\,\alpha_r.$$

Since $u_N^T(N)=1$,

$$\sum_{j\in C_q\backslash i} f_j^x(N,u_N^T,\mathcal{C}) = 1 - f_i^x(N,u_N^T,\mathcal{C}) - \sum_{C_r\in\mathcal{C}\backslash C_q} |C_r|\,\alpha_r.$$

It is not difficult to check that $1-\sum_{C_r\in\mathcal{C}\backslash C_q}|C_r|\,\alpha_r = \beta_q$ (defined in the previous case). Hence

$$\sum_{j\in C_q\backslash i} f_j^x(N,u_N^T,\mathcal{C}) = \beta_q - f_i^x(N,u_N^T,\mathcal{C}).$$

Replacing this expression in (2),

$$(|C_q|-1) f_i^x(N,u_N^T,\mathcal{C}) = \beta_q - f_i^x(N,u_N^T,\mathcal{C}) - \sum_{j\in C_q\backslash i} f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + \sum_{j\in C_q\backslash i} f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}).$$

Rearranging terms:

$$|C_q|\, f_i^x(N,u_N^T,\mathcal{C}) = \beta_q - \sum_{j\in C_q\backslash i} f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + \sum_{j\in C_q\backslash i} f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}).$$

And so, $f_i^x(N, u_N^T, \mathcal{C}) =$

$$\frac{1}{|C_q|}\left[\beta_q - \sum_{j\in C_q\backslash i} f_j^x(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) + \sum_{j\in C_q\backslash i} f_i^x(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})\right]$$

But by induction hypothesis:

$$f_j^1(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i}) = f_j^2(N\backslash i, u_{N\backslash i}^T, \mathcal{C}_{N\backslash i})$$

and

$$f_i^1(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j}) = f_i^2(N\backslash j, u_{N\backslash j}^T, \mathcal{C}_{N\backslash j})$$

for all $j \neq i$. Hence we conclude that $f_i^1(N, u_N^T, \mathcal{C}) = f_i^2(N, u_N^T, \mathcal{C})$. ■

Remark 20 *In parts a and c we need to add IS. Take for example the coalitional value F given by $F(N, v, \mathcal{C}) = Sh[Sh^\omega](N, v, \mathcal{C})$ if $\{1,2\} \notin \mathcal{C}$ or $\{3\} \notin \mathcal{C}$. When $\{1,2\}, \{3\} \in \mathcal{C}$, take $F_i(N, v, \mathcal{C}) = Sh[Sh^\omega]_i(N, v, \mathcal{C})$ for all $i \in N\backslash\{1,2\}$ and moreover*

$$\begin{aligned} F_1(N, v, \mathcal{C}) &= Sh[Sh^\omega]_1(N, v, \mathcal{C}) + v(\{3\}) \\ F_2(N, v, \mathcal{C}) &= Sh[Sh^\omega]_2(N, v, \mathcal{C}) - v(\{3\}). \end{aligned}$$

This coalitional value satisfies Eff, Lin, INC, BIC, Co and $ESUG$, but fails IS.

Analogously, define the coalitional value F' as before, but taking $Sh(Sh)$ instead of $Sh[Sh^\omega]$. Then, F' satisfies Eff, Lin, INC, $NBIC$, Co and $IPSUG$, but fails IS.

6 Independence of the axioms

In this section we show that the axioms used in Theorem 18 are independent.

The Aumann-Drèze value $Sh(In)$ satisfies Lin, INC, Co, BIC, $NBIC$, $IPSUG$, $ESUG$, IS and fails Eff.

Define the *bounded egalitarian value* BE as $BE_i(N, v) = v(N)/|Carr(N, v)|$ if $i \in Carr(N, v)$ and $BE_i(N, v) = 0$ otherwise.

$Sh(BE)$ satisfies Eff, INC, Co, BIC, $NBIC$, $IPSUG$, IS and fails Lin.

Define the *egalitarian value* E as $E_i(N,v) = v(N)/|N|$ for all $i \in N$.

$Sh(E)$ satisfies Eff, Lin, Co, BIC, $NBIC$, $IPSUG$, IS and fails INC.

Take the coalitional value G given by $G(N,v,\mathcal{C}) = Sh(Sh)(N,v,\mathcal{C})$ if $\{3,4\} \notin \mathcal{C}$, $1,2 \notin N$ or $1,2 \in N$ and they belong to the same coalition in $\mathcal{C}$. When $\{3,4\} \in \mathcal{C}$, $1,2 \in N$ and $1,2$ do not belong to the same coalition in $\mathcal{C}$, take $G_i(N,v,\mathcal{C}) = Sh(Sh)_i(N,v,\mathcal{C})$ for all $i \in N \setminus \{1,2\}$ and moreover

$$\begin{aligned} G_1(N,v,\mathcal{C}) &= Sh(Sh)_1(N,v,\mathcal{C}) + \delta^v_{\{1,2,3\}} \\ G_2(N,v,\mathcal{C}) &= Sh(Sh)_2(N,v,\mathcal{C}) - \delta^v_{\{1,2,3\}} \end{aligned}$$

where $\delta^v_{\{1,2,3\}} := v(\{1,2,3\}) - v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(1) + v(2) + v(3)$ is the *Harsanyi dividend* for $u_N^{\{1,2,3\}}$. This coalitional value G satisfies Eff, Lin, INC, BIC, $IPSUG$, IS and fails Co.

$E(Sh)$ satisfies Eff, Lin, INC, Co, $IPSUG$, IS and fails both BIC and $NBIC$.

$Sh\langle Sh^\omega \rangle$ satisfies Eff, Lin, INC, Co, BIC, IS and fails $IPSUG$.

$Sh[Sh^\omega]$ satisfies Eff, Lin, INC, Co, $NBIC$, IS and fails $IPSUG$.

The second coalitional value presented in Remark 20 satisfies Eff, Lin, INC, Co, $NBIC$, $IPSUG$ and fails IS.

Define the *weighted bounded egalitarian value* BE^ω as $BE_i^x(N,v) = x_i v(N) / \sum_{j \in Carr(N,v)} x_j$ if $i \in Carr(N,v)$ and $BE_i^x(N,v) = 0$ otherwise, for all $x \in \mathbb{R}^N_{++}$.

$Sh[BE^\omega]$ satisfies Eff, INC, Co, $NBIC$, $ESUG$, IS and fails Lin.

$Sh\langle BE^\omega \rangle$ satisfies Eff, INC, Co, BIC, $ESUG$ and fails Lin.

$Sh[E^\omega]$ satisfies Eff, Lin, Co, $NBIC$, $ESUG$, IS and fails INC.

$Sh\langle E^\omega \rangle$ satisfies Eff, Lin, Co, BIC, $ESUG$ and fails INC.

The coalitional Shapley value Sh satisfies Eff, Lin, INC, $NBIC$, BIC, $ESUG$, IS and fails Co.

$E[Sh^\omega]$ satisfies Eff, Lin, INC, Co, $ESUG$, IS and fails both $NBIC$ and BIC.

The Owen value $Sh(Sh)$ satisfies Eff, Lin, INC, Co, $NBIC$, BIC, IS and fails $ESUG$.

The coalitional value F presented in Remark 20 satisfies Eff, Lin, INC, Co, $NBIC$, $ESUG$ and fails IS.

In the following table we summarize the results presented in this Section:

	Eff	Lin	INC	BIC	$NBIC$	Co	$ESUG/IPSUG$	IS
$Sh\,(Sh)$	OK^{*+}	OK^{*+}	OK^{*+}	OK^{*}	OK^{+}	OK^{*+}	$IPSUG^{*+}$	OK^{+}
$Sh\,[Sh^{\omega}]$	OK^{*}	OK^{*}	OK^{*}	no	OK^{*}	OK^{*}	$ESUG^{*}$	OK^{*}
$Sh\,\langle Sh^{\omega}\rangle$	OK^{*}	OK^{*}	OK^{*}	OK^{*}	no	OK^{*}	$ESUG^{*}$	OK
$Sh\,(In)$	no	OK	OK	OK	OK	OK	BOTH	OK
$Sh\,(BE)$	OK	no	OK	OK	OK	OK	$IPSUG$	OK
$Sh\,(E)$	OK	OK	no	OK	OK	OK	$IPSUG$	OK
G	OK	OK	OK	OK	OK	no	$IPSUG$	OK
$E\,(Sh)$	OK	OK	OK	no	no	OK	$IPSUG$	OK
F'	OK	OK	OK	no	OK	OK	$IPSUG$	no
$Sh\,[BE^{\omega}]$	OK	no	OK	no	OK	OK	$ESUG$	OK
$Sh\,\langle BE^{\omega}\rangle$	OK	no	OK	OK	no	OK	$ESUG$	OK
$Sh\,[E^{\omega}]$	OK	OK	no	no	OK	OK	$ESUG$	OK
$Sh\,\langle E^{\omega}\rangle$	OK	OK	no	OK	no	OK	$ESUG$	OK
Sh	OK	OK	OK	OK	OK	no	$ESUG$	OK
$E\,[Sh^{\omega}]$	OK	OK	OK	no	no	OK	$ESUG$	OK
F	OK	OK	OK	no	OK	OK	$ESUG$	no

Table 1: Properties satisfied by the coalitional values. "*" (resp. "+") means that this property together with the others with "*" (resp. "+") in the line, characterizes the coalitional value.

7 Concluding remarks

In this paper we characterize three generalizations of the Shapley value. As for the Owen value, one of its most controversial properties is that of symmetry in the game among coalitions. In our characterization, this symmetry is in fact implied by $IPSUG$. Other characterizations of the Owen value also include some property that leads to this symmetry. This is the case of property A3 in the original characterization by Owen (1977); the *coalitional symmetry* in Winter (1989) and Albizuri (2008); the *intermediate game property* in Peleg (1989), called *game between coalitions property* in Winter (1992) and *quotient game property* in Vázquez-Brage et al. (1997); the property of *symmetry among coalitions* in Zhang (1995); the property of *block strong symmetry* in Amer and Carreras (1995), called *balanced contributions in the* *coalitions* in Calvo et al. (1996); the property of *symmetry* in Chae and Heidhues (2004); and the properties of *unanimity coalitional game*, *symmetry between exchangeable coalitions* and *coalitional symmetry* in the various characterizations presented in Bergantiños et al. (2007).

Hart and Kurz (1983) presented an alternative characterization of the Owen value without the property of symmetry in the game among coalitions. Instead, they used a property of *Carrier*, which implies that the value should not be affected by the presence of null players. Various axiomatic characterizations of the Owen value also use this property: Hamiache (1999 and 2001), Albizuri and Zarzuelo (2004), and Albizuri (2008).

One may wonder whether the Carrier axiom is a reasonable requirement in games with coalition structure. Since null players affect the size of the coalition, we should admit that they are not so null (as far as we accept that size is important). Take for example the unanimity game $\left(N, u_N^S\right)$ with $N = \{1,2,3\}$ and $S = \{1,2\}$. Take $\mathcal{C} = \{\{1\},\{2,3\}\}$. This game models the following situation, as described in Hart and Kurz (1983):

> As an everyday example of such a situation, "I will have to check this with my wife/husband" may (but not necessarily) lead to a better bargaining position, due to the fact that the other party has to convince *both* the player and the spouse.

The Owen value would simply ignore the presence of player 3:

$$Sh\,(Sh)\left(N, u_N^S, \mathcal{C}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

In this example, the role of the symmetry in the game among coalitions is clear: since both $\{1\}$ and $\{2,3\}$ are equally necessary to get a positive payoff, this payoff should be shared equally among them, irrespectively of their respective size. This idea is appropriate to describe situations where the negotiations take place among representatives with the same power of negotiation.

As opposed, $Sh\,[Sh^{\omega}]$ would assign twice as much to coalition $\{2,3\}$ than to coalition $\{1\}$, but still maintaining the null player property:

$$Sh\,[Sh^{\omega}]\left(N, u_N^S, \mathcal{C}\right) = \left(\frac{1}{3}, \frac{2}{3}, 0\right).$$

This idea is appropriate to describe situations where the power of negotiation among coalitions depend on their size. One may think for example on political parties that join forces in a Parliament, maintaining however their respective proposal prerogatives. In fact, Kalandrakis (2006) shows that proposal making has a very significant impact on outcomes.

Notice that player 2 would only expect to get $\frac{1}{2}$ in case player 3 be not present. Hence, the benefit of cooperation between players 2 and 3 is $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$. $Sh \langle Sh^{\omega} \rangle$ proposes to share this benefit equally between players 2 and 3:

$$Sh \langle Sh^{\omega} \rangle \left(N, u_N^S, \mathcal{C}\right) = \left(\frac{1}{3}, \frac{7}{12}, \frac{1}{12}\right).$$

In this case, the null player property is not satisfied. However, one may find examples of real situations where this null player property also fails. Consider the Basque Country[5] Parliament that arose in 2001 election. Five parties got representation: Coalition EAJ-PNV / EA, Partido Popular (PP), Partido Socialista de Euskadi - Euskadiko Ezquerra (PSE-EE / PSOE), Euskal Herritarrok (EH) and Ezker Batua-Izquierda Unida (EB-IU). The number of representatives is given in Table 2. The number of seats needed to win a vote is 38.

Party	Number of Seats
EAJ-PNV / EA	33
PP	19
PSE-EE / PSOE	13
EH	7
EB-IU	3

Table 2: Number of seats in the Basque Country Parliament.

Even though EB-IU is a null player in the associated voting game[6], a minority government was formed with the coalition of EAJ-PNV / EA and EB-IU. Whatever the reason for this decision could be, it suggests that null players can also play a significant role.

[5] Autonomous community of Spain.

[6] This game is defined as $v(S) = 1$ if the members of S sum up at least 38 seats, and $v(S) = 0$ otherwise.

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