Is Economics Entering its Post-Witchcraft Era?

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Abstract

Recently, an awareness is emerging in economics about the fact that important problems are not solvable algorithmically, that is, by any finite number of steps. This statement can be made mathematically exact and this paper reviews the contributions that have been made in this regard, related to standard topics in economics.

1 Introduction: on the Medieval Horror of Malleus Maleficarum

Quite a long time ago, when one of the authors was in school, he was asked to read the infamous late 15th century inquisitorial book *Malleus Maleficarum* and, beyond the endless gory details of tortures to be applied to witches, to comment on what he found as by far the most important feature of that text. As it happened, what he found was the most surprising perfect identity in their respective views of the ways the world functions which both the inquisitors and the witches manifestly shared with one another. Shared indeed, and did so of course, without in the very least being aware of that monumentally important fact.

And what they shared was nothing else but their fundamental—and never ever questioned— belief that for absolutely every problem in life there exists a clear cut and unquestionable solution, and on top of it, that solution is given by a finite number of equally clear and well defined steps which are to be implemented in one and only one well specified order.

Needless to say, the inquisitors who wrote that book used various finite sets of strictly theological and legal arguments, plus of course, and so fashionable at the time, thoroughly described tortures, each consisting of a finite number of horrible procedures. In their turn, the witches, in a perfectly similar manner, were using a large, and often hilarious, if not in fact, ridiculous variety of procedures, each of them consisting of a well defined finite number of steps, to be executed in one and only one strictly specified order. Witches and inquisitors were, therefore, united in their respective belief that a formulaic solution to their respective problem *exists*.

This reminds one of the authors of what he has always held to be the somewhat questionable guiding philosophy behind extramarital affairs[1]: if one intimate relationship does not make one happy, then two will perhaps ensure felicity. So much for an algorithmic view of the ways the world works...

[1] As opposed to pure adultery.

2 The Dismal Applied Science

Mathematical economics is undeniably a beautiful theory with considerable epistemic value. However, it was also a most practical thing when American soldiers were told to deliver food and other goods to the island of Vanuatu during World War II. But is it a good thing that islanders today worship a deity called *John Frum* ("John from Illinios/Kansas/Texas") who is supposed to return one day with salvation and... more cargo?

Economics is not the only science blighted by insistent application of simplistic, if not in fact, atavistic medieval views. Statistics, among other sciences, is equally damned. The government of South Africa takes comfort, for example, in the finding of a recent report [9], regarding possible perverse incentives in the Child Support Grant, or CSG, according to which

> the quantitative analysis suggests that the take-up rate of the CSG by teenage mothers remains low. Teenagers, that is, younger than 20 years, represent 5% of all CSG recipients registered at October 2005.

The report concludes, apparently largely on the basis of this single statistic[2], that there is no perverse incentive in the grant. The consideration that teenage mothers are sooner or later mothers of 30 with a 14 year old child, still receiving the grant, is absent. Buried in the same report, one will read that

> the proportion of mothers who were teenagers at the birth - not necessary at the application for CSG - of their eldest child to be registered on SOCPEN from 2000-2004, increased from 23% in 2000 to 32% in 2004.

This telling figure appears not to have influenced the conclusions of the report. And unless the minister for Social Development has read it very carefully, he will now sleep soundly at night. The activists who are proposing extending the payment of the grant until age 18—thereby creating an enabling, if not in fact, encouraging environment for two successive generations of one family to simultaneously receive it—will also be encouraged by the Department of Social Development's conclusion, rather than by the facts. Will our descendants look upon a social policy that actively promoted and rewarded a quarter of a million teenage pregnancies per annum with the same horror with which we regard the *Malleus*?

Mathematical economics still contributes much to the way that we see the world. Take the fashionable business of carbon emissions trading as an example. Carbon emission caps, and the tradability of the gap between an enterprise's actual emissions and its cap, have created a new asset class, called carbon accounting units, the market in which might be expected to approach a Walrasian equilibrium. By neoclassical economics we know that a equilibrium price exists, but we do not necessarily know whether there is a single equilibrium or many equilibria—nor whether there are any stable equilibria. Furthermore, there is no reason to believe that we can be more sure whether carbon trading units will be cheap or expensive next year, than we can be sure about the price of pork bellies in Chicago. It is rather amusing that individuals in favour of carbon caps (and trading in these new commodities) are sometimes stridently against so-called speculators and other investors. For some people carbon credits are clearly already very affordable, allowing Mr Albert Gore—for example—to effortlessly be a green prophet with a heated swimming pool.

Carbon trading can be better understood through a quotidian analogy: consider the possibility of trading highway speeding. A person could earn speed credits by driving, say, at 90km/h from Pretoria to Durban, and then sell these on the market—for example, to two people who want to each drive at an average of 135km/h from Durban to Pretoria. Monitoring will be necessary, but there will be a good incentive to drive slower than the speed limit of 120km/h. And there will be no need for speeding tickets, although, obviously, strict but non-violent monitoring of the road network will be required. The market will simply establish a price for speeding that matches supply and demand. This should at least be an improvement over the incorrect pricing observed so spectacularly in a study [8] of fines at Israeli day-care centers.

The existence proofs of mathematical economics have imbued theoretical economics with a semblence of the supposed rigid veracity of physics. The belief, erroneous in the minds of the authors, that the world in which we live is identical with an appropriate (nay, existing) model in physics has—in some places—also spread to economics. Without impugning the achievements of mathematical economics, this

[2] This is what the great physicist Richard Feynman has called [10] *cargo cult science*.

papers attemps to highlight some recent results—inspired by similar questions asked in mathematics—that reflect on the limits of the practical applications of the theory.

3 Voices from the Past

The founders of modern economics were well aware of problems related to the tractability of the mathematical models that they developed. Vilfredo Pareto has pointed out in his 1927 *Manuel d'économie politique* that the equations for a real economy would be so many as to be impossibly complex to solve. Of course he did not have computers in mind but it turns out that his intuition about this absolutely correct, as we shall explain later. In 1945 Friedrich Hayek remarked:

> The peculiar character of the problem of a rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exists in concentrated or integrated form but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess.

Hayek's comments can be interpreted to refer to availability of data only if one thinks of the "dispersed bits" that he refers to as in geographic dispersion. However, if one takes it to say simply that the necessary information or results cannot be succinctly described then his remark echoes the non-computability results of Richter and Wong which we briefly review below.

4 Enter Modern Economics

Computability enters economics in two very obvious ways.

- The computability of model outcomes: any practical applications would clearly require that the result be computable, i.e. that it could—even just in principle, because computers improve over time—be produced by a computer.
- As a limit on the actions of the agents: instead of allowing arbitrary preferences or utility functions, one could—reasonably—want to restrict the theory to computable objects, for example computable preferences on computable bundles of goods. A computable preference is simply one which can be calculated in a finite amount of time using a machine with a finite amount of resources and a computable bundle would be any bundle that can be coherently represented using only a finite number of symbols.

If it is assumed that the economic agents are anything like real human beings then it makes perfect sense for all the objects in the theory to be taken to be computable. For example, computable prices are the only prices that one can conceivable use in an MS-Excel[3] or Gnumeric[4] spreadsheet.

4.1 Computability in equilibrium theory

The general equilibrium theorem of Kenneth Arrow and Gérard Debreu, establishing the existence of a competitive equilibrium in an economy with a finite number of consumers and a finite number of goods, is proved using the Brouwer fixed-point theorem (BFPT). BFPT is of course also used in game theory and other areas of economics. In this section I denotes the standard unit interval $[0, 1]$ of real numbers.

Theorem 1 (Brouwer) *Any continuous function $f : I^2 \to I^2$ has a fixed point, i.e. there exists an $x \in I^2$ such that $f(x) = x$.*

The fact is that the BFPT is very non-constructive. In fact, there are functions f for which the fixed points themselves are *all* non-constructive when viewed as solitary objects. Given a function f, we do might not only not know how to find the fixed point x such that

$$f(x) = x$$

[3] The authors are not affiliated with Microsoft Corporation but would not reject donations or grants from Microsoft.
[4] Even Linux is limited by the theory of formal computability.

but it can also be a point which cannot be defined in an *effective* way at all. By this we mean that there exist functions f for which there exists no algorithm (using finite resources, e.g. a Turing machine) that can approximate any fixed point of f, given a margin of error ε, up to the given margin of error.

The exact definitions of computable real numbers and of computable functions on the natural numbers appear in Alan Turing's pioneering work of the 1930s. A *computable function* from the natural numbers ($\mathbb{N}_0$) to the natural numbers is a function that can be computed in principle by a Turing machine (which, for the purposes of this paper, the reader can assume to be a desktop computer with infinite memory). A computable real number is any number which can in principle be approximated by a Turing machine.

Definition 1 *A real number x is called* computable *if there exist computable functions $\phi_1^x, \phi_2^x, \phi_3^x : \mathbb{N}_0 \to \mathbb{N}_0$ such that for all $k \in \mathbb{N}_0$*

$$\left| x - (-1)^{\phi_1^x(k)} \frac{\phi_2^x(k)}{\phi_3^x(k)} \right| \quad \leq \quad 2^{-k}.$$

The approriate definition of computability for a real-valued function of real variables is beyond the scope of this paper and the reader is referred to the paper of Brattka [3], for example. In this paper, a computable real function f will be represented as a Turing-computable function that maps descriptions of computable reals x to descriptions of the computable reals $f(x)$. That is, f is called computable if there exists a computer program that transforms an approximation procedure for x to an approximation procedure for $f(x)$. This is a rather natural notation but the observant reader will remark that—as there are only countably many computable reals—such an f need not necessarily be defined outside a set of measure zero. Nevertheless, all of the continuous functions usually used in economics courses are computable in this sense. These functions are often called *computably coded*.

The computable counterexample to the Brouwer theorem is due to Orevkov [14] (cited i.a. in [12]) who worked in the Russian school of constructivism. Baigger [1] rewrote the Orevkov construction to define a computable function on I^2 which has no computable fixed point. A similar construction was described by Richter and Wong [17] in their paper that established the following

Theorem 2 (Richter and Wong [17, 16]) *For any $\ell \geq 3$ there exists a computable economy E with a finite number of agents and with ℓ goods for which no computable equilibrium exists.*

This theorem is a direct application of the Orevkov counter example and therefore says not only that the equilibrium cannot be found but also that the equilibrium price vector is an uncomputable one. The non-computability of the price vector implies not only that it cannot be found, but that it has no finite description in any formal computing scheme.

One can consider the situation where some additional information about the equilibria of an exchange economy are known, for example If f is a computable coded function with computable modulus of uniform continuity mapping computable points in I^2 to computable points in I^2 then f can be extended to a total function $f^*(x) : I^2 \to I^2$ and the following version of the BFPT can be obtained.

Theorem 3 (Hirst [12]) *If f^* has finitely many fixed points, then f has a computable fixed point.*

This result, together with Debreu's proof [6] that *regular* economies have finitely equilibria, appears to allow one to salvage a computable general equilibrium theorem. Certainly, if f^* has one fixed point then there appears to be a procedure for finding it. However, if f^* has more than one, but still finitely many fixed points, then the proof only infers the existence of a fixed point which is a computable real number, but in a non-constructive way. That is, it proves the *existence* of a computable x such that

$$f^*(x) = x$$

but we still have no indication of how to find it.

4.2 Computability in individual choice

In a recent interview[5], 2002 Nobel laureate Vernon L Smith—asked which economists had influenced him most—replied:

[5] With Russ Roberts, `http://www.econtalk.org/archives/2007/05/vernon_smith_on.html#highlights` (accessed 2007-05-23).

> At Harvard, Wassily Leontief, who had a certain amount of skepticism about economics and a great sense of humor. After two weeks of studying utility theory, student raised his hand and asked, "Professor, what is utility good for?" Leontief responded "It's good for teaching."

It would nevertheless obviously be nice of computable (in some sense) preferences should give rise to computable utility functions. The following theorems appeared to establish that this is the case.

Theorem 4 (Richter and Wong [18, 16]) *If* $\succ \subseteq \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$ *is a computable preference relation, then there exists a computable function* $u : \mathbb{R}^\ell_+ \to \mathbb{R}^\ell_+$ *that rationalizes* $\succ$*.*

However, Richard and Wong seem to have defined a computable preference relation through the characteristic function. This is in fact a bad definition since all computable functions are continuous and hence sets with computable characteristic functions are both open and closed (clopen). In Euclidean space however the only clopen sets are the empty set and the entire space. The preceding result (although correctly derived) appears to contradict an earlier result of Bridges and Richman [4] but only because of this specific use of the definition of a computable preference relation. The result of Bridges and Richman did in fact establish that a computable preference relation can give rise to an non-computable utility function.

4.3 Computability in social choice

The final remark concern Fishburn's treatment of the Arrow impossibility theorem.

Theorem 5 (Arrow) *If a decision-making body has at least two (but finitely many) members and at least three (but finitely many) options to decide among, then it is impossible to design a universal (it should create a deterministic, complete societal preference order from every set of individual preferences), non-imposing (every societal preference should be achievable) social choice function that satisfies all of the following three conditions.*

- *non-dictatorship;*
- *unanimity (if all individuals prefer* x *to* y *then so does the society); and*
- *independence of irrelevant alternatives.*

Peter Fishburn, recipient of the 1996 INFORMS von Neumann Theory Prize, showed in 1970 [7] that the three conditions in Arrow's Impossibility Theorem can in fact be satisfied by a social choice function if there are *infinitely* many decision makers. H. Reiju Mihara has however proved that Arrow's Theorem does in fact still hold for the case of a countably) infinite society if all coalitions are required to be computable and we require a pairwise-computable social welfare function [13]. In this case computability seems a highly reasonable requirement as it simply means that, for example, one can in practice identify the coalitions and that any two different options can be ranked in practice.

5 Conclusion

The intention of the preceding remarks is not to deny that, for example, markets function in practice. Indeed. prominent skeptics of equilibrium theory like experimental economist Vernon Smith [15] have shown how, in many instances, a kind of stability does emerge in simulated transactional environments. Words in Hayek's famous 1945 paper [11], already referred to above, remain appropriate.

> If we can agree that the economic problem of society is mainly one of rapid adaptation to changes in the particular circumstances of time and place, it would seem to follow that the ultimate decisions must be left to the people who are familiar with these circumstances, who know directly of the relevant changes and of the resources immediately available to meet them.

The question arises why an awareness of the futility of algorithmic solutions is still so much missing in management, not to mention the executive and legislative branches of government, and in general, in politics as such.

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