A note on the Exclusion Principle

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May 2006

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Abstract

According to the so-called Exclusion Principle (introduced by Baye *et alii*, 1993), it might be profitable for the seller to reduce the number of fully-informed potential bidders in an all-pay auction. We show that it does not apply if the seller regards the bidders' private valuations as belonging to the class of identical and independent distributions with a monotonic hazard rate.

Keywords: all-pay auctions, Exclusion Principle, monotonic hazard rate, economic theory of lobbying.

JEL *Classification*: D44, D72, D82.

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1. Introduction

Baye *et alii* (1993) demonstrate the following somewhat surprising result, the so-called Exclusion Principle. In an all-pay auction with complete information it might be in the best interest of the seller, if she is able to, to exclude some potential bidders from the short list of the auction participants. And in this case she should exclude those with the largest private valuations ("willingness to pay") for the (unique) object to be sold. The result can be applied to several social games, such as patent races and sports, and notably to lobbying games: see e.g. Hillman and Riley (1989).[1] It is due to the fact that the revenue expected (the bidding equilibrium is in mixed strategies) by the seller is decreasing in the largest valuation among bidders, call it v_1, while increasing with respect to the second-largest valuation, v_2 (the other bidders bid zero with probability 1). Excluding the "strongest" bidders induces (some of) the "weakest" ones to bid more and may increase the overall expenditure. In particular, it turns out that the expected total payment to the seller is $p(v_1, v_2) = v_2/2 + (v_2/v_1)(v_2/2) < v_2$, where the latter amounts are the expected payments of bidders 1 and 2 (those with the largest and the second largest valuations)[2] respectively. The object is assigned to bidders 1 and 2 respectively with probabilities $1 - v_2/v_1$ and v_2/v_1, and the former bidder expects $v_1 - v_2$ in the equilibrium (all the other bidders expect zero). The overall expected welfare is then $w(v_1, v_2) = p(v_1, v_2) + v_1 - v_2 < v_1$ (where $w > v_2$), and thus the outcome does not belong to the Core of the corresponding exchange game.

As indicated above, the quoted literature refers to the case of complete information,[3] which is a somewhat unusual assumption in auction theory. Moreover, the role and the information available to the designer (if any) of the auction are somehow left unexplained. In a companion paper (Bertoletti, 2005), we argue for example that the Exclusion Principle is affected by the implicit assumption that the auction "reserve price" is null.[4] Indeed, as far as the lobbying models are concerned, the only consistent justification for the adopted setting seems to be that the politician (the seller) who receives the lobbies' (bidders') contributions has very little bargaining power. However, the assumption that a fully informed seller can credibly exclude some bidder from her short list while she is unable to ask him a price not higher than his valuation does not appear generally palatable as a bargaining feature. More robust results should then be based on the explicit

[1] Che and Gale (1998) show a somehow related result, namely that the imposition of an exogenous cap on individual lobbying contributions may have the adverse effect of increasing total expenditure.

[2] The possibility of ties in the valuations is ignored here, since we assume that the valuations are ex ante continuously distributed (ties may imply the existence of multiple Nash equilibria which are not necessarily revenue equivalent: see Baye et alii, 1996).

[3] Hillman and Riley (1989: pp. 29-30) also deal with the case of incomplete information among contenders, and Che and Gale (1998: p. 648) claim that their result would hold even under incomplete information if there were asymmetry among bidders.

[4] In addition, there might also be other, possibly more efficient, ways to motivate the less favourite contenders (for example offering, if possible, multiple (divided) prizes: see Moldovanu and Sela, 2001).

assumption that the seller does not know the bidders' preferences.

Indeed, Menicucci (2005) strikingly shows that for some information structures the Exclusion Principle also applies to the case in which the seller regards the bidders' private valuations as identically and independently distributed (iid) and uses no reserve price. Namely, for the distributional structures that he considers, excluding from the all-pay auction with complete information among the bidders all but two of them (randomly selected) increases the seller's revenue. Menicucci's example uses a *discrete* distribution with "small" (the seller is almost certain about the bidders' valuations) uncertainty: however, his distribution can be easily made continuous. This then raises the obvious question of what distributional properties do sustain the Exclusion Principle in the indicated setting.

Notice that the Exclusion Principle is at odds with the positive value to the seller of additional symmetric (risk-neutral) bidders in private-value auctions with incomplete information: see e.g. Krishna (2002: chapter 2). In this note we show that actually it does not apply to the class of iid (continuous) distributions with a monotonic hazard rate. In particular, and somewhat more generally, we show that for the Exclusion Principle to apply the common distribution of valuations must be such that its so-called "mean residual life" (see e.g. Shaked and Shanthikumar, 1994: section 1.D) is somewhere increasing.

2. The setting and the result

Consider the following setting: m (risk-neutral) agents will possibly bid for a unique prize in an all-pay auction (there is no resale possibility). Bidder i's valuation of the prize is v_i ($i = 1, 2, \ldots, m$) and is ad interim (before bidding takes place) common knowledge among bidders, and we order them in such a way that $v_1 > v_2 > \ldots > v_{m-1} > v_m > 0$. The seller only knows that ex ante each valuation v is iid according to a common, strictly increasing and atomless, continuous cumulative distribution function $H(v)$ with support $[\underline{v}, \overline{v}]$, $\underline{v} \geq 0$.[5] From her point of view, then, the revenue she expects ex ante by (randomly) selecting n bidders ($2 \leq n \leq m$) to participate in the auction is given by $E\{p(v_1, v_2)\}$, where v_1 and v_2 are respectively the first (highest) and the second (second-highest) order statistics of n independent draws from $H(\cdot)$ (see e.g. Krishna, 2002: Appendix C). The following Proposition holds.

Proposition 1 *Consider an all-pay auction with complete information among bidders (no reserve price, no resale possibility). Suppose that the bidders' valuations are ex-ante identically and*

[5] These are, of course, the assumptions of the well-known Revenue Equivalence Theorem: see e.g. Klemperer (2004: p. 17).

independently distributed according to a strictly increasing, atomless continuous distribution $H(\cdot)$ with a monotonic hazard rate. In this case the seller maximizes her expected revenue by getting the largest possible set of actual participants.

The proof of Proposition 1 is presented in the Appendix. It depends on the fact that, with iid valuations, the conditional distribution of v_1 given v_2, $c(v_1|v_2)$, is just the distribution of v conditional on $v \geq v_2$, i.e., $c(v_1|v_2) = h(v_1)/(1 - H(v_2))$ (where $h(\cdot)$ is the density function which corresponds to $H(\cdot)$). We are then able to prove, by exploiting the convexity of $p(\cdot)$, that the conditional expectation $E_{v_1|v_2}\{p(v_1, v_2)\}$ is increasing with respect to v_2 *if* the hazard rate of $H(\cdot)$, $\lambda(\cdot) = h(\cdot)/(1 - H(\cdot))$, is monotonic. But since an increases in the number of symmetric participants to the auction first-order stochastically increases the distribution of v_2, it follows that it also raises $E\{p(v_1, v_2)\} = E_{v_2}\{ E_{v_1|v_2}\{p(v_1, v_2)\}\}$.

3. Conclusion

The intuition for the previous result is provided by the sign of the following derivative:[6]

$$\frac{\partial E_{v_1|v_2}\{v_1 - v_2)\}}{\partial v_2} = E_{v_1|v_2}\left\{\frac{\lambda(v_2)}{\lambda(v_1)} - 1\right\}. \qquad (1)$$

The expected value of the difference of the first and the second order statistics of the participants' valuations would change with the number of bidders according to the sign of (1) *if* this were constant. Moreover, a somewhere positive value for (1) is a *necessary* condition for the Exclusion Principle to apply to an ex-ante symmetric all-pay auction with complete information. That is, the addition of another identical bidder cannot decrease the seller's expected revenue if $E_{v_1|v_2}\{v_1 - v_2\}$ (i.e., if the "mean residual life" of v) is nowhere increasing. Since $E\{v_1 - v_2\} = E_{v_1}\{1/\lambda(v_1)\}$, and it is well-known that H_1^n first-order stochastically dominates H_1^{n-1} (where $H_1^n(\cdot)$ is the (unconditional) distribution function of v_1 in the case of n independent draws from $H(\cdot)$), if the hazard rate is monotonic the addition of another bidder does decrease $E\{v_1 - v_2\}$ and raises the seller's expected revenue.

[6] Note that, up to the second term of its Taylor expansion with respect to v_1 at the right of v_2, $p \approx v_2 - (v_1 - v_2)/2$.

Note that the expected welfare is given by $E\{w(v_1, v_2)\} = E\{p(v_1, v_2)\} + E\{v_1 - v_2\}$. So any bidder exclusion profitable for the seller would then raise the expected welfare by a trivial revealed-preference argument if it were also to increase $E\{v_1 - v_2\}$. But this can never be the case if the hazard rate is monotonic, and the impact on the expected welfare of increasing the number of bidders' set remains ambiguous even in such a case. However, it is easy to see that a *sufficient* condition for an expected welfare improvement to follow any bidder addition under a monotonic hazard rate is $\underline{v}h(\underline{v}) > 1$ ($E\{w(v_1, v_2)\}$ increases with respect to the number of bidders if $E_{v_1|v_2}\{0,5(1 + \lambda(v_2)/\lambda(v_1)) - 1/(v_2\lambda(v_1))\} \geq 0$ for any v_2). Perhaps interestingly, under the same assumptions no bidder exclusion (which always decreases expected welfare) through a positive reserve price would be optimal for the seller in a "standard" (see Klemperer, 2004: section 1.1.2) auction with incomplete information. More generally, the hazard rate of $H(\cdot)$ plays a role in the characterization of the optimal mechanism under incomplete information, because its monotonicity implies that the so-called "virtual valuation" $v - 1/\lambda(v)$ is an increasing function: see e.g. Krishna (2002: chapter 5).

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Appendix

Proof of Proposition 1. Since the density function of the joint distribution of the first and second order statistics (see e.g. Krishna, 2002: p. 267) is given by:

$$g(v_1, v_2) = (n^2 - n)(H(v_2))^{n-2} h(v_1) h(v_2) I_{(v_2,\infty)}(v_1) \quad \text{(A.1)}$$

(where $I_{(\cdot)}(\cdot)$ is the appropriate indicator function), the density function of v_1 conditional on v_2 is given by:

$$c(v_1|v_2) = \frac{g(v_1, v_2)}{n(n-1)(1 - H(v_2))(H(v_2))^{n-2} h(v_2)} = \frac{h(v_1)}{1 - H(v_2)} \quad \text{(A.2)}$$

on the support $[v_2, \bar{v}\,]$ (note that it does not depend on n). Now compute the derivative of:

$$E_{v_1|v_2}\{p(v_1, v_2)\} = \int_{v_2}^{\bar{v}} p(v_1, v_2) \frac{h(v_1)}{1 - H(v_2)} dv_1 \quad \text{(A.3)}$$

with respect to v_2:

$$\frac{\partial E_{v_1|v_2}\{p(v_1, v_2)\}}{\partial v_2} = \int_{v_2}^{\bar{v}} [\frac{\partial p(v_1, v_2)}{\partial v_2} \frac{1}{1 - H(v_2)} + \frac{h(v_2) p(v_1, v_2)}{[1 - H(v_2)]^2}] h(v_1) dv_1 - \frac{h(v_2) p(v_2, v_2)}{1 - H(v_2)}. \quad \text{(A.4)}$$

Then, by noting that $p(\cdot)$ is a convex function:

$$\frac{\partial E_{v_1|v_2}\{p(v_1, v_2)\}}{\partial v_2} \geq \frac{1}{1 - H(v_2)} \{ \int_{v_2}^{\bar{v}} \frac{\partial p(v_1, v_2)}{\partial v_2} h(v_1) dv_1 + \frac{h(v_2)}{1 - H(v_2)} \int_{v_2}^{\bar{v}} [p(v_2, v_2) + (v_1 - v_2) \frac{\partial p(v_2, v_2)}{\partial v_1}] h(v_1) dv_1$$

$$- h(v_2) p(v_2, v_2) \} = \frac{1}{1 - H(v_2)} \{ \int_{v_2}^{\bar{v}} (\frac{1}{2} + \frac{v_2}{v_1}) h(v_1) dv_1 - \frac{h(v_2)}{2(1 - H(v_2))} \int_{v_2}^{\bar{v}} (v_1 - v_2) h(v_1) dv_1 \}$$

$$= \int_{v_2}^{\bar{v}} [\frac{v_2}{v_1} + \frac{1}{2}(1 - \frac{\lambda(v_2)}{\lambda(v_1)})] \frac{h(v_1)}{1 - H(v_2)} dv_1 = E_{v_1|v_2} \{ \frac{v_2}{v_1} + \frac{1}{2}(1 - \frac{\lambda(v_2)}{\lambda(v_1)}) \}. \quad \text{(A.5)}$$

Thus $E_{v_1|v_2}\{p(v_1, v_2)\}$ is an increasing function of v_2 *if* the hazard rate is monotonic. Finally, recall that the (unconditional) distribution function of v_2, given n symmetric participants to the auction, is:

$$H_2^n(v_2) = n(H(v_2))^{n-1} - (n-1)(H(v_2))^n. \quad \text{(A.6)}$$

Since

$$H_2^n(v_2) - H_2^{n+1}(v_2) = n(H(v_2))^{n-1} - 2n(H(v_2))^n + n(H(v_2))^{n+1} \geq 0, \quad \text{(A.7)}$$

it follows that H_2^n first-order stochastically dominates H_2^{n-1}, and thus that any exclusion from the set of the potential bidders (strictly) decreases the expected revenue of the seller *if* the hazard rate of $H(\cdot)$ is monotonic. **QED**