ASYMPTOTICS OF SPECTRAL GAPS OF THE 1D SCHRODINGER OPERATOR WITH MATHIEU POTENTIAL

by

BERKAY ANAHTARCI

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Master of Science

Sabancı University
Fall 2010

ASYMPTOTICS OF SPECTRAL GAPS OF THE 1D SCHRODINGER OPERATOR WITH MATHIEU POTENTIAL

APPROVED BY:

Prof. Dr. Plamen Djakov ....
(Thesis Supervisor)

Prof. Dr. Cihan Saçlıoğlu ....

Asst. Prof. Gökhan Göğüş ....

DATE OF APPROVAL: 20/01/2011

ABSTRACT

The one-dimensional Schrödinger operator $L(y) = -y'' + v(x)y$, considered on $\mathbb{R}$ with π-periodic real-valued potential $v(x)$, is self-adjoint, and its spectrum has a gap-band structure- the intervals of continuous spectrum are separated by spectral gaps. In this thesis, we study the asymptotic behaviour of the spectral gaps of L. In the case of the Mathieu potential $v(x) = 2a\cos(2x)$, we give an alternative proof of the result of Harrell-Avron-Simon about the precise asymptotics of the lengths of spectral gaps.

ÖZET

$\mathbb{R}$ üzerinde π-periyodik reel potansiyel fonksiyonu $v(x)$ ile düşünülen, 1 boyutlu Schrödinger operatörü $L(y) = -y'' + v(x)y$ öz-eşleniktir ve spektrumu boşluklu yapıdadır- sürekli spektrumu spektral boşluklarla ayrılmıştır. Bu tezde, L operatörünün spektral boşluklarının asimtotik davranışını inceliyoruz. Mathieu potansiyel fonksiyonu $v(x) = 2a\cos(2x)$ durumunda, Harrell-Avron-Simon'ın spektral boşlukların uzunluklarıyla ilgili kesin asimtotik sonuçlarına eşdeğer bir ispat veriyoruz.

TABLE OF CONTENTS

1 Introduction

The one-dimensional Schrödinger operator $L(y) = -y'' + v(x)y$, considered on $\mathbb{R}$ with π-periodic real-valued potential $v(x)$, is self-adjoint, and its spectrum has a gap-band structure; namely, there are points

$$\lambda_0^+ < \lambda_1^- \le \lambda_1^+ < \lambda_2^- \le \lambda_2^+ < \lambda_3^- \le \lambda_3^+ < \lambda_4^- \le \lambda_4^+ < \cdots$$

such that

$$Sp(L) = \bigcup_{n=1}^{\infty} [\lambda_{n-1}^+, \lambda_n^-]$$

and the intervals of the spectrum are separated by the spectral gaps

$$(-\infty, \lambda_0^+), (\lambda_1^-, \lambda_1^+), \ldots, (\lambda_n^-, \lambda_n^+), \ldots .$$

Our goal is to investigate the asymptotic behaviour of the lengths of the spectral gaps

$$\gamma_n = \lambda_n^+ - \lambda_n^-, \quad n = 1, 2, \ldots .$$

First, we give some basics about general Floquet theory which is used to determine the points $\lambda_n^\pm$ as follows: for even n, the numbers $\lambda_n^\pm$ are eigenvalues of the eigenvalue problem

$$-y'' + v(x)y = \lambda y, \quad 0 \le x \le \pi,$$

subject to periodic boundary conditions

$$y(0) = y(\pi), \quad y'(0) = y'(\pi)$$

and for odd n, the numbers $\lambda_n^\pm$ are eigenvalues of the eigenvalue problem

$$-y'' + v(x)y = \lambda y, \quad 0 \le x \le \pi,$$

subject to antiperiodic boundary conditions

$$y(0) = -y(\pi), \quad y'(0) = -y'(\pi).$$

In the case of Mathieu potential

$$v(x) = 2a \cos 2x, \quad a \ real,$$

Levy and Keller [4] established the asymptotics of $\gamma_n = \gamma_n(a)$, for fixed n as $a \to 0$; namely

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}(1 + O(a)).$$

Almost 20 years later, Harrell [3] found, up to a constant factor, the asymptotics of the spectral gaps of the Mathieu potential as $n \to \infty$ (a fixed). Avron and Simon [1] gave another proof of Harrell's asymptotics and found the exact value of the constant factor, which led to the following formula:

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

In this thesis we give an alternative proof of the result of Harrell-Avron-Simon about the precise asymptotics of the lengths of spectral gaps using the method developed in [2].

2 Floquet Theory

In this section, we give some basics about Floquet theory, which are going to be used to establish the structure of spectral gaps.
A second-order linear differential equation

$$a_0(x)z''(x) + a_1(x)z'(x) + a_2(x)z(x) = 0, \quad a_0 \neq 0, \tag{2.1}$$

is called *Hill's equation* if the coefficients $a_i(x)$ are periodic, say $a_i(x+\pi) = a_i(x)$, for $i = 0, 1, 2$.

Lemma 1. *If $a_1(x)/a_0(x)$ have a piecewise continuous derivative, then* (2.1) *can be reduced to an equation of the form*

$$y''(x) + v(x)y(x) = 0, \tag{2.2}$$

where $v(x)$ is a real-valued periodic function.

Proof. Consider the substitution

$$z(x) = y(x)e^{-\frac{1}{2}\kappa(x)},$$

with

$$\kappa(x) = \int_0^x \frac{a_1(t)}{a_0(t)}\,dt.$$

Then, we have

$$z'(x) = e^{-\frac{1}{2}\kappa(x)} \left[y'(x) - \frac{1}{2} y(x) \frac{a_1(x)}{a_0(x)} \right], \tag{2.3}$$

and

$$\begin{aligned} z''(x) &= e^{-\frac{1}{2}\kappa(x)} \left[y''(x) - y'(x) \frac{a_1(x)}{a_0(x)} \right] \\ &+ e^{-\frac{1}{2}\kappa(x)} y(x) \left[\frac{1}{4} \left(\frac{a_1(x)}{a_0(x)} \right)^2 - \frac{1}{2} \left(\frac{a_1(x)}{a_0(x)} \right)' \right]. \end{aligned} \tag{2.4}$$

If (2.1) is multiplied by $e^{\frac{1}{2}\kappa(x)}\{a_0(x)\}^{-1}$, and (2.3),(2.4) are substituted inside, then the equation becomes

$$y''(x) + \left[\frac{a_2(x)}{a_0(x)} - \frac{1}{4} \left(\frac{a_1(x)}{a_0(x)} \right)^2 - \frac{1}{2} \left(\frac{a_1(x)}{a_0(x)} \right)' \right] y(x) = 0,$$

which has the form (2.2) since the coefficient of $y(x)$ is periodic. □

Consider the Hill's equation

$$-y''(x) + v(x)y(x) = 0, \tag{2.5}$$

where $v(x)$ is a real-valued $L^2([0,\pi])$-function and $v(x+\pi) = v(x)$.

From the *Existence-Uniqueness Theorem* for ordinary differential equations with L^1-coefficients [6], there are solutions $u_1(x)$ and $u_2(x)$ of (2.5) satisfying the initial conditions

$$u_1(0) = 1, \qquad u_1'(0) = 0, \tag{2.6}$$

$$u_2(0) = 0, \qquad u_2'(0) = 1. \tag{2.7}$$

Then every non-trivial solution $y(x)$ has the form $y(x) = c_1u_1(x) + c_2u_2(x)$, where c_1 and c_2 are not both zero's.

Let us look for non-trivial solutions of (2.5) with the property:

$$y(x+\pi) = \rho y(x), \quad \rho \neq 0. \tag{2.8}$$

In order to get the property (2.8), the following must hold:

$$c_1 u_1(x+\pi) + c_2 u_2(x+\pi) = \rho(c_1 u_1(x) + c_2 u_2(x)),$$
$$c_1 u_1'(x+\pi) + c_2 u_2'(x+\pi) = \rho(c_1 u_1'(x) + c_2 u_2'(x)).$$

Evaluation at the point $x = 0$ leads to the system:

$$M \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \rho \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{2.9}$$

where

$$M = \begin{pmatrix} u_1(\pi) & u_2(\pi) \\ u_1'(\pi) & u_2'(\pi) \end{pmatrix}. \tag{2.10}$$

The matrix M is known as the *Monodromy matrix.* Observe that (2.9) means that ρ is an eigenvalue of M and $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is an eigenvector of M corresponding to ρ.

The system (2.9) has a non-trivial solution $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ if and only if ρ is a root of the equation $\det(M - \rho I) = 0$.

Observe that

$$\det(M) = \begin{vmatrix} u_1(\pi) & u_2(\pi) \\ u_1'(\pi) & u_2'(\pi) \end{vmatrix} = W(u_1, u_2)(\pi) = W(u_1, u_2)(0) = 1,$$

in which $W(u_1, u_2)$ denotes the *Wronskian* of u_1 and u_2. Therefore, the equation $\det(M - \rho I) = 0$ can be written in the form

$$\rho^2 - [u_1(\pi) + u_2'(\pi)]\rho + 1 = 0. \tag{2.11}$$

Equation (2.11) is called the *characteristic equation.*

<u>*Case 2.1.*</u> First, consider the case where (2.11) has two distinct roots $\rho_1 \neq \rho_2$. Let $y_1(x)$ and $y_2(x)$ be, respectively, the solutions corresponding to ρ_1 and ρ_2 as in (2.8).

Then

$$y_k(x+\pi) = \rho_k y_k(x), \quad k = 1, 2.$$

By (2.11), $\rho_1\rho_2 = 1$, so $\rho_1, \rho_2 \neq 0$. Therefore, there are numbers τ_1, τ_2 such that $\rho_1 = e^{\pi\tau_1}$ and $\rho_2 = e^{\pi\tau_2}$. Notice that, $\tau_1 = -\tau_2$.

We set

$$\varphi_k(x) = y_k(x)e^{-\tau_k x}, \quad k = 1, 2. \tag{2.12}$$

Then,

$$\varphi_k(x+\pi) = y_k(x+\pi)e^{-\tau_k(x+\pi)} = \rho_k y_k(x)e^{-\tau_k x}\frac{1}{\rho_k} = \varphi_k(x), \tag{2.13}$$

which shows that $\varphi_k(x)$ is π-periodic and

$$y_k(x) = \varphi_k(x)e^{\tau_k x}, \quad k = 1, 2. \tag{2.14}$$

As a conclusion, there are two linearly independent solutions $y_1(x)$ and $y_2(x)$ of (2.5) such that $y_1(x) = \varphi_1(x)e^{\tau_1 x}$ and $y_2(x) = \varphi_2(x)e^{\tau_2 x}$, where τ_1, τ_2 are non-zero constants and $\varphi_1(x)$, $\varphi_2(x)$ are periodic with period π.

Case 2.2. Now, suppose that (2.11) has a repeated root ρ. Then, there exists a number τ so that $e^{\pi\tau} = \rho$.

Let y_1 be a non-trivial solution corresponding to ρ as in (2.8). Then,

$$y_1(x+\pi) = \rho y_1(x). \tag{2.15}$$

Let y_2 be any solution of (2.5) which is linearly independent of y_1. Since $y_2(x+\pi)$ also satisfies (2.5), there exists constants c_1, c_2 such that

$$y_2(x+\pi) = c_1 y_1(x) + c_2 y_2(x). \tag{2.16}$$

We now calculate c_2. By (2.15) and (2.16), we have

$$W(y_1, y_2)(x+\pi) = \begin{vmatrix} y_1(x+\pi) & y_2(x+\pi) \\ y_1'(x+\pi) & y_2'(x+\pi) \end{vmatrix} = \begin{vmatrix} \rho y_1(x) & c_1 y_1(x) + c_2 y_2(x) \\ \rho y_1'(x) & c_1 y_1'(x) + c_2 y_2'(x) \end{vmatrix},$$

which leads to

$$W(y_1, y_2)(x+\pi) = \rho c_2 \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \rho c_2 W(y_1, y_2)(x).$$

Since $W(y_1, y_2)$ is constant, it follows that $\rho c_2 = 1$. Since ρ is a double root of

(2.11), we have $\rho^2 = 1$, so it follows that $c_2 = \rho$.

Then, (2.16) gives us

$$y_2(x+\pi) = c_1 y_1(x) + \rho y_2(x). \tag{2.17}$$

Now, there are two cases to be considered.

First, if $c_1 = 0$, we have $y_2(x+\pi) = \rho y_2(x)$. This together with (2.15) shows that we have the same situation as in the *Case 2.1* but with $\rho_1 = \rho_2 = \rho$. Therefore, (2.5) has solutions

$$y_1(x) = \varphi_1(x)e^{\tau x}, \quad y_2(x) = \varphi_2(x)e^{\tau x},$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are periodic functions with period π.

Second, if $c_1 \neq 0$, define

$$\begin{aligned}
\psi_1(x) &= e^{-\tau x} y_1(x), \\
\psi_2(x) &= e^{-\tau x} y_2(x) - \frac{c_1}{\rho\pi} x \psi_1(x).
\end{aligned}$$

Then by (2.15) and (2.17),

$$\psi_1(x+\pi) = e^{-\tau(x+\pi)} y_1(x+\pi) = e^{-\tau x} e^{-\tau\pi} \rho y_1(x) = e^{-\tau x} y_1(x) = \psi_1(x),$$

and

$$\begin{aligned}
\psi_2(x+\pi) &= e^{-\tau(x+\pi)} y_2(x+\pi) - \frac{c_1}{\rho\pi}(x+\pi)\psi_1(x+\pi) \\
&= e^{-\tau x}\frac{1}{\rho} c_1 y_1(x) + e^{-\tau x} y_2(x) - \frac{c_1}{\rho\pi} x e^{-\tau x} y_1(x) - \frac{c_1}{\rho} e^{-\tau x} y_1(x) \\
&= e^{-\tau x} y_2(x) - \frac{c_1}{\rho\pi} x e^{-\tau x} y_1(x) \\
&= \psi_2(x)
\end{aligned}$$

which shows that $\psi_1(x)$ and $\psi_2(x)$ are both π-periodic. Therefore,

$$\begin{aligned}
y_1(x) &= e^{\tau x}\psi_1(x), \\
y_2(x) &= e^{\tau x}\left(\frac{c_1}{\rho\pi} x \psi_1(x) + \psi_2(x)\right)
\end{aligned}$$

are two linearly independent solutions where $\psi_1(x)$ and $\psi_2(x)$ are periodic with period π.

3 Stability Zones

In this section, we will relate the spectrum of the *Schrödinger* operator

$$L(y) = -y'' + v(x)y \tag{3.1}$$

on the real line with the eigenvalue equation

$$-y'' + v(x)y = \lambda y, \quad x \in [0, \pi] \tag{3.2}$$

subject to, respectively, periodic (Per^+) or antiperiodic (Per^-) boundary conditions where

$$Per^+ : \quad y(0) = y(\pi), \quad y'(0) = y'(\pi), \tag{3.3}$$

$$Per^- : \quad y(0) = -y(\pi), \quad y'(0) = -y'(\pi). \tag{3.4}$$

We will study the Hill's equation in the form (3.2), where λ is a parameter and $v(x)$ is a real valued periodic function with period π.

In order to indicate the dependence on λ which occurs in (3.2), we denote by $u_1(x, \lambda)$ and $u_2(x, \lambda)$ the solutions of (3.2) which satisfy the initial conditions

$$u_1(0, \lambda) = 1, \qquad u_1'(0, \lambda) = 0,$$
$$u_2(0, \lambda) = 0, \qquad u_2'(0, \lambda) = 1.$$

Then, the corresponding *characteristic equation* will be

$$\rho^2 - D(\lambda)\rho + 1 = 0, \tag{3.5}$$

where

$$D(\lambda) = u_1(\pi, \lambda) + u_2'(\pi, \lambda), \tag{3.6}$$

In order to investigate $Sp(L)$, we look at the operator

$$(\lambda - L) : Dom(L) \to L^2(\mathbb{R})$$

where $Dom(L) = \{y \in L^2(\mathbb{R}) : Ly \in L^2(\mathbb{R}), y'' \in L^2(\mathbb{R}), y'$ is absolutely continuous $\}$ denotes the domain of the operator L.

If $(\lambda - L)^{-1}$ exists, then for every f in $L^2(\mathbb{R})$, there exists an element y in the $Dom(L)$ so that $(\lambda - L)y = f$, which is equivalent to saying that

$$y'' + (\lambda - v(x))y = f \tag{3.7}$$

and we will get $y = (\lambda - L)^{-1} f$.

It is well known that if $y_1(x, \lambda)$ and $y_2(x, \lambda)$ are solutions of the homogenous differential equation

$$-y'' + (\lambda - v(x))y = 0, \tag{3.8}$$

then by *The Method of Variation of Constants*, the general solution $y(x, \lambda)$ of (3.7) has the form

$$y(x, \lambda) = \eta_1(x, \lambda)y_1(x, \lambda) + \eta_2(x, \lambda)y_2(x, \lambda), \tag{3.9}$$

where

$$\eta_1(x, \lambda) = \frac{1}{W} \int_x^{\infty} f(\xi)y_2(\xi, \lambda)\, d\xi \tag{3.10}$$

and

$$\eta_2(x, \lambda) = \frac{1}{W} \int_{-\infty}^{x} f(\xi)y_1(\xi, \lambda)\, d\xi, \tag{3.11}$$

with $W = W(y_1, y_2)(x, \lambda) = constant(\lambda)$. Then, $y(x, \lambda)$ will be

$$y(x, \lambda) = \frac{1}{W} \int_x^{\infty} y_1(x, \lambda)y_2(\xi, \lambda)f(\xi)\, d\xi + \frac{1}{W} \int_{-\infty}^{x} y_2(x, \lambda)y_1(\xi, \lambda)f(\xi)\, d\xi. \tag{3.12}$$

In order to determine the solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of (3.8), the roots of the characteristic equation (3.5) should be analyzed. Therefore, $y_1(x, \lambda)$ and $y_2(x, \lambda)$ depend on the values of $D(\lambda) = u_1(\pi, \lambda) + u_2'(\pi, \lambda)$.

Whether λ is real or complex, since $u_1(x, \lambda)$, $u_2(x, \lambda)$ and their x-derivatives are analytic functions of λ, for fixed x, $D(\lambda)$ is an analytic function of λ and in particular,

$D(\lambda)$ is a continuous function of λ.

In this section, unless stated otherwise, λ will be regarded as real and λ-dependence will not be presented explicitly in the formulas.

The roots of (3.5) are

$$\rho_{1,2} = \frac{D(\lambda) \pm \sqrt{D(\lambda)^2 - 4}}{2}.$$

<u>*Case 3.1.*</u> If $D(\lambda) > 2$, then both roots ρ_1, ρ_2 are real, positive and distinct but not equal to 1. Then, by *Case 2.1*, we have solutions

$$y_1(x) = e^{\tau x}\varphi_1(x), \qquad y_2(x) = e^{-\tau x}\varphi_2(x), \quad \tau > 0, \tag{3.13}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ periodic functions with period π.

Then, the solution $y(x)$ of (3.7) will be

$$y(x) = \frac{1}{W}\left(\int_x^\infty e^{\tau(x-\xi)}\varphi_1(x)\varphi_2(\xi)f(\xi)\,d\xi + \int_{-\infty}^x e^{\tau(\xi-x)}\varphi_1(\xi)\varphi_2(x)f(\xi)\,d\xi\right).$$

Now, define an operator S on $L^2(\mathbb{R})$ by $S(f) = y(x)$. We will show that the operator S is a continuous linear operator such that $y(x) \in Dom(L)$.

S is clearly a linear operator from its definition.

Since $\varphi_1(x)$ and $\varphi_2(x)$ are periodic functions on $\mathbb{R}$, they are bounded so there exist positive real numbers M_1 and M_2 such that

$$|\varphi_1(x)| \leq M_1 \quad \text{and} \quad |\varphi_2(x)| \leq M_2. \tag{3.14}$$

Furthermore, by *Cauchy-Schwarz inequality* we have

$$\begin{aligned}\int_{-\infty}^\infty \left|\int_x^\infty e^{\tau(x-\xi)}|f(\xi)|\,d\xi\right|^2 dx &\leq \int_{-\infty}^\infty \left(\int_x^\infty |f(\xi)|^2 e^{\tau(x-\xi)}\,d\xi\right)\left(\int_x^\infty e^{\tau(x-\xi)}\,d\xi\right) dx \\ &= \frac{1}{\tau}\int_{-\infty}^\infty \int_x^\infty |f(\xi)|^2 e^{\tau(x-\xi)}\,d\xi\,dx. \end{aligned}\tag{3.15}$$

On the other hand, by *Fubini's Theorem*, we obtain

$$\frac{1}{\tau}\int_{-\infty}^{\infty}\int_{x}^{\infty}|f(\xi)|^2 e^{\tau(x-\xi)}\,d\xi\,dx = \frac{1}{\tau}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\xi} e^{\tau(x-\xi)}\,dx\right)|f(\xi)|^2\,d\xi$$
$$= \frac{1}{\tau^2}\int_{-\infty}^{\infty}|f(\xi)|^2\,d\xi$$
$$= \frac{1}{\tau^2}\|f\|_2^2. \tag{3.16}$$

Then, combining results in (3.14), (3.15) and (3.16) shows that

$$\left\|\frac{1}{W}\int_{x}^{\infty} e^{\tau(x-\xi)}\varphi_1(x)\varphi_2(\xi)f(\xi)\,d\xi\right\|_2 \le \frac{1}{W}\frac{M_1M_2}{\tau^2}\|f\|_2. \tag{3.17}$$

Accordingly, a similar argument gives us

$$\left\|\frac{1}{W}\int_{-\infty}^{x} e^{\tau(\xi-x)}\varphi_1(\xi)\varphi_2(x)f(\xi)\,d\xi\right\|_2 \le \frac{1}{W}\frac{M_1M_2}{\tau^2}\|f\|_2. \tag{3.18}$$

Therefore, from (3.17) and (3.18), it follows that

$$\|Sf\|_2 \le \frac{2}{W}\frac{M_1M_2}{\tau^2}\|f\|_2, \tag{3.19}$$

which shows that $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a continuous linear operator which is the inverse operator of $\lambda - L$.

One can easily see that y' is absolutely continuous , $y'' \in L^2(\mathbb{R})$ and $Ly \in L^2(\mathbb{R})$. Therefore, $y(x) \in Dom(L)$.

Hence, we conclude that if $D(\lambda) > 2$, then $\lambda \notin Sp(L)$.

Case 3.2. If $D(\lambda) < -2$, then the situation is the same as in *Case 3.1* except both roots ρ_1 and ρ_2 are negative but not equal to -1. Hence, τ should be replaced by $\tau + i$ in (3.13).

Then, $\lambda \notin Sp(L)$.

Case 3.3. If $-2 < D(\lambda) < 2$, then both roots ρ_1, ρ_2 are non-real and distict. Since $\rho_1\rho_2 = 1$ and they are complex conjugates, by *Case 2.1*, we have solutions

$$y_1(x) = e^{i\tau x}\varphi_1(x), \quad y_2(x) = e^{-i\tau x}\varphi_2(x) \tag{3.20}$$

for some real number τ with $0 < \tau < 1$ where $\varphi_1(x)$ and $\varphi_2(x)$ are periodic functions with period π.

Consider the equation (3.9) with $\eta_1(x, \lambda)$ and $\eta_1(x, \lambda)$ as they are in (3.10) and (3.11), respectively.

Take $f(x) = \chi_{[0,1]}(x) \in L^2(\mathbb{R})$. Then, we have

$$\eta_1(x) = \begin{cases} C_1 & \text{if } x \leq 0 \\ \frac{1}{W}\int\limits_x^1 y_2(\xi)\, d\xi & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

and

$$\eta_2(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{W}\int\limits_0^x y_1(\xi)\, d\xi & \text{if } 0 < x < 1 \\ C_2 & \text{if } x \geq 1 \end{cases}$$

where C_1 and C_2 are constants.

Observe that if $x \in (-\infty, 0]$, by (3.9) and (3.20) we have

$$y(x) = C_1 y_1(x) = C_1 e^{i\tau x}\varphi_1(x),$$

for some real number τ with $0 < \tau < 1$ where $\varphi_1(x)$ is a periodic function with period π.

Accordingly, on $(-\infty, 0]$,

$$|y(x)| = C_1|\varphi_1(x)|.$$

Hence, $y(x) \notin L^2((-\infty, 0])$, which implies that $y(x) \notin L^2(\mathbb{R})$.

Therefore, if $-2 < D(\lambda) < 2$, then $\lambda \in Sp(L)$.

Case 3.4. If $D(\lambda) = 2$, then $\rho_1 = \rho_2 = 1$. Therefore, the Floquet solutions are periodic. In this case, for even n, the numbers $\lambda_n^\pm$ are eigenvalues of the eigenvalue problem

$$-y'' + v(x)y = \lambda y, \quad 0 \leq x \leq \pi$$

subject to periodic boundary conditions

$$y(0) = y(\pi), \quad y'(0) = y'(\pi).$$

If $D(\lambda) = -2$, then $\rho_1 = \rho_2 = -1$. Hence, the Floquet solutions are antiperiodic and for odd n, the numbers $\lambda_n^\pm$ are eigenvalues of the eigenvalue problem

$$-y'' + v(x)y = \lambda y, \quad 0 \leq x \leq \pi$$

subject to antiperiodic boundary conditions

$$y(0) = -y(\pi), \quad y'(0) = -y'(\pi).$$

The numbers $\lambda_n^\pm$ constitute the *boundary* of $Sp(L)$ in the light of *Case 3.1*, *Case 3.2* and *Case 3.3*. Since $Sp(L)$ is compact, it follows that $\lambda_n^\pm \in Sp(L)$.

Theorem 2. *(Oscillation Theorem) (see [5], Theorem 2.1)*
The periodic and antiperiodic spectra of L are discrete, and moreover, there is a sequence of real numbers

$$\lambda_0^+ < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \lambda_3^- \leq \lambda_3^+ < \lambda_4^- \leq \lambda_4^+ < \cdots$$

such that λ_n^+ and λ_n^- correspond to Per^+ if n is even and to Per^- if n is odd.

The intervals $(-\infty, \lambda_0)$ and $(\lambda_n^-, \lambda_n^+)$ are called *spectral gaps (instability zones)* in which (3.2) has unbounded solutions when λ lies in one of them, and $Sp(L)$ is the complement of union of these open intervals. They refer to *zero'th* and *n'th* spectral gaps, respectively, and the lenght of *n'th*-spectral gap is denoted by γ_n, i.e. $\gamma_n = \lambda_n^+ - \lambda_n^-$.

4 Projection Method (Lyapunov-Schmidt)

Consider the operator

$$L^0 y = -y'', \tag{4.1}$$

defined on $[0, \pi]$.

Let $L^0_{Per^+}$ and $L^0_{Per^-}$ denote, respectively, the operator $L^0 y = -y''$ considered, respectively, with periodic (Per^+), or antiperiodic (Per^-) boundary conditions.

Define

$$\begin{aligned} Dom(L^0_{Per^+}) = \{y \in L^2(\mathbb{R}) : y' &\text{ is absolutely continuous,} \\ y'' \in L^2([0, \pi]),\ &\text{ and } y \text{ satisfies } Per^+\} \end{aligned}$$

and

$$\begin{aligned} Dom(L^0_{Per^-}) = \{y \in L^2(\mathbb{R}) : y' &\text{ is absolutely continuous,} \\ y'' \in L^2([0, \pi]),\ &\text{ and } y \text{ satisfies } Per^-\}. \end{aligned}$$

We consider the eigenvalue problem $-y'' = \lambda y$ subject to periodic (Per^+), or antiperiodic (Per^-) boundary conditions.

A number $\lambda \in \mathbb{C}$ is eigenvalue of $L^0_{Per^+}$ if there exists y so that $-y'' = \lambda y$, $y \not\equiv 0$ with $y(0) = y(\pi)$ and $y'(0) = y'(\pi)$.

Similarly, a number $\lambda \in \mathbb{C}$ is eigenvalue of $L^0_{Per^-}$ if there exists y so that $-y'' = \lambda y$, $y \not\equiv 0$ with $y(0) = -y(\pi)$ and $y'(0) = -y'(\pi)$.

In the next proposition, we use the fact that every solution of the second order differential equation $y'' + \lambda y = 0$ is of the form $y = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$ with $c_1, c_2 \in \mathbb{C}$.

Proposition 3. *Let the operator L^0 be defined as in* (4.1)*. Then,*

(i) $Sp(L^0_{Per^+}) = \{n^2 : n \in 2\mathbb{N}\}$ *and* $Sp(L^0_{Per^-}) = \{n^2 : n \in 1 + 2\mathbb{N}\}$.

(ii) $Sp(L^0_{Per^+})$ *and* $Sp(L^0_{Per^-})$ *are both pure point spectrum.*

Proof. (i) Here, we only give a proof for the periodic case because a proof for the antiperiodic case could be handled similarly.

Any solution y satisfies Per^+ if and only if the coefficients c_1 and c_2 satisfy the system:

$$
\begin{aligned}
c_1 + c_2 &= c_1 e^{-\sqrt{-\lambda}\pi} + c_2 e^{\sqrt{-\lambda}\pi} \\
-c_1 + c_2 &= -c_1 e^{-\sqrt{-\lambda}\pi} + c_2 e^{\sqrt{-\lambda}\pi}
\end{aligned}
$$

Adding up and subtracting these equations gives, respectively, $c_2(1 - e^{\sqrt{-\lambda}\pi}) = 0$ and $c_1(1 - e^{-\sqrt{-\lambda}\pi}) = 0$. Since $y \not\equiv 0$ identically only if $e^{\sqrt{-\lambda}\pi} = 1$, it immediately follows that $\sqrt{-\lambda} = 2ki$ and $\lambda = (2k)^2, \quad k = 0, \pm 1, \pm 2, \dots$.

This shows that the periodic spectrum of L^0 is discrete and we have $Sp(L^0_{Per^+}) = \{n^2 : n \ even\}$. As mentioned before, a similar argument can be used to show that $Sp(L^0_{Per^-}) = \{n^2 : n \ odd\}$.

(ii) Indeed, it is possible to consider $L^0 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$. Let $f \in L^2([0,\pi])$, then

$$
f = \sum_{k \in 2\mathbb{Z}} f_k e^{ikx}.
$$

If $\lambda \notin Sp(L^0_{Per^+})$ then $(\lambda - L^0_{Per^+})^{-1}$ exists.

Hence,

$$
y = (\lambda - L^0_{Per^+})^{-1} f = \sum_{k \in 2\mathbb{Z}} \frac{f_k}{\lambda - k^2} e^{ikx}
$$

exists only if $\lambda \neq k^2$ with $k \in 2\mathbb{Z}$. Therefore, the periodic spectrum of L^0 coincides with its periodic point spectrum.

The result for the antiperiodic spectrum L^0 can be obtained by changing the boundary conditions to Per^- and changing the basis to $e_k = e^{ikx}$ with $k \in 1 + 2\mathbb{Z}$. $\square$

On the other hand, each eigenvalue $n^2 \neq 0$ is of multiplicity 2, and $e_{-n} = e^{-inx}$ and $e_n = e^{inx}$ are corresponding normalized eigenfunctions to n^2. Hence, if periodic boundary conditions are considered, then $\lambda = 0$ is the only eigenvalue of L^0 of multiplicity 1, and the constant function $e_0 = 1$ is the corresponding normalized eigenfunction.

If $L^2([0,\pi])$ is considered with the scalar product

$$(f,g) = \frac{1}{\pi}\int_0^\pi f(x)\overline{g(x)}\,dx,$$

then each of the families of functions $\{e_{-n}, e_n : n \in 2\mathbb{Z}\}$ and $\{e_{-n}, e_n : n \in 1+2\mathbb{Z}\}$ is an orthonormal basis in $L^2([0,\pi])$. The basis $\{e^{2kix} : k \in \mathbb{Z}\}$ (respectively $\{e^{(2k-1)ix} : k \in \mathbb{Z}\}$) is used when we study the periodic (respectively antiperiodic) spectra of L.

The Hill operator

$$L = L^0 + v(x) \tag{4.2}$$

can be considered as a perturbation of L^0 and it is possible to use the *Perturbation theory* of operators to study the spectrum of L.

Proposition 4. *(Localization of spectrum)(see [2], Proposition 1)*

(i) If $\|v\| \le 1/4$, *then*

$$|\lambda_0| \le 4\|v\| \quad \textit{and} \quad |\lambda_n^\pm - n^2| \le 4\|v\|, \quad \textit{for } n \in \mathbb{N}.$$

(ii) If $V(0) = \frac{1}{\pi}\int_0^\pi v(x)\,dx = 0$, *then there is a constant* $N_0 = N_0(v)$ *such that*

$$|\lambda_n^\pm - n^2| < 1, \quad \textit{for } n \ge N_0.$$

Here, V denotes the operator of multiplication by $v(x)$, i.e. $(Vy)(x) = v(x)y(x)$.

Note that the assumption that $V(0) = 0$ leads to no loss of generality, because any shift of the potential by a constant shifts the spectrum by the same constant, and thus the spectral gaps remain the same.

By *Proposition 4*, it follows that if $\|v\|$ is small then λ_0 is close to 0, and λ_n^+,λ_n^- are close to n^2.

The operator $L = L^0 + v(x)$ is considered on the Hilbert space $\mathscr{H} = L^2([0,\pi])$. Let $E_n^0 = Span\{e_{-n} = e^{-inx}, e_n = e^{inx}\}$ be the eigenspace of L^0 corresponding to the eigenvalue n^2, and let P_n^0 be the orthogonal projection onto E_n^0, i.e.

$$P_n^0 x = (x, e_{-n})e_{-n} + (x, e_n)e_n, \tag{4.3}$$

for $x \in \mathscr{H}$.

Set $Q_n^0 = 1 - P_n^0$, so $\mathscr{H} = E_n^0 \oplus H_n^1$, where H_n^1 is the range of Q_n^0 and the symbol $\oplus$ denotes the orthogonal sum of two spaces.

Consider the eigenvalue equation $(\lambda - L)f = 0$, where $\lambda = n^2 + z$ with $|z| < 1$. With $f_1 = P_n^0 f$ and $f_2 = Q_n^0 f$, this equation is equivalent to the system:

$$P_n^0(\lambda - L^0 - V)(f_1 + f_2) = 0,$$
$$Q_n^0(\lambda - L^0 - V)(f_1 + f_2) = 0.$$

Since

$$\begin{array}{lll} P_n^0 f_1 = f_1, & L^0 f_1 = n^2 f_1, & Q_n^0 f_1 = 0, \\ P_n^0 f_2 = 0, & L^0 f_2 \in H_n^1, & Q_n^0 f_2 = f_2, \end{array}$$

the system reduces to

$$z f_1 - P_n^0 V f_1 - P_n^0 V f_2 = 0, \tag{4.4}$$
$$Q_n^0 V f_1 + Q_n^0 V f_2 - (\lambda - L^0) f_2 = 0 \tag{4.5}$$

The restriction $n^2 + z - L^0$ on H_n^1 is invertible. We define an operator D_n by

$$D_n e_k = \begin{cases} \frac{1}{n^2 - k^2 + z} e_k & \text{if } k \neq \pm n \\ 0 & \text{if } k = \pm n. \end{cases}$$

Notice that $D_n|_{H_n^1} = [(n^2 + z - L)|_{H_n^1}]^{-1}$. The matrix representation of D_n is

$$(D_n)_{km} = \begin{cases} \frac{1}{n^2 - k^2 + z} \delta_{km} & \text{if } k, m \in (n + 2\mathbb{Z}) \setminus \{\pm n\} \\ 0 & \text{if } k, m = \pm n \end{cases} \tag{4.6}$$

where $\delta_{km} = 0$ for $k \neq m$ and $\delta_{km} = 1$ for $k = m$.

From (4.5), it follows that $f_2 = D_n Q_n^0 V f_1 + D_n Q_n^0 V f_2$.

Let us call

$$T_n := D_n Q_n^0 V. \tag{4.7}$$

Then, provided that $\|T_n\| < 1$;

$$f_2 = (1 - D_n Q_n^0 V)^{-1} D_n Q_n^0 V f_1. \tag{4.8}$$

Let $V : L^2([0,\pi]) \to L^2([0,\pi])$ be the operator of multiplication by the potential

$$v(x) = \sum_{k \in 2\mathbb{Z}} V(k) e^{ikx} \tag{4.9}$$

where $V(k)$ are the Fourier coefficients of $v(x)$, $x \in ([0,\pi])$.

Throughout the paper we assume that

$$V(0) = \frac{1}{\pi} \int_0^\pi v(x)\, dx = 0. \tag{4.10}$$

Lemma 5. *The matrix representation of the operator V, where $(Vy)(x) = v(x)y(x)$, is given by $V_{km} = V(k-m)$.*

Proof. Indeed,

$$\begin{aligned} V_{km} = (Ve_m, e_k) &= \frac{1}{\pi} \int_0^\pi v(x) e^{imx} e^{-ikx}\, dx \\ &= \frac{1}{\pi} \int_0^\pi v(x) e^{-i(k-m)x}\, dx = V(k-m) \end{aligned} \tag{4.11}$$

□

Now, it follows from (4.6), (4.7) and (4.11) that the matrix representation of T_n is given by

$$(T_n)_{km} = \begin{cases} \frac{V(k-m)}{n^2-k^2+z} & \text{if } k, m \in (n+2\mathbb{Z}) \setminus \{\pm n\} \\ 0 & \text{if } k = \pm n \end{cases} \tag{4.12}$$

and $T_n : \mathscr{H} \to H_n^1$.

Lemma 6. *For each $n \in \mathbb{N}$, and $|z| \le n$,*

$$\sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \frac{1}{|n^2 - k^2 + z|^2} < \frac{4}{n^2}.$$

Proof. Let $|z| \le n$, and $k \in (n+2\mathbb{Z}) \setminus \{\pm n\}$. Then, we have

$$|n^2 - k^2| = |n-k||n+k| \ge 2n,$$

which leads to

$$|n^2 - k^2 + z| \ge |n^2 - k^2| - |z| \ge \frac{1}{2}|n^2 - k^2|. \tag{4.13}$$

Therefore,

$$\begin{aligned}\sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \frac{1}{|n^2 - k^2 + z|^2} &\le \sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \frac{4}{|n^2 - k^2|^2} \\ &\le \frac{1}{n^2} \sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \left[\frac{2}{(n-k)^2} + \frac{2}{(n+k)^2}\right], \end{aligned} \tag{4.14}$$

which follows from the elementary identity

$$\frac{1}{n^2 - k^2} = \frac{1}{2n}\left(\frac{1}{n-k} + \frac{1}{n+k}\right),$$

and inequality

$$(a+b)^2 \le 2a^2 + 2b^2.$$

Since

$$\sum_{j \in \mathbb{Z}\setminus\{0\}} \frac{1}{j^2} = \frac{\pi^2}{3},$$

we have

$$\sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \left[\frac{2}{(n-k)^2} + \frac{2}{(n+k)^2}\right] \le 4 \sum_{\substack{m \neq 0 \\ m \in 2\mathbb{Z}}}^{\infty} \frac{1}{m^2} < \frac{\pi^2}{3}. \tag{4.15}$$

Then, by (4.14) and (4.15), we obtain

$$\sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \frac{1}{|n^2 - k^2 + z|^2} < \frac{\pi^2}{3n^2} < \frac{4}{n^2}. \tag{4.16}$$

□

Lemma 7. *Let the operator T_n be defined as in (4.7). Then*

$$\|T_n\| < 1, \quad \textit{for } n \ge 2\|v\|.$$

Proof. It is well-known that the ℓ^2-norm of an operator T does not exceed its Hilbert-

Schmidt norm which is defined as

$$\|T\|_{HS}^2 = \sum_k \|Te_k\|^2 = \sum_k \sum_m |(Te_k, e_m)|^2.$$

In view of (4.12),

$$\|T_n\|_{HS}^2 = \sum_{\substack{k \neq \pm n \\ k,m \in n+2\mathbb{Z}}} \sum \frac{|V(k-m)|^2}{|n^2 - k^2 - z|^2}.$$

For $|z| < 1$, by (4.16) and the *Parseval's identity*, it follows that

$$\|T_n\|^2 \leq \sum_{\substack{k \neq \pm n \\ k \in n+2\mathbb{Z}}} \frac{1}{|n^2 - k^2 + z|^2} \sum_{m \in n+2\mathbb{Z}} |V(k-m)|^2 \leq \frac{4\|v\|^2}{n^2} < 1$$

when $n \geq 2\|v\|$.

□

If we substitute (4.8) in the equation (4.4), we get

$$zf_1 - P_n^0 V f_1 - P_n^0 V(1 - T_n)^{-1} T_n f_1 = 0,$$

or equivalently

$$(z - S)f_1 = 0, \tag{4.17}$$

where

$$S = P_n^0 V + P_n^0 V(1 - T_n)^{-1} T_n : E_n^0 \to E_n^0, \tag{4.18}$$

Observe that $f_1 \neq 0$ (Otherwise, $f_2 = 0$, which implies that $f_1 + f_2 = 0 = f$).

By (4.18), we have

$$\begin{aligned} S &= P_n^0 V + P_n^0 V \sum_{m=0}^{\infty} T_n^{m+1} \\ &= P_n^0 V + \sum_{m=0}^{\infty} P_n^0 V T_n^m D_n Q_n^0 V. \end{aligned} \tag{4.19}$$

Since S is a 2-dimensional operator defined on E_n^0, it can be considered as a 2x2 matrix with entries;

$$S^{11} = (Se_{-n}, e_{-n}), \qquad S^{12} = (Se_n, e_{-n}), \tag{4.20}$$

$$S^{21} = (Se_{-n}, e_n), \qquad S^{22} = (Se_n, e_n). \tag{4.21}$$

Then, $f_1 \neq 0$ together with (4.17) implies;

$$\begin{vmatrix} z - S^{11} & S^{12} \\ S^{21} & z - S^{22} \end{vmatrix} = 0. \tag{4.22}$$

Lemma 8. *Entries of the operator S can be represented as follows:*

$$S^{ij}(n,z) = \sum_{k=0}^{\infty} S_k^{ij}(n,z), \quad i,j = 1,2,$$

where

$$S_0^{11} = S_0^{22} = 0, \quad S_0^{12} = V(-2n), \quad S_0^{21} = V(2n),$$

and for each $k = 1, 2, ...,$

$$S_k^{11}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}, \tag{4.23}$$

$$S_k^{22}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k-n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}, \tag{4.24}$$

$$S_k^{12}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k-n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}, \tag{4.25}$$

$$S_k^{21}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}. \tag{4.26}$$

Proof. By (4.20), (4.21), we have

$$S^{ij} = \sum_{k=0}^{\infty} S_k^{ij}, \quad i,j = 1,2, \tag{4.27}$$

where

$$S_k^{11} = (P_n^0 V T_n^k e_{-n}, e_{-n}), \qquad S_k^{12} = (P_n^0 V T_n^k e_n, e_{-n}), \tag{4.28}$$

$$S_k^{21} = (P_n^0 V T_n^k e_{-n}, e_n), \qquad S_k^{22} = (P_n^0 V T_n^k e_n, e_n). \tag{4.29}$$

Accordingly, we have

$$S_0^{11} = (P_n^0 V e_{-n}, e_{-n}), \qquad S_0^{12} = (P_n^0 V e_n, e_{-n}), \tag{4.30}$$

$$S_0^{21} = (P_n^0 V e_{-n}, e_n), \qquad S_0^{22} = (P_n^0 V e_n, e_n). \tag{4.31}$$

In light of (4.3) and (4.11), we calculate $P_n^0 V e_{-n}$ and $P_n^0 V e_n$, respectively, as follows:

$$e_{-n} \xrightarrow{V} \sum_{j_k} V(j_k+n)e_{j_k} \qquad \xrightarrow{P_n^0} V(0)e_{-n} + V(2n)e_n = V(2n)e_n,$$
$$e_n \xrightarrow{V} \sum_{j_k} V(j_k-n)e_{j_k} \qquad \xrightarrow{P_n^0} V(-2n)e_{-n} + V(0)e_n = V(-2n)e_{-n}.$$

Then by (4.30) and (4.31), we get

$$S_0^{11} = S_0^{22} = 0, \qquad S_0^{12} = V(-2n), \qquad S_0^{21} = V(2n). \tag{4.32}$$

Now, by making use of (4.6), (4.11) and (4.12) in view of (4.20) and (4.21), we look at how the operator

$$P_n^0 V T_n^k = P_n^0 V T_n^{k-1} D_n Q_n^0 V,$$

for $k \geq 1$, acts on an arbitrary base element e_m;

$$\begin{aligned}
e_m \xrightarrow{V} &\quad \sum_{j_k} V(j_k-m)e_{j_k} \\
\xrightarrow{Q_n^0} &\quad \sum_{j_k \neq \pm n} V(j_k-m)e_{j_k} \\
\xrightarrow{D_n} &\quad \sum_{j_k \neq \pm n} \frac{V(j_k-m)}{n^2-j_k^2+z} e_{j_k} \\
\xrightarrow{T_n^{k-1}} &\quad \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(j_1-j_2)\cdots V(j_k-m)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)} e_{j_1} \\
\xrightarrow{P_n^0 V} &\quad \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_k-n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)} e_n \\
&+ \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\cdots V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)} e_{-n}.
\end{aligned} \tag{4.33}$$

Accordingly, above calculations can be performed as replacing $m = n$ and $m = -n$.

Hence, by (4.33) and (4.29) it follows that

$$S_k^{22}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k-n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}.$$

for each $k = 1, 2, \dots$.

$S_k^{11}(n,z)$, $S_k^{12}(n,z)$, and $S_k^{21}(n,z)$ can be calculated in a similar way. $\square$

Lemma 9. *(i) For any (complex-valued) potential v,*

$$S^{11}(n,z) = S^{22}(n,z).$$

(ii) If v is a real-valued potential, then

$$S^{12}(n,z) = \overline{S^{21}(n,\overline{z})}.$$

Proof. *(i)* By (4.23) and (4.24), the change of summation indices

$$i_s = -j_{k+1-s}, \quad s = 1, ..., k,$$

explains that $S_k^{11}(n,z) = S_k^{22}(n,z)$. Thus, in view of (4.27) and (4.32), we get

$$S^{11}(n,z) = S^{22}(n,z).$$

(ii) If v is real-valued, we have for its Fourier coefficients $V(-m) = \overline{V(m)}$. By (4.32), it follows

$$S_0^{12}(n,z) = V(-2n) = \overline{V(2n)} = \overline{S_0^{21}(n,\overline{z})}.$$

By (4.25) and (4.26), for each $k = 1, 2, ...,$ the change of summation indices

$$i_s = -j_{k+1-s}, \quad s = 1, ..., k,$$

proves that $S^{12}(n,z) = \overline{S^{21}(n,\overline{z})}$. Hence, in view of (4.27), we get

$$S^{12}(n,z) = \overline{S^{21}(n,\overline{z})}.$$

□

Now, we consider the determinant (4.22). For convenience, let us denote

$$\alpha_n(z) := S^{11}(n,z) = S^{22}(n,z), \tag{4.34}$$

and

$$\beta_n^+(z) := S^{21}(n,z), \quad \beta_n^-(z) := S^{12}(n,z). \tag{4.35}$$

If z is real, then $\beta_n^-(z) = \overline{\beta_n^+(z)}$ and accordingly,

$$|\beta_n^-(z)| = |\beta_n^+(z)|. \tag{4.36}$$

We analyze the equation derived from (4.22):

$$(z - \alpha_n(z))^2 = \beta_n^-(z)\beta_n^+(z). \tag{4.37}$$

Hence, $\lambda = n^2 + z$ is an eigenvalue of L, with $|z| < 1$, if and only if z is a solution of (4.37).

Before we present estimates for the derivatives of $\alpha_n(z)$ and $\beta_n^{\pm}(z)$, let us recall a basic fact which gives an estimate for the derivative of a bounded analytic function.

Lemma 10. *Let U be an open set of the complex plane, $K \subset U$ be compact and $f : U \to \mathbb{C}$ be an analytic function. If*

$$\sup_{z\in U} |f(z)| \leq C < \infty,$$

then

$$\sup_{z\in K} |f'(z)| \leq \frac{C}{R},$$

where $R = dist(K, \partial U)$.

Proof. Let $z \in K$ and take a circle $\Gamma(t) = z + Re^{it}$, for $0 \leq t \leq 2\pi$. Then by *Cauchy integral formula*;

$$\sup_{z\in K} |f'(z)| = \sup_{z\in K} \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \leq \frac{1}{2\pi} \frac{C}{R^2} 2\pi R \leq \frac{C}{R}.$$

□

Lemma 11. *Let $\alpha_n(z)$ and $\beta_n^{\pm}(z)$ be defined as in* (4.34) *and* (4.35), *respectively. Then,*

(i)

$$\sup_{|z|\leq n/2} |\alpha_n(z)| \leq \frac{3}{n}\|v\|^2, \qquad \sup_{|z|\leq n/2} |\beta_n^{\pm}(z) - V(\pm 2n)| \leq \frac{3}{n}\|v\|^2,$$

(ii)

$$\sup_{|z|\leq n/2} \left|\frac{d}{dz}\alpha_n(z)\right| \leq \frac{6}{n^2}\|v\|^2, \qquad \sup_{|z|\leq n/2} \left|\frac{d}{dz}\beta_n^{\pm}(z)\right| \leq \frac{6}{n^2}\|v\|^2,$$

for $n \geq 4\|v\|$.

Proof. We prove both statements only for $\alpha_n(z)$ because the result related to $\beta_n^{\pm}(z)$ can be obtained by repeating the same argument.

(i) Recall from the definitions (4.27) and (4.23) that we have

$$\alpha_n(z) = S^{11}(n,z) = \sum_{k=1}^{\infty} S_k^{11}(n,z) \tag{4.38}$$
$$= \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}.$$

Let $|z| \le n$. Then, from (4.13) and the *Cauchy-Schwarz inequality*, it follows

$$\left| \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)} \right|^2 \le \Sigma_1 \Sigma_2, \tag{4.39}$$

where

$$\Sigma_1 = \sum_{j_1,\dots,j_k \neq \pm n} |V(-n-j_1)|^2 |V(j_1-j_2)|^2 \cdots |V(j_{k-1}-j_k)|^2,$$

and

$$\Sigma_2 = \sum_{j_1,\dots,j_k \neq \pm n} \frac{|V(j_k+n)|^2}{|n^2-j_1^2+z|^2 \cdots |n^2-j_k^2+z|^2}.$$

Observe that (4.9) implies

$$\Sigma_1 \le \sum_{j_1 \neq \pm n} |V(-n-j_1)|^2 \sum_{j_2 \neq \pm n} |V(j_1-j_2)|^2 \cdots \sum_{j_k \neq \pm n} |V(j_{k-1}-j_k)|^2 = \|v\|^{2k}. \tag{4.40}$$

On the other hand, by (4.9) and *Lemma 6*, we obtain

$$\Sigma_2 \le \left(\sum_{j_1 \neq \pm n} \frac{1}{|n^2-j_1^2+z|^2} \cdots \sum_{j_{k-1} \neq \pm n} \frac{1}{|n^2-j_{k-1}^2+z|^2} \right) \|v\|^2$$
$$= \left(\sum_{j \neq \pm n} \frac{1}{|n^2-j^2+z|^2} \right)^k \|v\|^2 \le \left(\frac{4}{n^2} \right)^k \|v\|^2. \tag{4.41}$$

All in all, using (4.40) and (4.41), which are the estimates for Σ_1 and Σ_2, respectively, the inequality (4.39) gives us

$$\left| \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)} \right|^2 \le \frac{4^k}{n^{2k}} \|v\|^{2k+2}. \tag{4.42}$$

Then, in view of (4.38), the inequality (4.42) leads to

$$|\alpha_n(z)|^2 \le \left(\sum_{k=1}^{\infty} \frac{4^k}{n^{2k}} \|v\|^{2k+2}\right) = \frac{4}{n^2}\|v\|^4 \frac{1}{1-\frac{4}{n^2}\|v\|^2} < \frac{8}{n^2}\|v\|^2,$$

for $n > 4\|v\|$.

Hence, we get

$$\sup_{|z|\le n/2} |\alpha_n(z)| \le \frac{3}{n}\|v\|^2. \tag{4.43}$$

(ii) In consideration of part *(i)* and *Lemma 10*, it follows from (4.43) that

$$\sup_{|z|\le n/2} \left|\frac{d}{dz}\alpha_n(z)\right| \le \frac{6}{n^2}\|v\|^2.$$

□

As a particular case, if we consider Mathieu potential, then we have a better estimate for $\alpha_n(z)$ which is expressed in the following lemma.

Lemma 12. *In the case of Mathieu potential, for fixed a,*

(i)

$$|\alpha_n(z_n^+)| \le C_1 \frac{a^2}{2n^2}, \quad C_1 \text{ constant.}$$

(ii)

$$|\beta_n^+(z_n^+)| \le C_2 \frac{a^n}{n^n}, \quad C_2 \text{ constant.}$$

Proof. *(i)* Recall the definition of $\alpha_n(z_n^+)$ expressed in *Lemma 11(i)*.

When $k = 1$, we have either

$$j_1 = n+2 \quad \text{or} \quad j_1 = -n-2.$$

Hence,

$$\sum_{j_1 \ne \pm n} \frac{V(-n-j_1)V(n+j_1)}{n^2 - j_1^2 + z_n^+} = \frac{a^2}{n^2-(n-2)^2+z_n^+} + \frac{a^2}{n^2-(n+2)^2+z_n^+}$$
$$= a^2\left(\frac{1}{4n-4+z_n^+} - \frac{1}{4n+4-z_n^+}\right)$$
$$= a^2 \frac{8-2z_n^+}{(4n)^2-(4-z_n^+)^2} \sim \frac{a^2}{2n^2}.$$

If we pick the first term out and repeat the same argument in *Lemma 11(i)*, we get

$$|\alpha_n(z_n^+)| \le C\frac{a^2}{2n^2}.$$

(ii) Recall from (4.26) and (4.27) that in the general case we have

$$\beta_n^+(z_n^+) = V(2n) + \sum_{k=1}^{\infty} \sum_{j_1,\ldots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}.$$

However, in the case of Mathieu potential

$$\beta_n^+(z_n^+) = \sum_{k=n}^{\infty} \sum_{j_1,\ldots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}.$$

because if we let

$$x_1 = n - j_1, \quad x_2 = j_2 - j_1, \quad x_3 = j_3 - j_2, \ldots, \quad x_n = j_{n-1} + n,$$

then there are no $x_1, x_2, \ldots, x_n \in \{-2, 2\}$ such that $x_1 + x_2 + \cdots + x_n = 0$.

When $k = n$, we have

$$\sum_{j_1,j_2,\ldots,j_n \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{n-1}-j_n)V(j_n+n)}{(n^2-j_1^2+z_n^+)\cdots(n^2-j_n^2+z_n^+)} \sim \frac{a^n}{n^n}.$$

Repeating the same argument in part *(i)* gives us

$$|\beta_n^+(z_n^+)| \le C\frac{a^n}{n^n}, \quad C \text{ constant}.$$

□

Theorem 13. *Let $\lambda_n^+ = n^2 + z_n^+$ and $\lambda_n^- = n^2 + z_n^-$. Then, for $n \ge n_0$,*

$$2|\beta_n^+(z_n^+)|)\left(1 - C\frac{\|v\|^2}{n^2}\right) \le \gamma_n \le 2|\beta_n^+(z_n^+)|\left(1 + C\frac{\|v\|^2}{n^2}\right), \tag{4.44}$$

with some absolute constant C.

Proof. Consider the function

$$\zeta(z) := z - \alpha_n(z). \tag{4.45}$$

Then, by (4.22), we have

$$\zeta(z_n^+)^2 = (z_n^+ - \alpha_n(z_n^+))^2 = \beta_n^-(z_n^+)\beta_n^+(z_n^+),$$
$$\zeta(z_n^-)^2 = (z_n^- - \alpha_n(z_n^-))^2 = \beta_n^-(z_n^-)\beta_n^-(z_n^-), \tag{4.46}$$

and

$$\gamma_n = \lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-. \tag{4.47}$$

First, we estimate γ_n from above.

By (4.46), we have

$$\zeta(z_n^+) - \zeta(z_n^-) = \int_{z_n^-}^{z_n^+} \left(1 - \frac{d}{dz}\alpha_n(z)\right) dz,$$

where by *Lemma 11(ii)*,

$$\left|\frac{d}{dz}\alpha_n(z)\right| \le \frac{C_1}{n^2}, \qquad |z| \le 1,$$

for $n \ge n_0$, and some constant $C_1 = 6\|v\|^2$.

Therefore,

$$|z_n^+ - z_n^-|\left(1 - \frac{C_1}{n^2}\right) \le |\zeta(z_n^+) - \zeta(z_n^-)| \le |z_n^+ - z_n^-|\left(1 + \frac{C_1}{n^2}\right). \tag{4.48}$$

Since $\lambda_n^\pm$, z_n^+ and z_n^- are real, by *Lemma 9(ii)*,

$$|\beta_n^-(z_n^\pm)| = |\beta_n^+(z_n^\pm)|,$$

which leads to

$$|\zeta(z_n^\pm)|^2 = |\beta_n^-(z_n^\pm)\beta_n^+(z_n^\pm)| = |\beta_n^+(z_n^\pm)|^2,$$

and accordingly

$$|\zeta(z_n^\pm)| = |\beta_n^+(z_n^\pm)|.$$

Therefore,

$$|\zeta(z_n^+) - \zeta(z_n^-)| \le |\zeta(z_n^+)| + |\zeta(z_n^-)| \le |\beta_n^+(z_n^+)| + |\beta_n^+(z_n^-)|$$
$$\le 2|\beta_n^+(z_n^+)| + \frac{C_1}{n^2}|z_n^+ - z_n^-|.$$

In view of (4.48), this implies

$$|z_n^+ - z_n^-| \left(1 - \frac{C_1}{n^2}\right) \le 2|\beta_n^+(z_n^+)| + \frac{C_1}{n^2}|z_n^+ - z_n^-|,$$

so

$$|z_n^+ - z_n^-| \left(1 - 2\frac{C_1}{n^2}\right) \le 2|\beta_n^+(z_n^+)|.$$

Hence,

$$\gamma_n = |z_n^+ - z_n^-| \le 2|\beta_n^+(z_n^+)| \left(1 + 4\frac{C_1}{n^2}\right), \quad n \ge n_1. \tag{4.49}$$

Now, we estimate γ_n from below.

By (4.46) and (4.47),

$$\zeta(z_n^+)^2 - \zeta(z_n^-)^2 = \int_{z_n^-}^{z_n^+} \frac{d}{dz}(\beta_n^-(z)\beta_n^+(z))\, dz.$$

Therefore,

$$|\zeta(z_n^+)^2 - \zeta(z_n^-)^2| \le |z_n^+ - z_n^-| \sup_{[z_n^-, z_n^+]} \left| \beta_n^+(z)\frac{d}{dz}\beta_n^-(z) + \beta_n^-(z)\frac{d}{dz}\beta_n^+(z) \right|. \tag{4.50}$$

By *Lemma 11(ii)*,

$$\sup_{[z_n^-, z_n^+]} \left| \frac{d}{dz}\beta_n^\pm(z) \right| \le \frac{C_1}{n^2}, \quad n \ge n_2.$$

From here it follows, for $z \in [z_n^-, z_n^+]$,

$$|\beta_n^\pm(z)| \le |\beta_n^\pm(z_n^+)| + |\beta_n^\pm(z_n^+) - \beta_n^\pm(z)|$$
$$\le |\beta_n^\pm(z_n^+)| + |z_n^+ - z|\frac{C_1}{n^2}$$
$$\le |\beta_n^\pm(z_n^+) + |z_n^+ - z_n^-|\frac{C_1}{n^2}.$$

By (4.48) and (4.50), it follows

$$|z_n^+ - z_n^-| \left(1 - \frac{C_1}{n^2}\right) |\zeta(z_n^+) + \zeta(z_n^-)| \leq |z_n^+ - z_n^-| \frac{C_1}{n^2} \left(|\beta_n^+(z_n^+)| + |\beta_n^-(z_n^+)| + |z_n^+ - z_n^-| \frac{C_1}{n^2}\right),$$

which implies

$$|\zeta(z_n^+) + \zeta(z_n^-)| \leq 2|\beta_n^+(z_n^+)| \frac{6C_1}{n^2}, \quad n \geq n_3.$$

Thus, from (4.48) we have

$$\begin{aligned} |z_n^+ - z_n^-| \left(1 + \frac{C_1}{n^2}\right) \geq |\zeta(z_n^+) - \zeta(z_n^-)| &\geq 2|\zeta(z_n^+)| - |\zeta(z_n^+) + \zeta(z_n^-)| \\ &\geq 2|\beta_n^+(z_n^+)| \left(1 - \frac{6C_1}{n^2}\right). \end{aligned}$$

Hence,

$$|\gamma_n| = |z_n^+ - z_n^-| \geq 2|\beta_n^+(z_n^+)| \left(1 - \frac{12C_1}{n^2}\right), \quad n \geq n_4. \tag{4.51}$$

Combining the inequalities (4.49) and (4.51), we obtain

$$2|\beta_n^+(z_n^+)|) \left(1 - C\frac{\|v\|^2}{n^2}\right) \leq \gamma_n \leq 2|\beta_n^+(z_n^+)| \left(1 + C\frac{\|v\|^2}{n^2}\right),$$

for some constant $C = 12$.

□

5 Asymptotics of Spectral Gaps in case of the Mathieu Potential

In this section, making use of the general asymptotic formula (4.44), we analyze the asymptotic behaviour of the spectral gaps of the Schrödinger operator L which is defined in (3.1), with the *Mathieu potential* $v(x) = 2a\cos(2x)$, where a is a real constant. In other words, we deal with the operator

$$L(y) = -y'' + 2a\cos(2x)y. \tag{5.1}$$

First, we find the asymptotics of $\gamma_n = \gamma_n(a)$ as $a \to 0$ (n fixed). Then we give the asymptotics of the spectral gaps of the Mathieu potential as $n \to \infty$ (a fixed).

On the way of finding both of the asymptotics, we use the results obtained in the previous section.

In order to apply the formula (4.44), we deal with $\beta_n^+(z)$. Recall that, in the general case, by (4.26) , (4.27) and (4.35), we have

$$\beta_n^+(z) = V(2n) + \sum_{k=1}^{\infty} \sigma_k(n,z), \tag{5.2}$$

where

$$\sigma_k(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n-j_1)V(j_1-j_2)\cdots V(j_{k-1}-j_k)V(j_k+n)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}. \tag{5.3}$$

Observe that each nonzero term in (5.3) correspond to a k-tuple of indices $(j_1,...,j_k)$ such that

$$(n+j_1)+(j_2-j_1)+\cdots+(j_k-j_{k-1})+(n-j_k) = 2n. \tag{5.4}$$

If we write the Mathieu potential in terms of Fourier coefficients with respect to basis elements, we have

$$v(x) = 2a\cos 2x = ae^{-2ix} + ae^{2ix},$$

which implies that

$$V(m) = \begin{cases} a & \text{if } m = \pm 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Therefore, we have a non-trivial term in (5.3) if and only if

$$(n+j_1), (j_2-j_1), \dots, (j_k-j_{k-1}), (n-j_k) \in \{\pm 2\}. \tag{5.6}$$

Consider all possible walks from $-n$ to n. Each such walk is determined by the sequence of its steps

$$x = (x_1, ..., x_{\nu+1}),$$

or by its vertices

$$j_k = -n + \sum_{i=1}^{k} x_i, \quad k = 1, ..., \nu. \tag{5.7}$$

Furthermore, if we know the vertices $j_1,...,j_\nu$ then the corresponding steps are given by the formula

$$x_1 = n + j_1; \quad x_i = j_i - j_{i-1}, \quad i = 2,...,\nu; \quad x_{\nu+1} = n - j_\nu.$$

Let X_n denotes the set of all walks from $-n$ to n that have no vertices $\pm n$, and no zero steps. Accordingly, let $X_n(p)$ denotes the set of all walks from $-n$ to n with p negative steps.

Consequently, in consideration of (5.4) and (5.6), there is one-to-one correspondence between the nonzero terms of $\sigma_\nu(n,z)$ and the walks $x = (x(t))_{t=1}^{\nu+1} \in X_n$.
Notice that, by (5.4),

$$\sum_{t=1}^{\nu+1} x(t) = 2n.$$

Therefore, it follows that

$$\beta_n^+(z) = \sum_{x\in X_n} h(x,z), \tag{5.8}$$

where

$$h(x,z) = \frac{V(x_1)V(x_2)\cdots V(x_{\nu+1})}{(n^2-j_1^2+z)(n^2-j_2^2+z)\cdots(n^2-j_\nu^2+z)}.$$

Clearly, $X_n(0)$ has only one element which is the walk of minimal number of steps, say

$$\xi = \xi(t)_{t=1}^n, \quad \xi(t) = 2. \tag{5.9}$$

In other words, $X_n(0) = \{\xi\}$.

Particularly, notice that in the Mathieu potential case, by (5.5), we have

$$V(x_i) = a, \qquad i = 1, \cdots, \nu+1. \tag{5.10}$$

In this section, unless stated otherwise, $v(x)$ will be regarded as the Mathieu potential. In other words, all computations and results will be based on the case when the potential $v(x) = 2a\cos(2x)$ with nonzero real number a. According to this, it will be the case that

$$h(x,z) = \frac{a^{\nu+1}}{(n^2-j_1^2+z)(n^2-j_2^2+z)\cdots(n^2-j_\nu^2+z)}. \tag{5.11}$$

Lemma 14. *Let $h(x,z)$ and ξ be defined as in* (5.11) *and* (5.9)*, respectively. Then,*

$$h(\xi,0) = \frac{4(a/4)^n}{[(n-1)!]^2}.$$

Proof. Since $\xi(t) = 2$, and $V(\xi(t)) = V(2) = a$, for $t = 1, ..., n$, we have

$$h(\xi, 0) = \frac{a^n}{(n^2 - j_1^2)(n^2 - j_2^2) \cdots (n^2 - j_{n-1}^2)}. \tag{5.12}$$

From (5.7), it follows that

$$j_k^2 = (-n + 2k)^2 = n^2 - 4nk + 4k^2,$$

for $k = 1, 2, ..., n - 1$. Then,

$$\begin{aligned}\prod_{k=1}^{n-1}(n^2 - j_k^2) &= \prod_{k=1}^{n-1} 4k(n-k) \\ &= 4^{n-1} \prod_{k=1}^{n-1} k \left(\prod_{k=1}^{n-1}(n-k) \right) \\ &= 4^{n-1}[(n-1)!]^2.\end{aligned}$$

Therefore, we obtain

$$h(\xi, 0) = \frac{a^n}{4^{n-1}[(n-1)!]^2} = \frac{4(a/4)^n}{[(n-1)!]^2}.$$

□

Proposition 15. *Let $\beta_n^+(z)$, $h(x, z)$ and ξ be defined as in* (5.8)*,* (5.11) *and* (5.9)*, respectively. Then,*

$$h(\xi, z) = h(\xi, 0)(1 + O(a)),$$

i.e. $h(\xi, 0)$ gives the main term of the asymptotics of $\beta_n^+(z)$ as $a \to 0$, for fixed n.

Proof. First, let us compute $\|v\|$.

Since we have $v(x) = 2a \cos 2x$, it follows that

$$\|v\| = 2|a| \left(\frac{1}{\pi} \int_0^\pi \cos^2 2x \, dx \right)^{1/2} = 2|a| \frac{1}{\sqrt{2}} = \sqrt{2}|a|.$$

Hence, for small a, we have $\|v\| \leq 1/4$. By *Proposition 4*, it follows that

$$|z_n^\pm| = |\lambda_n^\pm - n^2| \leq 4\|v\| \leq 4\sqrt{2}|a|, \quad n \in \mathbb{N}. \tag{5.13}$$

Now, we fix $n \in \mathbb{N}$, and work with the identity

$$\prod_{k=1}^{n-1} \frac{1}{n^2 - j_k^2 + z} = \prod_{k=1}^{n-1} \left(\frac{1}{n^2 - j_k^2} \frac{1}{1 + \frac{z}{n^2 - j_k^2}} \right). \tag{5.14}$$

By *Taylor's formula*, we have

$$\frac{1}{1 + \frac{z}{n^2 - j_k^2}} = 1 - \frac{z}{n^2 - j_k^2} + \frac{z^2}{(n^2 - j_k^2)^2} - \frac{z^3}{(n^2 - j_k^2)^3} + \cdots .$$

Hence, from (5.13) and (5.14), it follows that

$$\prod_{k=1}^{n-1} \frac{1}{n^2 - j_k^2 + z} = \prod_{k=1}^{n-1} \left(\frac{1}{n^2 - j_k^2} \right) (1 + O(a)), \quad z = z_n^{\pm}.$$

In particular, multiplying both sides with a^n, and in view of (5.11) and (5.12), we obtain

$$h(\xi, z) = h(\xi, 0)(1 + O(a)).$$

□

Now, we prove one of the main results in this section. We give the asymptotics of $\gamma_n = \gamma_n(a)$ as $a \to 0$ (n fixed).

Theorem 16. *Let γ_n, $n \in \mathbb{N}$, be the lengths of instability zones of the Hill operator*

$$Ly = -y'' + 2a\cos(2x)y, \quad a \in \mathbb{R}.$$

If n is fixed and $a \to 0$, then

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}(1 + O(a)).$$

Proof. Fix $n \in \mathbb{N}$, and take $x \in X_n(p)$. Then, by (5.10) and (5.11),

$$h(x, z) = \frac{a^{n+2p}}{(n^2 - j_1^2 + z) \cdots (n^2 - j_{n+2p-1}^2 + z)}. \tag{5.15}$$

Then, by (4.13) and (5.15),

$$|h(x, z)| \le \left| \frac{(2a)^{n+2p}}{(n^2 - j_1^2) \cdots (n^2 - j_{n+2p-1}^2)} \right|, \tag{5.16}$$

where $x \in X_n(p)$.

Therefore,

$$\sum_{x \in X_n \setminus \{\xi\}} |h(x,z)| = |h(\xi,0)| O(|a|^2), \tag{5.17}$$

where ξ is the walk defined in (5.9).

Then, in view of *Lemma 14* and *Proposition 15*, combining the results (5.8) and (5.17) leads to

$$\begin{aligned} |\beta_n^+(z)| = \left| \sum_{x \in X_n} h(x,z) \right| &= |h(\xi,0)|(1+O(a)) \\ &= \frac{4(a/4)^n}{[(n-1)!]^2}(1+O(a)). \end{aligned}$$

Hence, in consideration of *Theorem 13*, for fixed n, above equation yields

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}(1+O(a)), \quad \text{as } a \to 0.$$

□

Now, we analyze the asymptotic behaviour of the spectral gaps of the operator L with Mathieu potential $v(x)$, as it is defined in (5.1) when a is fixed and $n \to \infty$.

$$\sum_{x \in X_n} h(x,z) = \sum_{p=0}^{\infty} \sigma_p(n,z), \tag{5.18}$$

where

$$\sigma_p(n,z) = \sum_{x \in X_n(p)} h(n,z). \tag{5.19}$$

Lemma 17. *In the case of Mathieu potential, for large enough n,*

$$|z_n^{\pm}| \le \frac{C}{n^2}, \quad C \text{ constant.}$$

Proof. From the equalities (4.36) and (4.37) it follows that

$$|z_n^+ - \alpha_n(z_n^+)| = |\beta_n^+(z_n^+)|.$$

On the other hand, by *Lemma 12* we have

$$|\alpha_n(z_n^+)| \le \frac{C_1}{n^2} \quad \text{and} \quad |\beta_n^+(z_n^+)| \le \frac{C_2}{n}, \quad C_1, C_2 \textit{ constants}.$$

Hence,

$$|z_n^+| \le |z_n^+ - \alpha(z_n^+)| + |\alpha(z_n^+)| \le \frac{C}{n^2}, \quad C \textit{ constant}.$$

□

Lemma 18. *Let $h(x,z)$, ξ and $\sigma_p(x,z)$ be defined as in* (5.11)*,* (5.9) *and* (5.19)*, respectively. Then, as $n \to \infty$ (a fixed),*

$$h(\xi, z) = \sigma_0(n, z) = h(\xi, 0)\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

Proof. First, recall from *Lemma 14* that

$$h(\xi, 0) = \frac{a^n}{(n^2 - j_1^2)(n^2 - j_2^2)\cdots(n^2 - j_{n-1}^2)}.$$

Now, fix $a \in \mathbb{R}$.

As in *Proposition 15*, we use the identity (5.14), namely:

$$\prod_{k=1}^{n-1} \frac{1}{n^2 - j_k^2 + z} = \prod_{k=1}^{n-1}\left(\frac{1}{n^2 - j_k^2}\frac{1}{1 + \frac{z}{n^2 - j_k^2}}\right).$$

Since

$$\left|\frac{1}{1 + \frac{z}{n^2 - j_k^2}}\right| \le \frac{1}{1 - \frac{|z|}{n^2 - j_k^2}} \le 1 + 2\frac{|z|}{n^2 - j_k^2},$$

it follows that,

$$\prod_{0 \le |j_k| < n} \frac{1}{1 - \frac{|z|}{n^2 - j_k^2}} \le \prod_{0 \le |j_k| < n}\left(1 + 2\frac{|z|}{n^2 - j_k^2}\right) \le \exp\left(\sum_{0 \le |j_k| < n} \frac{2|z|}{n^2 - j_k^2}\right).$$

Hence, we have asymptotically

$$\sum_{j_k = -n + 2k} \frac{1}{n^2 - j_k^2} \sim \frac{\log n}{n}.$$

Therefore, by *Lemma 17* we get

$$\prod_{0\le|j_k|<n} \frac{1}{1-\frac{|z|}{n^2-j_k^2}} \sim 1 + C\frac{\log n}{n^3}, \quad C \text{ constant.}$$

As a consequence,

$$\prod_{k=1}^{n-1} \frac{1}{n^2-j_k^2+z} = \prod_{k=1}^{n-1} \left(\frac{1}{n^2-j_k^2}\right)\left(1+O\left(\frac{1}{n^2}\right)\right).$$

In particular, multiplying both sides with a^n, and in view of (5.11) and (5.12), we obtain

$$\sigma_0(n,z) = h(\xi,0)\left(1+O\left(\frac{1}{n^2}\right)\right).$$

□

Lemma 19. *For* $n \in \mathbb{N}$,

$$\sum_{k=2}^{n-1} \frac{1}{(k-1)k(n-k)(n+1-k)} \le \frac{1}{4n^2}.$$

Proof. Rewriting the quotients

$$\frac{1}{k(n-k)} = \frac{1}{n}\left(\frac{1}{k}+\frac{1}{n-k}\right),$$

and

$$\frac{1}{(k-1)(n+1-k)} = \frac{1}{n}\left(\frac{1}{k-1}+\frac{1}{n+1-k}\right),$$

leads to the following equality:

$$\sum_{k=2}^{n-1} \frac{1}{(k-1)k(n-k)(n+1-k)} = \frac{1}{n^2}\sum_{k=2}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right)\left(\frac{1}{k-1}+\frac{1}{n+1-k}\right). \tag{5.20}$$

We now investigate the series

$$\sum_{k=2}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right)\left(\frac{1}{k-1}+\frac{1}{n+1-k}\right). \tag{5.21}$$

Observe that the following two series telescope;

$$\sum_{k=2}^{n-1} \frac{1}{k(k-1)} = \sum_{k=2}^{n-1} \left(\frac{1}{k-1} - \frac{1}{k} \right) \tag{5.22}$$
$$= 1 - \frac{1}{n-1} \leq 1,$$

and

$$\sum_{k=2}^{n-1} \frac{1}{(n-k)(n+1-k)} = \sum_{k=2}^{n-1} \left(\frac{1}{n-k} - \frac{1}{n+1-k} \right) \tag{5.23}$$
$$= 1 - \frac{1}{n-1} \leq 1.$$

Now, we consider the series

$$\sum_{k=2}^{n-1} \frac{1}{k(n+1-k)} = \frac{1}{2(n-1)} + \cdots + \frac{1}{l(n+1-l)} + \cdots + \frac{1}{2(n-1)}, \quad l = \left\lfloor \frac{n+1}{2} \right\rfloor$$
$$\leq 2 \left(\frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots + \frac{1}{l(n+1-l)} \right)$$
$$\leq 2 \left(\frac{1}{2} \frac{l-1}{n+1-l} \right).$$

The term in the middle, $l \leq (n+1)/2$, leads to

$$\sum_{k=2}^{n-1} \frac{1}{k(n+1-k)} \leq \frac{\frac{n+1}{2} - 1}{n+1-\frac{n+1}{2}}$$
$$= \frac{n-1}{n+1} \leq 1. \tag{5.24}$$

Finally, we analyze the series

$$\sum_{k=2}^{n-1} \frac{1}{(k-1)(n-k)} = \frac{1}{n-2} + \cdots + \frac{1}{(l-1)(n-l)} + \cdots + \frac{1}{n-2}.$$

If n is even, say $n = 2m$, then

$$\begin{aligned}\sum_{k=2}^{n-1} \frac{1}{(k-1)(n-k)} &= 2\left(\frac{1}{n-2} + \frac{1}{2(n-3)} + \cdots + \frac{1}{(m-1)(n-m)}\right) \\ &= \frac{2}{n-2} + 2\left(\frac{1}{2}\frac{m-2}{n-m}\right) \\ &= 1 + \frac{2}{n-2} - \frac{4}{n} < 1.\end{aligned}$$

If n is odd, say $n = 2m+1$, then

$$\begin{aligned}\sum_{k=2}^{n-1} \frac{1}{(k-1)(n-k)} &= 2\left(\frac{1}{n-2} + \cdots + \frac{1}{(m-1)(n-m)}\right) + \frac{1}{m(n-(m+1))} \\ &= \frac{2}{n-2} + 2\left(\frac{1}{2}\frac{m-2}{n-m}\right) + \frac{1}{m(n-m-1)} \\ &= \frac{2}{n-2} + \frac{n-5}{n+1} + \frac{4}{(n-1)^2} < 1.\end{aligned}$$

Hence, in either cases we get

$$\sum_{k=2}^{n-1} \frac{1}{(k-1)(n-k)} < 1. \tag{5.25}$$

In view of (5.21), combining the inequalities (5.22), (5.23), (5.24) and (5.25) yields

$$\sum_{k=2}^{n-1} \left(\frac{1}{k} + \frac{1}{n-k}\right)\left(\frac{1}{k-1} + \frac{1}{n+1-k}\right) \leq 4.$$

From (5.20), we get

$$\sum_{k=2}^{n-1} \frac{1}{(k-1)k(n-k)(n+1-k)} \leq \frac{1}{4n^2}.$$

□

Lemma 20. *Let $\sigma_p(n,z)$ be defined as in* (5.19). *Then,*

$$\left|\frac{\sigma_p(n,z)}{\sigma_{p-1}(n,z)}\right| \leq \frac{a^2}{4n^2}, \tag{5.26}$$

Proof. First, consider the case $p = 1$.

Recall that

$$\sigma_0(n,z) = h(\xi, z),$$

where ξ is defined in (5.9). Since

$$\sigma_1(n,z) = \sum_{x \in X_n(1)} h(x,z) = \sum_{k=2}^{n-1} h(x_k, z),$$

where x_k denotes the walk with $k+1$'th step equal to -2.

Then,

$$x_k(t) = \begin{cases} 2 & \text{if } t \neq k \\ -2 & \text{if } t = k \end{cases}$$

for $1 \leq t \leq n+2$.

Now, we figure out the connection between vertices of ξ and x_k as follows:

$$j_\alpha(x_k) = j_\alpha(\xi), \quad \alpha = 1, 2, \ldots, k,$$

and

$$j_{k+1}(x_k) = j_{k-1}(\xi), \quad j_{k+2}(x_k) = j_k(\xi),$$

and

$$j_{\alpha+2}(x_k) = j_\alpha(\xi), \quad \alpha = k+1, \ldots, n-1.$$

Then, by (5.11), we have

$$\begin{aligned} h(x_k, z) &= \frac{a^{n+2}}{(n^2 - j_1(x_k)^2 + z) \cdots (n^2 - j_{n+1}(x_k)^2 + z)} \\ &= h(\xi, z) \frac{a^2}{(n^2 - j_{k-1}(\xi)^2 + z)(n^2 - j_k(\xi)^2 + z)}. \end{aligned} \tag{5.27}$$

Furthermore, from definition (5.7) of the vertices j_k's, we have

$$j_k(\xi) = -n + 2k, \quad k = 2, \ldots, n-1.$$

Since $x_k \in X_n(1)$ and $\xi \in X_n(0)$, from (5.27) it follows that

$$\sigma_1(n,z) = h(\xi, z) \sum_{k=2}^{n-1} \frac{a^2}{[n^2 - (-n+2k+z)^2][n^2 - (-n+2k-2+z)^2]}, \tag{5.28}$$

where $h(\xi, z) = \sigma_0(n, z)$.

Now, we estimate the quotient $|\sigma_1(n,z)/\sigma_0(n,z)|$.

Observe that,

$$\sum_{k=2}^{n-1} \frac{1}{[n^2-(-n+2k)^2][n^2-(-n+2k-2)^2]} = \frac{1}{16}\sum_{k=2}^{n-1} \frac{1}{(k-1)k(n-k)(n+1-k)},$$

since

$$n^2 - (-n+2k)^2 = 4k(n-k),$$

and

$$n^2 - (-n+2k-2)^2 = 4(k-1)(n+1-k).$$

Now, from *Lemma 19*, it follows that

$$\sum_{k=2}^{n-1} \frac{a^2}{[n^2-(-n+2k)^2][n^2-(-n+2k-2)^2]} \leq \frac{a^2}{4n^2}.$$

In consideration of (5.28), the equality (5.27) leads to

$$\left|\frac{\sigma_1(n,z)}{\sigma_0(n,z)}\right| \leq \frac{a^2}{4n^2}.$$

Next, we prove (5.26) for $p \geq 2$.

Consider a walk with p negative steps, say $x \in X_n(p)$. We wish to relate $x \in X_n(p)$ with $\tilde{x} \in X_n(p-1)$ and the vertex j_k where the first negative step of x is performed.

Define a map φ

$$\varphi : X_n(p) \longrightarrow X_n(p-1) \times I, \qquad I = \{-n+4, -n+6, \ldots, n-2\},$$

by $\varphi(x) = (\tilde{x}, j)$, where

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } 1 \leq t \leq k-1 \\ x(t+2) & \text{if } k \leq t \leq n+2p-2, \end{cases}$$

for $k = \min\{t : x(t) = 2,\ x(t+1) = -2\}$, and $j = -n + 2k$.

The map φ defined as above is clearly injective. Hence, any $x \in X_n(p)$ can be related with $\tilde{x} \in X_n(p-1)$ and the vertex j_k at which the first negative step of x is performed in a one-to-one manner.

Now, we analyze the relation between $\sigma_p(n,z)$ and $\sigma_{p-1}(n,z)$.

Since any walk from $-n$ to n with p negative steps can be related with a walk from $-n$ to n with $p-1$ negative steps, (in other words; a walk with one less negative step) and the vertex j_k where the first negative step of the former is taken, we have the following correspondence;

$$\begin{aligned} h(x,z) &= \frac{a^{n+2}}{(n^2 - j_1(x)^2 + z)\cdots(n^2 - j_{n+1}(x)^2 + z)} \\ &= h(\tilde{x},z)\frac{a^2}{(n^2 - j^2 + z)(n^2 - (j-2)^2 + z)}. \end{aligned} \tag{5.29}$$

Since the mapping φ is injective, from (5.29) it follows that

$$\begin{aligned} \sigma_p(n,z) &= \sum_{x \in X_n(p)} h(x,z) \\ &= \sum_{\tilde{x} \in X_n(p-1)} h(\tilde{x},z) \sum_{k=2}^{n-1} \frac{a^2}{[n^2 - (-n+2k)^2 + z][n^2 - (-n+2k-2)^2 + z]}. \end{aligned} \tag{5.30}$$

Now, recall that

$$\sum_{\tilde{x} \in X_n(p-1)} h(\tilde{x},z) = \sigma_{p-1}(n,z), \tag{5.31}$$

and estimate the quotient $|\sigma_p(n,z)/\sigma_{p-1}(n,z)|$.

Observe that we are now in the same situation which we consider the quotient $|\sigma_1(n,z)/\sigma_0(n,z)|$. Therefore, the same computations imply that

$$\left| \frac{\sigma_p(n,z)}{\sigma_{p-1}(n,z)} \right| \leq \frac{a^2}{4n^2}.$$

□

Theorem 21. *Let γ_n, $n \in \mathbb{N}$, be the spectral gaps of the Mathieu operator*

$$Ly = -y'' + 2a\cos(2x)y, \quad a \in \mathbb{R}.$$

If a is fixed and $n \to \infty$, then

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

Proof. First, recall the basic notions. From (5.8), we have

$$\beta_n^+(z) = \sum_{x \in X_n} h(x,z).$$

On the other hand, by (5.18) and (5.19), it follows that

$$\sum_{x \in X_n} h(x,z) = \sum_{p=0}^{\infty} \sum_{X_n(p)} h(x,z) = \sum_{p=0}^{\infty} \sigma_p(n,z).$$

Now, we analyze the result in *Lemma 20*. An inductive argument immediately leads to the following;

$$\left|\frac{\sigma_p(n,z)}{\sigma_0(n,z)}\right| \leq \left(\frac{a^2}{4n^2}\right)^p.$$

Therefore, it follows that

$$\left|\sum_{x \in X_n} h(x,z) - \sigma_0(n,z)\right| \leq \sum_{p=1}^{\infty} |\sigma_p(n,z)| \leq \sum_{p=1}^{\infty} \left[|\sigma_0(n,z)| \left(\frac{a^2}{4n^2}\right)^p\right] \tag{5.32}$$
$$\leq |\sigma_0(n,z)| \frac{a^2}{2n^2}, \quad n \geq n_0.$$

Hence,

$$\sum_{x \in X_n} h(x,z) = \sigma_0(n,z)\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

On the other hand, by *Lemma 18*;

$$h(\xi,z) = \sigma_0(n,z) = h(\xi,0)\left(1 + O\left(\frac{1}{n^2}\right)\right),$$

hence,

$$\sum_{x \in X_n} h(x,z) = h(\xi,0)\left(1 + O\left(\frac{1}{n^2}\right)\right). \tag{5.33}$$

In view of (5.8), the equality (5.33) shows that $h(\xi,0)$ is the main term of the asymptotics of $\beta_n^+(z)$, $z = z_n^+$, with respect to $n \to \infty$ (a fixed).

Therefore, in consideration of *Lemma 14* and *Lemma 18*, combining the results (5.8) and (5.33) leads to

$$\beta_n^+(z_n^+) = \frac{4(a/4)^n}{[(n-1)!]^2}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

Making use of *Theorem 13*, for fixed a, the above equation yields

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2}\left(1 + O\left(\frac{1}{n^2}\right)\right) \quad \text{as } n \to \infty.$$

□

References

[1] J. Avron and B. Simon, "The asymptotics of the gap in the Mathieu equation", *Annals of Physics* 134 (1981) 76-84.

[2] P. Djakov and B. Mityagin, "Asymptotics of instability zones of the Hill operator with a two term potential", *Journal of Functional Analysis* 242 (2007), 157-194.

[3] E. Harrell, "On the effect of the boundary conditions on the eigenvalues of ordinary differential equations ", *American Journal of Mathematics dedicated to P. Hartman*, John Hopkins Univ. Press, Baltimore, MD, 1981, pp. 139-150.

[4] D.M. Levy, J.B. Keller, "Instability intervals of Hill's equation", *Communications on Pure and Applied Mathematics* 16 (1963), 469-476.

[5] W. Magnus and S. Winkler, "Hill's Equation", *Interscience Tracts in Pure and Applied Mathematics* 20, New York, (1966).

[6] M. A. Naimark, "Linear Differential Operators. Part I: Elementary Theory of Linear Differential Operators", *Frederick Ungar Publishing Co.*, New York, (1967).