DIVIDEND OPTIMIZATION FOR A JUMP DIFFUSION MODEL

by

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Abstract

We consider a dividend optimization problem where the objective is to maximize the expected value of total dividends paid during the lifetime of a company. The capital process is assumed to be a jump-diffusion, and dividends are paid out continuously until the capital process hits a default barrier. At any time, the company may distribute dividends at full rate; however, this would bring the capital process closer to the ruin barrier. Hence, we need to find a strategy (from a given admissible set) that will resolve this trade-off optimally. Here, we show that the structure of the optimal policy depends on the parameters of the problem. We identify an optimal policy for different cases, and we show how to compute the value function of the problem.

SIÇRAMA DİFÜZYON MODELİ İÇİN KÂR PAYI EN İYİLEME

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Özet

Bu çalışmada bir şirketin ömrü boyunca ödediği toplam kâr paylarının beklenen değerinin maksimuma çıkarıldığı bir kâr payı eniyileme problemi incelenmektedir. Sermaye süreci sıçrama difüzyon olarak varsayılmakta ve kâr payları, sermaye süreci varsayılan bariyere değinceye kadar sürekli olarak dağıtılmaktadır. Herhangi bir zamanda şirket kâr paylarını maksimum oranda dağıtabilir; ancak bu, sermaye sürecini iflas bariyerine daha çok yakınlaştıracaktır. Bu yüzden, verilen bir kabul edilebilir kümeden öyle bir strateji seçilmelidir ki bu ödünleşim en iyi şekilde çözümlenebilsin. Bu çalışmada en iyi strateji yapısının problemin parametrelerine bağlı olduğu bulunmuştur. Farklı durumlar için en iyi strateji tanımlanmış ve problemin değer fonksiyonunun nasıl hesaplanacağı gösterilmiştir.

to universe...

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Introduction

Maximization of the expected cumulative discounted dividend pay-outs is a well-known problem that has been studied over decades. High dividend is desirable to increase the present value of the dividends. However, it also increases the likelihood of hitting the ruin barrier early. Hence, it is required to find a dividend payment strategy that will resolve this conflict optimally. For the compound Poisson model, this problem was solved by Gerber (1969), and the optimal policy is found as "barrier strategy", i.e. whenever the surplus exceeds some barrier level b, all the excess income is paid out as dividends and no dividends are paid out below that surplus level.

More recently, stochastic optimal control techniques have been used for the same problem. One of the important studies belongs to Gerber and Shiu (2004). They find explicit expressions for the expectation and the moment-generating function of sum of the discounted dividends for a barrier type of dividend policy. Another important study is done by Taksar (2000). In that paper, the maximization of dividend flow is considered with different types of conditions imposed upon a company and different types of re-insurances. In most of these cases, the optimal policy is found as a barrier strategy. Azcue and Miller (2005) also consider the dividend maximization problem considering both the re-insurance policy and the dividend distribution strategy. They proved the existence of a band strategy, which will be explained below.

The most related study with this thesis is carried out by Jeanblanc-Picqué and Shiryaev (1995). The objective of that article is to maximize the total discounted dividends from time zero to the bankruptcy time τ. Cash reserve is taken as the difference between the cumulative net earnings and the cumulative dividends. Let X_t denote the cash reserve at time t, and R_t denote the accumulated net revenues up through time t. For R_t the model which was used by Radner and Shepp (1996) is employed. According to this, R_t evolves as an arithmetic Brownian Motion. In other words,

$$dR_t = \begin{cases} \mu dt + \sigma dW_t & 0 < t < \tau \\ 0 & t > \tau \end{cases}$$

where μ is expected rate of net revenue; σ is volatility of these revenues; and W_t is standard Brownian motion. In that paper, impulse control theory is applied in order to obtain an optimal dividend policy. The problem is solved for three cases of dividend policies. First, the dividend is paid at a constant rate and this rate is taken as a bounded measurable function of capital and the optimal policy is found as a barrier strategy. In the second case, it is assumed that there is a fixed cost for paying dividend and the dividend process is taken as a multivariate point process, (T_i, ξ_i). Here T_i's are random times of payments of dividends, and ξ_i's are the non-negative amounts of dividends paid at time T_i's. In this case the optimal policy is not a barrier, but a band strategy. In other words, if the initial surplus exceeds some level b, $b - a$ amount of dividends is paid out where a is some other level to be determined. The strategy continues as the amount of initial surplus being a. Otherwise, the payment of the dividend is made whenever the capital process reaches level b, and there is no payment under this level. In the third case, the dividend policy is taken as an arbitrary non-negative, non-decreasing, non-anticipating process (whose paths are right-continuous with left-limits).

In addition to Jeanblanc-Picqué and Shiryaev (1995), Boguslavskaya (2003) solves the same problem for the same three cases, with the usage of some liquidation value, which is a salvage value of the firm's assets at the time of bankruptcy.

Paulsen and Gjessing (2006) find a solution of dividend optimization problem for two different cases. First the return on investments is constant and the surplus generating process is compound Poisson with exponentially distributed claims; secondly return on investments and the surplus generating process are Brownian motions with drift. Alvarez and Virtanen (2006) considers the same problem in the presence of cash flow uncertainty and transaction costs. Thonhauser and Albrecher (2006), solves the dividend optimization problem by introducing a value function not only considering the expected dividends but also the time value of ruin. Yin, Song, Yang (2009) finds an optimal barrier policy when the dividends are paid to the share holders according to a barrier strategy for more general surplus processes.

This thesis considers the dividend optimization problem in the case similar to the first case considered by Jeanblanc-Picqué and Shiryaev (1995). The main difference here is that the capital process is taken as a jump diffusion process. In other words, capital process is assumed to be a Brownian motion that jumps according to Poisson process. This way, the existed problem is extended to the case where firm's revenue may exhibit jumps, due to the financial crisis, etc.

We expect to have a barrier type of optimal strategy and with this intuition in mind the problem is modelled as a stochastic control problem. Dynamic programming approach is employed to solve it. We construct the value function and the optimal policy, up to proving the validity of some assumptions.

- In Chapter 1, general information about Brownian motion and stochastic calculus is provided as preliminaries.

- In Chapter 2, the problem and the required notation is introduced.

- In Chapter 3, a special case of the problem is considered. The optimal dividend policy is found as to pay at a constant rate.

- In Chapter 4, the problem is considered for a rather general case, and both the value function and the optimal policy are derived.

- In Chapter 5, one can find the conclusion and the possible future study of this thesis.

- In Chapter 6, additional approaches to prove some of the statements are provided.

Chapter 1

Preliminaries

1.1 Brownian Motion

Definition 1.1. A stochastic process $W = \{W_t\}_{t\in[0,\infty)}$ is called *Wiener Process* or *Brownian Motion* if the following conditions are satisfied:

- It starts at 0: $W_0 = 0$
- It has stationary independent increments.

  Having stationary increments mean $W_t - W_s$ and $W_{t+h} - W_{s+h}$ has the same distribution for all $t, s, h \geq 0$. Having independent increments mean for every choice of $t_i \geq 0$ with $t_1 < t_2 < ... < t_n$ and $n \geq 1$; $W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}$ are independent random variables.
- For every $t > 0$,W_t has normal distribution, with mean zero and variance t; i.e., $W_t \sim N(0, t)$.
- It has continuous sample paths.

Using the definition and the properties of the conditional expectation under natural filtration of Brownian motion, it is easy to see that Brownian motion is a martingale.

Brownian sample paths do not have bounded variation on any finite interval $[0, T]$. This means that

$$\sup_{\pi} \sum_{i=1}^{n} |W_{t_i}(w) - W_{t_{i-1}}(w)| = \infty,$$

where the supremum is taken over all possible partitions $\pi : 0 = t_0 < ... < t_n = T$ of $[0, T]$. On the other hand, quadratic variation of the Brownian Motion on interval $[0, T]$ is T; i.e.

$$\lim_{\|\pi\|\to 0} \sum_{i=1}^{n} |W_{t_i}(w) - W_{t_{i-1}}(w)|^2 = T$$

where $\|\pi\|$ is the mesh of partition π: $\|\pi\| = \max_i \|t_{i+1} - t_i\|$, and the convergence is in L^2-sense.

Sample paths of Brownian motion are nowhere differentiable. This can be proved by using the self-similarity property of Brownian motion. Unbounded variation and non-differentiability of Brownian sample paths are major reasons for the failure of classical integration methods.

1.2 Stochastic Calculus

Integrals with respect to Brownian motion are called Itô stochastic integrals and denoted as

$$\int_0^T f(t)dW_t.$$

We will make sense out of this integral for more general integrands later. First, consider the case where the integrand is a simple process.

Definition 1.2. An adapted process which is constant over a given partition $\pi : 0 = t_0 < t_1 < ... < t_n = T$ of $[0, T]$ is called a *simple process*.

Definition 1.3. Let Δ be a simple process, then stochastic integral of Δ with respect to Brownian motion is defined as

$$I(T) = \int_0^T \Delta_t dW_t := \sum_{i=1}^{n} \Delta_{t_i}(W_{t_{i+1}} - W_{t_i}).$$

Proposition 1.4. $\{I(t)\}_{t\geq 0}$ *is an adapted stochastic process, and it is a martingale.*

Proof. Since both Δ_t and W_t are adapted, $I(t)$ is also adapted. Now, check martingale property. Let $s < t$,

$$\mathbb{E}_s[I(t) - I(s)] = \mathbb{E}_s \sum_{s\leq t_i, t_{i+1}\leq t} \Delta_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) = \sum_{s\leq t_i, t_{i+1}\leq t} \mathbb{E}_s\Delta_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right).$$

Use the tower property of conditional expectation noting that $t_i \geq s$

$$\mathbb{E}_s[I(t) - I(s)] = \sum_{s\leq t_i, t_{i+1}\leq t} \mathbb{E}_s\mathbb{E}_{t_i}\left[\Delta_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)\right] = \sum_{s\leq t_i, t_{i+1}\leq t} \mathbb{E}_s\Delta_{t_i}\mathbb{E}_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right).$$

Note that $\mathbb{E}_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) = 0$. So, it is shown that $\mathbb{E}_s I(t) = I(s)$. □

Since $I(t)$ is a martingale and $I(0)$ is 0; we can conclude that $\mathbb{E}I(t) = 0$ for all $t \geq 0$.

Using the definition of stochastic integral, $I(t)$, for simple processes and following the similar tricks as it is done in the previous proof, one can easily show the following property of $I(t)$, so called *Itô isometry*

$$\mathbb{E}I^2(t) = \sum_i \Delta_{t_i}^2 (t_{i+1} - t_i) = \int_0^t \Delta_s^2 ds.$$

Now, consider the stochastic integration for general integrands. The approach to define the integral is to approximate the integrand Δ_t with simple processes $\{\Delta_n(t)\}_{n\in\,\mathbb{N}}$. Note that this approximation (in L^2 space) is possible if $\mathbb{E}\int_0^T \Delta_s^2 ds < \infty$. So, we have a sequence of simple processes $\{\Delta_n(t)\}_n$ such that

$$\lim_{n\to\infty} \mathbb{E} \int_0^T (\Delta_n(t) - \Delta(t))^2 \, dt = 0.$$

For each n the stochastic integral $I_n(T)$, is well defined since integrand is simple process, so we define $I(T)$ as the limit (in L^2) of $I_n(T)$'s as n tends to infinity, i.e.

$$I(T) = \int_0^T \Delta(s) dW_s := \lim_{n\to\infty} \int_0^T \Delta_n(s) dW_s.$$

Using this definition one can prove the following properties of Itô stochastic integral:

- $\{I(t)\}_{t\geq 0}$ is a continuous martingale.
- Linearity of $I(t)$:
$$\int_0^T (\Delta_t + Q_t) \, dW_t = \int_0^T \Delta_t dW_t + \int_0^T Q_t dW_t.$$
- Itô isometry:
$$\mathbb{E}I^2(t) = \mathbb{E} \int_0^T \Delta_t^2 dt.$$
- Quadratic variation of $I(T)$ is $\int_0^T \Delta_t^2 dt$.

Now, we will introduce Itô Rule as a tool of calculating stochastic integrals and proceeding with some operations on them. Assume that f is twice continuously differentiable function and write $\Delta W_t = W_{t+\Delta t} - W_t$ for the increment of W on $[t, t + \Delta t]$. The Taylor expansion gives:

$$f(W_t + \Delta W_t) - f(W_t) = f'(W_t)\Delta W_t + \frac{1}{2} f''(W_t)(\Delta W_t)^2 + ...$$

Note that if we take a differentiable function instead of W_t, the first term of this Taylor expansion gives the classical chain rule of differentiation. However for Brownian motion, we know that the quadratic variation is not negligible. Moreover, using the definition of the stochastic integration one can interpret $(dW_t)^2$ as dt.

In the light of this discussion, we can state the Itô formula for a twice continuously differentiable function f as

$$f(W_t) - f(W_s) = \int_s^t f'(W_u) dW_u + \frac{1}{2} \int_s^t f''(W_u) du,$$

for $s < t$. In differential form it can be written as

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

Definition 1.5. Let $\{\Delta_t\}_{t\geq 0}$ and $\{Q_t\}_{t\geq 0}$ be two adapted stochastic processes. A process X is called an *Itô process* if it has the following form:

$$X_t = X_0 + \int_0^t \Delta_s dW_s + \int_0^t Q_s ds,$$

for all $t \geq 0$. Note that in differential form, this can be written as $dX_t = \Delta_t dW_t + Q_t dt$ with given X_0.

Remark 1.6. Quadratic variation of X_t is $\int_0^t \Delta_s^2 ds$, in differential notation it is to write $(dX_t)^2 = \Delta_t^2 dt$.

For an Itô process X_t Itô formula gives

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2.$$

It can be written in terms of dW_t and dt as:

$$df(X_t) = \left[Q_t f'(X_t) + \frac{1}{2}\Delta_t^2 f''(X_t)\right] dt + \Delta_t f'(X_t)dW_t.$$

Finally, for a two dimensional function $f(t, x)$ whose partial derivatives f_x, f_t, and f_{xx} are continuous, the Itô formula is extended as

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2,$$

where X_t is an Itô process. Again, one can rewrite it such that it depends on dt and dW_t.

For these and other results, the reader may refer to Shreve (2004), Mikosch (2008) and Steele (2001).

Chapter 2

Problem Statement

In this chapter, the notation is introduced and the problem is stated briefly. First of all, the revenue process is denoted by $R \equiv \{R_t\}_{t\geq 0}$. Different from the model introduced by Jeanblanc-Picqué and Shiryaev (1995), jumps are added to this process, and R follows the following dynamics:

$$dR_t = \mu dt + \sigma dW_t - \alpha dN_t. \tag{2.1}$$

Here, μ, σ and W_t are used in the same way that was introduced in the Introduction above. In other words, μ is expected rate of net revenue; σ is the volatility of these revenues; and W_t is standard Brownian motion. Moreover, N is an independent simple Poisson process with rate Λ, and α is the loss occurring because of unfavorable events. In the case of an insurance company these undesirable events usually represent the claims the company is supposed to cover.

Throughout this thesis, it is assumed that the dividend is paid continuously at a rate of X_t and it is $u(X_t)$, where u is a measurable function bounded above by some $K > 0$. This upper bound can be considered as the maximum rate of distributing dividends the company can afford. According to this, company's capital process $X \equiv \{X_t\}_{t\geq 0}$, which was stated as the difference between the net revenue and the total dividends, now satisfies

$$dX_t = (\mu - u(X_t))dt + \sigma dW_t - \alpha dN_t, \quad \text{with } X_0 = x, \tag{2.2}$$

provided that the equation (2.2) admits a unique solution.

The objective of the company is to select the dividend process $u(\cdot)$ in order to maximize the present value of all the dividend payments over the lifetime of the company. Here, we assume that the company goes bankrupt at the first time the capital reserves hit the level zero. That is, we let τ denote this bankruptcy time, and we define it as

$$\tau := \inf\{t \geq 0 : X_t \leq 0\}. \tag{2.3}$$

The interest rate is denoted by λ, and for a fixed dividend policy $u(\cdot)$ expected total discounted dividends can be calculated as $\mathbb{E}^x \int_0^\tau e^{-\lambda t} u(X_t)dt$. Since we need to maximize the total discounted dividends, the objective is to compute

$$V(x) = \sup_{u \in \mathcal{D}} \mathbb{E}^x \int_0^\tau e^{-\lambda t} u(X_t) dt, \tag{2.4}$$

where $\mathcal{D}$ is the collection of all such *admissible* $u(\cdot)$'s, which use only the information generated by observations from X, and which are bounded in $[0, K]$.

Note that we expect to have a barrier type of solution. Also, we expect that the "barrier" b is determined according to the value function V in the following way: $u^*(x) = K$ if $V'(x) \leq 1$ ($x > b$), and $u^*(x) = 0$ if $V'(x) > 1$ ($x < b$). In the light of this, we first consider a very special case in which the optimal policy is to pay the maximum rate of dividend for all x. Therefore, we expect to have $V'(x) < 1$ for all x. Later, we consider a rather general case, in which, $V'(x)$ is allowed to be greater than 1.

Chapter 3

A Special Case

In this section, it is shown that the optimal dividend policy is to pay at the constant rate K under some assumptions.

Let us define the function

$$f_K(x) = \mathbb{E}^x \int_0^\tau e^{-\lambda t} K dt, \tag{3.1}$$

where $\tau := \inf\{t \geq 0 : X_t \leq 0\}$ and $dX_t = (\mu - K)dt + \sigma dW_t - \alpha dN_t$. Clearly, the function $f_K(\cdot)$ is non-decreasing. Also, if $x = 0$ then $\tau = 0$ by definition, so $f_K(0) = 0$. On the other hand, as x tends to infinity, τ also tends to infinity.

$$f_K(\infty) = \int_0^\infty e^{-\lambda t} K dt = K/\lambda$$

by monotone convergence theorem. Below we show that this function is the value function of our problem under the condition $f'(\cdot) \leq 1$, on $\mathbb{R}_+$.

Remark 3.1. On $(0, \infty)$, f_K satisfies

$$-\lambda f_K(x) + (\mu - K) f_K'(x) + \frac{1}{2}\sigma^2 f_K''(x) + \Lambda\Big(f_K(x-\alpha) - f_K(x)\Big) + K = 0. \tag{3.2}$$

To prove this remark, one may follow the steps provided in the next chapter in order to calculate the value function for general case. The only difference is that here the dividend policy u is taken as K, while in the next chapter we consider the value function where u is taken as the optimal policy. Further explanation is provided in the Appendix.

Lemma 3.2. *If $f_K'(x) \leq 1$ for all $x \geq 0$, then $u^*(x) = K$ and the value function coincides with the function $f_K(\cdot)$.*

Proof. First note that if $f_K'(x) \leq 1$, then for any admissible strategy $u(\cdot) \in [0, K]$ we have

$$-\lambda f_K(x) + \mu f_K'(x) + \frac{1}{2}\sigma^2 f_K''(x) + \Lambda\Big(f_K(x-\alpha) - f_K(x)\Big) + u(x)(1 - f_K'(x)) \leq 0. \tag{3.3}$$

One can see this by rewriting (3.2) as

$$-\lambda f_K(x) + \mu f_K'(x) + \frac{1}{2}\sigma^2 f_K''(x) + \Lambda\Big(f_K(x-\alpha) - f_K(x)\Big) + K(1 - f_K'(x)) = 0.$$

$1 - f_K'(x) \geq 1$ and $0 \leq u(\cdot) \leq K$ imply the inequality (3.3).

Now, let $u(\cdot)$ be an admissible policy, and let $X^{(u)}$ denote the corresponding capital process. That is,

$$dX_t^{(u)} = (\mu - u(X_{t-}^{(u)}))dt + \sigma dW_t - \alpha dN_t.$$

Define $\tau_n := \inf\{t \geq 0 : X_t^{(u)} \geq n\}$. Let $\tau^{(u)}$ denotes the ruin time for this particular process.

Now, we write $d\big(e^{-\lambda t} f_K(X_t)\big)$. First, we consider the case without jump and write it using Itô rule:

$$\begin{aligned} d\big(e^{-\lambda t} f_K(X_t)\big) &= -\lambda e^{-\lambda t} f_K(X_t)dt + e^{-\lambda t}\Big(f_K' dX_t + \frac{1}{2} f_K''(dX_t)^2\Big) \\ &= -\lambda e^{-\lambda t} f_K dt + e^{-\lambda t}\Big[\Big(f_K'(\mu - u) + \frac{1}{2}\sigma^2 f_K''\Big)dt + \sigma f_K' dW_t\Big] \\ &= e^{-\lambda t}\Big[\Big(-\lambda f_K + (\mu - u) f_K' + \frac{1}{2}\sigma^2 f_K''\Big)dt + \sigma f_K' dW_t\Big] \end{aligned}$$

Secondly, it is written for a jump time, say T_n.

$$e^{-\lambda T_n} f_K(X_{T_n}) - e^{-\lambda T_n} f_K(X_{T_n-}) = e^{-\lambda T_n}\Big(f_K(X_{T_n-} - \alpha) - f_K(X_{T_n-})\Big).$$

Thus, in general

$$\begin{aligned} d\big(e^{-\lambda t} f_K(X_t)\big) &= e^{-\lambda t}\Big[-\lambda f_K(X_t) + (\mu - u) f_K'(X_t) + \frac{1}{2}\sigma^2 f_K''(X_t)\Big]dt \\ &\quad + e^{-\lambda t}\sigma f_K'(X_t)dW_t + e^{-\lambda t}\Big[f_K(X_t - \alpha) - f_K(X_t)\Big]dN_t. \end{aligned} \tag{3.4}$$

Then, we write dN_t as $d\big(N_t - \Lambda t\big) + d\big(\Lambda t\big)$ in order to use compensated Poisson process. Note that $\big[N_t - \Lambda t\big]_{t \geq 0}$ is a martingale. This can be seen by considering $\mathbb{E}_s\big[(N_{t+s}) - \Lambda(s+t)\big]$. By no memory property of Poisson process, it is known that

$$\mathbb{E}_s(N_{t+s}) = N_s + \Lambda t$$

Thus, it is now easy to see that

$$\mathbb{E}_s\big[(N_{t+s}) - \Lambda(s+t)\big] = N_s + \Lambda t - \Lambda s - \Lambda t = N_s - \Lambda s.$$

In order to use the properties of martingale processes we now rewrite (3.4) as:

$$
\begin{aligned}
d\big(e^{-\lambda t} f_K(X_t)\big) &= e^{-\lambda t}\Big[-\lambda f_K(X_t) + (\mu - u) f_K'(X_t) + \frac{1}{2}\sigma^2 f_K''(X_t)\Big]dt + e^{-\lambda t}\sigma f_K'(X_t) dW_t \\
&\quad + e^{-\lambda t}\Big[f_K(X_t - \alpha) - f_K(X_t)\Big] d\big[N_t - \Lambda t\big] \\
&\quad + \Lambda e^{-\lambda t}\Big[f_K(X_t - \alpha) - f_K(X_t)\Big] dt.
\end{aligned}
$$

As a result we get

$$
\begin{aligned}
& e^{-\lambda(t\wedge\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{t\wedge\tau^{(u)}\wedge\tau_n}) - f_K(x) = \\
& \qquad \int_0^{(t\wedge\tau^{(u)}\wedge\tau_n)} e^{-\lambda s}\Big(-\lambda f_K(X^{(u)}_{s-}) + (\mu-u) f_K'(X^{(u)}_{s-}) + \frac{1}{2}\sigma^2 f_K''(X^{(u)}_{s-}) \\
& \qquad\qquad + \Lambda\big(f_K(X^{(u)}_{s-} - \alpha) - f_K(X^{(u)}_{s-})\big)\Big) ds \\
& \qquad + \int_0^{t\wedge\tau^{(u)}\wedge\tau_n} e^{-\lambda s}\big(f_K(X^{(u)}_{s-} - \alpha) - f_K(X^{(u)}_{s-})\big)\big[dN_s - \Lambda ds\big] \\
& \qquad + \int_0^{t\wedge\tau^{(u)}\wedge\tau_n} e^{-\lambda s}\sigma f_K'(X^{(u)}_{s-}) dW_s.
\end{aligned}
$$

Note that $\{N_t - \Lambda t\}_{t\geq 0}$ is martingale and f_K is bounded. Thus, the integral with respect to $\{N_t - \Lambda t\}$ is also martingale. On the other hand, f_K' is assumed to be bounded. Then, the integral with respect to W_t is also a martingale. Therefore, the expectations of these two integrals will be zero, and taking the expectations we obtain

$$
\begin{aligned}
& \mathbb{E}^x e^{-\lambda(t\wedge\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{t\wedge\tau^{(u)}\wedge\tau_n}) - f_K(x) = \\
& \qquad \mathbb{E}^x \int_0^{(t\wedge\tau^{(u)}\wedge\tau_n)} e^{-\lambda s}\Big(-\lambda f_K(X^{(u)}_{s-}) + (\mu-u) f_K'(X^{(u)}_{s-}) + \frac{1}{2}\sigma^2 f_K''(X^{(u)}_{s-}) \\
& \qquad\qquad + \Lambda\big(f_K(X^{(u)}_{s-} - \alpha) - f_K(X^{(u)}_{s-})\big)\Big) ds \\
& \qquad \leq \mathbb{E}^x \int_0^{t\wedge\tau^{(u)}\wedge\tau_n} e^{-\lambda s}\big[-u(X^{(u)}_{s-})\big] ds.
\end{aligned}
\tag{3.5}
$$

Here the inequality follows thanks to (3.3).

Note that as $t \to \infty$ we have

$$
\begin{aligned}
e^{-\lambda(t\wedge\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{t\wedge\tau^{(u)}\wedge\tau_n}) &\to e^{-\lambda(\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{\tau^{(u)}\wedge\tau_n}) \\
\int_0^{(t\wedge\tau^{(u)}\wedge\tau_n)} e^{-\lambda s}\big[-u(X^{(u)}_{s-})\big] ds &\to \int_0^{(\tau^{(u)}\wedge\tau_n)} e^{-\lambda s}\big[-u(X^{(u)}_{s-})\big] ds.
\end{aligned}
$$

So applying the bounded convergence theorem, we obtain

$$\mathbb{E}^x e^{-\lambda(\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{\tau^{(u)}\wedge\tau_n}) - f_K(x) \le \mathbb{E}^x \int_0^{\tau^{(u)}\wedge\tau_n} e^{-\lambda s}\big[-u(X^{(u)}_{s-})\big]ds. \tag{3.6}$$

When $n\to\infty$, we have $\tau_n\to\infty$ since $X^{(u)}_t<\infty$, for $t<\infty$. Moreover, we have

$$e^{-\lambda(\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{\tau^{(u)}\wedge\tau_n}) = 1_{\{\tau^{(u)}<\infty\}} e^{-\lambda(\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{\tau^{(u)}\wedge\tau_n}) + 1_{\{\tau^{(u)}=\infty\}} e^{-\lambda(\tau_n)} f_K(X^{(u)}_{\tau_n}).$$

Hence if we let $n\to\infty$, we have

$$1_{\{\tau^{(u)}<\infty\}} e^{-\lambda(\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{\tau^{(u)}\wedge\tau_n}) \to 1_{\{\tau^{(u)}<\infty\}} e^{-\lambda(\tau^{(u)})} f_K(X^{(u)}_{\tau^{(u)}}) = 0$$

since $\tau^{(u)}$ is defined as the bankrupt time for the policy u and $f(0)=0$. Similarly,

$$1_{\{\tau^{(u)}=\infty\}} e^{-\lambda(\tau_n)} f_K(X^{(u)}_{\tau_n}) \to 0 \qquad \text{as } n\to\infty.$$

Note that we use the fact that f_K is bounded when we take this limit. Then by bounded convergence theorem, we get

$$\mathbb{E}^x e^{-\lambda(\tau^{(u)}\wedge\tau_n)} f_K(X^{(u)}_{\tau^{(u)}\wedge\tau_n}) \to 0, \qquad \text{as } n\to\infty.$$

Also note that the monotone convergence theorem gives

$$\mathbb{E}^x \int_0^{\tau^{(u)}\wedge\tau_n} e^{-\lambda s}\big[-u(X^{(u)}_{s-})\big]ds \to \mathbb{E}^x \int_0^{\tau^{(u)}} e^{-\lambda s}\big[-u(X^{(u)}_{s-})\big]ds, \qquad \text{as } n\to\infty.$$

Finally, letting $n\to\infty$ in (3.6) yields

$$f_K(x) \ge \mathbb{E}^x \int_0^{\tau^{(u)}} e^{-\lambda s} u(X^{(u)}_{s-})ds.$$

Since this inequality holds for any admissible strategy $u(\cdot)$, it follows that $u^*(x)=K$, and $f_K(\cdot)$ is the value function of the problem. $\square$

Chapter 4

A Rather General Case

In this chapter, we study the problem without the assumption that $f'_K(\cdot) \leq 1$ on $\mathbb{R}_+$, where $f_K(\cdot)$ is defined in (3.1).

4.1 Dynamic programming equation

Note that the first arrival time T_1 of the Poisson process N is a regeneration time for the problem in (2.4). In other words, assume a company starts with the optimum dividend policy and continue with it until $\tau \wedge T_1$. If bankrupt happens before T_1, the company stops; and if the bankrupt happens after T_1, it continues with optimal policy where X_0 is taken as the value of $X_{T_1} = X_{T_1-} - \alpha$. Then, it must be same as the company applies the optimal policy starting with $X_0 = x$ and continue until the end, i.e. τ. Hence, we expect the value function to satisfy

$$V(x) = \sup_{u(\cdot)\in[0,K]} \mathbb{E}^x \left[\int_0^{\tau\wedge T_1} e^{-\lambda t} u(X_t)dt + 1_{\{T_1 \leq \tau\}} e^{-\lambda T_1} V(X_{T_1-} - \alpha) \right].$$

Until the first arrival time, T_1, the process X coincides with a pure diffusion process Y satisfying

$$dY_t = (\mu - u(Y_t)) + \sigma dW_t, \quad \text{with } Y_0 = x. \tag{4.1}$$

Note that this process is independent from N. Moreover, on the event $\{\tau < T_1\}$, τ coincides with the ruin time $\widetilde{\tau}$ of the process Y. Thus, value function can be rewritten as:

$$V(x) = \sup_{u(\cdot)\in[0,K]} \mathbb{E}^x \left[\int_0^{\widetilde{\tau}\wedge T_1} e^{-\lambda t} u(Y_t)dt + 1_{\{T_1 \leq \widetilde{\tau}\}} e^{-\lambda T_1} V(Y_{T_1-} - \alpha) \right]. \tag{4.2}$$

Now, organize the right hand side above. First, consider the first term:

$$\mathbb{E}^x \int_0^{\widetilde{\tau}\wedge T_1} e^{-\lambda t} u(Y_t)dt = \mathbb{E}^x \int_0^{\infty} 1_{\{t<T_1\}} 1_{\{t<\widetilde{\tau}\}} e^{-\lambda t} u(Y_t)dt.$$

Using Fubini theorem, we can rewrite it as:

$$\int_0^\infty \mathbb{E}^x \left[1_{\{t<T_1\}}1_{\{t<\widetilde{\tau}\}}e^{-\lambda t}u(Y_t)\right] dt.$$

Note that Y_t is independent from the Poisson process N. Therefore, it is possible to write the first term as

$$\int_0^\infty \mathbb{E}^x \left[1_{\{t<T_1\}}\right] \mathbb{E}^x \left[1_{\{t<\widetilde{\tau}\}}e^{-\lambda t}u(Y_t)\right] dt.$$

Now, we can use the fact that the first expectation above is $e^{-\Lambda t}$, and we can write

$$\int_0^\infty \left(e^{-\Lambda t}\mathbb{E}^x \left[1_{\{t<\widetilde{\tau}\}}e^{-\lambda t}u(Y_t)\right]\right) dt.$$

Finally, using the Fubini theorem and the property of indicator function, the first term is written as

$$\mathbb{E}^x \int_0^{\widetilde{\tau}} e^{-(\Lambda+\lambda)t}u(Y_t)dt. \tag{4.3}$$

Now, consider the second term of (4.2) and rewrite it using tower property of conditional expectation.

$$\mathbb{E}^x \left(1_{\{T_1\le\widetilde{\tau}\}}e^{-\lambda T_1}V(Y_{T_1-}-\alpha)\right) = \mathbb{E}^x \left(\mathbb{E}^x \left[1_{\{T_1\le\widetilde{\tau}\}}e^{-\lambda T_1}V(Y_{T_1-}-\alpha)|W\right]\right).$$

Given Brownian motion W, the distribution of T_1 is exponential with parameter Λ since T_1 is independent from W. Using the definition of expectation, we write it as

$$\mathbb{E}^x \int_0^\infty \left(\Lambda e^{-\Lambda t}1_{\{t\le\widetilde{\tau}\}}e^{-\lambda t}V(Y_t-\alpha)\right) dt = \mathbb{E}^x \int_0^{\widetilde{\tau}} \left(\Lambda e^{-(\Lambda+\lambda)t}V(Y_t-\alpha)\right) dt. \tag{4.4}$$

Thus, we can now use (4.3) and (4.4) to rewrite (4.2).

$$V(x) = \sup_{u(\cdot)\in[0,K]} \mathbb{E}^x \left[\int_0^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t}[u(Y_t)+\Lambda V(Y_t-\alpha)]dt\right] =: JV(x)$$

where the operator J is defined as

$$Jw(x) := \sup_{u(\cdot)\in[0,K]} \mathbb{E}^x \left[\int_0^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t}[u(Y_t)+\Lambda w(Y_t-\alpha)]dt\right] \tag{4.5}$$

in terms of a given bounded function $w(\cdot)$ defined on $\mathbb{R}_+$ ($w(\cdot)$ is set to zero on $\mathbb{R}_-$.) Note that the problem in (4.5) is in terms of the process Y defined in (6.1).

Remark 4.1. *(i)* For two bounded functions $w_1(\cdot) \le w_2(\cdot)$, we have $Jw_1(\cdot) \le Jw_2(\cdot)$.
(ii) If $w(\cdot)$ is non-decreasing on $\mathbb{R}_+$, then so is $Jw(\cdot)$.
(iii) If $0 \le ||w(\cdot)|| \le K/\lambda$, then we have $0 \le ||Jw(\cdot)|| \le K/\lambda$.

Proof. The first claim is immediate by the definition of the operator Jw in (4.5). The monotonicity of $x \mapsto Jw(x)$ on $\mathbb{R}_+$ is also obvious.

Next, for a bounded function $0 \leq ||w(\cdot)|| \leq K/\lambda$, we have

$$\begin{aligned}
0 \leq Jw(x) &= \sup_{u(\cdot)\in[0,K]} \mathbb{E}^x \left[\int_0^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t} [u(Y_t) + \Lambda w(Y_t - \alpha)] dt \right] \\
&\leq \mathbb{E}^x \left[\int_0^{\infty} e^{-(\lambda+\Lambda)t} \left[K + \Lambda \frac{K}{\lambda} \right] dt \right] \\
&= \frac{K(\lambda+\Lambda)}{\lambda} \int_0^{\infty} e^{-(\lambda+\Lambda)t} dt \\
&= \frac{K(\lambda+\Lambda)}{\lambda} \frac{1}{\lambda+\Lambda} = \frac{K}{\lambda}.
\end{aligned}$$

□

4.2 Further properties of the operator J in (4.5)

Below we study the maximization problem in (4.5) and study the properties of the function $Jw(\cdot)$ under the following assumption.

Assumption 4.2. Below, $w(\cdot)$ is some given continuous function which is zero on $\mathbb{R}_-$. On $\mathbb{R}_+$, it is non-decreasing, and bounded above by K/λ.

Note that we are looking for the value function satisfying $JV = V$, so we want w to satisfy the properties of the value function. Remember that the value function is defined as:

$$V(x) = \sup_{u\in\mathcal{D}} \mathbb{E}^x \int_0^{\tau} e^{-\lambda t} u(X_t) dt.$$

It is clear that this function is non-decreasing, since τ is non-decreasing with respect to initial surplus x. Also, one can show that $\lim_{x\to\infty} V(x) \leq \frac{K}{\lambda}$ since τ tends to ∞ as x tends to ∞; and u is bounded above by K.

Now, since we expect to have a barrier type of optimal dividend policy, let us introduce the dividend policy $u_r(x) := K 1_{\{x>r\}}$ for some $r > 0$. Besides, denote the respective capital process with $Y^{(r)}$ and the ruin time of it with $\widetilde{\tau}^{(r)}$, and let us define the operator

$$H_r w(x) := \mathbb{E}^x \left[\int_0^{\widetilde{\tau}^{(r)}} e^{-(\lambda+\Lambda)t} [u_r(Y_t^{(r)}) + \Lambda w(Y_t^{(r)} - \alpha)] dt \right], \tag{4.6}$$

which is the expected reward until the process $Y^{(r)}$ hits the default barrier.

Lemma 4.3. *On $[0, r]$, the function $H_r w(\cdot)$ in (4.6) has the explicit form*

$$H_r w(x) = C(e^{\rho_1 x} - e^{\rho_2 x}) - e^{\rho_2 x} \int_0^x \frac{2\Lambda e^{-\rho_2 y} w(y-\alpha)}{\sigma^2(\rho_2 - \rho_1)} dy + e^{\rho_1 x} \int_0^x \frac{2\Lambda e^{-\rho_1 y} w(y-\alpha)}{\sigma^2(\rho_2 - \rho_1)} dy \tag{4.7}$$

where $\rho_1 < 0 < \rho_2$ are the roots of the equation

$$-(\lambda + \Lambda) + \mu\rho + \frac{1}{2}\sigma^2\rho^2 = 0.$$

On $[r, \infty)$ we have

$$H_r w(x) = \frac{K}{\lambda + \Lambda} + e^{r_1 x} \left[B + \int_r^x \frac{2\Lambda e^{-r_1 y} w(y-\alpha)}{\sigma^2(r_2 - r_1)} dy \right] + e^{r_2 x} \int_x^\infty \frac{2\Lambda e^{-r_2 y} w(y-\alpha)}{\sigma^2(r_2 - r_1)} dy \tag{4.8}$$

where $r_1 < 0 < r_2$ are the roots of the equation

$$-(\lambda + \Lambda) + (\mu - K)r + \frac{1}{2}\sigma^2 r^2 = 0.$$

In (4.7-4.8) above, the constants B, C are set such that the function $H_r w(\cdot)$ satisfies the linear system (in B, C)

$$H_r w(r+) = H_r w(r-) \quad \text{and} \quad H_r' w(r+) = H_r' w(r-). \tag{4.9}$$

The function $H_r w(\cdot)$ satisfies $H_r w(0) = 0$, $H_r w(\infty) = \frac{K + \Lambda w(\infty)}{\lambda + \Lambda}$. It is smooth on $\mathbb{R}_+$, has bounded derivative, and solves

$$-(\lambda + \Lambda) H_r w(x) + (\mu - u_r(x))(H_r w)'(x) + \frac{1}{2}\sigma^2 (H_r w)''(x) + u_r(x) + \Lambda w(x - \alpha) = 0, \tag{4.10}$$

except possibly at $x = r$.

Proof. Define $\phi(x)$ as

$$\phi(x) := C(e^{\rho_1 x} - e^{\rho_2 x}) - e^{\rho_2 x} \int_0^x \frac{2\Lambda e^{-\rho_2 y} w(y-\alpha)}{\sigma^2(\rho_2 - \rho_1)} dy + e^{\rho_1 x} \int_0^x \frac{2\Lambda e^{-\rho_1 y} w(y-\alpha)}{\sigma^2(\rho_2 - \rho_1)} dy$$

for $0 < x < r$, and

$$\phi(x) := \frac{K}{\lambda + \Lambda} + e^{r_1 x} \left[B + \int_r^x \frac{2\Lambda e^{-r_1 y} w(y-\alpha)}{\sigma^2(r_2 - r_1)} dy \right] + e^{r_2 x} \int_x^\infty \frac{2\Lambda e^{-r_2 y} w(y-\alpha)}{\sigma^2(r_2 - r_1)} dy$$

for $x > r$. We can choose B and C such that $\phi(r+) = \phi(r-)$ and $\phi'(r+) = \phi'(r-)$.

Note that $\phi(x)$ also satisfies $\phi(0) = 0$, which is obvious by definition of $\phi(x)$. To show $\lim_{x\to\infty} \phi(x) =: \phi(\infty) = [K + \Lambda w(\infty)]/(\lambda + \Lambda)$, take limit as x tends to ∞:

$$\lim_{x\to\infty}\phi(x) = \frac{K}{\lambda+\Lambda} + \lim_{x\to\infty}\frac{B+\int_r^x \frac{2\Lambda e^{-r_1 y}w(y-\alpha)}{\sigma^2(r_2-r_1)}dy}{e^{-r_1 x}} + \lim_{x\to\infty}\frac{\int_x^\infty \frac{2\Lambda e^{-r_2 y}w(y-\alpha)}{\sigma^2(r_2-r_1)}dy}{e^{-r_2 x}}$$

Using L'Hôpital's rule it is easy to calculate the following.

$$\begin{aligned}\lim_{x\to\infty}\phi(x) &= \frac{K}{\lambda+\Lambda} - \frac{2\Lambda w(x-\alpha)}{\sigma^2(r_2-r_1)r_1} + \frac{2\Lambda w(x-\alpha)}{\sigma^2(r_2-r_1)r_2} \\ &= \frac{K}{\lambda+\Lambda} + \frac{2\Lambda w(x-\alpha)}{\sigma^2(r_2-r_1)r_1}\frac{r_1-r_2}{r_1.r_2}.\end{aligned}$$

Since $r_1.r_2 = \frac{-2(\Lambda+\lambda)}{\sigma^2}$,

$$\phi(\infty) = \frac{K+\Lambda w(\infty)}{\lambda+\Lambda}.$$

Now, we will show that ϕ satisfies the differential equation given by 4.10. Consider the case where $x < r$. Since $u_r(x) = 0$, one now needs to show:

$$-(\lambda+\Lambda)\phi(x) + \mu\phi'(x) + \frac{1}{2}\sigma^2\phi''(x) + \Lambda w(x-\alpha) = 0. \tag{4.11}$$

Using the definition of $\phi(x)$ for $x < r$, $\phi'(x)$ and $\phi''(x)$ can be easily found as:

$$\begin{aligned}\phi'(x) = C(\rho_1 e^{\rho_1 x} - \rho_2 e^{\rho_2 x}) &- \rho_2 e^{\rho_2 x}\int_0^x \frac{2\Lambda e^{-\rho_2 y}w(y-\alpha)}{\sigma^2(\rho_2-\rho_1)}dy \\ &+ \rho_1 e^{\rho_1 x}\int_0^x \frac{2\Lambda e^{-\rho_1 y}w(y-\alpha)}{\sigma^2(\rho_2-\rho_1)}dy.\end{aligned}$$

$$\begin{aligned}\phi''(x) = C(\rho_1^2 e^{\rho_1 x} - \rho_2^2 e^{\rho_2 x}) &- \rho_2^2 e^{\rho_2 x}\int_0^x \frac{2\Lambda e^{-\rho_2 y}w(y-\alpha)}{\sigma^2(\rho_2-\rho_1)}dy \\ + \rho_1^2 e^{\rho_1 x}\int_0^x &\frac{2\Lambda e^{-\rho_1 y}w(y-\alpha)}{\sigma^2(\rho_2-\rho_1)}dy - \frac{2\Lambda}{\sigma^2}w(x-\alpha).\end{aligned}$$

The direct calculation and also using the fact that ρ_1 and ρ_2 satisfy the equation

$$-(\lambda+\Lambda) + \mu\rho + \frac{1}{2}\sigma^2\rho^2 = 0,$$

it is easy to show that 4.11 holds.

Similarly, for $x > r$, the differential equation that $\phi(x)$ needs to satisfy becomes:

$$-(\lambda+\Lambda)H_r w(x) + (\mu-K)H_r' w(x) + \frac{1}{2}\sigma^2 H_r'' w(x) + K + \Lambda w(x-\alpha) = 0. \tag{4.12}$$

Again we calculate $\phi'(x)$ and $\phi''(x)$, this time for $x > r$; then plug them into the equation 4.12. It is obvious to see that equation holds.

Now, consider $d\big(e^{-(\lambda+\Lambda)t}\phi(Y_t^{(r)})\big)$. Remember that $Y^{(r)}$ is defined as the capital process when there is no jump and when the dividend policy is taken as u_r. Thus, it evolves according to:

$$dY^{(r)} = (\mu - u_r(Y^{(r)}))dt + \sigma dW_t$$

Using Itô rule, we can write the following.

$$\begin{aligned}
d\big(e^{-(\lambda+\Lambda)t}\phi\big) &= -(\lambda+\Lambda)e^{-(\lambda+\Lambda)t}\phi dt + e^{-(\lambda+\Lambda)t}\left(\phi' dY_t^{(r)} + \frac{1}{2}\phi''(dY_t^{(r)})^2\right)\\
&= -(\lambda+\Lambda)e^{-(\lambda+\Lambda)t}\phi dt + e^{-(\lambda+\Lambda)t}\left[\left(\phi'(\mu-u) + \frac{1}{2}\sigma^2\phi''\right)dt + \sigma\phi' dW_t\right]\\
&= e^{-(\lambda+\Lambda)t}\left[\left(-(\lambda+\Lambda)\phi + (\mu-u)\phi' + \frac{1}{2}\sigma^2\phi''\right)dt + \sigma\phi' dW_t\right]
\end{aligned}$$

Define $\widetilde{\tau}_n^{(r)}$ as the exit time for this process, i.e. $\widetilde{\tau}_n^{(r)} := \inf\{t \geq 0 : Y_t^{(r)} \geq n\}$ and using the differential equation above, write

$$\begin{aligned}
&e^{-(\lambda+\Lambda)(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) - \phi(x) =\\
&\quad\int_0^{(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})} e^{-(\lambda+\Lambda)s}\big(-(\lambda+\Lambda)\phi(Y_{s-}^{(r)}) + (\mu-u_r)\phi'(Y_{s-}^{(r)}) + \frac{1}{2}\sigma^2\phi''(Y_{s-}^{(r)})\big)ds\\
&\qquad + \int_0^{t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}} e^{-\lambda s}\sigma\phi'(Y_{s-}^{(r)})dW_s.
\end{aligned}$$

Since the last term is a stochastic integral, taking expectation yields

$$\begin{aligned}
&\mathbb{E}^x e^{-(\lambda+\Lambda)(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) - \phi(x) =\\
&\quad\mathbb{E}^x\int_0^{(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})} e^{-(\lambda+\Lambda)s}\big(-(\lambda+\Lambda)\phi(Y_{s-}^{(r)}) + (\mu-u_r)\phi'(Y_{s-}^{(r)}) + \frac{1}{2}\sigma^2\phi''(Y_{s-}^{(r)})\big)ds.
\end{aligned}$$

But it is proved that ϕ satisfies 4.10. Therefore, we can rewrite the last equation as

$$\begin{aligned}
&\mathbb{E}^x e^{-(\lambda+\Lambda)(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) - \phi(x) =\\
&\qquad\mathbb{E}^x\int_0^{(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})} e^{-(\lambda+\Lambda)s}\big(-u_r(Y_{s-}^{(r)}) - \Lambda w(Y_{s-}^{(r)} - \alpha)\big)ds.
\end{aligned}$$

Note that as $t \to \infty$ we have

$$e^{-(\lambda+\Lambda)(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y_{t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) \to e^{-(\lambda+\Lambda)(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}})$$

$$\begin{aligned}
&\int_0^{(t\wedge\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})} e^{-(\lambda+\Lambda)s}\big(-u_r(Y_{s-}^{(r)}) - \Lambda w(Y_{s-}^{(r)} - \alpha)\big)ds \to\\
&\qquad\int_0^{(\widetilde{\tau}^{(r)}\wedge\tau_n)} e^{-(\lambda+\Lambda)s}\big(-u_r(Y_{s-}^{(r)}) - \Lambda w(Y_{s-}^{(r)} - \alpha)\big)ds
\end{aligned}$$

So applying the bounded convergence theorem, we obtain

$$\mathbb{E}^x e^{-(\lambda+\Lambda)(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) - \phi(x) =$$
$$\mathbb{E}^x \int_0^{(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})} e^{-(\lambda+\Lambda)s}\big(- u_r(Y^{(r)}_{s-}) - \Lambda w(Y^{(r)}_{s-} - \alpha)\big)ds. \quad (4.13)$$

Note that we have

$$e^{-(\lambda+\Lambda)(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) =$$
$$1_{\{\widetilde{\tau}^{(r)}<\infty\}} e^{-(\lambda+\Lambda)(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) + 1_{\{\widetilde{\tau}^{(r)}=\infty\}} e^{-(\lambda+\Lambda)(\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}_n^{(r)}}).$$

As $n \to \infty$, we have $\tau_n \to \infty$ since $Y_t^{(r)} < \infty$, for $t < \infty$. Hence, we have

$$1_{\{\widetilde{\tau}^{(r)}<\infty\}} e^{-(\lambda+\Lambda)(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) \to 1_{\{\widetilde{\tau}^{(r)}<\infty\}} e^{-(\lambda+\Lambda)\widetilde{\tau}^{(r)}}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}}) = 0,$$

since $\widetilde{\tau}^{(r)}$ is defined as the bankrupt time for the policy u_r and $\phi(0) = 0$. Similarly,

$$1_{\{\widetilde{\tau}^{(r)}=\infty\}} e^{-(\lambda+\Lambda)(\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}_n^{(r)}}) \to 0,$$

as n tends to ∞. Note that we use the fact that ϕ is bounded when we take this limit. Then by bounded convergence theorem, we get

$$\mathbb{E}^x e^{-(\lambda+\Lambda)(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})}\phi(Y^{(r)}_{\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)}}) \to 0, \qquad \text{as } n \to \infty.$$

Also note that the monotone convergence theorem gives

$$\mathbb{E}^x \int_0^{(\widetilde{\tau}^{(r)}\wedge\widetilde{\tau}_n^{(r)})} e^{-(\lambda+\Lambda)s}\big(- u_r(Y^{(r)}_{s-}) - \Lambda w(Y^{(r)}_{s-} - \alpha)\big)ds \to$$
$$\mathbb{E}^x \int_0^{\widetilde{\tau}^{(r)}} e^{-(\lambda+\Lambda)s}\big(- u_r(Y^{(r)}_{s-}) - \Lambda w(Y^{(r)}_{s-} - \alpha)\big)ds, \qquad \text{as } n \to \infty.$$

We can conclude that $n \to \infty$ in (4.13) results in

$$\phi(x) = \mathbb{E}^x \int_0^{\widetilde{\tau}^{(r)}} e^{-(\lambda+\Lambda)s}\big(u_r(Y^{(r)}_{s-}) + \Lambda w(Y^{(r)}_{s-} - \alpha)\big)ds = H_r w(x).$$

□

Note that in this proof, we directly define $\phi(x)$ and by plugging it in the differential equations and using Itô calculus we showed that it actually equals $H_r w(x)$. However, the intuition that how $\phi(x)$ is constructed have not been introduced. In the Appendix, one can find a less rigorous approach which also gives that intuition.

Assumption 4.4. There exists a unique positive number $r[w] > 0$ such that $(H_{r[w]}w)'(r[w]) = 1$. Moreover, $(H_{r[w]}w)'(\cdot) > 1$ on $(0, r[w])$, and $(H_{r[w]}w)'(\cdot) < 1$ on $(r[w], \infty)$.

The validity of this assumption has been studied, it has not been finalized yet. It is an ongoing work while this thesis is being written. The solution approach that we have employed to show it can be found in the appendix.

Corollary 4.5. *The function* $H_{r[w]}(\cdot)$ *satisfies*

$$-(\lambda+\Lambda)H_{r[w]}w(x)+(\mu-u(x))(H_{r[w]}w)'(x)+\frac{1}{2}\sigma^2(H_{r[w]}w)''(x)+u(x)+\Lambda w(x-\alpha)\le 0, \tag{4.14}$$

for any admissible $u(\cdot)\in[0,K]$. *Moreover, with* $u(x)=u_{r[w]}(x):=K1_{\{x>r[w]\}}$, *the inequality becomes an equality.*

Proof. Write (4.10) for $r=r[w]$:

$$\begin{aligned}0=&-(\lambda+\Lambda)H_{r[w]}w(x)+\mu H_{r[w]}w'(x)+\frac{1}{2}\sigma^2H_{r[w]}w''(x)\\&+\left[1-H_{r[w]}w'(x)\right]u_{r[w]}(x)+\Lambda w(x-\alpha)\end{aligned}$$

On $(0,r[w])$, $H_{r[w]}w'(\cdot)>1$ by Assumption 4.4 and also for any admissible u, $u\ge u_{r[w]}=0$ is true. Thus, inequality holds. Similarly, on $(r[w],\infty)$, $H_{r[w]}w'(\cdot)<1$ and $u\le u_{r[w]}=K$ result in the inequality.

□

Lemma 4.6. *We have* $Jw(\cdot)=H_{r[w]}w(\cdot)$, *and the optimal control process in* (4.5) *has the form* $u_{r[w]}(x)=K1_{\{x>r[w]\}}$.

Proof. Let $u(\cdot)$ be an arbitrary admissible control, $Y^{(u)}$ be the corresponding capital process; and define the exit time for this process as $\widetilde{\tau}_n^{(u)}:=\inf\{t\ge0;Y_t^{(u)}\ge n\}$ and the ruin time of it as $\widetilde{\tau}$. Then by Itô rule, we have

$$\begin{aligned}&\mathbb{E}^x e^{-(\lambda+\Lambda)(\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)})}H_{r[w]}w(Y^{(u)}_{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}})-H_{r[w]}w(x)=\\&\qquad\mathbb{E}^x\int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}}e^{-(\lambda+\Lambda)t}\Big(-(\lambda+\Lambda)H_{r[w]}w(Y_t^{(u)})+(\mu-u(Y_t^{(u)}))(H_{r[w]}w)'(Y_t^{(u)})\\&\qquad\qquad+\frac{1}{2}\sigma^2(H_{r[w]}w)''(Y_t^{(u)})\Big)dt\\&\qquad\le\mathbb{E}^x\int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}}-e^{-(\lambda+\Lambda)t}\left[u(Y_t^{(u)})+\Lambda w(Y_t^{(u)}-\alpha)\right]dt\end{aligned}$$

thanks to (4.14). Then the following inequality also holds.

$$\begin{aligned}&\lim_{n\to\infty}\left[\mathbb{E}^x e^{-(\lambda+\Lambda)(\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)})}H_{r[w]}w(Y^{(u)}_{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}})-H_{r[w]}w(x)\right]\\&\qquad\le\lim_{n\to\infty}\left[\mathbb{E}^x\int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}}-e^{-(\lambda+\Lambda)t}\left[u(Y_t^{(u)})+\Lambda w(Y_t^{(u)}-\alpha)\right]dt\right].\end{aligned}\tag{4.15}$$

Note that $\widetilde{\tau}_n^{(u)} \to \infty$ as $n \to \infty$. First, consider the left hand side of the inequality and rewrite it using bounded convergence theorem:

$$\begin{aligned}
\lim_{n\to\infty} &\left[\mathbb{E}^x e^{-(\lambda+\Lambda)(\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)})} H_{r[w]}w(Y_{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}}) - H_{r[w]}w(x)\right] \\
&= \mathbb{E}^x \lim_{n\to\infty} e^{-(\lambda+\Lambda)(\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)})} H_{r[w]}w(Y^{(u)}_{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}}) - H_{r[w]}w(x) \\
&= \mathbb{E}^x \lim_{n\to\infty} 1_{\{\widetilde{\tau}<\infty\}} e^{-(\lambda+\Lambda)(\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)})} H_{r[w]}w(Y^{(u)}_{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}}) - H_{r[w]}w(x) \\
&= \mathbb{E}^x 1_{\{\widetilde{\tau}<\infty\}} e^{-(\lambda+\Lambda)\widetilde{\tau}} H_{r[w]}w(Y^{(u)}_{\widetilde{\tau}}) - H_{r[w]}w(x) \\
&= 0 - H_{r[w]}w(x)
\end{aligned}$$

Now, consider the right hand side of (4.15), and rewrite it using monotone convergence theorem

$$\begin{aligned}
\lim_{n\to\infty} &\left[\mathbb{E}^x \int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}} -e^{-(\lambda+\Lambda)t}\left[u(Y_t^{(u)}) + \Lambda w(Y_t^{(u)} - \alpha)\right] dt\right] \\
&= \mathbb{E}^x \left[\lim_{n\to\infty} \int_0^{\widetilde{\tau}\wedge\widetilde{\tau}_n^{(u)}} -e^{-(\lambda+\Lambda)t}\left[u(Y_t^{(u)}) + \Lambda w(Y_t^{(u)} - \alpha)\right] dt\right] \\
&= \mathbb{E}^x \int_0^{\widetilde{\tau}} -e^{-(\lambda+\Lambda)t}\left[u(Y_t^{(u)}) + \Lambda w(Y_t^{(u)} - \alpha)\right] dt.
\end{aligned}$$

Thus, we have

$$\mathbb{E}^x \int_0^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t}\left[u(Y_t^{(u)}) + \Lambda w(Y_t^{(u)} - \alpha)\right] dt \le H_{r[w]}w(x). \tag{4.16}$$

Taking supremum over u yields $Jw(x) \le H_{r[w]}w(x)$. In particular, when we repeat the steps above with the policy $u_{r[w]}$, we get an equality in (4.16) and this concludes the proof. □

Corollary 4.7. *Since $Jw(\cdot) = H_{r[w]}w(\cdot)$, Corollary 4.5 and Lemma 4.6 imply that*

$$-(\lambda+\Lambda)Jw(x) + (\mu - u(x))(Jw)'(x) + \frac{1}{2}\sigma^2(Jw)''(x) + u(x) + \Lambda w(x-\alpha) \le 0,$$

for any admissible $u(\cdot) \in [0, K]$. In particular, we have an equality with $u(x) = u_{r[w]}(x)$.

4.3 A sequential construction

Using the operator J in (4.5) above, we define a sequence of functions

$$v_0 \equiv 0 \quad \text{and} \quad v_{n+1}(\cdot) = Jv_n(\cdot), \quad n \in \mathbb{N}. \tag{4.17}$$

Remark 4.8. The sequence $(v_n)_{n\in\mathbb{N}}$ defined in (4.17) is non-decreasing, and each element v_n is bounded as $0 \le ||v_n(\cdot)|| \le K/\lambda$. The limit function $v_\infty(x) := \sup_{n\in\mathbb{N}} v_n(x)$ exists point-wise and satisfies $0 \le ||v_\infty(\cdot)|| \le K/\lambda$.

Proof. Since $v_0 \equiv 0$, we have $v_0 \le v_1$ by Remark 4.1. Now assume that $v_{n-1} \le v_n$, then again by Remark 4.1, we have $v_n = Jv_{n-1} \le Jv_n = v_{n+1}$. By induction, it follows that v_n is non-decreasing in $n \in \mathbb{N}$, and therefore the point-wise limit v_∞ exists.

Since $v_0 \equiv 0$ each element v_n is bounded as $0 \le ||v_n(\cdot)|| \le K/\lambda$ again by induction thanks to Remark 4.1. Obviously, these bounds also hold for their limit v_∞. □

Lemma 4.9. *The limit function v_∞ satisfies $v_\infty = Jv_\infty$.*

Proof. Using bounded convergence theorem we obtain

$$\begin{aligned} v_\infty(x) = \sup_n v_n(x) = \sup_n Jv_{n-1}(x) &= \sup_n \sup_{u(\cdot)} \mathbb{E}^x \left[\int_0^{\tilde{\tau}} e^{-(\lambda+\Lambda)t}[u(Y_t) + \Lambda v_{n-1}(Y_t - \alpha)]dt \right] \\ &= \sup_{u(\cdot)} \sup_n \mathbb{E}^x \left[\int_0^{\tilde{\tau}} e^{-(\lambda+\Lambda)t}[u(Y_t) + \Lambda v_{n-1}(Y_t - \alpha)]dt \right] \\ &= \sup_{u(\cdot)} \mathbb{E}^x \left[\int_0^{\tilde{\tau}} e^{-(\lambda+\Lambda)t}[u(Y_t) + \Lambda v_\infty(Y_t - \alpha)]dt \right] = Jv_\infty(x). \end{aligned}$$

□

Lemma 4.10. *The sequence $(v_n)_{n\in\mathbb{N}}$ converges to v_∞ uniformly on $\mathbb{R}_+$ with the explicit bounds*

$$v_\infty(\cdot) - v_n(x) \le \frac{K}{\lambda}\left(\frac{\Lambda}{\lambda+\Lambda}\right)^n, \quad \text{for all } n \ge 0. \tag{4.18}$$

Proof. The bounds in (4.18) hold for $n = 0$ since $v_0(\cdot) = 0$. Assume now that it holds for some $n \ge 0$. Then, by Lemma 4.9 and by induction hypothesis we have

$$\begin{aligned} v_\infty(x) = Jv_\infty(x) &= \sup_{u(\cdot)} \mathbb{E}^x \left[\int_0^{\tilde{\tau}} e^{-(\lambda+\Lambda)t} \left[u(Y_t) + \Lambda v_\infty(Y_t - \alpha)\right] dt \right] \\ &\le \sup_{u(\cdot)} \mathbb{E}^x \left[\int_0^{\tilde{\tau}} e^{-(\lambda+\Lambda)t} \left[u(Y_t) + \Lambda \left(v_n(Y_t - \alpha) + \frac{K}{\lambda}\left(\frac{\Lambda}{\lambda+\Lambda}\right)^n \right)\right] dt \right] \\ &\le \sup_{u(\cdot)} \mathbb{E}^x \left[\int_0^{\tilde{\tau}} e^{-(\lambda+\Lambda)t} \left[u(Y_t) + \Lambda v_n(Y_t - \alpha)\right] dt \right] + \int_0^\infty e^{-(\lambda+\Lambda)t} \Lambda \frac{K}{\lambda}\left(\frac{\Lambda}{\lambda+\Lambda}\right)^n dt \\ &= v_{n+1}(x) + \frac{K}{\lambda}\left(\frac{\Lambda}{\lambda+\Lambda}\right)^{n+1}. \end{aligned}$$

This proves (4.18) for $n + 1$, and the proof is complete by induction. □

Remark 4.11. As the uniform limit of the functions $(v_n)_{n\in\mathbb{N}}$, v_∞ is also continuous (see Lemma 4.10). Hence, it satisfies the properties in Assumption 4.2. Moreover, it solves

$$v_\infty(\cdot) = Jv_\infty(\cdot).$$

Then, by Corollary 4.7, we have

$$-\lambda v_\infty(x) + (\mu - u(x))v'_\infty(x) + \frac{1}{2}\sigma^2 v''_\infty(x) + u(x) + \Lambda[v_\infty(x-\alpha) - v_\infty(x)] \leq 0, \tag{4.19}$$

for any $u(\cdot) \in [0, K]$. Moreover, (4.19) becomes an equality with $u(x) = u_{r[v_\infty]}(x) := K1_{\{x>r[v_\infty]\}}$.

Lemma 4.12. *The limit function v_∞ is the value function of the problem in (2.4). That is; $v_\infty(\cdot) = V(\cdot)$, and $u_{r[v_\infty]}(x) = K1_{\{x>r[v_\infty]\}}$ is an optimal policy.*

Proof. As in the proof of Lemma 3.2, let $u(\cdot)$ be an admissible policy, and let $X^{(u)}$ be the corresponding capital process. That is,

$$dX_t^{(u)} = (\mu - u(X_{t-}))dt + dW_t - \alpha dN_t.$$

Note that v'_∞ is bounded (see Lemmas 4.3, 4.6, and 4.7). Then, in terms of the exit time $\tau_n := \inf\{t \geq 0 : X_t^{(u)} \geq n\}$ and the ruin time τ, Ito rule gives

$$\begin{aligned}
&\mathbb{E}^x e^{-\lambda(t\wedge\tau\wedge\tau_n)} v_\infty(X^{(u)}_{t\wedge\tau\wedge\tau_n}) - v_\infty(x) \\
&= \mathbb{E}^x \int_0^{(t\wedge\tau\wedge\tau_n)} e^{-\lambda s}\Bigg(- \lambda v_\infty(X_{s-}^{(u)}) + (\mu - u(X_{s-}^{(u)}))v'_\infty(X_{s-}^{(u)}) + \frac{1}{2}\sigma^2 v''_\infty(X_{s-}^{(u)}) \\
&\qquad\qquad\qquad\qquad + \Lambda\Big(v_\infty(X_{s-}^{(u)} - \alpha) - v_\infty(X_{s-}^{(u)})\Big)\Bigg)ds \\
&\leq \mathbb{E}^x \int_0^{t\wedge\tau\wedge\tau_n} e^{-\lambda s}\big[-u(X_{s-}^{(u)})\big]ds,
\end{aligned} \tag{4.20}$$

where the inequality is thanks to (4.19). Letting first $t \to \infty$, and then $n \to \infty$ yields (as in the proof of Lemma 3.2)

$$v_\infty(x) \geq \mathbb{E}^x \int_0^\tau e^{-\lambda s} u(X_{s-}^{(u)})ds. \tag{4.21}$$

This implies that $v_\infty \geq V(\cdot)$

Remark 4.11 implies that with $u(x) = u_{r[v_\infty]}(x) := K1_{\{x>r[v_\infty]\}}$, we have equalities in (4.20-4.21). □

Chapter 5

Conclusion

In this thesis, we considered a dividend optimization problem. We tried to find the optimal dividend payment policy to maximize the expected total discounted dividends. The distinction of this study from the existed literature is that we consider a jump-diffusion model for the capital process. Note that for most of the cases in the literature, the optimal dividend policy is found as a barrier type of function. Therefore, we also looked for a barrier type of optimal dividend policy.

After modeling the problem, we tried to solve it for two cases. In the first part, we considered a special case and the optimal dividend payment strategy is found as to pay at a constant rate of dividend. In the second part, a rather general case is considered. As a result, it is shown that a barrier type of dividend payment is an optimal policy. The barrier, which determines if the payment is necessary, depends on the parameters of the problem. Also, the rate of the dividend payment -if the capital is greater than the barrier- equals to the maximum rate, which is given as the upper bound for all admissible dividend policies. Moreover, the value function is constructed in each cases.

Even though the optimal dividend payment policies and the value functions are constructed for each cases, these constructions depend on some assumptions that we have claimed to hold, but have not been able to prove yet. It is an ongoing work for us to show the validity of them.

As a further study, one may deal with the same problem for different cases of dividend policies. In other words, the maximization of total discounted dividend payments in the presence of jumps in the capital process, can be considered for different type of dividend policies, rather than considering policies that we have employed in this thesis. For example, the dividend policies may be taken as point processes, or as more general type of dividend policies like Jeanblanc-Picqué and Shiryaev (1995) have used in the second case and third case of their study, respectively.

Chapter 6

Appendix

6.1 Prooving Remark 3.1

Similar to the approach considered for general case, we first use dynamic programming idea to write f_K as

$$f_K(x) = \mathbb{E}^x \left[\int_0^{\widetilde{\tau} \wedge T_1} e^{-\lambda t} K dt + 1_{\{T_1 \leq \widetilde{\tau}\}} e^{-\lambda T_1} f_K(Y_{T_1-} - \alpha) \right]$$

where Y is a pure diffusion process satisfying

$$dY_t = (\mu - K) + \sigma dW_t, \quad \text{with } Y_0 = x, \tag{6.1}$$

and $\widetilde{\tau}$ is the the ruin time of the process Y.

If we organize the function following the steps done for the general case, we obtain

$$f_K(x) = \mathbb{E}^x \left[\int_0^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t} [K + \Lambda f_K(Y_t - \alpha)] dt \right] =: H_0 f_K(x)$$

where the operator H_0 is defined as

$$H_0 w(x) := \mathbb{E}^x \left[\int_0^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t} [K + \Lambda w(Y_t - \alpha)] dt \right] \tag{6.2}$$

in terms of a given bounded function $w(\cdot)$ defined on $\mathbb{R}_+$ ($w(\cdot)$ is set to zero on $\mathbb{R}_-$.)

Note that this is the same operator with H_r defined in (4.6), with dividend policy is taken as $u = K$, instead of $u_r = K 1_{\{x > r\}}$. Therefore, Lemma 4.3 holds for H_0 with $r = 0$. In other words, on $[0, \infty)$ we have

$$H_0 w(x) = \frac{K}{\lambda + \Lambda} + e^{r_1 x} \left[B + \int_0^x \frac{2\Lambda e^{-r_1 y} w(y - \alpha)}{\sigma^2 (r_2 - r_1)} dy \right] + e^{r_2 x} \int_x^\infty \frac{2\Lambda e^{-r_2 y} w(y - \alpha)}{\sigma^2 (r_2 - r_1)} dy \tag{6.3}$$

where $r_1 < 0 < r_2$ are the roots of the equation

$$-(\lambda + \Lambda) + (\mu - K) r + \frac{1}{2} \sigma^2 r^2 = 0$$

and it solves

$$-(\lambda+\Lambda)H_0w(x)+(\mu-K)(H_0w)'(x)+\frac{1}{2}\sigma^2(H_0w)''(x)+K+\Lambda w(x-\alpha)=0. \quad (6.4)$$

In the next step, using the a sequential construction in which $w_0\equiv 0$ and $w_{n+1}(\cdot)=H_0w_n(\cdot)$, for $n\in\mathbb{N}$, one can show that w_∞ satisfies $H_0w_\infty=w_\infty$, the sequence $(w_n)_{n\in\mathbb{N}}$ converges to w_∞ uniformly on $\mathbb{R}_+$ and finally $f_K=w_\infty$. Thus, f_K satisfies (6.4),

$$-(\lambda+\Lambda)f_Kw(x)+(\mu-K)(f_Kw)'(x)+\frac{1}{2}\sigma^2(f_Kw)''(x)+K+\Lambda f_K(x-\alpha)=0.$$

6.2 A heuristic approach for Lemma 4.3

For $x=0$, $\widetilde{\tau}=0$ and so $H_rw(0)=0$. Also, as x tends to ∞, $\widetilde{\tau}$ also tends to ∞, so

$$H_rw(\infty)=\int_0^\infty e^{-(\lambda+\Lambda)t}\left[K+\Lambda w(\infty)\right]dt=\frac{K+\Lambda w(\infty)}{\lambda+\Lambda}<\infty.$$

First rewrite $H_rw(x)$ as (for h small)

$$\begin{aligned}H_rw(x)=\mathbb{E}^x&\int_0^h e^{-(\lambda+\Lambda)t}\left[u_r(Y_t)+\Lambda w(Y_t-\alpha)\right]dt\\ +\mathbb{E}^x&\int_h^{\widetilde{\tau}} e^{-(\lambda+\Lambda)t}\left[u_r(Y_t)+\Lambda w(Y_t-\alpha)\right]dt.\end{aligned}$$

Now, we can write second term as $\mathbb{E}^x e^{-(\lambda+\Lambda)h}H_rw(Y_h)$. Thus the above equality now can be written as

$$0=\mathbb{E}^x\int_0^h e^{-(\lambda+\Lambda)t}\left[u_r(Y_t)+\Lambda w(Y_t-\alpha)\right]dt+\mathbb{E}^x\left[e^{-(\lambda+\Lambda)h}H_rw(Y_h)-H_rw(x)\right].$$

Note that Itô rule gives us

$$\begin{aligned}&e^{-(\lambda+\Lambda)h}H_rw(Y_h)-H_rw(x)=\\ &\quad\int_0^h e^{-(\lambda+\Lambda)t}\left[-(\lambda+\Lambda)H_rw(Y_t)+(\mu-u_r(Y_t))(H_rw)'(Y_t)+\frac{1}{2}\sigma^2(H_rw)''(Y_t)\right]dt\\ &\qquad+\int_0^h e^{-(\lambda+\Lambda)t}(H_rw)'(Y_t)dW_t.\end{aligned}$$

The integral with respect to W gives us a martingale with zero expectation. Thus, the equality becomes

$$\begin{aligned}0=\mathbb{E}^x&\int_0^h e^{-(\lambda+\Lambda)t}\left[u_r(Y_t)+\Lambda w(Y_t-\alpha)\right]dt\\ +\mathbb{E}^x&\int_0^h e^{-(\lambda+\Lambda)t}\left[-(\lambda+\Lambda)H_rw(Y_t)+(\mu-u_r(Y_t))(H_rw)'(Y_t)+\frac{1}{2}\sigma^2(H_rw)''(Y_t)\right]dt\end{aligned}$$

Since h is small enough we can use linear approximation around zero to write the equation as:

$$0 = h\Big[u_r(x) + \Lambda w(x-\alpha) - (\lambda+\Lambda)H_r w(x) + (\mu - u_r(x))H_r w'(x) + \frac{1}{2}\sigma^2 H_r w''(x)\Big].$$

Thus, $H_r w(x)$ should satisfy:

$$0 = -(\lambda+\Lambda)H_r w(x) + (\mu - u_r(x))H_r w'(x) + \frac{1}{2}\sigma^2 H_r w''(x) + u_r(x) + \Lambda w(x-\alpha). \tag{6.5}$$

Now, for $x \in [0, r)$, $u_r(x) = 0$ and (6.5) becomes

$$0 = -(\lambda+\Lambda)H_r w(x) + \mu H_r w'(x) + \frac{1}{2}\sigma^2 H_r w''(x) + \Lambda w(x-\alpha)$$

which is a non-homogenous second order linear equation. Let $y := H_r w$ and y_p, y_h be the particular and homogenous solutions respectively. It is clear that $y_h = C_1 e^{\rho_1 x} + C_2 e^{\rho_2 x}$ for some C_1 and C_2, where ρ_1 and ρ_2 are the roots of $0 = \frac{1}{2}\sigma^2\rho^2 + \mu\rho - (\lambda+\Lambda)$ i.e.

$$\rho_{1,2} = \frac{-\mu \mp (\mu^2 + 2\sigma^2(\lambda+\Lambda))^{\frac{1}{2}}}{\sigma^2}$$

For the particular solution, y_p, take $y_1 := e^{\rho_1 x}$, $y_2 := e^{\rho_2 x}$ as two solutions for the homogenous part. Then y_p will be in the form

$$y_p(x) = -e^{\rho_1 x}\int_0^x \frac{e^{\rho_2 z}(-\Lambda w(z-\alpha))}{\frac{\sigma^2}{2}W(y_1,y_2)(z)}dz + e^{\rho_2 x}\int_0^x \frac{e^{\rho_1 z}(-\Lambda w(z-\alpha))}{\frac{\sigma^2}{2}W(y_1,y_2)(z)}dz$$

where W is Wronskian of two functions, so $W(y_1, y_2)(x) = (\rho_2 - \rho_1)e^{(\rho_1+\rho_2)x}$.

Thus, y_p is

$$y_p(x) = \frac{2}{\sigma^2(\rho_2-\rho_1)}\Big[e^{\rho_1 x}\int_0^x e^{-\rho_1 z}w(z-\alpha)dz - e^{\rho_2 x}\int_0^x e^{-\rho_2 z}w(z-\alpha)dz\Big]$$

and $H_r w \equiv y = y_h + y_p$ becomes

$$H_r w(x) = C_1 e^{\rho_1 x} + C_2 e^{\rho_2 x} + \frac{2}{\sigma^2(\rho_2-\rho_1)}\Big[e^{\rho_1 x}\int_0^x e^{-\rho_1 z}w(z-\alpha)dz - e^{\rho_2 x}\int_0^x e^{-\rho_2 z}w(z-\alpha)dz\Big].$$

Since $H_r w(0) = 0$, we have $C_1 + C_2 = 0$. Let, $C := C_1 = -C_2$ and for $x \in [0, r)$, $H_r w(x)$ is in the form

$$H_r w(x) = C\big[e^{\rho_1 x} - e^{\rho_2 x}\big] + \frac{2}{\sigma^2(\rho_2-\rho_1)}\Big[e^{\rho_1 x}\int_0^x e^{-\rho_1 z}w(z-\alpha)dz - e^{\rho_2 x}\int_0^x e^{-\rho_2 z}w(z-\alpha)dz\Big].$$

Now, consider the case where $x \in (r, \infty)$. Clearly, $u_r(x) = K$ and (6.5) becomes

$$0 = -(\lambda+\Lambda)H_r w(x) + \mu(H_r w)'(x) + \frac{1}{2}\sigma^2(H_r w)''(x) + \Lambda w(x-\alpha) + K - K(H_r w)'(x)$$

which also is a non-homogenous second order linear equation. Similar to the previous case let $y := H_r w$; and y_p, y_h be the particular and homogenous solutions, particularly. Then, $y_h = D_1 e^{r_1 x} + D_2 e^{r_2 x}$ for some D_1 and D_2, where $r_1 < 0 < r_2$ are the roots of $0 = \frac{1}{2}\sigma^2 r^2 + (\mu - K)r - (\lambda + \Lambda)$, i.e.

$$r_{1,2} = \frac{K - \mu \mp ((K-\mu)^2 + 2\sigma^2(\lambda + \Lambda))^{\frac{1}{2}}}{\sigma^2}.$$

For the particular solution, choose the two solutions as the previous case, so that the Wronskian will be same. Thus, particular solution will be in the form

$$y_p(x) = -e^{r_1 x}\int_0^x \frac{e^{r_2 t}(-K - \Lambda w(t-\alpha))}{\frac{\sigma^2}{2}(r_2 - r_1)e^{(r_1+r_2)t}}dt + e^{r_2 x}\int_0^x \frac{e^{r_1 t}(-K - \Lambda w(t-\alpha))}{\frac{\sigma^2}{2}(r_2 - r_1)e^{(r_1+r_2)t}}dt.$$

Thus, y_p is

$$\begin{aligned} y_p(x) &= \frac{2}{\sigma^2(r_2 - r_1)}\left[e^{r_1 x}\int_r^x e^{-r_1 t}(K + \Lambda w(t-\alpha))dt - e^{r_2 x}\int_r^x e^{-r_2 t}(K + \Lambda w(t-\alpha))dt\right] \\ &= \frac{2}{\sigma^2(r_2 - r_1)}\left[\frac{e^{r_1(x-r)}}{{r_1}^2} - \frac{1}{r_1} - \frac{e^{r_2(x-r)}}{{r_2}^2} + \frac{1}{r_2} + e^{r_1 x}\int_r^x e^{-r_1 t}\Lambda w(t-\alpha)dt \right. \\ &\qquad \left. - e^{r_2 x}\int_r^x e^{-r_2 t}\Lambda w(t-\alpha)dt\right]. \end{aligned}$$

Now, in order to find $H_r w(x)$ for $x \in (r, \infty)$, add y_h and y_p and also let the new coefficients of $e^{r_1 x}$, and $e^{r_2 x}$ be B_1 and B_2 respectively. Note that there is a constant $\frac{2k}{\sigma^2(r_2 - r_1)}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$ which turns out to be $\frac{K}{\lambda + \Lambda}$ since $r_1 r_2 = \frac{-2(\lambda+\Lambda)}{\sigma^2}$.

$$\begin{aligned} H_r w(x) &= B_1 e^{r_1 x} + B_2 e^{r_2 x} + \frac{K}{\lambda + \Lambda} \\ &\quad + \frac{2\Lambda}{\sigma^2(r_2 - r_1)}\left[e^{r_1 x}\int_r^x e^{-r_1 t}\Lambda w(t-\alpha)dt - e^{r_2 x}\int_r^x e^{-r_2 t}\Lambda w(t-\alpha)dt\right]. \end{aligned}$$

Using the boundary condition $H_r w(\infty) = \frac{K + \Lambda w(\infty)}{\lambda + \Lambda}$ one can easily show that

$$B_2 = \int_r^\infty \frac{e^{-r_2 t} 2\Lambda w(t-\alpha)}{\sigma^2(r_2 - r_1)}dt.$$

When we replace B_2, organize the equation, and define $B := B_1$ we have

$$\begin{aligned} H_r w(x) = \frac{K}{\lambda + \Lambda} &+ e^{r_1 x}\left[B + \frac{2\Lambda}{\sigma^2(r_2 - r_1)}\int_r^x e^{-r_1 t} w(t-\alpha)dt\right] \\ &+ e^{r_2 x}\frac{2\Lambda}{\sigma^2(r_2 - r_1)}\int_x^\infty e^{-r_2 t} w(t-\alpha)dt. \end{aligned}$$

6.3 An approach to show validity of Assumption 4.4

Remember that

$$H_r w(x) = \begin{cases} C(e^{\rho_1 x} - e^{\rho_2 x}) - e^{\rho_2 x}\int_0^x \frac{2\Lambda e^{-\rho_2 y} w(y-\alpha)}{\sigma^2(\rho_2-\rho_1)}dy + e^{\rho_1 x}\int_0^x \frac{2\Lambda e^{-\rho_1 y} w(y-\alpha)}{\sigma^2(\rho_2-\rho_1)}dy, & 0 < x < r, \\ \\ \frac{K}{\lambda+\Lambda} + e^{r_1 x}\left[B + \int_r^x \frac{2\Lambda e^{-r_1 y} w(y-\alpha)}{\sigma^2(r_2-r_1)}dy\right] + e^{r_2 x}\int_x^\infty \frac{2\Lambda e^{-r_2 y} w(y-\alpha)}{\sigma^2(r_2-r_1)}dy, & x > r. \end{cases}$$

We know that this function satisfies $H_r w(r+) = H_r w(r-)$, $H_r w'(r-) = H_r w'(r+)$ and we are looking for some $r[w]$ such that $H_{r[w]} w'(r[w]) = 1$. To simplify the notation we will denote $r[w]$ with $\hat{x}$.

To find the three unknowns B, C, and $\hat{x}$ we will use three equations listed below.

1. $H_{\hat{x}} w(\hat{x}+) = H_{\hat{x}} w(\hat{x}-)$:

$$C(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}}) - \frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}\left[e^{\rho_2 \hat{x}}\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy - e^{\rho_1 \hat{x}}\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\right] =$$

$$\frac{K}{\lambda+\Lambda} + e^{r_1 \hat{x}} B + \frac{2\Lambda}{\sigma^2(r_2-r_1)} e^{r_2 \hat{x}}\int_{\hat{x}}^\infty e^{-r_2 y} w(y-\alpha)dy$$

2. $(H_{\hat{x}} w)'(\hat{x}-) = 1$:

$$1 = C(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}) - \frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}\left[\rho_2 e^{\rho_2 \hat{x}}\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy - \rho_1 e^{\rho_1 \hat{x}}\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\right]$$

3. $(H_{\hat{x}} w)'(\hat{x}+) = 1$:

$$1 = r_1 e^{r_1 \hat{x}} B + r_2 e^{r_2 \hat{x}} \frac{2\Lambda}{\sigma^2(r_2-r_1)}\int_{\hat{x}}^\infty e^{-r_2 y} w(y-\alpha)dy$$

From item 2, C is found as

$$C = \frac{1 + \frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}\left[\rho_2 e^{\rho_2 \hat{x}}\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy - \rho_1 e^{\rho_1 \hat{x}}\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\right]}{(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}})}.$$

and from item 3, B is found as

$$B = \frac{1 - r_2 e^{r_2 \hat{x}} \frac{2\Lambda}{\sigma^2(r_2-r_1)}\int_{\hat{x}}^\infty e^{-r_2 y} w(y-\alpha)dy}{r_1 e^{r_1 \hat{x}}}.$$

We plug these into the equation given in item 1. Below, we calculate the left hand side, LHS, and the right hand side, RHS, of that equation separately.

$$LHS = \frac{(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}})}\Big(1 + \frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}\big[\rho_2 e^{\rho_2 \hat{x}}\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy$$
$$-\rho_1 e^{\rho_1 \hat{x}}\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\big]\Big)$$

$$-\frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}\Big[e^{\rho_2 \hat{x}}\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy - e^{\rho_1 \hat{x}}\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\Big].$$

$$LHS = \frac{(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}})} + \frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}e^{\rho_2 \hat{x}}\big[\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy\big]\left(\frac{\rho_2(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}} - 1\right)$$
$$-\frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}e^{\rho_1 \hat{x}}\big[\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\big]\left(\frac{\rho_1(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}} - 1\right)$$

$$= \frac{(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}})} + \frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}e^{\rho_2 \hat{x}}\big[\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy\big]\left(\frac{\rho_2 e^{\rho_1 \hat{x}} - \rho_1 e^{\rho_1 \hat{x}}}{\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}}\right)$$
$$-\frac{2\Lambda}{\sigma^2(\rho_2-\rho_1)}e^{\rho_1 \hat{x}}\big[\int_0^{\hat{x}} e^{-\rho_1 y} w(y-\alpha)dy\big]\left(\frac{\rho_2 e^{\rho_1 \hat{x}} - \rho_1 e^{\rho_1 \hat{x}}}{\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}}\right)$$

$$= \frac{(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}})} + \frac{2\Lambda}{\sigma^2}\frac{e^{(\rho_2+\rho_1)\hat{x}}}{\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}}\big[\int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy - \int_0^{\hat{x}} e^{-\rho_2 y} w(y-\alpha)dy\big].$$

Now, write the right hand side of the equation given by item 1.

$$RHS = \frac{K}{\lambda+\Lambda} + \frac{1}{r_1} - \frac{r_2}{r_1}e^{r_2 \hat{x}}\frac{2\Lambda}{\sigma^2(r_2-r_1)}\int_{\hat{x}}^{\infty} e^{-r_2 y} w(y-\alpha)dy$$
$$+\frac{2\Lambda}{\sigma^2(r_2-r_1)}e^{r_2 \hat{x}}\int_{\hat{x}}^{\infty} e^{-r_2 y} w(y-\alpha)dy$$

$$= \frac{K}{\lambda+\Lambda} + \frac{1}{r_1} - \frac{2\Lambda}{\sigma^2 r_1}e^{r_2 \hat{x}}\int_{\hat{x}}^{\infty} e^{-r_2 y} w(y-\alpha)dy.$$

After rearranging the equation $LHS = RHS$, we have

$$\frac{K}{\lambda+\Lambda} + \frac{1}{r_1} = \frac{(e^{\rho_1 \hat{x}} - e^{\rho_2 \hat{x}})}{(\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}})} + \frac{2\Lambda}{\sigma^2} \frac{e^{(\rho_2+\rho_1)\hat{x}}}{\rho_1 e^{\rho_1 \hat{x}} - \rho_2 e^{\rho_2 \hat{x}}} \int_0^{\hat{x}} (e^{-\rho_2 y} - e^{-\rho_2 y}) w(y-\alpha) dy$$
$$+ \frac{2\Lambda}{\sigma^2} \frac{e^{r_2 \hat{x}}}{r_1} \int_{\hat{x}}^{\infty} e^{-r_2 y} w(y-\alpha) dy := F(\hat{x}).$$

We are looking for the unique $\hat{x}$ satisfying $F(\hat{x}) = \frac{K}{\lambda+\Lambda} + \frac{1}{r_1}$. Note that

$$\rho_{1,2} = \frac{-\mu \mp \sqrt{\mu^2 + 2\sigma^2(\Lambda+\lambda)}}{\sigma^2}$$

and

$$r_{1,2} = \frac{-\mu \mp \sqrt{(\mu-K)^2 + 2\sigma^2(\Lambda+\lambda)}}{\sigma^2}.$$

We assume $K > \frac{\sigma^2(\Lambda+\lambda)}{2\mu}$ so that $\frac{K}{\lambda+\Lambda} + \frac{1}{r_1} > 0$.

First look at the value of F at $x = 0$ and as x tends to ∞.

$$F(0) = \frac{2\Lambda}{\sigma^2} \frac{1}{r_1} \int_0^{\infty} e^{-r_2 y} w(y-\alpha) dy \leq 0.$$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \Bigg[\frac{e^{\rho_1 x}}{(\rho_1 e^{\rho_1 x} - \rho_2 e^{\rho_2 x})} - \frac{e^{\rho_2 x}}{(\rho_1 e^{\rho_1 x} - \rho_2 e^{\rho_2 x})} + \frac{2\Lambda}{\sigma^2} \frac{e^{r_2 x}}{r_1} \int_x^{\infty} e^{-r_2 y} w(y-\alpha) dy +$$
$$\frac{2\Lambda}{\sigma^2} \frac{e^{(\rho_2+\rho_1)x}}{\rho_1 e^{\rho_1 x} - \rho_2 e^{\rho_2 x}} \int_0^x (e^{-\rho_2 y} - e^{-\rho_2 y}) w(y-\alpha) dy \Bigg]$$

$$= \frac{1}{\rho_2} + \frac{2\Lambda}{r_1 \sigma^2} \lim_{x\to\infty} \frac{\int_x^{\infty} e^{-r_2 y} w(y-\alpha) dy}{e^{-r_2 x}} + \frac{2\Lambda}{\sigma^2} \lim_{x\to\infty} \frac{\int_0^x (e^{-\rho_2 y} - e^{-\rho_2 y}) w(y-\alpha) dy}{\rho_1 e^{-\rho_2 x} - \rho_2 e^{-\rho_1 x}}.$$

Using L'Hôpital's rule, and the definitions of r_1, r_2, ρ_1, and ρ_2, this limit can be rewritten as

$$\begin{aligned}
\lim_{x\to\infty} F(x) &= \frac{1}{\rho_2} + \frac{2\Lambda}{\sigma^2} \lim_{x\to\infty} \left[\frac{e^{-\rho_2 x} w(x-\alpha)}{\rho_1\rho_2(e^{-\rho_1 x} - e^{-\rho_2 x})} - \frac{e^{-\rho_1 x} w(x-\alpha)}{\rho_1\rho_2(e^{-\rho_1 x} - e^{-\rho_2 x})} + \frac{(x-\alpha)}{r_1 r_2} \right] \\
&= \frac{1}{\rho_2} - \frac{2\Lambda}{\sigma^2} \lim_{x\to\infty} \left[\frac{w(x-\alpha)}{\rho_1\rho_2} + \frac{w(x-\alpha)}{r_1 r_2} \right] \\
&= \frac{1}{\rho_2}
\end{aligned}$$

Below, we will check if $\frac{1}{\rho_2} > \frac{1}{r_1} + \frac{K}{\Lambda+\lambda}$. By simple calculations one can show that the inequality holds if and only if

$$\frac{\sigma^2}{-\mu+\sqrt{\mu^2+2\sigma^2(\Lambda+\lambda)}} > \frac{K}{\Lambda+\lambda} - \frac{\sigma^2}{\mu-K+\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)}}$$

$$\Leftrightarrow$$

$$\sqrt{\mu^2+2\sigma^2(\Lambda+\lambda)}+\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)} > K$$

$$\Leftrightarrow$$

$$\sqrt{\mu^2+2\sigma^2(\Lambda+\lambda)} > K-\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)}$$

$$\Leftrightarrow$$

$$\mu^2+2\sigma^2(\Lambda+\lambda) > K^2+(\mu-K)^2+2\sigma^2(\Lambda+\lambda)+2K\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)}$$

$$\Leftrightarrow$$

$$0 > 2K^2-2K\mu-2K\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)}$$

$$\Leftrightarrow$$

$$0 > 2K^2-2K\mu-2K\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)}$$

$$\Leftrightarrow$$

$$\sqrt{(\mu-K)^2+2\sigma^2(\Lambda+\lambda)} > K-\mu$$

which is obviously true since Λ and λ are positive.

Hence, it is known that

$$F(0) < 0 < \frac{1}{r_1}+\frac{K}{\Lambda+\lambda} < \frac{1}{\rho_2} = \lim_{x\to\infty} F(x),$$

which means that there exists at least one $\hat{x}$ satisfying

$$F(\hat{x}) = \frac{1}{r_1}+\frac{K}{\Lambda+\lambda}.$$

In the next step, the aim is to show the uniqueness of such $\hat{x}$. We tried several ways to show it. First of all, we checked if F is increasing by looking at the derivative of it. Unfortunately, the expressions are very cumbersome. Then, we tried to show that F is increasing at the point $\hat{x}$, i.e. $F'(\hat{x}) > 0$. This would also work since if there are more than one $\hat{x}$ satisfying the condition then, at least at one of them, F will have zero or negative derivative. Again, we have not proved or disproved inequality yet.Note that after showing the uniqueness part one also need to show that $H_{\hat{x}}w(x) > 1$, for $x < \hat{x}$ and $H_{\hat{x}}w(x) < 1$, for $x < \hat{x}$.

Bibliography

[1] E. Boguslavskaya, On optimization of dividend flow for a company in the presence of liquidation value, (2003) Avaible at http://www.boguslavsky.net/fin/dividendflow.pdf.

[2] G. Yin, Q.S. Song, H. Yang, Stochastic optimization algorithms for barrier dividend strategies, *Journal of Computational and Applied Mathematics* **223**, (2009) 240-262.

[3] H. Gerber, Entscheidungskriterien fr den zusammengesetzten Poisson-Proze, *Mitt. Ver. Schweiz. Vers. Math* **69** (1969) 185-228.

[4] H. Gerber, E. Shiu, Optimal dividends: analysis with Brownian motion, *North American Actuarial Journal* **8:1** (2004).

[5] J. M. Steele, Stochastic Calculus and Financial Applications, *Springer-Verlag, New York, USA* (2001).

[6] J. Paulsen, H. K. Gjessing, Optimal choice of dividend barriers for a risk process with stochastic return on investments, *Insurance: Mathematics and Economics* **20**, (1997) 215-223.

[7] J. Zou, Z. Zhang, J. Zhang, Optimal dividend payouts under jump-diffusion risk processes, *Stochastic Models* **25** (2009) 332-347.

[8] L.H.R. Alvarez, J. Virtanen, A class of solvable stochastic dividend optimization problems: on the general impact of flexibility on valuation, *Economic Theory* **28**, (2006) 373-398.

[9] M. I. Taksar, Optimal risk and dividend distribution control models for an insurance company, *Mathematical Methods for Operations Research* **51** (2000) 142.

[10] M. Jeanblanc-Picqué, A. N. Shiryaev, Optimization of the flow of dividends, *Russian Math. Surveys* **50:2** (1995) 257277.

[11] P. Azcue, N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, *Mathematical Finance* **15:2** (2005) 261308.

[12] R. Rodner, L. Shepp, Risk vs. profit potential: A model for corporate strategy, *Journal of Economic Dynamics and Control* **20** (1996) 1373-1393.

[13] S. E. Shreve, Stochastic Calsulus for Finance II; Continuous-time models, *Springer, Pittsburg, USA* (2004).

[14] S. Thonhauser, H. Albrecher, Dividend maximization under consideration of the time value of ruin, *Insurance: Mathematics and Economics* **41**, (2007) 163-184.

[15] T. Mikosch, Elementary Stochastic Calculus with Finance in View, *World Scientific, Singapore,* (1998).