Pairing Games and Markets*

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Abstract

Pairing Games or *Markets* that we study here are a generalization of the assignment game where players are not a priori partitioned into two sides and utility realizations are NTU. We identify a necessary and sufficient condition for the nonemptiness of the *core,* equivalently the set of *competitive equilibria*. We define *semistable* and *pseudostable* allocations and show that there exists a semistable and a pseudostable allocation when the core is empty. We also show that pseudostable allocations belong to the Bargaining Set of a Pairing Game. Our approach is constructive and utilizes *solitary-minimal matchings* that we introduce. We give a *Market Procedure* that reaches the Equilibrium Set and show several properties that the Equilibrium Set has.

Keywords : Stable Matching, Market Design, NTU Assignment Game, Roommate Problem, Coalition Formation, Bargaining Set, Bilateral Transaction, Gallai-Edmonds Decomposition

1 Introduction

Matching models in economics mostly have a *two-sided* structure, e.g., workers and firms, buyers and sellers. In this paper we study *pairing games* or *pairing markets* where an arbitrary set of players partition into pairs and singletons. Each pair of players has a continuum of activities to jointly choose from if they form a pair - call it a *partnership* or a *bilateral transaction.* We are interested in outcomes that are stable or in competitive equilibrium and in designing a procedure to achieve them.

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Our model is a generalization of the assignment game (Shapley and Shubik (1972)) in two ways. First, players are not a priori partitioned into two sides. Second, utility realizations permit income effects and are not restricted to the *transferable utility* domain.

The assignment game has been very fruitful in modelling a wide range of economic situations, e.g., markets for indivisible objects, marriage, fair allocations, principal-agent matching.[1] An important property of the assignment game is the existence and coincidence of core and competitive equilibrium allocations. Also, two sidedness has permitted the design of rather simple *coordinated* market procedures[2] for attaining desired outcomes, and the results carry over to more general cases. For example, players' preferences may belong to the general *nontransferable utility* domain[3], players on one side may have *multiple partners* if preferences satisfy *gross substitutability*[4], and players on both sides may have multiple partners if preferences are *additive separable*.[5]

Yet many markets are not two-sided : For example many mergers occur among firms that are alike. Likewise, acquisitions and joint ventures.[6] Various swap markets are example to the multiple partners version of our model.[7] So are organized markets for bilateral contracting in electricity where some players are seller to one partner and buyer to another.[8] It is only recently on the other hand that Pairing Games and Markets are being explored. This is in contrast to the discrete counterpart of our model, the *roommate problem* (Gale and Shapley (1962)), which has a fairly substantial literature including the interesting application in market design for *kidney exchange*.[9]

One reason why non-two-sided models have not been much considered is the possible nonexistence of core or competitive equilibrium allocations. It is well known that this possibility is not uncommon. For example, in the three-player Game where two players may share a cake and none of the cakes is sufficiently large in comparison to the other two cakes, the odd-man-out will be able to lure away one of the partners in any pair that forms. So there is no core allocation, equivalently, no competitive equilibrium in partnership prices. The three-player Game was taken up by Binmore (1985) for a study of bargaining with pair formation. As Binmore demonstrated, actually, there

[1] As in Demange, Gale and Sotomayor (1986), Becker (1973), Alkan, Demange and Gale (1991), Dam and Perez-Castrillo (2006) respectively.

[2] The multi-item auctions in Crawford and Knoer (1981), Demange, Gale and Sotomayor (1986), Perez-Castrillo and Sotomayor (2002).

[3] Alkan (1989,1992,1997), Alkan and Gale (1990).

[4] Kelso and Crawford (1982), Gul and Stachetti (2000).

[5] Sotomayor (1992,2009).

[6] Gong et al (2007) report that most joint ventures especially those succesful are bilateral.

[7] Our main results in this paper would carry over to the multiple partners model under additive separability.

[8] As in the Free Contract Market ACL in Brazil.

[9] E.g., Irving (1985), Tan (1990,1991), Diamantoudi, Miyagawa and Xue (2004), Inarra, Larrea and Molis (2008), Klaus, Klijn and Walzl (2011), and Roth, Sonmez and Unver (2005).

are three particular outcomes in this Game that are “stable” under a broader consideration - more precisely, a unique *endogenous outside-option* vector that “generates” these three outcomes.[10] The three-player Game is of course special and Binmore remarked that “the four-player game is less easily dealt with” citing “combinatorial difficulties intrinsic to the problem.”

In this paper we offer a comprehensive analysis of all Pairing Games and Markets. We first identify a condition that is necessary and sufficient for the emptiness of the *core* - the set of allocations that are individually rational and *stable* under pairwise blocking. Core allocations are equivalently *competitive equilibria* in partnership prices as in the assignment game. We address the situations when the core is empty through two alternative extensions. We show that there exists, for every Market/Game, an *Equilibrium Set* that coincides with the set of core and competitive equilibrium allocations when they exist and generates two “extended solutions” when they do not exist. We also show that the Equilibrium Set is reachable by a fairly simple coordinated market procedure and that it has several nice properties. On the transferable utility domain, “constant” players aside, it is nearly identical in structure to the two-sided Equilibrium Set of an assignment game. On the nontransferable utility domain, it has a generalized convexity and median property but is not necessarily two-sided.

One of the extensions involves allowing *half-partnerships* and we call an allocation *semistable* if it is immune to blocking under this possibility. The other extension involves strengthening the blocking condition as in the definition of a Bargaining Set[11] for cooperative games and we call an allocation *pseudostable* if it is immune to blocking in this sense. These two extensions pertain to two different institutional environments; mathematically, semistable and pseudostable allocations are closely related. To illustrate, in the three-player Game, the allocation where each player is half-partner to the other two players and the cakes are shared “equally” is semistable. And each of the three allocations where two players share the cake “equally” and the third player gets nothing - the outcomes Binmore (1985) identified - is pseudostable.

Let us give a more detailed account : We look at *aspirations*[12] and *demand*. The standard definitions are as follows : An aspiration is a payoff vector that gives for each player the maximum-utility she can achieve in *some* partnership. The payoffs in an aspiration are mutually determined and can be seen as the *prices* the players ask for entering into partnership. The set of players with any of whom a player can achieve her aspiration payoff is the demand set of that player. A player is said to be *active* at an aspiration if her payoff is strictly above her stand alone utility, in other words,

[10] Binmore’s demonstration has additional aspects; in particular, he shows that each of the three allocations arises as the unique subgame perfect equilibrium of a noncooperative sequential offer game.

[11] The definition allows for *counterobjections* to objections. See Aumann and Maschler (1964), Zhou (1994).

[12] Albers (1974), Bennett (1983).

if she *strictly* supplies herself for partnership. A *stable* or *competitive equilibrium allocation* is an aspiration where there is a demand compatible *matching* that leaves no active player unmatched. We call such an aspiration *realizable*.

Aspirations are many and may be highly nonrealizable. Consider for example a Market where initially a large number of players "aspire" for one particular player only. Such a situation would not be uncommon. The aspiration payoff vector in this situation is highly nonrealizable since none among the many suitors except possibly one will be able to achieve her aspiration payoff. Naturally then many players will look for lowering their aspiration levels and including other potential partners into their demand sets. It is evident that this is a complex process especially when players are not two-sided. Quite likely, pairs will form in some sequential occurrence, with inefficiencies, before aspirations are patiently awaited to settle down. Our quest in this paper is for an aspiration where a maximum number of pairs may form simultaneously - compatible with demand - and leave unmatched a *minimum* number of active players.[13] In the three-player Game a third of the population remains unmatched at any aspiration. As we show, this is a worst case among all games.

We show that, for any Pairing Market, there always exist aspirations - that we call *settled* - at which it is possible to match all active players in a demand compatible way, provided half-partnerships are viable, and otherwise, it is possible to match at least two thirds of all active players. A special but relevant case is when players come in *types*. Then, it is possible to match all the active players in any type with an even number of players and all but one of the active players in any type with an odd number of players. So, in the worst case, the proportion of active players who necessarily remain unmatched is small in large populations.[14] The market procedure we give describes how such outcomes may be reached in a coordinated way. Settled aspirations constitute the Equilibrium Set we have mentioned.

To be more precise, our characterization result says : There is no stable allocation if and only if there is a nonrealizable aspiration which has no *bilateral submarket* consisting of more "buyers" than "sellers". (We give the definition of a "submarket" in the next section but let us emphasize that whether a player is a "buyer" or a "seller" or neither is with reference to an aspiration and may change from one aspiration to another.) We call a bilateral submarket with more buyers than sellers a *seller-market* and call an aspiration with no seller-market a *settled aspiration*. More concisely, then, there is no stable allocation if and only if there is a settled aspiration that is nonrealizable. This characterization gives a local condition for the nonexistence of stable allocations and a stopping rule for our market procedure.

We show then that any settled aspiration that is nonrealizable - i.e., that is not a stable allocation

[13]Then a *residual market* may form among the unmatched.

[14]Chiappori, Galichon and Salanie (2012) show the same for the transferable utility case.

- generates a semistable and a pseudostable allocation. The method is constructive : we construct the semistable and pseudostable allocations by making use of certain maximum-cardinality matchings. These matchings we have named *solitary-minimal*. Solitary-minimal matchings stand behind and unify most of our results. For instance we identify a seller-market by way of a solitary-minimal matching. And our market procedure traces a path of aspirations noting a solitary-minimal matching at each aspiration.

The organization of the paper is as follows : In the subsection below, we make additional remarks about our work and the existing literature. In Section 2, we give formal definitions and introduce the notion of a seller-market. In Section 3, we give our existence results along with a characterization of seller-markets via solitary-minimal matchings. In Section 4, we show several properties of the Equilibrium Set including the fact that there exists at each settled aspiration a *minimum* number of unmatchable active players. In Section 5, we first show that settled aspirations generate *maximum-stable allocations*[15] and define pseudostable allocations via solitary-minimal matchings. We then give a characterization of the Demand Bargaining Set[16] in our context, show that it contains all pseudostable allocations and point at the need for a coordinated procedure for achieving a settled-aspiration-allocation. Section 6 spells out the Market Procedure in thorough detail. Section 7 contains concluding remarks. Proofs omitted are in the Appendix.

1.1 Additional Remarks and Current Literature

Let us say more about solitary-minimal matchings and our contribution from a mathematical point of view. The mathematical apparatus we use is that of maximum-cardinality or *maximal* matchings in graphs and some of our results are closely related to the Gallai-Edmonds Decomposition Theorem[17] although we nowhere use it explicitly. This Theorem says that, in any demand graph, players partition into three types - let us say, “independent”, “central”, “substitutable” - such that (i) every maximal matching pairs an independent player with an independent player, a central player with a substitutable player, and leaves unmatched only a subset of the substitutable players, and that (ii) each unmatched player resides in an *odd-cycle* defined with respect to the matching. An odd-cycle may consist of a single unmatched player. The solitary-minimal matchings are those maximal matchings where the number of such *solitary* players is minimum.[18] We do not know whether solitary-minimal matchings have been defined or utilized elsewhere.

[15] As in the definition given by Tan (1991) for the roommate problem.

[16] Introduced for TU games by Morelli and Montero (2003) who show that it is a subset of the Zhou Bargaining Set (Zhou (1994)). We show that this is also true in our context.

[17] Edmonds (1965), Gallai (1963,1964).

[18] More precisely, in our case, where the number of *active* unmatched singleton players is minimum.

The Gallai-Edmonds Theorem pertains to a single graph.[19] Our work involves viewing demand graphs continuously in the neighborhood of an aspiration. The demand graph changes at a finite number of aspirations on any path of aspirations. Our Market Procedure traces a path of aspirations over which certain properties of the Gallai-Edmonds decompositions are lexicographically monotone. This involves identifying a unique *Seller-Market* at each aspiration and keeping track of it on the traced path. We spell out in Section 6.1 how this can be done recursively by a judicious selection of successive solitary-minimal matchings.

The existing literature on Pairing Games and Markets is on the transferable utility domain. There are several characterizations given for the existence of stable allocations and some procedures for finding them when they exist : Eriksson and Karlander (2001) give a characterization for stable allocations at a *given* matching, that is similar to the characterization for roommate problems by Tan (1991), and then use linear programming duality for optimal matchings. Talman and Yang (2011) also give a characterization that uses linear programming duality. Sotomayor (2005) gives a characterization that makes use of "simple outcomes" and is of a nonconstructive nature. Chiappori, Galichon and Salanie (2012), as already mentioned, consider populations with types. More recently, Biro et al (2012) describe an algorithm for finding a stable allocation that is based on satisfying blocking pairs at unstable allocations, and Andersson et al (2013) give a market procedure for finding a stable allocation that is based on constructing allocations with equal division and using overdemanded sets.

Our analysis and our results nowhere involve or require interpersonal comparison of utility. We handle the NTU aspect essentially through the Direction Lemma in Section 3 which we have adapted from our earlier work on the NTU assignment game.[20] For this reason, in a sense, the additional complexity we incur in covering the NTU domain has essentially a "one-shot" cost that is associated with how the Direction Lemma works. There are though several aspects that require particular attention on the NTU domain and significant differences between the two models. The Equilibrium Set, for example, has special properties on the TU domain that do not hold on the NTU domain. Still a substantial part of our work would nearly be the same if restricted to the TU domain - for example, our characterization of the Seller-Market and identifying it on the Procedure path or our Bargaining Set analysis.

In comparison to the existing literature, our work (i) covers the NTU domain, (ii) gives a characterization for the existence of stable allocations via seller-markets and aspirations, (iii) explores

[19]Utilized as such by Roth, Sonmez and Unver (2005) and Yilmaz (2011).

[20]Where it has sometimes been referred to as the Perturbation Lemma : Alkan (1989, 1992, 1997), Alkan, Demange and Gale (1991), Alkan and Gale (1990). There are few other papers on the NTU assignment game : Moldovanu (1990), Kucuksenel (2011).

in detail what may happen when stable allocations do not exist and describes two independent extended solutions, (iv) shows several properties of the Equilibrium Set and (v) gives a Market Procedure fully spelled out. The analysis we offer is entirely self-contained and utilizes a characterization of Seller-Markets via solitary-minimal matchings.

2 Model and Basic Definitons

A *Pairing Game* is a triplet (N, r, f) where N is a finite set of *players*, the vector $r = (r_i)$ gives the *stand alone* utilities of players, and the array $f = (f_{ij})$ consists of *partnership* functions for pairs of players : $f_{ij}(u_j)$ is the utility u_i which i achieves as partner of j when j achieves the utility u_j. In particular

$$f_{ij} = f_{ji}^{-1}.$$

We assume f_{ij} are continuous decreasing functions and $f_{ij}(r_j) < \infty$.

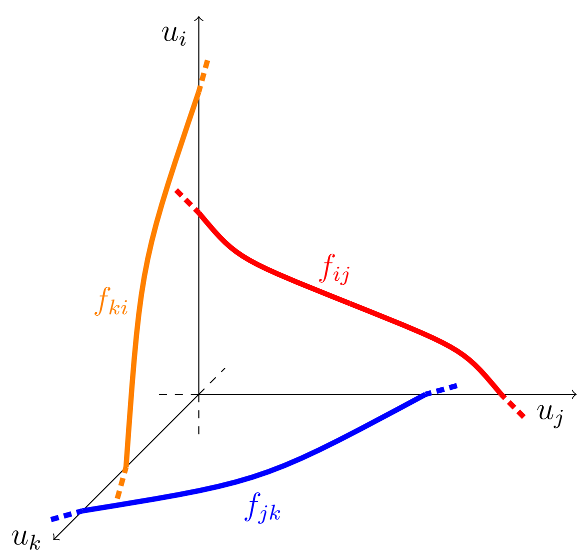


Figure 1: Partnership Functions

A *matching* is a set of pairs where each player is in at most one pair. A player who does not belong to any pair in a matching μ is *unmatched* at μ. A *payoff* is a vector $u \in R^N$ that assigns a utility to each player. We say that a payoff u is *realizable* if there is a matching μ where

$$u_i = f_{ij}(u_j) \text{ for } ij \in \mu$$

and $u_i = r_i$ for i unmatched at μ. An *allocation* is a pair (u, μ) where u is a payoff realizable by the matching μ. We will mostly suppress the particular matching μ at an allocation (u, μ) and refer to the "realizable payoff" u as an allocation.

An allocation u is *individually rational* if $u \geq r$. A pair ij is a *blocking pair* at an allocation u if there exists $(u'_i, u'_j) > (u_i, u_j)$ satisfying $u'_i = f_{ij}(u'_j)$. An allocation u is a *stable* (or *core*) allocation if it is individually rational and there is no blocking pair at u.

The triplet (N, r, f) is at the same time a *Pairing Market* : An allocation u is a *competitive equilibrium* allocation if it is individually rational and $u_i \geq f_{ij}(u_j)$ for every j. We show below that a stable allocation is equivalently a competitive equilibrium allocation. The notion of an "aspiration" is essential :

An *aspiration* is a payoff u where

$$u_i = \max\{r_i, \max f_{ij}(u_j)\} \text{ for all } i.$$

Thus, an aspiration is a vector that assigns to each player the *maximum-utility* (or *price*) she can achieve, through some partnership or by standing alone, given all the other maximum-utilities (or prices.) One can construct an aspiration in $|N|$ simple steps : Order the players in any way and let N_k be the top k players in that order. Let u_1 be the stand alone utility r_1 of the first player and step by step let $u_k = \max\{r_k, \max_{j\in N-N_k} f_{ij}(u_j)\}$ for the remaining players.

By definition an aspiration that is realizable is a competitive equilibrium allocation. And, a competitive equilibrium allocation u is a stable allocation - since otherwise $(u'_i, u'_j) > (u_i, u_j)$ for some $u'_i = f_{ij}(u'_j)$ but then $u_i < u'_i = f_{ij}(u'_j) < f_{ij}(u_j)$. Also, a stable allocation is an aspiration (otherwise there is a blocking pair) that is realizable. Thus, a stable allocation, a competitive equilibrium allocation and a realizable aspiration are mutually equivalent.

We let $r = 0$ with no loss of generality and regard (N, f) as describing a Pairing Game and a Pairing Market fixed in the rest of the paper.

2.1 An Extension : Half-Partnerships

Stable or competitive equilibrium allocations do not always exist. Here we give an extension of our model where, as we will show, they exist and are equivalent.

The extension is in the notion of realizability : We allow a player to have at most *two* half-partners as an alternative to *one* full-partner, understanding that half-partnership is reciprocal, namely, a player i is half-partner to j if and only if j is half-partner to i. We will assume that a pair of players i, j can achieve the "half-partnership utilities" $(v_{ij}, v_{ji}) = (h_{ij}(v_{ji}), h_{ji}(v_{ii}))$ through the "half-partnership functions" h_{ij} that satisfy

$$h_{ij}(z) = f_{ij}(2z)/2 \text{ for all } z$$

(*constant-returns-to-scale*) and that the utility of a player with two half-partners is the sum of her half-partnership utilities (*separability.*) Under these assumptions, the definition of an *aspiration* carries over "unchanged": a player i "aspires" to have player j as a half-partner (under the expectation that she will have a second half-partner) or as a full-partner if and only if $f_{ij}(u_j) \geq 0$ and

$$f_{ij}(u_j) \geq f_{ij'}(u_{j'}) \text{ for all } j'.$$

We note that there will exist a "full-partnership" blocking pair at any allocation which is not an aspiration. In particular, any allocation u where there is a player i with only one half-partner (and $u_i > 0$) is not an aspiration (since $2u_i > u_i$) therefore will be blocked by some pair ij. Therefore, a player who has only one half-partner at a "stable" allocation can only be a nonactive player. We will show that if there is no stable allocation then there is a "stable" allocation where every player has one full-partner, two half-partners or no partner. For simplicity we will just assume that allocations do not admit single half-partnerships.

Formally, a *half-matching* χ is a nonempty set of pairs where every player in χ belongs to two pairs, in other words, has two distinct *half-partners*. A payoff u is an *allocation* if it is realizable under half-partnership, that is to say, if there is a matching ν and a half-matching χ that have no player in common and an array (v_{ij}) of half-partnership utilities, such that

$$u_i = f_{ij}(u_j) \text{ for } ij \in \mu,$$

$$u_i = h_{ij}(v_{ji}) + h_{ij'}(v_{j'i}) \text{ for } ij, ij' \in \chi,$$

and $u_i = 0$ for i unmatched at (ν, χ). We call an allocation (u, ν, χ) *semistable* if there is no blocking pair at u. It is easily seen that a semistable allocation u is equivalently a realizable aspiration, i.e., a competitive equilibrium allocation. In particular, a player i in half-partnership with j, j' at a semistable allocation is *indifferent* between half-partnership and having either half-partner as full-partner, i.e.,

$$u_i = h_{ij}(v_{ji}) + h_{ij'}(v_{j'i}) = f_{ij}(u_j) = f_{ij'}(u_{j'}).$$

A player in half-partnership belongs to a unique "cycle" of players where each player is half-partner to her two neighbors in the cycle.

The notion of a half-matching was introduced by Pulleyblank (1973) and Tan (1990) in the context of the roommate problem. Half-matchings and fractional matchings have recently been more closely incorporated into economic modelling and design. Half-time and part-time relationships are a natural example as Biro and Fleiner (2012) and Manjunath (2011) discuss in more detail.

2.2 Active-Minimal Matchings

Let u be an aspiration. We say that players i and j *demand* each other at u if

$$u_i = f_{ij}(u_j).$$

The set of all pairs ij who demand each other at u, $\mathcal{D}(u)$, is the *demand graph* at u. We say that a matching

$$\mu \subset \mathcal{D}(u)$$

is a *matching at* u. For any $S \subset N$, we denote $\mathcal{D}_S(u) = \{ij \in \mathcal{D}(u) | i \in S\}$. We call $D_i(u) = \{j | ij \in \mathcal{D}(u)\}$ the *demand set* of i. $D_S(u)$ is the union of the demand sets of S-players. A player i is *active* at u if

$$u_i > 0.$$

Let μ be a matching at u. We say that a player i is *active-unmatched* at $[u, \mu]$[21] if i is active at u and unmatched at μ. We denote A^μ the set of active-unmatched players at $[u, \mu]$ and define

$$\alpha(u) = \min_{\mu} |A^\mu| .$$

We say μ is *active-minimal* at u if $|A^\mu| = \alpha(u)$. That is, a matching at u is active-minimal if it leaves a minimum number of active players unmatched, equivalently, if it matches a maximum number of active players.[22]

We note that an aspiration u is realizable - i.e., u is an allocation, hence a stable allocation - if and only if active players are matchable, i.e.,

$$\alpha(u) = 0.$$

We will say that an aspiration u is *nonrealizable* if $\alpha(u) > 0$. Our approach is based on identifying "seller-markets" at nonrealizable aspirations which we introduce in the next subsection.

2.3 Submarkets

For any matching μ and *player set* $S \subset N$, we denote the set of all μ-partners of S-players

$$\mu(S) = \{j | ij \in \mu, i \in S\}$$

and say that μ *matches* S (*into* T and *to* T, respectively) if

$$|\mu(S)| = |S|$$

(and $\mu(S) \subset T$ and $\mu(S) = T$, respectively.) We say S is *matchable* (*into* T and *to* T, respectively) if there is a matching μ that matches S (*into* T and *to* T, respectively.)

[21] Notice the use of "brackets" : $[u, \mu]$ may or may not be an allocation (u, μ). When there is no possibility of confusion, we will say "at μ" instead of "at $[u, \mu]$".

[22] A matching at u is *maximal* if it contains a maximum number of pairs. An active-minimal matching is maximal *unless* it can be augmented to a matching that contains *two* additional *nonactive* players. There is always a maximal matching which is active-minimal.

Definition 1 *A pair of player sets (B, S) is a* submarket *at u if (i) B consists of active players, (ii) the demand set of every B-player is in S, and (iii) S is matchable into B.*

We make frequent use of the definitions in the paragraph below which are quite standard :

Let μ be a matching at an aspiration u and i be an active-unmatched player. We say j is *μ-reachable from i* if there is a sequence of distinct players

$$i_0, i_1, ..., i_{n-1}, j_1, ..., j_n$$

where $i_0 = i, j_n = j$, $i_{k-1}j_k \in \mathcal{D}(u)$, and

$$i_k j_k \in \mu,$$

for every $k \leq n-1$. Let $i_0, i_1, ..., i_{n-1}, j_1, ..., j_n$ be such a sequence from $i = i_0$. If j_n is unmatched, then μ can be *augmented* to the matching that contains the pairs $i_{k-1}j_k$ (instead of the pairs $i_k j_k$) and matches at least one more active player. If j_n is matched with a nonactive player, then the matching μ can be *alternated* to the matching that contains the pairs $i_{k-1}j_k$ (instead of the pairs $i_k j_k$ and $j_n\mu(j_n)$) and matches one more active player.[23]

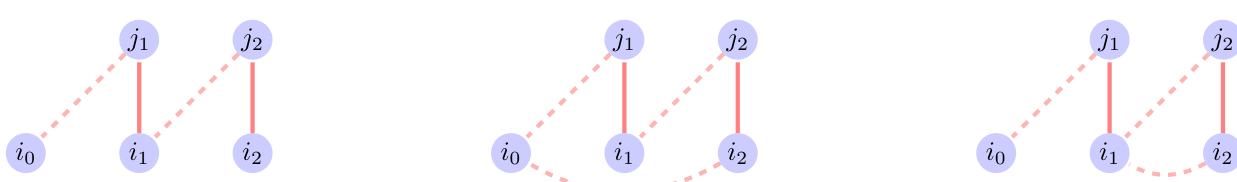
Let μ be an active-minimal matching at an aspiration u and i be an active-unmatched player.

We refer to the sequence $i_0, i_1, ..., i_n, j_1, ..., j_n$ (where $i_0 = i, j_n = j$, $i_{k-1}j_k \in \mathcal{D}(u), i_k j_k \in \mu$) as a *$\mu$-sequence* from i_0; we say it is *cycle-free* if there is no player i_m such that

$$i_m i_n \in \mathcal{D}(u),$$

and call it *cyclic* or a *$\mu-cycle$* if

$$i_0 i_n \in \mathcal{D}(u).$$



(a) cycle-free (b) cyclic (c) not cycle-free

Figure 2: μ-sequences

[23] A characterization statement for active-minimal matchings, similar to the characterization for maximal matchings by Berge (1957), would say : A matching μ is active-minimal if and only if every μ-reachable player from an active-unmatched player is matched with an active player.

The pair of player sets (I, J) where J is the set of all μ-reachable players from i and $I = i \cup \mu(J)$ is clearly a submarket at u. This is because (i) if I contained any nonactive player then μ could be alternated to match an additional active player, and by "reachability" (ii) the demand sets of I-players are in J, (iii) $\mu(J) \subset I$. We refer to the submarket (I, J) as the *μ-market* from i or as the *μ^i-market* at u.

We call a submarket (B, S) at u *bipartite* if $B \cap S$ is empty, that is, if the players $B \cup S$ partition into *buyers* and *sellers*. (Note that a bipartite submarket is not exactly a "two-sided buyers-sellers" market because a seller may demand a seller.) Bipartite submarkets and bipartite μ-markets play a central role in our work.

Example 1 *Suppose there are three players i_0, i_1, j_1 all active at an aspiration u where i_0, j_1 and i_1, j_1 demand each other but i_0, i_1 do not. Consider the matching $\mu = \{i_1 j_1\}$. The μ-sequence i_0, i_1, j_1 reaches j_1 from i_0. In fact the μ^{i_0}-market is $(I, J) = (\{i_0, i_1\}, \{j_1\})$ which is bipartite. Now suppose i_0, i_1 demand each other as well at u. In this case, the μ^{i_0}-market is $(\{i_0, i_1, j_1\}, \{i_1, j_1\})$ and not bipartite (and there exists no stable allocation.)*

We will make use of the following straightforward fact.

Proposition 1 *A μ^i-market is bipartite if and only if every μ-sequence from i is cycle-free.*

It will also be useful to note the following fact.

Lemma 1 *If (B, S) is a bipartite submarket at u and μ is an active-minimal matching, then μ leaves no S-player unmatched.*

Proof. Otherwise μ is not active-minimal : By definition, there is a matching ν that matches S into B. The matching μ' that agrees with ν for S-players and with μ for other players matches more active players than μ does. ■

Definition 2 *We call $|B| - |S|$ the* excess *in (B, S). A* seller-market *at u is a bipartite submarket with positive excess.*[24] *A* balanced-market *at u is a bipartite submarket with zero excess.*

If there is a seller-market at u, we say that u *has* a seller-market or that u is an aspiration *with* a seller-market.

It is clear that if u has a seller-market then it is not a stable allocation. As we will show, on the other hand, a seller-market at u points the way to an aspiration with no seller-market.

We will show that a μ-market is bipartite hence a seller-market, provided μ is "solitary-minimal", a property for active-minimal matchings that we introduce in the next section.

[24] This definition is closely related to the definition of an *overdemanded set* in Demange, Gale and Sotomayor (1986).

3 Existence of Settled Aspirations : Stable and Semistable Allocations

Here we introduce "settled aspirations" and prove their existence. We then give a characterization for the existence of stable allocations and show that if there is no stable allocation then there is always a semistable allocation. In between, we introduce the notion of a solitary-minimal matching and characterize "the Seller-Market" at an aspiration by a solitary-minimal matching. We also show that stable allocations always exist when there are an even number of players in each type.

What a stable allocation and a semistable allocation have in common is having no seller-market. We call an aspiration *settled* if it has no seller-market.

3.1 Settled Aspirations

Theorem 1 *There exists a settled aspiration.*

The proof uses the key result below.

We call a nonzero vector $d \in R^N$ a *feasible direction* at u if $u + \lambda d$ is an aspiration for all sufficiently small $\lambda > 0$. We say (N, f) is *piecewise linear* if all the partnership functions f_{ij} are piecewise linear.

Lemma 2 *(*Direction Lemma*) Let (N, f) be piecewise linear. If (B, S) is a bipartite submarket at an aspiration u, then there is a feasible direction d with*

$$d_i < 0 \text{ for } i \in B,$$

$$d_i > 0 \text{ for } i \in S,$$

$$d_i = 0 \text{ for } i \in N - B \cup S$$

such that (B, S) is a bipartite submarket at $u + \lambda d$ for all sufficiently small $\lambda > 0$.

Proof. (Theorem 1) Suppose (N, f) is piecewise linear. For any aspiration u and any seller-market (B, S) at u, let $g_S(u)$ be the sum of u_i for $i \in S$, and let $g(u)$ be the maximum of $g_S(u)$ over all seller-markets at u. Since the set of aspirations is nonempty and closed, there is an aspiration u^* such that $g(u^*)$ is maximum among all aspirations. Then u^* has no seller-market, for otherwise by the Direction Lemma, there is an aspiration u' with $g(u') > g(u^*)$ contradicting maximality of u^*. So there exists a settled aspiration for every piecewise linear (N, f) and by uniform approximation for (N, f). ■

We define, for any two aspirations u, u', the following two disjoint sets

$$N^+_{uu'} = \{i | u_i > u'_i\},$$

$$N^-_{uu'} = \{i | u_i < u'_i\}.$$

Note

$$N^+_{u'u} = N^-_{uu'}.$$

The demand set of any $N^+_{uu'}$-player is in $N^-_{uu'}$:

Lemma 3 $D_{N^+_{uu'}}(u) \subset N^-_{uu'}$.

Proof. If i demands j at u and u' is an aspiration with $u'_i < u_i$, then $u'_j \geq f_{ji}(u'_i) > f_{ji}(u_i) = u_j$. ■

Lemma 4 *Let u be a settled aspiration and u' be any aspiration. Then $N^+_{uu'}$ is matchable into $N^-_{uu'}$ at u.*

Proof. Every player in $N^+_{uu'}$ is active (otherwise $u'_i < 0$ for some $i \in N^+_{uu'}$ hence u' is not an aspiration.) Suppose $N^+_{uu'}$ is not matchable into $N^-_{uu'}$ at u. Let μ be a matching at u that matches a maximum number of players in $N^+_{uu'}$ and let i be a player unmatched. Let (B, S) be the μ^i-market at u. By Lemma 3 and maximality of μ, using induction, $S \subset N^-_{uu'}$ and $\mu(S) \subset N^+_{uu'}$. But then (B, S) is a seller-market at u. Contradiction. ■

We now have the following result as analog of the "Decomposition Lemma" in the two-sided market literature.

Proposition 2 *Let u, u' be any two settled aspirations. Then $(N^+_{uu'}, N^-_{uu'})$ is a balanced-market at u.*

Proof. By Lemma 4, $N^+_{uu'}$ is matchable into $N^-_{uu'}$ at u and symmetrically $N^+_{u'u}$ is matchable into $N^-_{u'u}$ at u'. Then, $(N^+_{uu'}, N^-_{uu'})$ and $(N^+_{u'u}, N^-_{u'u})$ are bipartite submarkets at u and u' respectively, so $|N^+_{uu'}| = |N^-_{uu'}|$, therefore they are balanced-markets. ■

Proposition 3 *Let u, u' be any two settled aspirations. Then u is a stable allocation if and only if u' is a stable allocation.*

Proof. Suppose (u, μ) is a stable allocation. Then μ matches $N^+_{uu'}$ and $N^-_{uu'}$ to each other (otherwise μ leaves a player i in $N^+_{uu'}$ unmatched, which is not possible, because i is active). So

$$\mu(N^0_{uu'}) \subset N^0_{uu'},$$

where $N^0_{uu'} = N - (N^+_{uu'} \cup N^-_{uu'})$. Let μ_0 be the set of all pairs $ij \in \mu$ with $i, j \in N^0_{uu'}$. By Proposition 2, there is a matching ν at u' that matches $N^+_{uu'}$ and $N^-_{uu'}$ to each other. The matching that agrees with μ_0 for $N^0_{uu'}$-players and with ν otherwise is u'-compatible and leaves no active player unmatched. So u' is a stable alloaction. ■

A settled aspiration may be nonrealizable as in Example 1. The statement below gives a characterization for the existence of stable allocations; the "if" part is a restatement of Theorem 1 and the "only if" part follows from Proposition 3.

Theorem 2 *There is a stable allocation if and only if there is no nonrealizable settled aspiration.*

This characterization to be sure is not based on the primitives in our model. Still it identifies a local occurrence that is conclusive regarding the existence of a stable allocation. It provides the stopping rule in our Market Procedure (Section 6.)

In Section 3.3, we show by construction that a nonrealizable settled aspiration is a semistable allocation. The construction utilizes "solitary-minimal" matchings that we introduce in the subsection below.

3.2 Solitary-Minimal Matchings and the Seller-Market

The definitions and results we give here play a fundamental role throughout our paper.

Let u be an aspiration. Let μ be an active-minimal matching at u. Consider the set of active-unmatched players A^μ at μ.

We call a player $i \in A^\mu$ *nonsolitary* or *solitary* at μ depending on whether there is a μ^i-cycle or no μ^i-cycle at u. We denote $A^\mu_{\mathbf{s}}$ the set of all solitary players at μ and define

$$\sigma(u) = \min \left\{ |A^\mu_{\mathbf{s}}| \, |\mu \text{ is active-minimal at } u \right\}.$$

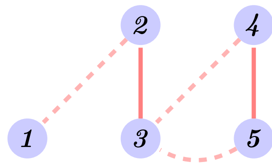
We say μ is *solitary-minimal* at u if $|A^\mu_{\mathbf{s}}| = \sigma(u)$. Thus a solitary-minimal matching is an active-minimal matching at which the number of solitary players is minimum. We will call $A^\mu_{\mathbf{s}}$ the *solitary set* at μ.

We will show that there is a seller-market at u if and only if there is a solitary player at some solitary-minimal matching at u, i.e., $\sigma(u) > 0$. Equivalently,

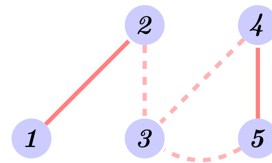
Theorem 3 *An aspiration u is settled if and only if $\sigma(u) = 0$.*

Theorem 3 follows from Theorem 4 which we state and prove below.

We call a seller-market (B, S) *unitary* if it has unit excess ($|B| - |S| = 1$) and S is matchable to $B - i$ for every $i \in B$.



(a) not solitary-minimal



(b) solitary-minimal

Figure 3: Active-Minimal Matchings

Proposition 4 *Let μ be a solitary-minimal matching at u. A μ-market from a solitary player is a unitary seller-market at u.*

It is clear that if a μ-market from a solitary player is bipartite then it is unitary. The proof of Proposition 4 essentially amounts to showing bipartiteness and is obtained by putting together the fact that solitary-minimal μ-sequences from solitary players are cycle-free (shown in the proof below), Proposition 1 and the following observation.

Lemma 5 *Let μ, μ' be any two active-minimal matchings at u. A player who is nonsolitary (solitary) at μ is either matched or nonsolitary (solitary) at μ'.*

Proof. (Proposition 4) Let μ be a solitary-minimal matching at u and i a solitary player at μ. Suppose the Proposition is false. Then the μ^i-market is not bipartite. So by Proposition 1 there is a μ-sequence $i_0, i_1, ..., i_n, j_1, ..., j_n$ from $i_0 = i$ and

$$i_m i_n \in \mathcal{D}(u)$$

for some player $i_m \neq i_0$. Alternate μ to μ' which matches i_0 but not i_n. Now i_n is nonsolitary at μ' because the μ'-sequence $i'_0, i'_1, ..., i'_{n-m}, j'_1, ..., j'_{n-m}$ from $i'_0 = i_n$ where $i'_{n-m} = i_m$ is cyclic. Note that, except for i_0 and i_n, the players who are unmatched at μ and μ' are identical, hence by Lemma 5, the players who are solitary at μ and μ' are identical except i. Thus, μ' has one less solitary player than μ, contradicting the fact that μ is solitary-minimal. ■

We now prepare for Theorem 4 below which is our main result in this subsection.

Let μ be any solitary-minimal matching at u. Let (B^μ, S^μ) be the union of all μ-markets from $A^\mu_{\mathbf{s}}$-players. We refer to a μ-market from an $A^\mu_{\mathbf{s}}$-player as a *solitary-player-market*. Thus (B^μ, S^μ) is the union of all solitary-player-markets at μ. It is straightforward to see that S^μ is the set of all μ-reachable players from the solitary set $A^\mu_{\mathbf{s}}$ and $B^\mu = A^\mu_{\mathbf{s}} \cup \mu(S^\mu)$. In particular, (B^μ, S^μ) is a seller-market.

Now let (B^*, S^*) be the union of all unitary seller-markets at u. It is easily seen that (B^*, S^*) is a seller-market. (It is in general not true that the union of two seller-markets is a seller-market : Consider, for example, an aspiration u where $\mathcal{D}(u) = \{i_1i_3, i_2i_3, i_3i_4, i_4i_5\}$ among five active players. Both $(\{i_1, i_2, i_4\}, \{i_3, i_5\})$ and $(\{i_1, i_2, i_5\}, \{i_3, i_4\})$ are seller-markets but not their union $(\{i_1, i_2, i_4, i_5\}, \{i_3, i_4, i_5\})$. Note that each of the two seller-markets contains the unitary seller-market $(\{i_1, i_2\}, \{i_3\})$.)

Theorem 4 says that, at any aspiration, the union of all unitary seller-markets is equal to the union of all solitary-player-markets at any solitary-minimal matching. We will need the following fact.

Lemma 6 *Let (B, S) be any bipartite submarket at u. (B, S) has no nonsolitary player at any active-minimal matching μ at u.*

Proof. Suppose to the contrary that a bipartite submarket (B, S) at u has a nonsolitary player i at μ, where μ is an active-minimal matching at u. By Lemma 1 every S-player is matched at μ therefore $i \in B$. So there is a μ-cycle $i_0, i_1, ..., i_n, j_1, ..., j_n$ from $i_0 = i$ where $j_1 \in S$. Then, by alternation, there is an active-minimal matching μ' at u that leaves j_1 unmatched, contradicting Lemma 1. ■

Theorem 4 *The seller-market (B^*, S^*) at u is identical to the seller-market (B^μ, S^μ) for any solitary-minimal matching μ at u.*

The proof follows from Proposition 4 and Lemma 6 :

Proof. Let μ be any solitary-minimal matching at u. By Proposition 4 $(B^\mu, S^\mu) \subset (B^*, S^*)$. We will show that (B^μ, S^μ) contains every unitary seller-market which completes the proof.

Let (B, S) be any unitary seller-market at u, B_0 be the set of all B-players unmatched at μ, S' be the set of all μ-reachable players from B_0-players and $B' = B_0 \cup \mu(S')$. By Lemma 6 $B_0 \subset A^\mu_{\mathbf{s}}$ so $(B', S') \subset (B^\mu, S^\mu)$. We will show $(B, S) = (B', S')$.

By construction, no B'-player has demand for any player in $S - S'$ (since $S - S'$ is "unreachable" from B_0). Also $\mu(B - B') \subset (S - S')$ since $\mu(S') \subset B'$. Therefore μ matches $B - B'$ to $S - S'$ (otherwise S is not matchable into B.) Then $B - B'$ and $S - S'$ must be empty because (B, S) is unitary (otherwise S is not matchable to $B - i$ for some $i \in B - B'$.). ■

Theorem 3 directly follows from Theorem 4.

Theorem 4 holds only in the aggregate. A unitary seller-market need not be a solitary-player-market :

Example 2 *Consider an aspiration u where $\mathcal{D}(u) = \{i_1i_4, i_2i_4, i_3i_4\}$ among four active players i_1, i_2, i_3, i_4. Let $B = \{i_1, i_2\}$ and $S = \{i_4\}$. Then (B, S) is a unitary seller-market but not a solitary-player-market at the solitary-minimal matching $\mu = \{i_3i_4\}$.*

We call (B^*, S^*) the *Seller-Market* at u. The Market Procedure we give in Section 6 for finding a settled aspiration is a Seller-Market tracing procedure. We will use the following fact in proving its convergence.

The excess in the Seller-Market is equal to the size of a solitary set :

Corollary 1 $|B^*| - |S^*| \ = \sigma(u)$.

The statement below gives a characterization for solitary-minimal matchings ; the "only if" part follows from the proof of Proposition 4 and "if" part follows from Theorem 4.

Corollary 2 *An active-minimal matching μ is solitary-minimal if and only if all the μ-sequences from solitary players are cycle-free.*

3.3 Stable and Semistable Allocations

Let u be a nonrealizable settled aspiration and μ be a solitary-minimal matching at u. By Theorem 3, active-unmatched players are nonsolitary. Let C_i be a μ^i-cycle for each nonsolitary player at μ. Then

$$C_i \cap C_{i'} = \varnothing$$

for $i \neq i'$, since otherwise μ can be augmented to a matching where i, i' are matched, contradicting the fact that μ is active-minimal.

Proposition 5 *A nonrealizable settled aspiration is a semistable allocation.*

The proof is constructive :

Proof. Let u be a nonrealizable settled aspiration and μ be a solitary-minimal matching at u. Then there is at least one nonsolitary player at μ. For every nonsolitary i, pick a μ-cycle $C_i = i_0, i_1, ..., i_n, j_1, ..., j_n$. Let μ_i and ν_i respectively be the matchings that consist of all pairs i_kj_k and $i_{k-1}j_k$ with $j_k \in C_i$. Let $\nu_i' = \nu_i \cup i_0i_n$ and μ^+, ν^+ respectively be the union of μ_i, ν_i' over all unmatched i. Denote ν the matching $\mu - \mu^+$ and χ the half-matching $\mu^+ \cup \nu^+$. Then (ν, χ) matches every active player at u. So u is a semistable allocation. ■

We note that the construction in the proof above gives a unique semistable allocation (u, ν, χ) if and only if there is a unique μ-cycle from every nonsolitary player i at μ.

As we noted earlier, half-matched players at a semistable allocation (u, ν, χ) partition into cycles of players. The construction in the proof above always gives rise to *half-partner cycles* that contain an *odd* number of players. But *even* half-partner cycles may also exist : Consider, for example, an aspiration u where $\mathcal{D}(u) = \{i_1i_2, i_2i_3, i_3i_4, i_4i_1\}$ among four players and it is possible for every player to be half-matched. In particular there may exist a stable allocation (u, μ) and a semistable allocation (u, ν, χ).

Another example to coexistence of stable and semistable allocations is provided by an aspiration u where $\mathcal{D}(u) = \{i_1i_2, i_2i_3, i_3i_1\}$ and i_1, i_2 are active while i_3 is nonactive. Here there is a stable allocation with the full-partnership i_1i_2 as well as a semistable allocation with the half-partnerships i_1i_2, i_2i_3, i_3i_1.

We call a semistable allocation (u, ν, χ) *essential* if it has a minimum number of half-partner cycles among all allocations at u. So if there is a stable allocation then no semistable allocation is essential.

From Proposition 5 and Theorem 1, it follows that there exists a stable allocation or a semistable allocation. We now have the following stronger version :

Corollary 3 *There exists either a stable allocation or an essential semistable allocation.*

3.3.1 Even Populations

Let us say that players i and i' are of the same *type* if

$$f_{ij} = f_{i'j}$$

for all players j other than i' and i.[25]

Theorem 5 *There is a stable allocation if there are an even number of players of each type.*

Proof. By Proposition 1, there is a settled aspiration, say u. We will show that u is a stable allocation. Suppose not.

Let μ be a solitary-minimal matching at u. Since u is nonrealizable, μ leaves an active player unmatched, say i. Note that i is *nonsolitary* at μ, because otherwise by Proposition 4 u has a seller-market and is not settled.

Let C be a μ-cycle from i which consists of a maximum number of players. Since there are an *odd* number of players in C, there must be two same-type players, say j, j', such that j is in C and j' is not in C. We claim

$$u_j = u_{j'}.$$

[25]If there are only two players in a type, say i, i', then $f_{ii'}$ may be any partnership function. If there are more than two, then $f_{ii'}$ is neccessarily "symmetric" with respect to equal utility realization.

If not, then $u_j < u_{j'}$ (otherwise no C-player demands j at u contradicting $j \in C$). Then no player other than j demands j' at u. So $D_{j'}(u) = \{j\}$ (otherwise $D_{j'}(u)$ is empty but j' is active at u). Hence j' is unmatched at u. But then μ is augmentable contradicting the fact that μ is active-minimal. End of claim.

Therefore there is a C-player who demands j' at u. Then j' must be matched at u (otherwise μ is augmentable therefore not active-minimal). But then there is a μ-cycle from i (obtained by "adding" the pair $(j', \mu(j'))$ to C) which contains a greater number of players. Contradiction. ■

When there are an even number of players in each type, it is easily seen that, there is a stable allocation where same-type players get the same payoff. This is not true at every stable allocation, as is evident by considering a two-player Game. When there are more than two players in each type, though, it is easily shown that same-type players get the same payoff at every stable allocation.

Theorem 5 generalizes to the NTU domain the main result of Chiappori, Galichon and Salanie (2012). (In a general population, clearly, leaving out *any* one player in each type with an odd number of players would give a Pairing Game with an even population for which Theorem 5 holds.) We would like to mention that Theorem 5 can be gotten in two other ways. One of these involves the fact that the stable allocations of a two-fold Pairing Market/Game coincide with the stable allocations of the two-sided Market/Game which has one copy of each type.[26] The other way is to set up a similar equivalence in the extended model with half-partnerships and use Proposition 5.

4 Properties of the Set of Settled Aspirations

We have shown that a settled aspiration always exists and that a settled aspiration is either a stable allocation (with full-partnerships) or an essential semistable allocation (with some or all half-partnerships.) Each settled aspiration is at the same time a competitive equilibrium allocation. It is appropriate to think of the set of all settled aspirations as the *Extended Core* or the *Equilibrium Set*. Let us denote this set U. By Corollary 3, U consists entirely of stable allocations or of essential semistable allocations. For simplicity we will assume that all semistable allocations that form are essential.

In this section we show some properties that U has independent of whether it consists of stable or semistable allocations. We also show that each settled aspiration is "active-minimal" in the sense that the number of unmatchable active players is minimum among all aspirations.

Clearly U is a closed bounded set.

[26] As Chiappori, Galichon and Salanie (2012) do on the TU domain.

4.1 Virtual Convexity

Say that a vector $\overline{z}$ is *between* two vectors z, z' if

$$\overline{z}_i \in (\min\{z_i, z'_i\}, \max\{z_i, z'_i\})$$

in case $z_i \neq z'_i$ and $\overline{z}_i = z_i = z'_i$ otherwise. Call an arbitrary set Z in R^N *virtually convex* if for every $z, z' \in Z$ there is a $\overline{z} \in Z$ that is between z, z'.

Proposition 6 *The Equilibrium Set is a virtually convex set.*

Proof. Suppose U is the Equilibrium Set of a piecewise linear Game. Take any u, u' in U. By Proposition 2 the pair of player sets $(N^+_{uu'}, N^-_{uu'})$ is a balanced-market and by Lemma 2 there exist payoffs between u, u' that belong to U. By uniform approximation, the result holds for any Game. ■

A virtually convex set is equivalently (i) a set such that any two elements in the set are connected by a continuous "*monotone*" path that lies in the set, or (ii) a set such that any point outside can be *separated* from the set by an *orthant*. See Alkan and Gale (1990). It is straightforward that U is a convex polyhedral set if the partnership functions f_{ij} are linear.

4.2 Constant vs Free Players and Median Settled Aspirations

Let us call a player i a *constant player* if $u_i = u'_i$ for every u, u' in U and call i a *free player* otherwise.

Proposition 7 *Every stable or semistable allocation at any settled aspiration fully matches a free player with a free player.*[27]

Let us denote the sets of constant and free players C and F respectively. Proposition 7 says that, at any equilibrium, each player in half-partnership is in C and each player in F is in full-partnership with a player in F. It is worthwhile to add the following observation : Consider the Market/Game restricted to constant players, i.e., (C, f). It is easily seen that the Equilibrium Set of (C, f) is identical to $U^C = \{(u_i)|i \in C, u \in U\}$. The Equilibrium Set of (F, f) on the other hand is in general a superset of $U^F = \{(u_i)|i \in F, u \in U\}$. For example, when $N = \{\mathbf{1,2,3}\}$ and the worth of a pair is 3 for $\{\mathbf{1,2}\}$ and 1 otherwise, $U = \{x, 3-x, 0\}$ where $1 \leq x \leq 2$. $F = \{\mathbf{1,2}\}$ and the Equilibrium set of (F, f) is $\{x, 3-x\}$ where $0 \leq x \leq 3$.

[27] Recall the players labelled *independent* in the Gallai-Edmonds Decomposition Theorem. Every free player is an independent player except possibly at the boundary of U.

We next show that the Equilibrium Set U always contains a settled aspiration that is a *generalized median* of any finite collection of settled aspirations. As Schwarz and Yenmez (2011) have shown, in the two-sided TU case, U always contains the median of any odd collection and the upper and lower median of any even collection.[28]

Let K be any finite set consisting of k payoff vectors. Let $m = k/2$ for k even and $m = (k+1)/2$ for k odd. For every player i, order her k payoffs over K in any nonincreasing way. Let $\overline{v}_i$ be her mth (weakly) highest payoff. Now let v_i be her $m+1$st highest payoff for k even and mth highest payoff for k odd. Note that $v_i \leq \overline{v}_i$ in general and that $v_i = \overline{v}_i$ for k odd. We define

$$med(K)_i = [v_i, \overline{v}_i].$$

Proposition 8 *Given any finite collection of settled aspirations K, there is a settled aspiration $u \in U$ with $u_i \in med(K)_i$ for every i.*

4.3 Stable Bipartitions

For further insight on U, we ask whether free players separate into two sides anywhere at the variable Equilibrium Set $U^F = \{(u_i)|i \in F, u \in U\}$. We let $N = F$ below for simplicity.

We say that (N_1, N_2) is a *stable bipartition* at u if $N = N_1 \cup N_2$ and

$$\mu(N_1) = N_2$$

for every stable allocation (u, μ) and that (N_1, N_2) is a stable bipartition at U if it is a stable bipartition at every u in U.

Lemma 7 *There exist a stable bipartition at any u in U.*

Proof. Suppose there is no stable bipartition of N at some u in U. Let u' be any allocation where $N^+_{uu'}$ and $N^-_{uu'}$ contain a maximum number of players. Then, there exist a free player i such that $u_i = u'_i$ (otherwise $(N^+_{uu'}, N^-_{uu'})$ is a stable bipartition at u by Proposition 2). Let u'' be an allocation in U such that $u''_i \neq u'_i$. Pick any allocation $\overline{u}$ that is between u' and u'' and sufficiently close to u'. Since $u'_j \neq u_j$ for every j in $N^+_{uu'} \cup N^-_{uu'}$, by Proposition 6 $\overline{u}_j \neq u_j$ for every j in $N^+_{uu'} \cup N^-_{uu'}$. Also, $u_i \neq \overline{u}_i$ (since $u'_i = u_i$ and $u''_i \neq u_i$). Contradiction. ■

We next show that there always exists a stable bipartition at U on the transferable utility domain, that is, for *quasilinear* Games (N, f) where the partnership have the form

$$f_{ij}(u_j) = c_{ij} - u_j.$$

[28] Eriksson and Karlander (2001) show the median property for any *three* stable allocations.

Lemma 8 *If (u,μ) is a stable allocation at some u in U, then (u',μ) is a stable allocation at every u' in U.*

Proposition 9 *There exist a stable bipartition at U if (N,f) is quasilinear.*

Proof. By Lemma 7, let (N_1,N_2) be a stable bipartition at some u in U. By Lemma 8, (N_1,N_2) is a stable bipartition at U. ■

Proposition 9 tells us that, on the TU domain, the Equilibrium Set of a non-two-sided Market/Game is, constant players aside, two-sided and has essentially the same properties as the Equilibrium Set of an assignment game. In particular, with reference to Schwarz and Yenmez (2011) again, the Equilibrium Set of every non-two-sided TU Market/Game has a unique median settled aspiration.

On the NTU domain, on the other hand, Proposition 9 does not hold and Proposition 8 is particularly relevant. Below is a *heterogenously linear* (N,f) where there is no stable bipartition at U.

Example 3 *There are six players in $N=\{\boldsymbol{1},\boldsymbol{2},\boldsymbol{3},\boldsymbol{4},\boldsymbol{5},\boldsymbol{6}\}$. The partnership functions f_{ij} satisfy*

$$u_i = f_{ij}(u_j) = c_{ij} - q_{ij}u_j$$

where the pair (c_{ij},q_{ij}) is equal to

$$(15,2) \text{ for } ij \in \{\boldsymbol{12},\boldsymbol{23},\boldsymbol{31}\} \text{ and } (15/2,1/2) \text{ for } ij \in \{\boldsymbol{21},\boldsymbol{32},\boldsymbol{13}\}$$

$$(30,10) \text{ for } ij \in \{\boldsymbol{46},\boldsymbol{65},\boldsymbol{54}\} \text{ and } (3,1/10) \text{ for } ij \in \{\boldsymbol{64},\boldsymbol{56},\boldsymbol{45}\}$$

$$(10,1) \text{ for } ij \in \{\boldsymbol{14},\boldsymbol{41},\boldsymbol{25},\boldsymbol{52},\boldsymbol{36},\boldsymbol{63}\}$$

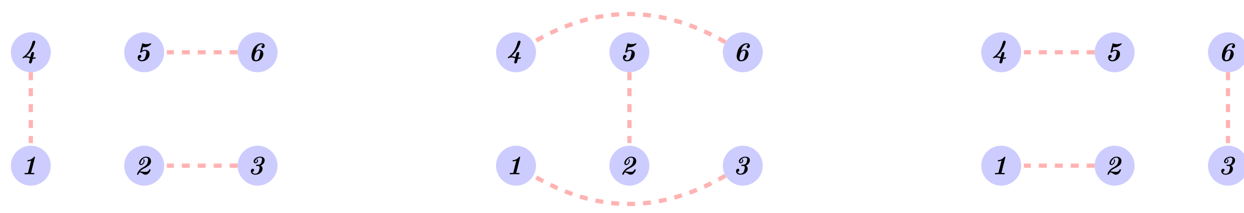
and $(0,0)$ otherwise. It is straightforward to check that the demand graphs at the three allocations

$$u=[7,9,3,3,2,10], u'=[3,7,9,10,3,2], u''=[9,3,7,2,10,3]$$

are

$$\mathcal{D}(u)=\{(\boldsymbol{14}),(\boldsymbol{23}),(\boldsymbol{56})\}, \mathcal{D}(u')=\{(\boldsymbol{13}),(\boldsymbol{25}),(\boldsymbol{46})\}, \mathcal{D}(u'')=\{(\boldsymbol{12}),(\boldsymbol{45}),(\boldsymbol{36})\}$$

respectively (see Figure 4) and that each allocation is realizable by a unique matching. It is also straightforward to check that there is no partition of N to two sides such that each of these matchings matches one side to the other.



(a) $\mathcal{D}(u)$ (b) $\mathcal{D}(u')$ (c) $\mathcal{D}(u'')$

Figure 4: Example 3

4.4 Settled Aspirations are Active-Minimal

Here we show that, at every settled aspiration, there is a full-partnership matching that leaves a minimum number of active players unmatched among all aspirations. We will make use of this property in the next section.

Formally, let us recall that a matching μ at an aspiration u is active-minimal if μ leaves unmatched a minimum number of active players, in other words, if $\alpha(u) = |A^{\mu}(u)|$. We call an aspiration u *active-minimal* if $\alpha(u) \leq \alpha(u')$ for every aspiration u'.

Proposition 10 *Every settled aspiration is active-minimal.*

There may exist active-minimal aspirations that are not settled : In fact consider any aspiration u in a three-player Market/Game where each player is active and $\mathcal{D}(u)$ consists of two pairs. Clearly u is not settled but active-minimal.

5 Pseudostable Allocations and Bargaining Set Stability

So far we have looked at Pairing Games from the point of view of the core and competitive equilibrium as solution concepts. We have shown that, in an environment where half-partnerships are viable, stable or semistable allocations always exist. More precisely, we showed that settled aspirations always exist and generate stable or semistable allocations, that are at the same time competitive equilibrium allocations.

In this section we adopt a broader perspective and consider what may happen in a Pairing Game when half-partnerships are not viable - in particular when there is no stable allocation. We will show that two particular types of allocations - “maximum-stable” and “pseudostable” allocations - have distinctive properties. The first of these is grounded on the active-minimality property we defined in the previous section. The latter is intimately related - as semistable allocations are - to solitary minimal matchings at nonrealizable settled aspirations. (Pseudostable allocations are maximum-stable.)

5.1 Maximum-Stable and Pseudostable Allocations

We need to define "aspiration-allocations" and "restricted-stable" allocations.

We use the following notation : For any payoff z and matching μ, we denote z^μ the payoff where

$$z_i^\mu = z_i \text{ for } i \in \mu(N) \text{ and } z_i^\mu = 0 \text{ for } i \notin \mu(N).$$

An allocation (v, μ) is an *aspiration-allocation* if v is *extendible* to an aspiration u, in other words, if there is an aspiration u such that

$$u^\mu = v.$$

Aspiration-allocations are abundant : Every aspiration u and every matching μ at u generates an aspiration-allocation u^μ.

Let (v, μ) be an allocation and consider any set of players T that contains or is equal to the set of all matched players $\mu(N)$. Note $v_i = 0$ for any T-player i not in $\mu(N)$. We say (v, μ) is *T-restricted-stable* if the restriction of v to T is a stable allocation in the "restricted" Game (T, f). (Note that players in $T - \mu(N)$ are nonactive.) We call (v, μ) *restricted-stable* if it is T-restricted-stable for some $T \supseteq \mu(N)$. A straighforward yet significant observation is the following which we state without proof :

Proposition 11 *An allocation is restricted-stable if and only if it is an aspiration-allocation.*

Now let (v, μ) be an allocation and T be the largest player set containing $\mu(N)$ such that (v, μ) is T-restricted-stable. Call the players in $N - T$ the *outcasts* of (v, μ).

Definition 3 *An allocation is* maximum-stable *if it is a restricted-stable allocation that has the minimum number of outcasts among all restricted-stable allocations.*

Maximum-stable allocations are the counterpart of the maximum stable matchings defined and proposed by Tan (1990) as a solution concept for roommate problems with empty core.[29]

Now let us recall the definition of an *active-minimal aspiration* given in the previous section. From Proposition 11, we have the following observation.

Corollary 4 *An allocation is maximum-stable if and only if it is an aspiration-allocation u^μ where u is active-minimal and μ is active-minimal.*

As mentioned above, aspiration-allocations form a very large class. It will be useful to identify certain (nested) subclasses of aspiration-allocations (see Figure 5) : Let us say that an allocation

[29] Also see Inarra, Larrea and Molis (2008).

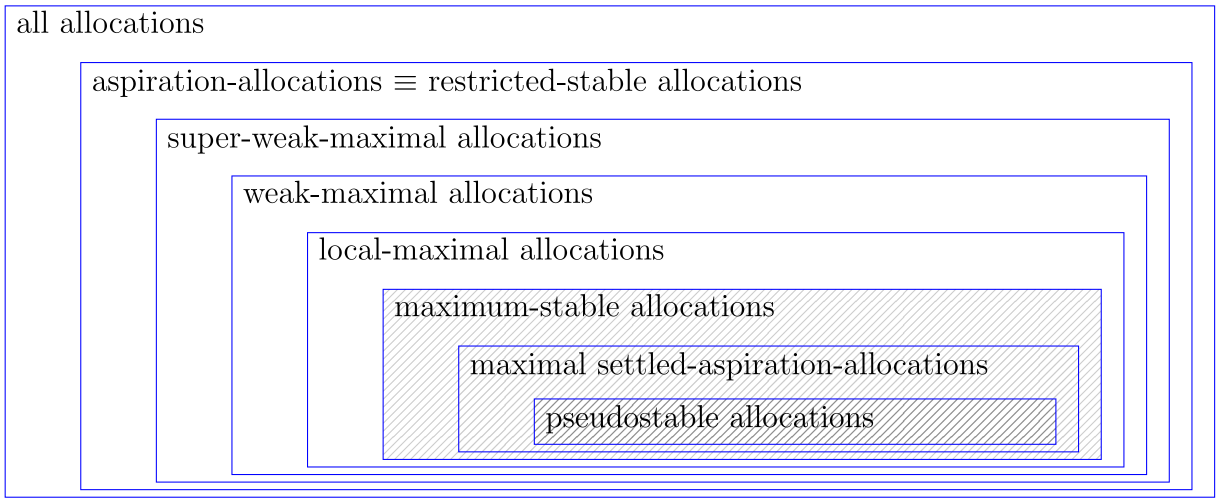


Figure 5: Allocation Types

(v, μ) is *weak-maximal* if v is extendible to an aspiration u where $u_i = u_j = 0$ for any $ij \in \mathcal{D}(u)$ such that $i, j \notin \mu(N)$. It is easily seen that a weak-maximal allocation has a *unique* extension to an aspiration. In words, then, a weak-maximal allocation is an aspiration-allocation where there is no active-unmatched player who has demand for an unmatched player. Weak-maximal allocations form a large class. An important subclass is the set of aspiration-allocations u^μ where μ is active-minimal; we call them *local-maximal*.[30]

We will refer to a local-maximal aspiration-allocation u^μ where u is settled as a *maximal settled-aspiration-allocation*. Since settled aspirations are active-minimal (Proposition 10), from Corollary 4 we have :

Corollary 5 *Every maximal settled-aspiration-allocation is a maximum-stable allocation.*

It follows from the observation following Proposition 10 that a maximum-stable allocation need not be a settled-aspiration-allocation.

We finally introduce the subclass of maximal settled-aspiration-allocations we are particularly interested in :

Definition 4 *An aspiration-allocation u^μ* pseudostable *if u is a nonrealizable settled aspiration and μ is solitary-minimal.*

In the subsection below, we look at a Pairing Game and pseudostable allocations from a Bargaining Set perspective.

[30] A subclass of local-maximal aspiration-allocations are the ones associated with solitary-minimal matchings. The Market Procedure we give in the next section operates on this class.

Remark 1 *Not surprisingly, pseudostable and semistable allocations are closely related. We now briefly describe this relationship while emphasizing that they pertain to different institutional environments : By definition, every nonrealizable settled aspiration u in U generates the set of pseudostable allocations*

$$PS(u) = \{u^{\mu} | \mu \text{ solitary minimal at } u\}.$$

It can be seen from the construction of a semistable allocation (in the proof of Proposition 5) that $PS(u)$ is related to the set of semistable allocations at u in the following manner : Let (u, ν, χ) be an essential semistable allocation at u and $H \subset N$ be the players in the half-matching χ. As previously noted, H partitions into n odd-cycles C_k where n is equal to the number of (nonsolitary) active-unmatched players at any solitary-minimal matching μ at u. In fact, μ is a solitary-minimal matching at u if

$$\mu = \nu \cup \nu_1 ... \cup \nu_n$$

where ν_k is a matching in C_k that leaves one player in C_k unmatched. In particular, there is a solitary-minimal matching μ at u for every selection of n players from $C_1 \times ... \times C_n$. Associated with each essential semistable allocation (u, ν, χ) then, we obtain a set of $|C_1| \times ... \times |C_n|$ solitary-minimal matchings or pseudostable allocations at u. $PS(u)$ is their union over all the essential semistable allocations at u.

5.2 Bargaining Set Stability

The three-person Pairing Game has been intensively studied for bargaining with pair formation and extensions of the core. To summarize briefly, let us first note that in *any* three-person Game where there is no stable allocation, there is a unique settled aspiration $u = (u_1, u_2, u_3)$ and - the *null allocation* $(0, 0, 0)$ aside - the aspiration-allocations generated by u are

$$(u_1, u_2, 0), (u_1, 0, u_3), (0, u_2, u_3).$$

Binmore (1985) put forward these three allocations as the "stable set" : He demonstrated that the triplet (u_1, u_2, u_3) is the only mutually consistent endogenous *outside-option* vector - when any two players may bargain and the outside player is a potential partner in case they cannot agree - from which follows the unique realizability of the three allocations above.[31] There is also the following two-step farsighted stability or Bargaining Set argument : Each of the three allocations if realized would survive - because a *prudent* player would not be lured into forming a blocking pair with

[31] See Footnote 10. Bennett (1997) has shown that there always exist "consistent endogenous outside-options" in general cooperative games, that they are aspirations, and that a *wide* set of aspirations turn out as the subgame perfect equilibrium outcomes of sequential offer games.

the odd-man-out since he could in turn become the odd-man-out. And no other allocation would survive.

The three-person case is of course a very special one. Our pursuit here is in the direction of whether there is a natural generalization of the three-person "stable set" for Pairing Games with any number of players. We give a partial answer. We show that pseudostable allocations always belong to the (two) Bargaining Sets that we consider. We then show that there are some non-pseudostable allocations that also belong and observe their properties.

There are several definitions and variants of what a Bargaining Set is. We employ two : the Zhou Bargaining Set (Zhou (1994)) and the Demand Bargaining Set proposed by Morelli and Montero (2003). The former has the following definition :

Let u be an allocation. An *objection* from a coalition T against u is a pair (T, u') where u' is an allocation for the restricted Game (T, f) and

$$u'_i > u_i \text{ for all } i \in T.$$

A *counterobjection* from a coalition Q to (T, u') is a pair (Q, u'') where u'' is an allocation for the restricted Game (Q, f) such that

$$Q - T \neq \varnothing,\ T - Q \neq \varnothing,\ T \cap Q \neq \varnothing$$

and

$$u''_i \geq u_i \text{ for } i \in Q - T \text{ and } u''_i \geq u'_i \text{ for } i \in T \cap Q.$$

An objection against u is *justified* if there is no counterobjection to it. An allocation is in the *Zhou Bargaining Set* $\boldsymbol{Z}$ if there is no justified objection against it.

It is well known that a Bargaining Set - $\boldsymbol{Z}$ included - is typically "large" and not sufficiently exclusive in describing bargaining outcomes. One of our reasons in considering the *Demand Bargaining Set* $\boldsymbol{D}$ in addition to $\boldsymbol{Z}$ is that $\boldsymbol{D}$ is more exclusive than $\boldsymbol{Z}$. We give the definition of $\boldsymbol{D}$ by stating the differences it has with the definition of $\boldsymbol{Z}$.[32] There are four differences :

(i) the allocation under consideration is an aspiration-allocation u^μ,[33]
(ii) $u''_i = u_i$ for $i \in Q$,
(iii) $u''_i > u'_i$ for $i \in T \cap Q$ and
(iv) $Q - T$ or $T - Q$ may be empty.

[32]The definition of $\boldsymbol{D}$ in Morelli and Montero (2003) is for TU games.

[33]Morelli and Montero (2003) allow more general allocations but show that the Demand Bargaining Set consists of aspiration-allocations.

There will in general be many allocations that belong to $\boldsymbol{Z}$ but not to $\boldsymbol{D}$. This is not only because $\boldsymbol{D}$ admits aspiration-allocations only but also because counterobjection is highly restricted in the definition of $\boldsymbol{D}$ in comparison to $\boldsymbol{Z}$ - primarily on account of condition (ii). Let us also note that if condition (iv) were excluded then $\boldsymbol{D}$ would be a subset of $\boldsymbol{Z}$. We show below that condition (iv) is in fact vacuous and $\boldsymbol{D}$ is a subset of $\boldsymbol{Z}$:[34]

Lemma 9 *$\boldsymbol{D} \subset \boldsymbol{Z}$.*

We now show that pseudostable allocations are in $\boldsymbol{D}$ and therefore in $\boldsymbol{Z}$ as well. We actually do so by first giving a characterization for $\boldsymbol{D}$ which we also use in the second example below and which is of independent interest.

Let us say that an aspiration-allocation u^μ is *super-weak-maximal* if there is no demand between any two players who are active at u and unmatched by μ.[35]

Proposition 12 *An aspiration-allocation u^μ is in $\boldsymbol{D}$ if and only if it is super-weak-maximal and u has no balanced market (B, S) where every B-player is unmatched at μ.*

Proposition 13 *Every pseudostable allocation is in the Demand Bargaining Set and therefore in the Zhou Bargaining Set.*

Proof. Suppose u^μ is a pseudostable allocation not in $\boldsymbol{D}$. By Proposition 12, there is a balanced market (B, S) at u such that $B \subset N - \mu(N)$. But then, by Theorem 3, B-players are nonsolitary at $[u, \mu]$. This contradicts with Lemma 6. ■

Proposition 13 says that pseudostable allocations are "stable" from an "exclusive" Bargaining Set perspective. As we mentioned, on the other hand, the Demand Bargaining Set $\boldsymbol{D}$ may contain non-pseudostable allocations. We show in the examples below that these may in fact be various not fitting into a classification at hand.

Example 4 shows that $\boldsymbol{D}$ may contain *non-pseudostable* maximal-settled-aspiration-allocations but not all. Example 5 shows that $\boldsymbol{D}$ may contain maximal *nonsettled*-aspiration-allocations. Example 5 also displays *dominated* (local-maximal) allocations that are in the Zhou Bargaining Set $\boldsymbol{Z}$ and excluded by $\boldsymbol{D}$. It is worth adding that, in our context, the null-allocation is not in $\boldsymbol{D}$ but may be in $\boldsymbol{Z}$, for instance, in any three-person Game where there is no stable allocation.

Example 4 *There are five players in $N = \{\boldsymbol{1,2,3,4,5}\}$. The worth of a partnership is* 2 *for the pairs in*

$$\{\boldsymbol{12,13,23,34,45}\}$$

[34]Morelli and Montero (2003) show the same for TU games.

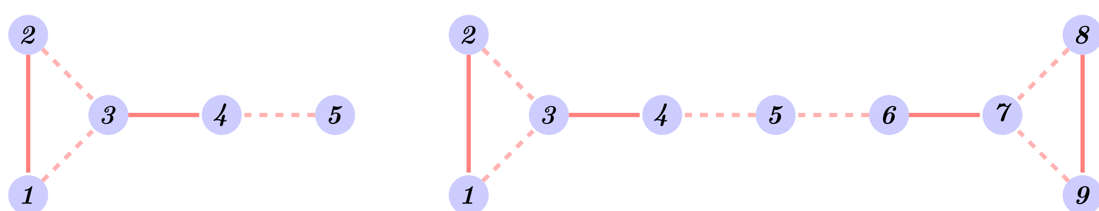
[35]Note that weak-maximal allocations are super-weak-maximal.

and 0 *otherwise. Let* $u = (1,1,1,1,1)$ *and* $\mu = \{\boldsymbol{12,34}\}$ *(see Figure 6(a)). Clearly* u *is a settled aspiration and* μ *is an active-minimal but not solitary-minimal matching at* u*. Let* $T = \{\boldsymbol{4,5}\}$ *and* $u' = (u'_{\boldsymbol{4}}, u'_{\boldsymbol{5}}) = (1+\epsilon, 1-\epsilon)$ *where* $0 < \epsilon < 1$*. It is easily checked that* (T, u') *is a justified objection to* u^μ *so* u^μ *is not in* $\boldsymbol{Z}$ *and therefore not in* $\boldsymbol{D}$*.*

Now consider the extended game with four additional players $\{\boldsymbol{6,7,8,9}\}$ *where the worth of a partnership is* 2 *for the pairs in*

$$\{\boldsymbol{12,13,23,34,45,56,67,78,79,89}\}$$

and 0 *otherwise. Let* $u = (1,1,1,1,1,1,1,1,1)$ *and* $\mu' = \mu \cup \{\boldsymbol{67,89}\}$ *(see Figure 6(b)). Clearly, again,* u *is a settled aspiration and* μ' *is an active-minimal but not solitary-minimal matching at* u*. Using Proposition 12, it is easy to see that* $u^{\mu'}$ *is in* $\boldsymbol{D}$ *and therefore in* $\boldsymbol{Z}$*.*



(a) (b)

Figure 6: Example 4

Example 5 *There are two sets - I and J - of same-type players where* $|I| = n \geq 3$ *and* J *consists of two players say* j, j'*. The worth of a pair with one player from each set is* 2 *and with both players from* I *is* $2 - 2\epsilon$ *(where* $0 < \epsilon < 1$*.) The players* j, j' *cannot form a pair with each other.*

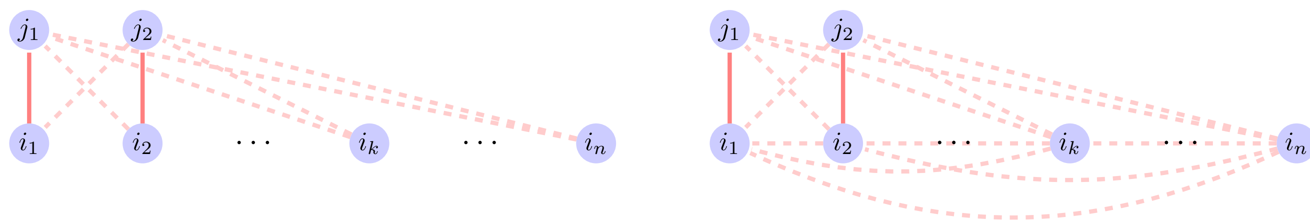
The payoff u *where* u_i *is equal to* $1-\delta$ *for* $i \in I$ *and* $1+\delta$ *for* $j \in J$ *is an aspiration for every* $\delta \leq \epsilon$*. (It is a nonsettled aspiration for every* $\delta < \epsilon$ *and a settled aspiration for* $\delta = \epsilon$*.) The demand graph* $\mathcal{D}(u)$ *is equal to* $\{ij, ij' | i \in I\}$ *for* $\delta < \epsilon$ *(see Figure 7(a)) and equal to* $\{ij, ij' | i \in I\} \cup \{ii' | i, i' \in I\}$ *for* $\delta = \epsilon$ *(see Figure 7(b)). Let* μ *be any matching that consists of two pairs* ij *and* $i'j'$ *where* $i, i' \in I$ *and* $j, j' \in J$*.*

It is easily seen that u^μ *is in* $\boldsymbol{Z}$ *for any* odd n *for all* $\delta \leq \epsilon$*. Note that* u^μ *is dominated for* $n \geq 5$ *: There are* $n-2$ unmatched I*-players all but one of whom can form a pair with another unmatched* I*-player and achieve a payoff equal to* $1-\epsilon$ *strictly above her stand alone utility.*

On the other hand, by Proposition 12, u^μ *is not in* $\boldsymbol{D}$ *for* $n > 3$ *for any* $\delta \leq \epsilon$*. To see this, let* $I_0 = \{i, i'\} \subset I$ *be any two unmatched players. In case* $\delta < \epsilon$*,* (I', J) *is a balanced market with* I_0

$\subset N - \mu(N)$, *and in case* $\delta = \epsilon$, μ *is not super-weak-maximal at* u, *so in both cases* u^μ *is not in* $\boldsymbol{D}$ *by Proposition 12.*

It is easily checked that u^μ *is in* $\boldsymbol{D}$ *for* $n = 3$ *for any* $\delta \leq \epsilon$.



(a) $\delta < \epsilon$ (b) $\delta = \epsilon$

Figure 7: Example 5

We have shown that pseudostable allocations are "stable" from a Bargaining Set perspective. But some other allocations also are. Even the exclusive Demand Bargaining Set may contain nonsettled-aspiration-allocations.[36] "Market forces" are not enough - allocations realized at seller-markets may survive even when a competitive equilibrium exists.

In the next section, we give a *coordinated* Market Procedure that always arrives at an aspiration where there is no seller-market.

6 Market Procedure

In this section we describe a Procedure for finding a settled aspiration. The Procedure works for all piecewise linear partnership functions. For simplicity, we restrict our presentation to *heterogenously linear* partnership functions that have the form $f_{ij}(u_j) = c_{ij} - q_{ij}u_j$. The Procedure starts from any aspiration, generates a piecewise linear path of aspirations, and stops in a bounded number of steps at a settled aspiration.

Here is a preview : The Procedure is coordinated by a Center that displays an *aspiration* at each moment and players register their demand sets at that aspiration. (Since demand is reciprocal, i registers j if and only if j registers i.) The Center observes all the demand sets (i.e., the demand graph) and stops if there is no seller-market. Otherwise, the Center chooses a seller-market and *adjusts* the aspiration along a suitable *direction*. The demand graph *changes* at a number of aspirations. The Center collects all the changes and resets the direction when a "critical" change

[36] This may be compared with Klijn and Masso (2003) who show that the core in the discrete two-sided case is essentially equivalent to the Zhou Bargaining Set.

occurs. In resetting the direction, the Center interacts with a particular subset of players about their "marginal" demand sets. On the quasilinear domain (where $q_{ij} = 1$) resetting the direction requires no such interaction.

The aspiration is adjusted for those players who constitute the seller-market at that aspiration. Sellers' payoffs increase and buyers' payoffs decrease. The other players remain unaffected. The Center can actually choose any seller-market. In the Procedure we present here, it is the "grand" *Seller-Market* - the union of all *unitary seller-markets* - that is chosen at each aspiration. The Center is able to keep track of the Seller-Market *continuously* on the path of aspirations, by an algorithm that is based on the fact that the Seller-Market is also the union of all *solitary-player-markets* (Theorem 4).

There is a single criterion for *admitting* a direction d at any aspiration u on the path, namely the requirement that the Seller-Market at $u + \lambda d$ is identical to the Seller-Market at u for all sufficiently small $\lambda > 0$. When the Seller-Market changes at an aspiration, the Center needs to find a Seller-Market preserving direction. On the quasilinear domain, the direction that has the entry $+1$ for every Seller, -1 for every Buyer and 0 for all other players ensures this. On the more general domain, the Center implements the *Direction Procedure* we describe below.

To conclude the preview, there is actually one other situation where the Center has to reset the direction. This occurs when the path arrives at an aspiration where the Seller-Market has not changed but would change for any continuation along the "current" direction. Such a situation may arise only when a new demand is registered by a Buyer Seller pair, in particular, not on the quasilinear domain.

Formally, let u be an aspiration and d be a feasible direction at u.

Note that, by linearity of the partnership functions, the demand graph $\mathcal{D}(u + \lambda d)$ is identical for all sufficiently small $\lambda > 0$. We denote this graph

$$\mathcal{D}^+(u, d)$$

and call it the *outgoing directional* demand graph. We will say that a direction d is *Seller-Market-Preserving* at u if the Seller-Market at u is identical to the Seller-Market in $\mathcal{D}^+(u, d)$.

Next note that $\mathcal{D}(u - \lambda d)$ is similarly identical for all sufficiently small $\lambda > 0$. We denote this *incoming directional* demand graph

$$\mathcal{D}^-(u, d).$$

Likewise the set of active players $A(u - \lambda d)$ is identical for all sufficiently small $\lambda > 0$ which we denote

$$A^-(u, d).$$

Clearly, the demand graph *changes* at u if and only if

$$\mathcal{D}(u) \neq \mathcal{D}^{-}(u,d) \text{ or } A(u) \neq A^{-}(u,d).$$

It is important to note this may happen finitely often and when it does

$$\begin{aligned} \mathcal{D}(u) &\supset \mathcal{D}^{-}(u,d), \\ A(u) &\subset A^{-}(u,d). \end{aligned}$$

Market Procedure

Step 0 : Take any aspiration $u = u^1$.

Step t : End if there is no seller-market at u^t. Otherwise, find a Seller-Market-Preserving direction d^t by the Direction Procedure below. Then, display the aspiration

$$u^t + \lambda d^t$$

as λ increases above 0 and let the Buyers in the Seller-Market register the changes in their demand sets. Stop at the earliest $u^t + \lambda^* d^t$ where d^t is not Seller-Market-Preserving. Set

$$u^{t+1} = u^t + \lambda^* d^t.$$

The Center's role is to keep track of the Seller-Market, and ensure that aspirations follow a Seller-Market-Preserving path, by implementing the Direction Procedure below when the path needs to be reoriented. On the heterogenous domain where $f_{ij}(u_j) = c_{ij} - q_{ij}u_j$, the Direction Procedure utilizes the information

$$f'_{ij} = -q_{ij}.$$

We have adapted the Direction Procedure below from Alkan (1997) where it is given for arbitrary piecewise linear partnership functions.[37] The Procedure is a "multiplicative" analog of the well-known Demange, Gale and Sotomayor (1986) multi-item auction and has identical convergence properties.

Let (B_t, S_t) be the Seller-Market at the aspiration u^t. Let us suppress reference to u^t and write $\mathcal{D} = \mathcal{D}(u^t)$ and $\mathcal{D}^+(e) = \mathcal{D}^+(u^t, e)$. Also let $B^* = B_t$ and $S^* = S_t$.

[37] See the proof of Lemma 2 in the Appendix. Convergence is easily seen.

The Direction Procedure

Step 0 : Set the initial direction to be the vector e^1 where e^1_i is equal to 1 if $i \in S^*$, $\min_{j\epsilon D_i}\{q_{ij}\}$ if $i \in B^*$, and 0 otherwise.

Step k : End if the Seller-Market (B^*, S^*) in the demand graph $\mathcal{D}$ is the Seller-Market in the directional demand graph $\mathcal{D}^+_{B^*}(e^k)$ and set $d^t = e^k$. Otherwise, find the Seller Set S^k in $\mathcal{D}^+_{B^*}(e^k)$ and set $e^k_i(\delta)$ equal to

$$(1+\delta)e^k_i \text{ for } i\epsilon S^k,$$

$$e^k_i \text{ for } i\epsilon(S^* - S^k),$$

$$\min_{j\epsilon D_i}\{q_{ij}e^k_j\} \text{ for } i\epsilon B^*.$$

Then alter the direction $e^k(\delta)$ by increasing δ continuously above 0 up to δ^* where a new pair joins $\mathcal{D}^+_{B^*}(e^k(\delta))$. Set $e^{k+1} = e^k(\delta^*)$.

Theorem 6 *The Market Procedure reaches a settled aspiration in a finite number of steps.*

It is immediate from the stopping rule that Market Procedure ends at a settled aspiration. Finite[38] convergence follows essentially from Lemma 10 below.

Lemma 10 *Let (u^t) be any sequence of aspirations generated by the Market Procedure, (B_t, S_t) be the Seller-Market at u^t and $a_t = |B_t| - |S_t|, b_t = |S_t|$. Then, for all t,*

$$a_{t+1} \leq a_t,$$

and if $a_{t+1} = a_t$ then

$$b_{t+1} \geq b_t,$$

moreover if $a_{t+1} = a_t$ and $b_{t+1} = b_t$ then

$$(B_{t+1}, S_{t+1}) = (B_t, S_t).$$

We give the proof of Lemma 10 following Proposition 15 at the end of the subsection below.

Proof. (Theorem 6) Clearly (a_t) is bounded below and (b_t) bounded above. Therefore Theorem 6 would fail to hold only if there is a T such that $a_t = a_{t+1}$ and $b_t = b_{t+1}$ for all $t \geq T$. In that case, the Seller-Market remains unaltered while only the direction changes for all $t \geq T$. But by linearity there are only a finite number of directions that can be encountered for all $t \geq T$. Therefore it must be that $d^\tau = d^{\tau'}$ at two distinct steps $\tau < \tau'$. However, this is impossible because then Step τ need not have stopped at $u^{\tau+1}$. ■

[38] In fact, polynomially bounded.

6.1 Identifying the Seller Market

It is fundamental in the Market Procedure that the Center observes how the Seller-Market changes along the Procedure path. Below we give a "dynamic" algorithm that identifies the Seller-Market at an aspiration given the Seller-Market at a "previous" aspiration. This enables the Center to identify the Seller-Market *continuously* along the Procedure path. We then show that there is a lexicographic monotonicity in three attributes of the Seller-Market along the Procedure path - (i) the excess in the Seller-Market, (ii) the number of Sellers, (iii) the Seller-Market itself - which gives the convergence of Market Procedure.

Let us call any two aspirations u, u' *successive* if demands at u are demands at u', i.e.,

$$\mathcal{D}(u) \subset \mathcal{D}(u'),$$

and active players at u' are active at u, i.e.,

$$A(u) \supset A(u').$$

Clearly, any two aspirations on the same linear segment of the Procedure path are successive.

Let u, u' be any two successive aspirations and denote

$$\mathcal{D} = \mathcal{D}(u), \mathcal{D}' = \mathcal{D}(u'), A = A(u), A' = A(u').$$

The *Seller-Market Algorithm* we present here finds a solitary-minimal matching μ' and the solitary set $A_{\mathbf{s}}^{\mu'}$ in $(\mathcal{D}', A')$ given a solitary-minimal matching μ and the solitary set $A_{\mathbf{s}}^{\mu}$ in $(\mathcal{D}, A)$. The Seller-Market at u' is then identified as in Theorem 4.

Thus the Algorithm consists of two routines, which we name the *Solitary-Minimal-Matching Routine* and the *Solitary-Set Routine*, the first for finding a solitary-minimal matching and the second for finding a solitary set at that matching.

We call two successive payoffs u, u' *consecutive* if $A - A'$ consists of a single player and $\mathcal{D} = \mathcal{D}'$ or $\mathcal{D}' - \mathcal{D}$ consists of a single pair and $A = A'$. It will be sufficient to give the Algorithm for consecutive aspirations.

The Routine below simply names the obvious steps in finding a solitary-minimal matching.

Solitary-Minimal Matching Routine

Let u, u' be any two consecutive aspirations and μ be a solitary-minimal allocation in $(\mathcal{D}, A)$.

Step 1 : If μ is not active-minimal in $(\mathcal{D}', A')$ then augment/alter μ to an active-minimal matching μ_1.[39] Otherwise, let $\mu_1 = \mu$.

[39]See Footnote 23.

Step 2 : If μ_1 is not solitary-minimal in $(\mathcal{D}', A')$ then alter μ_1 to a solitary-minimal matching μ_2.[40] Otherwise, let $\mu_2 = \mu_1$.

Then let $\mu' = \mu_2$.

We say that a matching μ' is a *successor* to μ if μ' can be found by using the Routine above given u, μ and u'.

Lemma 11 *If u, u' are consecutive aspirations and (B, S) is a unitary seller-market at u' then (B, S) is a seller-market at u.*

Proof. Since $A' \subset A$ and $\mathcal{D} \subset \mathcal{D}'$, we only need to show that S is matchable into B in $\mathcal{D}$. Let ν be a matching that matches S into B in $\mathcal{D}'$. In all cases except when the pair ij in $\mathcal{D}' - \mathcal{D}$ is in ν and $i \in B, j \in S$, it is clear that ν also matches S into B in $\mathcal{D}$. In the remaining case, let ν' be a matching that matches S into $B - i$ in $\mathcal{D}'$, then ν' is in $\mathcal{D}$. ■

It is worth pointing out that the result above holds neither for unitary seller-markets nor for seller-markets consecutively. We use it below in showing that consecutive solitary sets are nested, that is to say, a player who is not solitary at $[u, \mu]$ cannot become solitary at $[u', \mu']$.

Let $A_{\mathbf{s}}, A'_{\mathbf{s}}$ be the solitary sets at any two solitary-minimal matchings μ, μ' at the consecutive aspirations u, u' respectively, where μ' is a successor to μ, and $A^-_{\mathbf{s}}$ be the set of all $A_{\mathbf{s}}$-players who are active and unmatched in μ'.

Lemma 12 $A'_{\mathbf{s}} \subset A^-_{\mathbf{s}} \subset A_{\mathbf{s}}$.

Proof. A player $i \in A'_{\mathbf{s}}$ is in A' and therefore in A. Also, i is unmatched at the matching μ_1 that is constructed in Solitary-Minimal Matching Routine (otherwise $i \notin A'_{\mathbf{s}}$), and therefore also unmatched at μ (otherwise $i \notin A'$). By Proposition 4 i belongs to a unitary seller-market in $(\mathcal{D}', A')$, and therefore by Lemma 11, to a seller-market in $(\mathcal{D}, A)$. Therefore $i \in A_{\mathbf{s}}$ by Lemma 6. ■

We next show that, excluding a particular occurrence, the solitary set $A'_{\mathbf{s}}$ is equal to $A^-_{\mathbf{s}}$: In other words, a solitary player at $[u, \mu]$ remains solitary at $[u', \mu']$ except when she becomes matched or nonactive. In the remaining "particular" case, a solitary player becomes nonsolitary and all other solitary players (if any) remain solitary. In precise detail this occurs as follows : It takes place when the "new" demand in $\mathcal{D}' - \mathcal{D}$ is a pair $b_1 b_2$ where b_1, b_2 are two Buyers (in B^*) at u, and the matching μ that is solitary-minimal at u remains solitary-minimal at u', that is to say, when the Solitary-Minimal-Matching Routine finds $\mu' = \mu$ (so $b_1 b_2 \notin \mu'$.) Then, there is a μ'-cycle C to which b_1, b_2 belong and the set $A_{\mathbf{s}} - A'_{\mathbf{s}}$ consists of the player who is the unmatched player in C

[40] See Corollary 2.

(who is, to be more precise, the player b_1 if $b_1 \in A_{\mathbf{s}}, b_2 \in (B^* - A_{\mathbf{s}})$ and a player other than b_1, b_2 if $b_1, b_2 \in (B^* - A_{\mathbf{s}})$ otherwise.) The Solitary-Set Routine below is a statement of these assertions :

Solitary Set Routine

Let u, u' be any two consecutive aspirations and μ, μ' solitary-minimal matchings at u, u' where μ' is a successor to μ. Let $A_{\mathbf{s}}$ be the solitary set at $[u, \mu]$ and $A_{\mathbf{s}}^-$ be the set of all $A_{\mathbf{s}}$-players who are active-unmatched at $[u', \mu']$. If

$$\mu' = \mu \text{ and } \mathcal{D}' - \mathcal{D} = b_1 b_2$$

where b_1, b_2 are two Buyers at u, then the unmatched player b in the μ'-cycle that contains b_1, b_2 is nonsolitary at $[u', \mu']$ and

$$A'_{\mathbf{s}} = A_{\mathbf{s}} - b,$$

otherwise

$$A'_{\mathbf{s}} = A_{\mathbf{s}}^-.$$

Proposition 14 *Let u, u' be any two consecutive aspirations and μ, μ' solitary-minimal matchings at u, u' where μ' is a successor to μ. The solitary set at $[u', \mu']$ is obtained from the solitary set at $[u, \mu]$ as in the Solitary Set Routine.*

Proposition 14 follows directly from Lemma 13 in Appendix.

We complete our presentation of the Seller-Market Algorithm for consecutive aspirations with the following result to be used in proving the convergence of Market Procedure.

Proposition 15 *Let $A_{\mathbf{s}}, A'_{\mathbf{s}}$ be the solitary sets and $(B^*, S^*), (B^{*\prime}, S^{*\prime})$ be the Seller Markets at any two consecutive aspirations u, u'. Then $A'_{\mathbf{s}} \subset A_{\mathbf{s}}$. Also, if $A'_{\mathbf{s}} = A_{\mathbf{s}}$ then $(B^*, S^*) \subset (B^{*\prime}, S^{*\prime})$. Moreover, if $A'_{\mathbf{s}} = A_{\mathbf{s}}$ and $S^{*\prime} = S^*$ then $(B^{*\prime}, S^{*\prime}) = (B^*, S^*)$.*

The extension of the Seller-Market Algorithm to successive aspirations is straightforward because any aspiration successive to another aspiration can be reached from the latter by a sequence of consecutive aspirations in any order. In particular, Proposition 15 holds for successive aspirations and therefore all along the Procedure path except possibly when the direction is reset. Since any direction that is reset is Seller-Market-Preserving, the Algorithm keeps track of the Seller-Market - and Proposition 15 holds - continuously along the Procedure path. In view of Corollary, then, Lemma 10 holds and the proof of Theorem 6 is completed.

7 Concluding Remarks

We have looked at Pairing Market/Games from both a coalitional game and a market equilibrium perspective. It is worth highlighting that essential blocking coalitions in our context are pairs. Relatedly, core and competitive equilibrium allocations coincide when they exist. We have looked for extended solution concepts under nonexistence. Our search has delivered a rather complete picture on the structure of Pairing Market/Games and two extended solution concepts. In one of these - half-partnerships and semistable allocations - core and competitive equilibrium allocations coincide.

In the second extension - prudent blocking and pseudostable allocations - coalitions of all sizes may be essential, coincidence breaks down and “market forces” may be ineffective. It is worth pointing out that the Demand Bargaining Set (Morelli and Montero 2003), which contains the pseudostable allocations as we have shown, is in a way market-based. This is so because the Demand Bargaining Set excludes allocations that are not aspiration-allocations and aspirations are market-prices. Bennett (1983,1997) and Bennett and Zame (1988) have elaborated on this in their work on general coalitional games. Indeed the Demand Bargaining Set in our context does filter out many aspiration-allocations that belong to other Bargaining Sets. As we have shown, on the other hand, it may contain allocations that are away from market equilibrium when there is one. It is of interest what additional criteria would narrow down the Demand Bargaining Set or characterize pseudostable allocations. Pairing Games are surely a relatively tractable class of coalitional games. Our work here shows that they are at the same time an interesting class for reviewing the various Bargaining Set solution concepts.

From a mathematical point of view, our work uncovers a *continuous* version of the Gallai Edmonds Decomposition Theorem. We see as one of our main contributions the introduction and utilization of solitary-minimal matchings in this context. We mention again that they give us our main results : characterization of the Seller-Market at an aspiration, the definition of semistable and pseudostable allocations, the Equilibrium Set, and the Market Procedure for reaching the Equilibrium Set.

Finally a remark about a limiting case of our model : The partnership functions in our model do not allow “flats” that arise under budget constraints for example. The broader model that allows for flats can be uniformly approximated by our model and existence results would carry over. On the other hand, some of our results on the properties of the Equilibrium Set do not and designing a Market procedure appears more involved.[41]

[41] Alkan, Anbarci and Sarpça (2012) is an exercise in this domain.

APPENDIX: OMITTED PROOFS

PROOF OF PROPOSITION 1 : The "only if" part is clear from Example 1. For the "if" part, note that by active-minimality of μ, the demand set of every I-player is in J. So it remains to show that I and J are disjoint. Suppose not : Then there are two μ-sequences $i_0, i_1, ..., i_n, j_1, ..., j_n$ and $i'_0, i'_1, ..., i'_m, j'_1, ..., j'_m$ from $i_0 = i'_0 = i$ and a smallest index k such that (say)

$$j_k = i'_{k'}$$

for some $1 \leq k \leq n$ and $1 \leq k' \leq m$. Then $k \neq 1$ for otherwise the μ-sequence $i'_0, i'_1, ..., i'_{k'}, j'_1, ..., j'_{k'}$ is cyclic. For $k \geq 2$, the μ-sequence $i''_0, i''_1, .., i''_{n''}, j''_1, ..., j''_{n''}$ where $i''_0, i''_1, .., i''_{k'} = i'_0, i'_1, .., i'_{k'}$ and $j''_1, ..., j''_{k'} = j'_1, ..., j_{k'}$ also $i''_{k'+1}, .., i''_{n''} = j_{k-1}, ..., j_1$ and $j''_{k'+1}, ..., j''_{n''} = i_{k-1}, .., i_1$ is cyclic because $i''_{n''} = j_1$. Contradiction.

PROOF OF LEMMA 2 : Take any direction vector d such that

$$d_i > 0 \text{ for } i \in S,$$

$$d_i = \max_{j \in D_i(u)} \{f'_{ij}(u_j)d_j\} \text{ for } i \in B,$$
$$d_i = 0 \text{ for } i \in N - B \cup S.$$

Clearly, d is a feasible direction at u. Let ν be any active-minimal matching in the *directional demand graph* $\mathcal{D}^+_B(u, d)$, namely the demand graph $\mathcal{D}(u + \lambda d)$ which is identical for all sufficiently small $\lambda > 0$ (see Section 6.) If ν matches every S-player, then (B, S) is a bipartite submarket at $u + \lambda d$ for all sufficiently small $\lambda > 0$. Therefore, suppose ν does not match every S-player. Let B' be the set of all unmatched B-players and S' be the set of all S-players which are ν-reachable from B'-players in $\mathcal{D}^+_B(u, d)$.

We claim that there is a feasible direction d^* such that $\nu \subset \mathcal{D}^+_B(u, d^*)$ and (i) $\mathcal{D}^+_B(u, d^*)$ contains a matching of greater cardinality than ν *or else* (ii) the set of all S-players, say S'^*, which are ν-reachable from B'-players in $\mathcal{D}^+_B(u, d^*)$ has a greater cardinality than S'. By recursion, this will prove the lemma since S is a finite set.

Set $d_i(\delta)$ equal to

$$(1 + \delta)d_i \text{ for } i \epsilon S' \text{ and } d_i \text{ for } i \epsilon S - S',$$

$$\max_{j \in D_i(u)} \{f'_{ij}(u_j)d_j(\delta)\} \text{ for } i \epsilon B,$$
$$0 \text{ for } i \in N - B \cup S.$$

Alter the direction $d(\delta)$ by increasing δ continuously above 0 up to δ^* where a new pair ij joins $\mathcal{D}^+_B(u, d(\delta))$. Set $d^* = d(\delta^*)$. Note that $\mathcal{D}^+_B(u, d) \subset \mathcal{D}^+_B(u, d^*)$ and hence $\nu \subset \mathcal{D}^+_B(u, d^*)$. Let

$\overline{B} = B' \cup \nu(S')$. See that $(i,j) \in \overline{B} \times (S - S')$. Therefore, player j is ν-reachable from B', i.e., $j \in S'^*$. If j is unmatched at ν, then ν is not active-minimal at $\mathcal{D}^+_B(u, d^*)$, in which case claim (i) holds. Otherwise, S'^* has a greater cardinality than S' since $j \in S'^* - S'$ and $S' \subset S'^*$. In this case, claim (ii) holds. End of claim.

PROOF OF LEMMA 5 : Let μ, μ' be any two active-minimal matchings. If a player i is nonsolitary at μ, then there is a μ-cycle $i_0, i_1, ..., i_n, j_1, ..., j_n$ from $i_0 = i$. Let ν be an active-minimal matching where i_n is unmatched. Suppose i is unmatched at μ' and consider the μ'-sequence $C = i'_0, i'_1, ..., i'_m, j'_1, ..., j'_m$ from $i'_0 = i$ with $j'_k = \upsilon(i'_{k-1})$ and maximum length m. If i'_m is matched in ν, say $\nu(i'_m) = j$, then either μ' is not active-minimal (when j is unmatched in μ') or m is not maximum length. Therefore, i'_m must be unmatched in ν and then $i'_m = i_n$ (otherwise i'_m is ν-reachable from i_n and so ν is not active-minimal). Hence, C is cyclic and so i is nonsolitary at μ'.

PROOF OF PROPOSITION 7 : Let i be a free player and let u be any aspiration in U. Take any $u' \in U$ such that $u_i \neq u'_i$. By Proposition 2, $(N^+_{uu'}, N^-_{uu'})$ is a balanced-market at u.

Case (i) : Let (u, μ) be a stable allocation. Every $N^+_{uu'}$-player is active at u so in $\mu(N)$. Then μ matches $N^+_{uu'}$ to $N^-_{uu'}$. Recall i is in $N^+_{uu'} \cup N^-_{uu'}$. So $\mu(i)$ is also in $N^+_{uu'} \cup N^-_{uu'}$ hence a free player.

Case (ii) : Let (u, ν, χ) be a semistable allocation. Any player with whom an $N^-_{uu'}$-player is in half-partnership or full-partnership must be in $N^+_{uu'}$, for otherwise by balancedness there would be an $N^+_{uu'}$-player unmatched or single-half-matched contradicting (u, ν, χ) is semistable. In particular $\nu(N^-_{uu'}) \subset N^+_{uu'}$. Also no $N^-_{uu'}$-player is in half-partnership because otherwise there would be an even half-partner cycle in $N^+_{uu'} \cup N^-_{uu'}$ contradicting (u, ν, χ) is essential. Then $\nu(N^-_{uu'}) = N^+_{uu'}$ because otherwise an $N^+_{uu'}$-player is unmatched contradicting (u, ν, χ) is semistable. Thus $\nu(i)$ is in $N^+_{uu'} \cup N^-_{uu'}$ hence a free player.

PROOF OF PROPOSITION 8 : Consider any piecewise linear Market/Game. Let K be any finite collection of settled aspirations. Let u be any settled aspiration in U at which $u_i \in med(K)_i$ for a maximum number of players. We claim $u_i \in med(K)_i$ for every i. Suppose not. Then the sets $B = \{i \in N|\ v_i < u_i\}$ and $S = \{i \in N|\ u_i < \overline{v}_i\}$ cannot both be empty. Note that players in B and S are free players.

Define $U' = \{u' \in U | v_i \leq u'_i$ for $i \in B$, $u'_i \leq \overline{v}_i$ for $i \in S$, and $u'_i = u_i$ for $i \notin B \cup S\}$. Clearly U' is nonempty and closed. So there is a $u^* \in U'$ such that $\sum\limits_{i \in B} u^*_i \leq \sum\limits_{i \in B} u'_i$ for every $u' \in U'$. If $u^*_i = v_i$ for some $i \in B$ or $u^*_i = \overline{v}_i$ for some $i \in S$, then there would be an additional player i with $u_i \in med(K)_i$. Contradiction. So $B = \{i \in N|\ u^*_i > v_i\}$ and $S = \{i \in N|\ u^*_i < \overline{v}_i\}$. By Proposition 7, there is a matching at u^*, say μ, that matches all the free players (among each other.)

Let $n = m$ when k is odd and $n = m + 1$ when k is even.

Let i be any player in S and $j = \mu(i)$. By Proposition 2, $u'_j < u^*_j$ for every $u' \in U$ such that

$u'_i > u^*_i$. Since at least n elements of K give a higher payoff to i than u^*, at least n elements of K give a lower payoff to j than u^*. Hence $j \in B$. Thus S is matchable into B at u^*.

Let i be any player in B and $j \in D_i(u^*)$. By Lemma 3, $u'_j > u^*_j$ for any $u' \in U$ such that $u'_i < u^*_i$. Since at least n elements of K give a lower payoff to i than u^*, at least n elements of K give a higher payoff to j than u^*. Hence $j \in S$ and $D_B(u^*) \subset S$.

Thus (B, S) is a balanced-market at u^*. But then, by Lemma 2, there exists $u^{**} \in U'$ such that $\sum_{i \in B} u^{**}_i < \sum_{i \in B} u^*_i$. Contradiction. This proves our claim and the Proposition 8 for any piecewise linear Market/Game. Proposition 8 holds for any Market/Game by uniform approximation.

PROOF OF LEMMA 8 : Let $(u, \mu), (u', \mu')$ be any two stable allocations. We claim $\mu \subset \mathcal{D}(u')$ which completes the proof.

By Proposition 2, μ, μ' both match $N^+_{uu'}$ and $N^-_{uu'}$ to each other. Suppose the claim is not true. Then there is a pair $(i, j) \in (N^+_{uu'}, N^-_{uu'})$ such that $ij \in \mu - \mathcal{D}(u')$. Let $i_1 = i$ and $I = \{i_1, ..., i_n\}, J = \{j_1, ..., j_n\}$ be the player sets defined recursively by setting $j_k = \mu(i_k)$ and $i_{k+1} = \mu'(j_k)$. Then

$$(i_{n+1}, j_{n+1}) = (i_1, j_1).$$

Clearly $I \subset N^+_{uu'}$ and $J \subset N^-_{uu'}$ (since $\mu(N^+_{uu'}) = \mu'(N^+_{uu'}) = N^-_{uu'}$.) Then $c_{i_k j_{k-1}} - u'_{j_{k-1}} \geq c_{i_k j_k} - u'_{j_k}$ since $i_k j_{k-1} \in \mathcal{D}(u')$ and $c_{i_k j_k} - u_{j_k} \geq c_{i_k j_{k-1}} - u_{j_{k-1}}$ since $i_k j_k \in \mathcal{D}(u)$ for all k. So

$$u'_{j_{k-1}} - u_{j_{k-1}} \leq u'_{j_k} - u_{j_k}$$

for all k. But then $u'_{j_1} - u_{j_1} = u'_{j_n} - u_{j_n}$. Therefore $i_1 j_1 \in \mathcal{D}(u')$ (since $i_1 j_1 \in \mathcal{D}(u)$, $i_1 j_n \in \mathcal{D}(u')$). Contradiction.

PROOF OF PROPOSITION 10 : Let u be a settled aspiration and u' be any aspiration. Let μ' be any matching at u' and

$$\mu'_+ = \left\{ ij \in \mu' | i \in N^+_{uu'} \right\}.$$

By Lemma 3 $D_{N^-_{uu'}}(u') \subset N^+_{uu'}$. So the matching $\mu'_0 = \mu' - \mu'_+$ contains only players in $N^0_{uu'} = N - (N^+_{uu'} \cup N^-_{uu'}) = \{i \in N | u_i = u'_i\}$. Using Lemma 4, let μ_+ be a matching at u that matches $N^+_{uu'}$ into $N^-_{uu'}$. It is clear that

$$\mu = \mu'_0 \cup \mu_+$$

is a matching at u. We show below that there are at least as many active-unmatched players at $[u', \mu']$ as there are at $[u, \mu]$.

Since $D_{N^-_{uu'}}(u') \subset N^+_{uu'}$,

$$\mu'(N^-_{uu'}) \subset N^+_{uu'}.$$

Let A be the set of all players in $N - N^0_{uu'}$ who are active-unmatched at $[u, \mu]$. By definition of μ, $A \subset N^-_{uu'} - \mu(N^+_{uu'})$. Hence

$$\left|N^-_{uu'}\right| \geq \left|\mu(N^+_{uu'})\right| + |A| = \left|N^+_{uu'}\right| + |A| .$$

Let A^0 be the set of $N^0_{uu'}$-players who are active-unmatched at $[u, \mu]$ but matched at $[u', \mu']$. Then $\mu'(A^0) \subset N^+_{uu'}$. Therefore

$$\left|\mu'(N^-_{uu'})\right| \leq \left|N^+_{uu'}\right| - \left|A^0\right| .$$

So $\left|N^-_{uu'}\right| - \left|\mu'(N^-_{uu'})\right| \geq |A| + |A^0|$. Recall that $N^-_{uu'}$-players are active at u'.

PROOF OF LEMMA 9 : Suppose to the contrary that there is an aspiration-allocation u^μ in $\boldsymbol{D}$ but not in $\boldsymbol{Z}$. Then, there is an objection (T, u') to u^μ such that any counterobjection (Q, u) to (T, u') satisfies either $Q \subset T$ or $T \subset Q$.

There can be no counterobjection (Q, u) to (T, u') with $Q \subset T$: Otherwise $u_i > u'_i > u^\mu_i$ for $i \in Q$ so (Q, u) is a justified objection to u^μ, implying $u^\mu \notin \boldsymbol{D}$. Contradiction.

Consider now any counterobjection (Q, u) to (T, u') with $T \subset Q$. Then $u_i > u'_i > u^\mu_i$ for $i \in T$. Now let (u, μ') be any allocation for the restricted Game (Q, f) and i be any player in T. If $\mu'(i) \in T$, then $((i, j), (u_i, u_j))$ is a justified objection to u^μ, implying $u^\mu \notin \boldsymbol{D}$. Contradiction. If $\mu'(i) \notin T$, then $((i, \mu'(i)), (u_i, u_{\mu'(i)}))$ is a counterobjection to (T, u'), but then it is not true that $T \subset \{i, \mu'(i)\}$ since there are at least two players in T. Contradiction.

PROOF OF PROPOSITION 12 : $(\Rightarrow)$ Let u^μ be a super-weak-maximal aspiration-allocation and suppose there is no balanced-market (B, S) at u such that $B \subset N - \mu(N)$. Suppose to the contrary that there is a justified objection (T, u') to u^μ.

Let (u', μ') be an allocation for the restricted market (T, f). (Recall that $A^\mu(u)$ denotes the set of active-unmatched players at $[u, \mu]$.) Let $B = \{i \in T | u'_i < u_i\}$. Then $u_i > u'_i > u^\mu_i$ for all $i \in B$, so $B \subset A^\mu(u)$.

We claim $|T - B| \leq |B|$. Suppose not. Since (u', μ') is an allocation for (T, f), there is a pair $ij \in \mu'$ such that $i, j \in T - B$. So $(u'_i, u'_j) \geq (u_i, u_j)$. Since u is an aspiration, $(u'_i, u'_j) = (u_i, u_j)$. Therefore $ij \in \mathcal{D}(u)$ and $(u_i, u_j) > (u^\mu_i, u^\mu_j)$, saying of u^μ is not super-weak-maximal. End of claim.

Now let $S = D_B(u)$. Suppose $S \subsetneq T$. Then, there is a pair ij in $\mathcal{D}(u)$ such that $i \in B$ and $j \in S - T$. So $((i, j), (u_i, u_j))$ is a counterobjection to (T, u'). Contradiction. Therefore, $S \subset T$. By super-weak-maximality of μ, $S \cap A^\mu(u) = \varnothing$. Hence $S \subset T - B$. So from the claim above $|S| \leq |B|$.

Finally let ν be any active-minimal matching in $\mathcal{D}_B(u)$. It is not possible that ν matches B to S for otherwise (B, S) would be a balanced-market at u where $B \subset A^\mu(u)$. Therefore, since $|S| \leq |B|$ as shown above, it must be that ν leaves a player i in B unmatched. Let (B_i, S_i) be the ν^i-market. Then $(B_i, S_i) \subset (B, S)$ and $(B_i - i, S_i)$ is a balanced-market at u where $B_i - i \subset A^\mu(u)$. Contradiction.

($\Leftarrow$) Suppose μ is not super-weak-maximal at u. Then there is a pair $ij \in \mathcal{D}(u)$ where $i, j \in A^\mu(u)$. Then $((i,j),(u_i,u_j))$ is a justified objection to u^μ. Hence $u^\mu \notin \boldsymbol{D}$.

Consider any piecewise linear Market/Game. Suppose there is a balanced-market (B,S) at u where $B \subset A^\mu(u)$. By Lemma 2, let $u' = u + \lambda d$, where d is a feasible direction such that $d_i > 0$ for all $i \in S$ and $d_i < 0$ for all $i \in B$. It is clear that $(B \cup S, u')$ is a justified objection to u^μ for sufficiently small $\lambda > 0$. Hence $u^\mu \notin \boldsymbol{D}$. This holds for for any Market/Game by uniform approximation.

Lemma 13 *Let $A_{\mathbf{s}}, A'_{\mathbf{s}}$ be the solitary sets at any two solitary-minimal matchings μ, μ' at the consecutive aspirations u, u' respectively, where μ' is a successor to μ, and $A^-_{\mathbf{s}}$ be the set of all $A_{\mathbf{s}}$-players who are active and unmatched at $[u', \mu']$. If $A'_{\mathbf{s}}$ is not equal to $A^-_{\mathbf{s}}$, then*

$$\mu' = \mu,$$

$\mathcal{D}' - \mathcal{D} = b_1 b_2$ where b_1, b_2 are Buyers at u, and

$$A'_{\mathbf{s}} = A_{\mathbf{s}} - b,$$

where b is the unmatched player in the cyclic μ'-sequence that contains b_1, b_2.

Proof. Suppose $A'_{\mathbf{s}}$ is not equal to $A^-_{\mathbf{s}}$. Then, by Lemma 12, $A'_{\mathbf{s}}$ is contained in but not equal to $A^-_{\mathbf{s}}$. Let i be any player in $A^-_{\mathbf{s}} - A'_{\mathbf{s}}$ and (B^*_i, S^*_i) be the μ^i-market in $\mathcal{D}$. We will show that (Claim 1) $\mathcal{D}' - \mathcal{D} = b_1 b_2$ where $b_1, b_2 \in B^*_i$, (Claim 2) $\mu = \mu'$ and $A'_{\mathbf{s}} = A_{\mathbf{s}} - i$ (in particular, there is exactly one player in $A^-_{\mathbf{s}} - A'_{\mathbf{s}}$.)

Note i is nonsolitary in $[u', \mu']$ (since i is in $A^-_{\mathbf{s}} - A'_{\mathbf{s}}$), so there is a μ'-cycle $C = i_0, i_1, ..., i_n, j_1, ..., j_n$ in $\mathcal{D}'$ from $i_0 = i$. Since $\mathcal{D}' - \mathcal{D}$ is at most a singleton, $i_0 j_1$ or $i_0 i_n$ is in $\mathcal{D}$. Then j_1 or i_n is in S^*_i (since $i \in B^*_i$). Say $j_1 \in S^*_i$. Let ν be an active-minimal matching in $\mathcal{D}'$ that leaves j_1 unmatched.

Claim 1 is true, because otherwise the demand set of every B^*_i-player except possibly one (say player k) in $\mathcal{D}'$ would be in S^*_i, implying $(B^*_i - k, S^*_i)$ is a bipartite submarket at u' (since S^*_i is matchable into $B^*_i - k$ in $\mathcal{D}$ and so in $\mathcal{D}'$), and contradicting active-minimality of ν by Lemma 1.

We prove Claim 2 in two steps :

Step (i) μ is active-minimal at u' : Otherwise, since μ is active-minimal in $\mathcal{D}$ and $b_1 b_2$ is the only demand in $\mathcal{D}' - \mathcal{D}$, any matching that is active-minimal at u' would necessarily contain $b_1 b_2$. But consider the active-minimal matching ν constructed above and let $\nu_{S^*_i}, \mu_{S^*_i}$ be the restriction of ν, μ respectively to the pairs that have a player in S^*_i. Note that the matching $(\nu - (\nu_{S^*_i} \cup b_1 b_2)) \cup \mu_{S^*_i}$ is active-minimal (because $\nu_{S^*_i} \cup b_1 b_2$ and $\mu_{S^*_i}$ have equal cardinality and contain an equal number of active players) but does not contain $b_1 b_2$. Contradiction.

Step (ii) $\mu = \mu'$ and $A'_{\mathbf{s}} = A_{\mathbf{s}} - i$: By using $\mathcal{D} \neq \mathcal{D}'$, $A' = A$. Then, any μ-cycle in $\mathcal{D}$ is also a μ-cycle in $\mathcal{D}'$ because $\mathcal{D} \subset \mathcal{D}'$. Therefore, $A_{\mathbf{s}}$ contains the solitary set at $[u', \mu]$. By Lemma 5, i

is nonsolitary in $[u', \mu]$ since i is nonsolitary at $[u', \mu']$. Therefore there is a μ-cycle C in $\mathcal{D}'$ from i. Note that $b_1b_2 \in C$ (otherwise C is in $\mathcal{D}$ and so $i \notin A_{\mathbf{s}}$). Take any $i' \in A_{\mathbf{s}} - i$ and let C' be any μ-sequence from i' in $\mathcal{D}'$. Then, $b_1b_2 \notin C'$ (since $C \cap C' = \varnothing$ by active-minimality of μ in $\mathcal{D}'$) and so C' is in $\mathcal{D}$. Then, C' is cycle-free (otherwise $\mu^{i'}$-market is not bipartite by Proposition 1 in $\mathcal{D}$ and so $i' \notin A_{\mathbf{s}}$ by Proposition 4). Then, $A_{\mathbf{s}} - i$ is the solitary set at $[u', \mu]$ and by Corollary 2 μ is solitary-minimal at u'. So, $\mu' = \mu$ (recall Solitary-Minimal Matching Routine) and $A'_{\mathbf{s}} = A_{\mathbf{s}} - i$.

$$\text{Case 1} : b_1 \in A_{\mathbf{s}}, b_2 \in (B^* - A_{\mathbf{s}}),$$

$$\text{Case 2} : b_1, b_2 \in (B^* - A_{\mathbf{s}}).$$

Note that, in Case 2, both b_1, b_2 are matched in μ and there exists a player

$$b_3 \in A_{\mathbf{s}}$$

such that b_1 and b_2 are both μ'-reachable in $\mathcal{D}'$ from only b_3 in $A_{\mathbf{s}}$. Now we are ready to state cases:

Case 1) If i is b_1 or b_2, say b_1. Then, $A'_{\mathbf{s}} = A_{\mathbf{s}} - b_1$.

Case 2) Otherwise, say i is b_3. Then, $A'_{\mathbf{s}} = A_{\mathbf{s}} - b_3$. ■

PROOF OF PROPOSITION 15 : Let $A_{\mathbf{s}}, A'_{\mathbf{s}}$ be the solitary sets at any two solitary-minimal matchings μ, μ' at the consecutive aspirations u, u' respectively, where μ' is a successor to μ. $A'_{\mathbf{s}} \subset A_{\mathbf{s}}$ by Lemma 12.

Suppose $A'_{\mathbf{s}} = A_{\mathbf{s}}$ and $(B^*, S^*) \subsetneq (B^{*\prime}, S^{*\prime})$. Let $B = B^* - B^{*\prime}$ and $S = S^* - S^{*\prime}$. If B is empty, then $D_{B^*}(u') \subset D_{B^{*\prime}}(u') = S^{*\prime}$ since $B^* \subset B^{*\prime}$. By using $\mathcal{D} \subset \mathcal{D}'$,

$$S^* = D_{B^*}(u) \subset D_{B^*}(u') \subset D_{B^*}(u') = S^{*\prime}$$

and hence $S^* \subset S^{*\prime}$. Then, $(B^*, S^*) \subset (B^{*\prime}, S^{*\prime})$. Contradiction. Hence, B is nonempty.

No player in $B^* - B$ demands an S-player in $\mathcal{D}'$ and so in $\mathcal{D}$ since $\mathcal{D} \subset \mathcal{D}'$. Therefore, μ matches S into B since $\mu(S^*) \subset B^*$ and in particular $|S| \leq |B|$. If $|S| = |B|$, then μ matches S to B and $S^* - S$ into $B^* - B$. Then, $A_{\mathbf{s}} \subset B^* - B$ and so any μ-market from any player in $A_{\mathbf{s}}$ is in $(B^* - B, S^* - S)$ since $D_{B^*-B}(u) = S^* - S$ and $\mu(S^* - S) \subset B^* - B$. By Theorem 4 B is empty. Contradiction. Thus, $|S| < |B|$.

Using $A'_{\mathbf{s}} = A_{\mathbf{s}}$, it must be that $|B^{*\prime}| - |S^{*\prime}| = |B^*| - |S^*|$. Then, $|S^{*\prime} - S^*| < |B^{*\prime} - B^*|$ since $|S| < |B|$. The demand set of each player in $B^{*\prime} - B^*$ is in $S^{*\prime}$ in $\mathcal{D}'$. Then by using $\mathcal{D} \subset \mathcal{D}'$, $D_{B^{*\prime}-B^*}(u) \subset S^{*\prime}$. By using the fact that μ matches S^* into B^*, there is a player $i \in B^{*\prime} - B^*$ unmatched at $[u, \mu]$. Player i is active-unmatched at $[u, \mu]$ since $A' \subset A$. By Theorem 4, player i is in a unitary seller-market at u' and then in a seller-market at u by Lemma 11. Therefore $i \in A_{\mathbf{s}}$ at $[u, \mu]$ by Lemma 6. By Theorem 4, $i \in B^*$. Contradiction. Thus, $(B^*, S^*) \subset (B^{*\prime}, S^{*\prime})$.

If $A'_{\mathbf{s}} = A_{\mathbf{s}}$ and $S^{*\prime} = S^*$, then it must be that $B^{*\prime} = B^*$ since otherwise $|B^{*\prime}| - |S^{*\prime}| > |B^*| - |S^*|$.

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