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Conservation Laws, Equivalence Principle and Forbidden Radiation Modes

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There are standard proofs showing there can be no monopole electromagnetic radiation and no dipole gravitational radiation. We supplement these with a global topological argument for the former, and a local argument based directly on the principle of equivalence for the latter.

1. Introduction: Wave and Particle Pictures of Radiation

By radiation, we mean an electromagnetic or gravitational signal that, after being produced at a source, takes a finite amount of time to arrive at a final observation point. Static Coulomb or gravitational fields are not of this type and will not be discussed here. What we wish to do is to take a new look at some familiar selection rules on modes of such radiation. While doing this, we will adopt particle and field descriptions, or global versus local viewpoints, depending on the issue at hand.

Let us start with the particle picture, according to which both electromagnetic and gravitational forces are mediated by massless particles, of spin one and spin two, respectively. If a particle is massive, one can always go to a frame where it is at rest, and choose any direction as the z-axis. Quantum mechanics dictates that a particle of spin s must have $2s+1$ spin states, corresponding to the possible values of the z-component of spin ranging from s to $-s$ in descending integer steps. A free massless photon or graviton, on the other hand, can never be viewed this way as it is impossible to transform to a frame traveling at the speed of light. In the $m = 0$ case, the velocity vector defines a special axis in space,

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and the only two spin states allowed are $\pm s$, according to whether the spin is entirely parallel or anti-parallel to the velocity. This alignment can be understood in terms of a classical mechanical particle picture based on the behavior of angular momentum under Lorentz transformations. Consider a closed system with some internal motion which generates angular momentum $\mathbf{L}_0$ in the frame K_0, where the total momentum vanishes. The angular momentum components $L_i (i = 1, 2, 3)$, seen in a frame K moving along the z-axis with velocity V, are related to the 'rest-frame' components via

$$L_z = L_{z0}, \;\; L_y = L_{y0}\sqrt{1 - V^2/c^2}, \;\; L_x = L_{x0}\sqrt{1 - V^2/c^2}. \tag{1}$$

The proof of (1) can be found in [1], but there is an intuitive interpretation: Imagine viewing, from K, a spinning sphere with its center at rest in K_0. As V approaches c, Lorentz contraction will make it appear flattened to a disk spinning in the xy-plane, which means both L_x and L_y have to go to zero. The spin is now forced to being either parallel or antiparallel to the total momentum of the disk along the z-axis, which is consistent with the earlier statement that a massless particle of spin s can only have $s_z = \pm s$.

It is important to note that in the case of a massive particle, rotations and Lorentz boosts (transformations between inertial frames moving at different constant velocities) can be used to change the angle between the momentum $\mathbf{p}$ and the spin $\mathbf{s}$ to any desired value. This is impossible in the massless case, where the helicity $\mathbf{s} \cdot \{\mathbf{p}/|\mathbf{p}|$ is not just invariant under rotations, but also under Lorentz transformations. To see the difference between the two cases, consider a massive particle first. In its rest frame, h is undefined. If its spin and momentum happen to be parallel in some other frame, one can always find yet another frame moving faster than the particle, where the momentum will appear to be $-\mathbf{p}$, The fundamental reason why massless photons and gravitons of both helicities exist in Nature is that electromagnetic and gravitational interactions happen to respect mirror-reflection symmetry

while the spin component along the momentum necessarily remains the same. The helicity thus becomes $-h$. With a massless particle, on the other hand, to be able to see a state with helicity $+h$ turn to $-h$, one would have to go flying past it at some $V > c$, which is impossible. Indeed, as Wigner has noted [2], the $+s$ and $-s$ states can only be transformed into each other via improper Lorentz transformations involving mirror reflections, which, by definition, reverse the momentum, but not the spin. These two opposite helicity states really represent different particles for all practical purposes. The fundamental reason why massless photons and gravitons of both helicities exist in Nature is that electromagnetic and gravitational interactions happen to respect mirror-reflection symmetry, or parity invariance. Strong interactions are also parity-invariant, but these are experimental facts and not the result of an inescapable a priori rule for all interactions, as was uncritically believed until Lee and Yang [3] examined the evidence. Indeed, experiments [4] since 1955 have shown that weak interactions do not respect parity invariance, and spin one-half fermions of opposite helicities couple differently to the W and Z particles mediating weak interactions.

Let us now examine the 'wave-picture' counterparts of some of the above points. The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ of a wave in empty space satisfy the wave equations

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{E} = 0 \quad , \tag{2}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\mathbf{B} = 0. \tag{3}$$

The simplest such classical electromagnetic radiation field is a plane wave of the form $\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \mathbf{B} = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t)$, with $\mathbf{B} \cdot \mathbf{E} = 0$ and $E = B$

(in Gaussian units). This actually represents a quantum superposition built from an indefinite number of photons of momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$; the classical field energy density is proportional to $|E_0|^2$. In the quantum picture, this energy density is given by the product of photon number density and the energy per photon, as first pointed out by Einstein in his 1905 'photoelectric effect' paper.

Returning to the classical description, Maxwell's equations in free space guarantee that the fields are always perpendicular to the local propagation direction. The equation pair $\nabla\cdot\mathbf{E} = 0,\ \ \nabla\cdot\mathbf{B} = 0$ become the transversality conditions $\mathbf{k}\cdot\mathbf{E} = 0,\ \ \mathbf{k}\cdot\mathbf{B} = 0$ for plane waves. Coherent superposition of infinitely many photons of the same helicity produces a circularly polarized state. In this state, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ remain perpendicular to each other in the plane transverse to the direction of propagation $\mathbf{k}$, while rotating together in the clockwise or counter-clockwise sense. Superposing these two states, one can obtain linearly polarized waves as well, represented by a constant $\mathbf{E}_0$.

With gravitational plane waves, one starts by writing the space-time metric $g_{\alpha\beta}(\alpha,\beta = 0,1,2,3$ for time and the three space components, respectively) as $\eta_{\alpha\beta} + h_{\alpha\beta}$, where $\eta_{00} = 1,\ \ \eta_{11} = \eta_{22} = \eta_{33} = -1$, with all other components zero, is the flat Minkowski spacetime metric. The part $h_{\alpha\beta}$, representing small fluctuations around the flat metric, can be shown to obey the wave equations in (2) and (3). Furthermore, the wave ripples $h_{\alpha\beta}$ are transverse to the direction of wave propagation just like $\mathbf{E}$ and $\mathbf{B}$. To see why this is so, use the quantum particle and classical wave descriptions in parallel. We argued that when the velocity is $\mathbf{V} = c\mathbf{z}$, a particle of non-zero spin s could have at most two quantum states $s_z = \pm s$, and this corresponded to the transverse circular polarization modes of the classical plane wave. A 'massless' longitudinal wave mode, where the oscillation is along the direction of propagation, is then only possible for $s = 0$. Now, how does the number of states come out to be 2 while $h_{\alpha\beta}$, being a symmetric second-rank tensor, has $4 \times 5/2 = 10$ components? The answer again lies in the fact that 'motion' in the spacetime directions $z + ct$ and $z - ct$ is frozen, and the only possible dynamics is in the x and y dimensions. This means that the actual propagating components of the rank 2 symmetric tensor should be counted not in 4, but in the 2 transverse dimensions. The result is then $2 \times 3/2 = 3$. But this still includes the longitudinal mode, which, in the massless case, can only be associated with an $s = 0$ particle. Discarding it, we are left with the 2 transverse polarizations. Mathematically, the last operation amounts to leaving out the trace of the matrix $h_{\alpha\beta}$. The latter, being invariant under rotations, is indeed an $s = 0$ object. These points will be discussed in greater detail in Section 3.

2. A Global Implication of Transversality

The above discussion of transversality is purely local, but one can also give a global topological argument showing there can be no transverse electromagnetic radiation in the monopole mode, where the source charge/current would have to move in a spherically symmetric way. The wave fronts for monopole radiation, if it were allowed, would be perfect spheres with the source at the center. This actually happens with pressure waves of sound. A Helmholtz resonator produces such a wavefront. A naturally occurring resonator of this kind is seen in frogs which inflate pouches around their necks and use them as spherical loudspeakers. To return to electromagnetic waves let us consider a specific example – the form of a linearly polarized electromagnetic wave at the equator, with the electric field pointing north, the magnetic field west and $\mathbf{k}$, the direction of propagation, radially outwards. This transversal right-handed triad structure has to be maintained everywhere on the spher-

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ical wavefront. Hence if we start from some point on the equator and move **E** (say tangent to the meridian) and **B** (say tangent to the parallel) northwards along a meridian, we get a certain orientation of **E** and **B**, still mutually perpendicular, at the north pole. However, a different starting point on the equator would give us different **E** and **B** directions at the same final point. The same problem occurs if we move from the same two equatorial initial points towards the south pole, proving that the attempt to define such fields globally fails at the two poles. A transversal monopole wave field configuration is thus impossible. This is simply a manifestation of the so-called 'hairy-ball theorem', first stated by Poincaré and proven by Brouwer [5]. An informal statement of the theorem is that "one cannot comb the hair on a coconut". On the other hand, one *can* comb the hair on a *doughnut*, which is the wavefront of an electric dipole source at a fixed distance from the source; the field lines correspond to the 'hair'. At a given instant, one can find lines of **E** tangential to circles parallel to the circular hole in the middle, intersected by circles of **B** running perpendicular to them. Obviously, there is no global conflict between transversality of the fields and the shape of the wavefront in this case.

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An interesting point is that radiation fields are transversal in all dimensions, while the global topological argument above does not work for S^3, the 3-dimen- sional spherical hypersurface embedded in 4 space dimensions. This is because S^3 enjoys the rare property of being 'parallelizable', which is the formal way of saying it allows 'combable hair'. It is a mathematical theorem that the only other such spheres are S^1 and S^7.

There are also arrangements where dipole fields are suppressed, but not for topological reasons. A well-known example is described below.

3. A Situation where Electric and Magnetic Dipole Radiation is Forbidden

There can be no electric or magnetic dipole radiation from a closed system if all the particles in it have the same charge to mass ratio e/m. The proof below is standard [1], but we will nevertheless reproduce it to prepare for the gravitational analogy to follow later. In these modes, the electric and magnetic radiation fields are proportional to the second time derivatives $\ddot{\mathbf{d}}$ and $\ddot{\mathbf{m}}$ of the electric and magnetic dipole moments, respectively. The appearance of the second derivative in the electric dipole case can be understood partially by noting that a charge moving at constant velocity will appear at rest when viewed from a co-moving frame, where it obviously cannot radiate. The principle of relativity then ensures that it cannot radiate in any inertial frame (the same conclusion can be extended to magnetic dipole radiation by invoking the symmetry between electricity and magnetism in the absence of free charges, but such arguments are not sufficient for explaining why, for example, third time derivatives appear in the quadrupole mode). In Gaussian units the dipole moments are

$$\mathbf{d} = \sum_{a=1}^{N} e_a \mathbf{r}_a, \quad \mathbf{m} = \frac{1}{2c} \sum_{a=1}^{N} e_a \mathbf{r}_a \times \mathbf{v}_a \quad . \tag{4}$$

Here, e_a is the charge, $\mathbf{r}_a$ is the position, and $\mathbf{v}_a$ is the velocity of the ath particle. When all particles have the same e/m, one can multiply and divide each term in the sum by m_a, pull out the common e/m, which turns (4) into

$$\mathbf{d} = \frac{e}{m} \sum_{a=1}^{N} m_a \mathbf{r}_a, \quad \mathbf{m} = \frac{e}{2mc} \sum_{a=1}^{N} m_a \mathbf{r}_a \times \mathbf{v}_a \quad . \tag{5}$$

Thus the electric and magnetic dipole moments are seen to be proportional to the position $\mathbf{R}$ of the center of mass and the total angular momentum $\mathbf{L}$ of the system. The first time derivative of $\mathbf{R}$ is the center of mass velocity, which is simply the total linear momentum divided by the total mass. The latter is constant for a closed system (which is by definition not subject to a net external force), so its time derivative, which is proportional to $\ddot{\mathbf{d}}$, must be zero. Similarly, $\ddot{\mathbf{m}}$ is proportional to the second time derivative of the total angular momentum, which vanishes in the absence of external torques. This means the total momentum and angular momentum both have vanishing second time derivatives, proving the claim at the beginning of the section. Hence when the particles in a closed system have the same e/m, the lowest radiation mode is quadrupole.

4. The Principle of Equivalence and Gravitational Radiation Modes

Einstein's theory of General Relativity, having passed all experimental tests, including a very recent one [6], is considered to be the correct description of gravitational phenomena at the classical level. The fundamental physical input on which the theory rests is the very accurately tested equality (or, more precisely, proportionality – it can be turned into an equality by the choice of G) of the gravitational mass M_{g} appearing in $F = GM_{\mathrm{g1}}M_{\mathrm{g2}}/r^2$, and the inertial mass M_{I} in $F = M_{\mathrm{I}}a$. The former determines how strongly the gravitational field couples to an object and hence is the gravitational equivalent of the charge e, while the latter is just a measure of the object's inertia, i.e., its resistance to acceleration when acted upon by a force. It is the exact cancelation of the two kinds of masses from the equations of motion that makes possible a purely geometric description of motion in a gravitational field. It is perhaps worth mentioning here that there is as yet no satisfactory explanation of this very remarkable equality of the two kinds of mass, although Mach [7] made the plausible suggestion that the inertial mass of a test object must result from its gravitational interactions with the rest of

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the universe. Mach's idea or principle inspired Einstein in his search for a theory of gravity, but it has never been precisely formulated as a testable statement. Einstein realized that the equality of the two kinds of mass allows one to make 'gravitational fields' locally appear or disappear by a choice of accelerating frame. For example, astronauts see no effects of gravity inside an orbiting cabin because it is falling freely towards the earth at every instant. A detailed account of how this 'Equivalence Principle' and a few other reasonable assumptions can be exploited to arrive at General Relativity is explained masterfully by Chandrasekhar in [8].

There is one very fundamental respect in which gravitational radiation in General relativity and electromagnetic radiation in classical electromagnetism are dissimilar: The former is inherently non-linear. In physical terms, this means that gravitons interact with other gravitons, while photons do not. Mathematically, only the fields and not their higher powers appear in the field equations of electromagnetism, whereas higher powers of the metric show up in Einstein's theory. Thus, for weak fields, i.e., small deviations $h_{\alpha\beta}(\alpha, \beta = 0, 1, 2, 3)$ of the metric $g_{\alpha\beta}$ from the flat space-time form $\eta_{\alpha\beta} = diag(1, -1, -1, -1)$, neglect of quadratic and higher field terms amounts to ignoring the self-interactions of gravitons. It is then not surprising (although mathematically beyond the level of this article to prove) that the $h_{\alpha\beta}$ obey the same wave equation (with the same speed c) as the electromagnetic fields in Maxwell's theory. In EM, the sources are electric charge and electric current, evaluated at the retarded time $t - R/c$, where t is the time at the field observation point and R the distance between the observation point and the location of the charge or current at the retarded time. The gravitational analog of charge and current is the energy-momentum tensor of matter. To get a feeling for why this is so, let us recall that the analog of electrical charge in Newtonian

Suggested Reading

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theory is the gravitational mass. Now, since relativity says that mass and energy are really the same thing (actually, energy is the more fundamental quantity since one can have energy without mass, but not the other way around), it is reasonable to expect kinetic energies will also act as sources of gravitational fields. Indeed, the energy-momentum tensor is constructed out of the densities, positions and velocities of the mass/energy distribution, just as electromagnetic source terms are built from the densities, positions and velocities of the charges. Another fundamental similarity is transversality: the $h_{\alpha\beta}$ oscillate in a plane perpendicular to the direction of propagation.

The lowest gravitational radiation mode is quadrupole – there is no gravitational radiation in dipole modes.

However, even in this linearized form, an important difference remains: The lowest gravitational radiation mode is quadrupole – there is no gravitational radiation in dipole modes. While the possibility of radiation in the lower dipole modes is automatically bypassed in the full General theory of relativity, it is instructive to define corresponding 'gravielectric' and 'gravimagnetic' dipole moments, and evaluate their time derivatives in analogy with equations (4) and (5). For example, if we considered the possibility of gravitational radiation before we knew about the General theory of relativity, these would have to be looked at as the lowest possible modes in analogy with electromagnetism. The counterpart of the charge e_a would be the gravitational mass $m_{\mathrm{g}a}$. However, since the equivalence principle asserts that this is equal to the inertial mass $m_{\mathrm{I}a}$, the common 'e/m' factor for all the terms in the sum is just $m_{\mathrm{g}a}/m_{\mathrm{I}a} = 1$! Hence the two dipole moments become the center of mass coordinate and total angular momentum, both of which have vanishing second time derivatives for a closed system.

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