Entropic Selection of Nash Equilibrium*

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Abstract

This study argues that Nash equilibria with less variations in players' best responses are more appealing. To that regard, a notion measuring such variations, the entropic selection of Nash equilibrium, is presented: For any given Nash equilibrium, we consider the cardinality of the support of a player's best response against others' strategies that are sufficiently close to the behavior specified. These cardinalities across players are then aggregated with a real-valued function on whose form we impose no restrictions apart from the natural limitation to nondecreasingness in order to obtain equilibria with less variations. We prove that the entropic selection of Nash equilibrium is non-empty and admit desirable properties. Some well-known games, each of which display important insights about virtues / problems of various equilibrium notions, are considered; and, in all of these games our notion displays none of the criticisms associated with these examples. These examples also show that our notion does not have any containment relations with other associated and well-known refinements, perfection, properness and persistence.

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1 Introduction

The concept of Nash equilibrium is central in the theory of games, and as put by Myerson (1978), "it is one of the most important and elegant ideas in game theory". On the other hand, Nash's pointwise stability may create multiplicity of equilibria some of which do not satisfy local stability and produce outcomes which can be criticized on grounds of not corresponding to intuitive notions about how plausible behavior should look like. In order to alleviate these problems, important refinements of Nash equilibrium have been developed. Indeed, perfection by Selten (1975), properness by Myerson (1978), and persistence by Kalai and Samet (1984), among others, have been standards in the theory of games.

However, below we display that, some undesirable variations in the best responses of players still remain in a perfect and proper equilibria. In this game, none of the pure

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I

$1\backslash 2$	I	II
I	$(1,1,0)$	$(1,0,1)$
II	$(1,1,1)$	$(0,0,1)$

II

$1\backslash 2$	I	II
I	$(1,0,1)$	$(0,1,0)$
II	$(0,1,0)$	$(1,0,0)$

Table 1: A 3-player game: Player 1 chooses rows, player 2 columns and player 3 matrices

strategies is weakly dominated for any of the players, and it can easily be verified that this game has two Nash equilibria, $s^1 = (II, I, I)$ and $s^2 = (I, 1/2I + 1/2II, 1/2I + 1/2II)$; and, only s^2 is perfect and proper.[1]

We argue that this particular undominated Nash equilibrium, which is neither perfect nor proper, is more desirable on account of displaying less variations in players' best responses. This is because of the following: Every players' only best response to any of the other players' strategies that are sufficiently close to the one given in s^1, is as given in s^1. Thus, the number of (pure) actions in any player's best response against the other players'

[1]Perfection follows because (1) no matter what the levels of player 2 and 3's mistakes (around mixing I and II with equal probabilities) are, action I for player 1 is the only best response, as long as these mistakes are strictly positive; and (2) every finite normal form game has to possess a perfect equilibrium, which, in fact, has to be a Nash equilibrium as well.

strategies that are sufficiently close to the one given in s^1, is 1, showing that players' best responses do not exhibit any variations around s^1. On the other hand, when s^2 is considered, the players choosing mixed strategies (players 2 and 3) clearly have 2 (pure) actions in their best responses no matter how close the others' strategies are. Therefore, player 2 and 3's best responses are exhibiting more variations around s^2.

In 2 player coordination games, the mixed strategy Nash equilibrium which is perfect and proper (but, not persistent) displays the same type of undesirable variations. Kalai and Samet (1984) refers to this observation as the lack of strong neighborhood stability.[2] The example borrowed from Kalai and Samet (1984) and given in table 2, illustrates this particular point. Only pure strategy Nash equilibria satisfy our criteria and the perfect

$1\backslash 2$	I	II
I	$(1,1)$	$(0,0)$
II	$(0,0)$	$(1,1)$

Table 2: A 2-player coordination game

and proper equilibrium with totally mixed behavior does not. This is because in the pure strategy Nash equilibria, the best response of any player against the other's strategies that are sufficiently close to the one specified, involves only the pure strategies specified in these equilibria. Thus, the number of (pure) actions in a player's best response against the

[2]Adopting rationalistic interpretation in normal form games (for a formal discussion of these ideas, see Aumann and Brandenburger (1995) and Kuhn (1996)) delivers a very counterintuitive observation associated with the mixed strategy Nash equilibrium in coordination games: Even though this game is played once and there are no communication phases, a player perfectly anticipates the mixed behavior of the other due to which that player becomes indifferent between his actions and can choose a specific mixed strategy which in turn makes the other be able to randomize between his. In order to eliminate this counterintuitive aspect, Nash (1950) offers a second interpretation, that is referred to as *the mass action interpretation*, and is less demanding on players: "[i]t is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes." What is assumed is that there is a population of participants for each position in the game, which will be played throughout time by participants drawn at random from the different populations. If there is a stable average frequency with which each pure strategy is employed by the "average member" of the appropriate population, then this stable average frequency constitutes a Nash equilibrium. We refer the reader to Barlo and Carmona (2011) for more on this subject.

other player's strategy that is sufficiently close to the one given in any pure strategy Nash equilibria, is 1. Hence, players' best responses do not exhibit any variations around those Nash equilibria. On the other hand, the mixed strategy Nash equilibrium clearly involve 2 (pure) actions in each of the player's best responses against any strategy of the other no matter how close it is to the specified one. Therefore, the mixed strategy Nash equilibrium involves more variations in players' best responses.

In order to formalize these ideas, we propose a low-cost technology, the notion of *entropic selection of Nash equilibrium*.[3] Given any Nash equilibrium, we evaluate its *entropy*, measuring its variations, as follows: For any player, we consider the cardinality of that player's best response when the others can choose any one of the strategies that are arbitrarily close to the one specified in that particular Nash equilibrium. In fact, given any neighborhood from which the others' strategies can be drawn, a player's best response is a probability distribution. Following the intuition borrowed from information theory (we refer the reader to the original source by Shannon (1948)), we use a weaker form of entropy to measure the variations of players' best responses around a given strategy profile. To be precise, we consider the cardinality of the support of each player's best response against others' strategies that are sufficiently close to the one specified. It turns out that we do not need to employ the standard definition of classic entropy, and this weaker version suffices to deliver the desired results.

By employing finiteness and linearity, we show that for any given strategy profile (not necessarily in equilibrium) the entropy of a player's best response is nonincreasing in how close the neighborhoods considered are. It is also established that this particular integer achieves a lowerbound (greater or equal to one). Thus, inherently every strategy involves an entropy figure, this particular lowerbound, separately for each player. Then, this entropy vector is aggregated with a real-valued function. A natural restriction, on which our results do not depend, is that this *aggregation function* has to be nondecreasing, in order to obtain a strategy with *less* variations. Consequently, the resulting complete and continuous preorder defined on the set of strategy profiles, thence, calls for the strategy with a lower aggregated

[3] The notion of entropic selection does not necessarily have to be used in conjunction with the set of Nash equilibrium. In fact, our existence theorem does not put any limitations on the set on which this notion is desired to be employed.

entropy figure to be ranked higher than one with a higher entropy figure. Using these results, we establish that for any given aggregation function and any non-empty subset of the strategies, an entropic selection exists.

We consider many well-known examples, each of which display some important insights about virtues / problems of various equilibrium notions. It is shown that in all of these games the concept of entropic selection of Nash equilibrium displays none of the criticisms associated with these examples. That is, our notion shows solid performances. In fact, using these examples we also show that the entropic selection of Nash equilibrium does not have any containment relations with the notions of perfection, properness and persistence.[4]

Following the display of these solid performances of our notion, a serious drawback of entropic selection of Nash equilibria is identified: It may be consist of a unique Nash equilibrium which involves the play of a weakly dominated strategy. That is why, in what follows the notion of the *entropic selection of undominated Nash equilibrium* is considered, and we show that all of the above results, including existence and solid performances, continue to hold. Moreover, an additional desirable property is obtained: If the set of undominated Nash equilibrium contains a pure strategy profile, then that pure strategy Nash equilibrium has to be in the entropic selection of the undominated Nash equilibrium, and there cannot be any other mixed strategy Nash equilibria in that selection.

The organization of the paper is as follows: The next section presents the preliminaries. In section 3, we characterize the entropic selection of Nash equilibria, and after proving its existence its desirable properties are displayed. In section 4, a problem related to weak domination is illustrated, and its solution, the entropic selection of undominated Nash equilibrium is presented. In the same section, we prove its existence and establish that all of the desirable properties are maintained.

[4] It should be mentioned that when solving these examples we often use the symmetric aggregation function obtained by the simple addition of the entropy figures. But, it is important to emphasize that these observations do not depend on this particular specification of the aggregation function, and hold for any strictly increasing aggregation function.

2 Definitions and Notations

Let $\Gamma = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite normal–form game, where A_i is a finite non-empty set of actions of player $i \in N$, and $u_i : \times_{i\in N} A_i \to \mathbb{R}$ is agent i's von Neumann Morgenstern utility function. We keep the standard convention that $A = \times_{i\in N} A_i$ and $A_{-i} = \times_{j\neq i} A_j$. A mixed strategy of player i is represented by $s_i \in \Delta(A_i) \equiv S_i$, where $\Delta(A_i)$ denotes the set of all probability distributions on A_i and $s_i(a_i) \in [0, 1]$ denotes the probability that s_i assigns the pure strategy a_i with the obvious restriction that $\sum_{a_i\in A_i} s_i(a_i) = 1$. Hence, a strategy profile is denoted by $s \in \times_{i\in N} S_i \equiv S$. Again, $S_{-i} \equiv \times_{j\neq i} S_j$.

Define the best response of player i to s_{-i} by $\mathcal{BR}_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \forall s'_i \in S_i\}$. s^* is a *Nash equilibrium* if for every $i \in N$, $s^*_i \in \mathcal{BR}_i(s^*_{-i})$. The set of Nash equilibria of Γ is denote by $\mathcal{N}(\Gamma) \subset S$. An action $a_i \in A_i$ is *weakly dominated* for player i, if there exists $a'_i \in A_i$ with $u_i(a_i, a_{-i}) \leq u_i(a'_i, a_{-i})$ for all $a_{-i} \in A_{-i}$ and this inequality holds strictly for some $a_{-i} \in A_{-i}$. A strategy profile $s \in S$ is undominated if $s_i(a_i) = 0$ for any $a_i \in A_i$ that is weakly dominated. The set of *undominated Nash equilibrium* of Γ is denote by $\mathcal{N}^d(\Gamma) \subset S$. For any given $\varepsilon > 0$, we say that $s \in S$ is an *ε–perfect equilibrium* if $s_i(a_i) > \varepsilon$ for all $i \in N$ and $a_i \in A_i$, and for all $i \in N$ we have $u_i(s) \geq u_i(\tilde{s}_i, s_{-i})$ for all $\tilde{s}_{-i} \in S_i$. On the other hand, s is *ε–proper equilibrium* if it is ε–perfect and for every pair of $a_i, a'_i \in A_i$ the following condition holds: $u_i(a_i, s_{-i}) > u_i(a'_i, s_{-i})$ implies $s_i(a_i) \geq s_i(a'_i)$. s is *perfect* (*proper*), if there exists $\{\varepsilon^k\}$ and $\{s^k\}$ with the property that $\varepsilon^k > 0$ and $\lim_k \varepsilon^k = 0$ and s^k an ε^k–perfect (ε–proper, respectively) equilibrium for each k and $\lim_k s^k = s$. R is a *retract* of S if $R = \times_{i\in N} R_i$ where for any $i \in N$, R_i is a non-empty convex and closed subset of S_i. For any given $K \subset S$, it is said that R *absorbs* K if for every $s \in K$, for any $i \in N$ it must be that $\mathcal{BR}_i(s_{-i}) \cap R_i \neq \emptyset$. Any retract absorbing itself is a *Nash retract* and a retract is an *essential Nash retract* if it absorbs a neighborhood of itself. Moreover, it is said to be a *persistent retract*, if it is an essential Nash retract and is minimal with respect to this property. $s \in S$ is a *persistent equilibrium* if it is a Nash equilibrium contained in a persistent retract.

For any given $s \in S$ and $\varepsilon > 0$, define $\mathbf{B}_\varepsilon(s_{-i}) \equiv \{s'_{-i} \in S_{-i} : |s'_{-i} - s_{-i}| < \varepsilon\}$. Moreover, we let $\mathbb{S}^\varepsilon_i(s) \equiv \left\{a_i \in A_i : a_i \in \mathcal{BR}_i(s'_{-i}) \text{ where } s'_{-i} \in \mathbf{B}_\varepsilon(s_{-i})\right\}$, and $\mathbb{S}^\varepsilon(s) \equiv (\mathbb{S}^\varepsilon_i(s))^N_{i=1}$; $\mathbb{E}^\varepsilon_i(s) \equiv |\mathbb{S}^\varepsilon_i(s)|$, and $\mathbb{E}^\varepsilon(s) \equiv (\mathbb{E}^\varepsilon_i(s))^N_{i=1}$.

For any given $f : \mathbb{N}^N \to \mathbb{R}$, that we call the *aggregation function*, we let the *f–induced entropic order* be denoted by $\succcurlyeq_E^f \subset S \times S$, and defined by $s \succcurlyeq_E^f s'$, if there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$, $f(\mathbb{E}^\varepsilon(s)) \leq f(\mathbb{E}^\varepsilon(s'))$. For any non-empty $K \subseteq S$, s is said to be in the *f–induced entropic selection* of K, if $s \succcurlyeq_E^f s'$ for all $s' \in K$. Moreover, the set of f–induced entropic selection of $K \subset S$ is denoted by $\mathcal{E}^f(K)$.

Lemma 1 *The following hold:*

1. *For any $s \in S$ and for any $\eta, \eta' > 0$ with $\eta < \eta'$, it must be the case that $\mathbb{E}^\eta(s) \leq \mathbb{E}^{\eta'}(s)$.*
2. *For any $s \in S$, there exists $\bar{\eta} > 0$ such that for all $\eta, \eta' \in (0, \bar{\eta})$, $\mathbb{E}^\eta(s) = \mathbb{E}^{\eta'}(s)$.*
3. *Suppose that $K \subseteq S$ is non–empty and compact. Then, $\mathcal{E}^f(K)$ is non–empty and compact.*

Proof. The first two parts are due to the finiteness of Γ and linearity of the expected utility function. Hence, $\succcurlyeq_E^f \subset S \times S$ is complete and continuous preorder. Thus, the third part follows from Berge's Theorem of Maximum. ■

The following is a general existence theorem:

Theorem 1 *$\mathcal{E}^f(K)$ is non–empty for any non–empty $K \subset S$.*

Proof. For any given $f : \mathbb{N}^N \to \mathbb{R}$ and any non–empty $K \subset S$, let $s \in K$ and from the second part of Lemma 1 there exists $\bar{\varepsilon} > 0$ such that for all $\eta, \eta' \in (0, \bar{\varepsilon})$, $\mathbb{E}^\eta(s) = \mathbb{E}^{\eta'}(s) \equiv \mathbb{E}(s) \in \mathbb{N}^N$, where $\mathbb{E}_i(s) \leq |A_i|$, hence, $V \equiv \cup_{s \in K} \mathbb{E}(s)$ is a finite set in $\mathbb{N}^N$. Thus, $f(V) \subset \mathbb{R}$ is finite.

Recall that, f–induced entropic order is defined by $s \succcurlyeq_E^f s'$, if there exists $\bar{\varepsilon} > 0$ such that for all $\eta < \bar{\varepsilon}$, $f(\mathbb{E}^\eta(s)) \leq f(\mathbb{E}^\eta(s'))$. On the other hand, we also have that $s \succcurlyeq_E^f s'$ if and only if $f(\mathbb{E}(s)) \leq f(\mathbb{E}(s'))$. This follows from (1) if $f(\mathbb{E}(s)) \leq f(\mathbb{E}(s'))$, then there exists $\bar{\varepsilon}_1, \bar{\varepsilon}_2 > 0$ such that $f(\mathbb{E}(s)) = f(\mathbb{E}^{\eta_1}(s)) = f\left(\mathbb{E}^{\eta'_1}(s)\right) \leq f(\mathbb{E}(s')) = f(\mathbb{E}^{\eta_2}(s')) = f\left(\mathbb{E}^{\eta'_2}(s')\right)$ for all $\eta_k, \eta'_k \in (0, \min_{k=1,2} \bar{\varepsilon}_k)$, hence, $s \succcurlyeq_E^f s'$; and (2) the reverse from our construction.

Because that $f(V)$ is a finite set in $\mathbb{R}$, it possesses a minimal element, $f(\mathbb{E}(s^*))$, $s^* \in K$. Hence, by the above $s^* \succcurlyeq_E^f s$ for all $s \in K$, thus, $s^* \in \mathcal{E}^f(K)$. ■

$1\backslash 2$	I	II
I	(1,1)	(0,0)
II	(0,0)	(0,0)

Table 3: A 2-player game with a weakly dominated Nash equilibrium

Naturally, the choice of $f : \mathbb{N}^N \to \mathbb{R}$ determines the selection of elements in $\mathcal{E}^f(K)$, and insisting on equilibria with "less" variations calls for the aggregation function to be strictly increasing. Furthermore, in this study, we often treat all players and the cardinality of the support of their best responses for any given $\varepsilon > 0$ and $s \in S$, symmetrically. Hence, we often concentrate on the following particular aggregation function: for any $x \in \mathbb{N}^N$, $f(x) = \sum_{i \in N} x_i$. In fact, we suppress the notation of f whenever this particular aggregation function is employed. We wish to emphasize that other plausible methods of aggregations may also be constructed.[5]

3 Entropic Selection of Nash equilibrium

In this section, we consider the entropic selection of Nash equilibria, $\mathcal{E}^f(\mathcal{N}(\Gamma))$. It is worthwhile to point out that due to Theorem 1, $\mathcal{E}^f(\mathcal{N}(\Gamma)) \neq \emptyset$. On the other hand, because that $\mathcal{N}(\Gamma)$ is non-empty and compact, Lemma 1 can be employed to obtain the following result, which is presented without any proof.

Theorem 2 $\mathcal{E}^f(\mathcal{N}(\Gamma)) \subset S$ *is non–empty and compact.*

Below, we present some well known examples in which the entropic selection of Nash equilibrium eliminate non-plausible Nash equilibria.

The example presented in table 3, often referred to in the analysis of perfection due to Selten (1975), displays that entropic selection of Nash equilibrium eliminates weakly dominated Nash equilibria. The Nash equilibria are s, s' with $s_i(I) = 1$ and $s_i(II) = 0$, and

[5]That is, concerns such as those put forth by Myerson in the analysis of properness (see Myerson (1978)) and those raised by Kohlberg and Mertens in the analysis of stable subsets of equilibria (refer to Kohlberg and Mertens (1986)), may be included in the formulation of f.

$1\backslash 2$	I	II
I	$(3,3)$	$(0,0)$
II	$(0,0)$	$(1,1)$
III	$(2,2)$	$(2,2)$

Table 4: The Kohlberg Example

$s'_i(I) = 0$ and $s'_i(II) = 1$, $i = 1, 2$. Because that for any $\varepsilon \in (0,1)$ and any $s''_{-i} \in \mathbf{B}_\varepsilon(s_{-i})$

$$\mathcal{BR}_i(s''_{-i}) = \begin{cases} (1,0) & \text{if } s''_{-i}(I) > 0, \\ (\alpha, 1-\alpha) & \text{otherwise,} \end{cases}$$

we have that $\mathbb{E}^\varepsilon(s) = (1,1)$ and $\mathbb{E}^\varepsilon(s') = (2,2)$. Thus, only s is in the entropic selection of Nash equilibria.[6]

It is important to emphasize that the notion of entropic selection of Nash equilibrium does not display the type of weakness identified in the Kohlberg Example, which is depicted in table 4. In this game II is a strictly dominated action for player 1, and without it, the pure action profile (III, II) is not perfect, because II for player 2 is weakly dominated by I. Yet, with the addition of II for player 1, it becomes perfect. However, the only equilibrium in the entropic selection of Nash equilibrium is the pure action profile (I, I): There are two types of Nash equilibria $s^1 = ((1,0,0),(1,0))$, and $s^\alpha = ((0,0,1),(\alpha, 1-\alpha))$, $\alpha \leq 2/3$. Clearly, $\mathbb{E}^\varepsilon(s^1) = (1,1)$, and for any $\alpha > 0$, $\mathbb{E}^\varepsilon(s^\alpha) = (1,2)$. Moreover, when $\alpha = 0$, i.e.

[6] A similar conclusion holds in a well-known example, given below, displaying a situation where an "incredible threat" (by player 2) results in (a continuum of) Nash equilibria that are weakly dominated. In fact, this game is the normal form version of the following extensive form game: Player 1 chooses first, and if his choice is I, then the game ends and player 1 obtains a return of 1 and player 2 a payoff of 2. On the other hand, if player 1 chooses II, then depending on player 2's choice, players obtain a payoff vector of $(2,1)$ if player 2's choice is I, and $(0,0)$ otherwise.

$1\backslash 2$	I	II
I	$(1,2)$	$(1,2)$
II	$(2,1)$	$(0,0)$

The set of Nash equilibria of this game is given by (II, I) and $(I, \alpha I + (1-\alpha)II)$, $\alpha \leq 1/2$; and, for $s = ((0,1),(0,1))$ we have $\mathbb{E}^\varepsilon(s) = (1,1)$ for all $\varepsilon < 1/2$, and $\mathbb{E}^{\varepsilon'}(s') = (1,2)$ for any other Nash equilibrium s' and $\varepsilon' > 0$ sufficiently small. Hence, the set of entropic selection of Nash equilibrium equals (II, I).

pure action (III, II) is considered notice that $\mathbb{E}_1^\varepsilon(s^0) = 1$, and for $\varepsilon > 0$ arbitrarily small let $\tilde{s}_1 = (\eta_1, \eta_2, 1-\eta_1-\eta_2)$ for $\eta_1, \eta_2 \geq 0$ such that $\tilde{s}_1 \in \mathbf{B}_\varepsilon(s_1^0)$. If $\tilde{s}_1$ is given by $\eta_1 > 0$ and $\eta_2 = 0$, then I will be player 2's only best response. On the other hand, if $\eta_1 = 0$ and $\eta_2 > 0$ (for η_2 small) II would be player 2's best response. Finally, when $\eta_1, \eta_2 > 0$, then both I and II could be in player 2's best response, depending on the relation between η_1 and η_2. Hence, $\mathbb{E}_2^\varepsilon(s^0) = 2$. Thus, s^0 is also not in the entropic selection of Nash equilibrium.

Entropic selection of Nash equilibrium displays a similar performance in the game often referred to in the discussion of "imperfections of perfection" (see Myerson (1978)), an example given in table 5. It is basically the same game given in table 3 with a strictly dominant strategy added for both of the players. The critical aspect is that, the addition of the strictly dominated strategies results in pure strategy II not being weakly dominated. In this game,

$1\backslash 2$	I	II	III
I	$(1,1)$	$(0,0)$	$(1,-1)$
II	$(0,0)$	$(0,0)$	$(2,-1)$
III	$(-1,1)$	$(-1,2)$	$(-1,-1)$

Table 5: Myerson's 2-player game depicting "imperfections of perfection"

the set of Nash and perfect equilibria coincide and are equal to $s^1 = ((1,0,0),(1,0,0))$ and $s^2 = ((0,1,0),(0,1,0))$. On the other hand, following the same reasoning as displayed in the discussion about the Kohlberg Example, one can show that $\mathbb{E}^\varepsilon(s^1) = (1,1)$ and $\mathbb{E}^\varepsilon(s^2) = (2,2)$. Thus, the only element of the entropic selection of Nash equilibrium is s^1.

The game given in table 8, due to Kalai and Samet (1984), makes a significant observation displaying a serious problem of perfectness and properness. Namely, that "very specific set of trembles is needed to justify equilibria". We will show that our notion, like persistence, is free of this shortcoming. In this game it can be shown that there are three Nash equilibria, $s^1 = (I,I,I)$, $s^2 = (II,II,I)$, and $s^3 = (II,II,II)$. Only s^3 involves a weakly dominated strategy for player 3, and the perfect and proper equilibria are given by s^1 and surprisingly s^2, which in particular needs very specific trembles. Clearly, s^2 is undesirable. But, when we consider the entropic selection of Nash equilibrium, it is easy to observe that $\mathbb{E}^\varepsilon(s^1) = (1,1,1)$ and $\mathbb{E}^\varepsilon(s^3) = (1,1,2)$ (because s^3 is weakly dom-

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1\2	I	II
	I	
I	$(1,1,1)$	$(0,0,0)$
II	$(0,0,0)$	$(0,0,0)$

1\2	I	II
	II	
I	$(0,0,0)$	$(0,0,0)$
II	$(0,0,0)$	$(1,1,0)$

Table 6: The 3-player game of Kalai and Samet (1986)

1\2	I	II	III
I	$(1,1)$	$(2,-2)$	$(-2,2)$
II	$(1,1)$	$(-2,2)$	$(2,-2)$
III	$(0,0)$	$(1,1)$	$(1,1)$

Table 7: A 2-player game displaying persistence and ...

inated) for $\varepsilon > 0$ sufficiently small. Next, consider s^2, and for any given and sufficiently small $\varepsilon > 0$, let $\eta_2, \eta_3 \geq 0$ with $\tilde{s}_{-1} = ((\eta_2, 1-\eta_2), (1-\eta_3, \eta_3)) \in \mathbf{B}_\varepsilon(s^2_{-1})$. Then, I is player 1's only best response whenever $\eta_2(1-\eta_3) > (1-\eta_2)\eta_3$, and similarly II whenever $\eta_2(1-\eta_3) < (1-\eta_2)\eta_3$. Thus, $\mathbb{E}^\varepsilon_1(s^2) = 2$, so s^2 is not in the entropic selection of Nash equilibrium because $\mathbb{E}^\varepsilon(s^1) = (1,1,1)$. To sum up, the only element in the entropic selection of Nash equilibrium is given by s^1.

The notions of entropic selection of Nash equilibrium and perfection and properness do not have any containment relations. This observation has been presented in the introduction in table 1.

Under the light of these discussions, one may wonder about the relation of our notion with persistence. That is why, in the next two games, we display that the entropic selection of Nash equilibrium does not have any containment relations with the notion of persistence.

Consider the 2 player game in table 7, a coordination game, where one of the pure actions, in which players are not coordinated, is replaced by a matching pennies: Here, the set of Nash equilibrium equals the set of persistent equilibria, and is given by $s^1 = ((1/4, 1/4, 1/2), (1/2, 1/4, 1/4))$, $s^2 = ((0,0,1), (0, 1/4, 3/4))$, $s^3 = ((0,0,1), (0, 3/4, 1/4))$, $s^3 = ((3/4, 1/4, 0), (1,0,0))$, $s^5 = ((1/4, 3/4, 0), (1,0,0))$. It can be observed that in this game s^k, $k = 2, 3, 4, 5$, correspond to the pure strategy Nash equilibria of a standard coordination game, while s^1 can be associated with the totally mixed one. In this example, because that the entropy of s^1 is given by $(3, 3)$, and those of the others are either $(1, 2)$ or $(2, 1)$, s^1 is not in the entropic selection of Nash equilibria.

Next, we display that there is an element in the entropic selection of Nash equilibrium that is not persistent. To that regard consider the following game. It has 4 players, where

$1\backslash 2$	I	II
I	$(1, 1)$	$(0, 0)$
II	$(0, 0)$	$(1, 1)$

$3\backslash 4$	III	IV
III	$(1, 0)$	$(0, 1)$
IV	$(0, 1)$	$(1, 0)$

$3\backslash 4$	III	IV
III	$(2, 2)$	$(2, 0)$
IV	$(0, 2)$	$(0, 0)$

Table 8: A 4-player game

first and second players are playing the game on the left in table 8, independent of the choices of player 3 and player 4. Whereas player 3 and player 4 play the game in the middle in table 8 when first and second players choose (I, I) or (II, II), and the game on the right in table 8 when first and second choose (I, II) or (II, I). We will show that in this example there exists an equilibrium which is not persistent but in the entropic selection of Nash equilibrium.

This equilibrium is $s = (1/2I+1/2II, 1/2I+1/2II, III, III)$. Note that first and second players will not deviate since their payoffs are not dependent on other players actions, and they play the mixed strategy Nash Equilibrium of the game on the left in table 8. Third and fourth players are playing the equally weighted mixtures of games in the middle and on the right in table 8 and so their best-response are (III, III).

Now, we will show that it is not persistent. For any $\varepsilon > 0$, the tremble $(((1/2 - \varepsilon)I + (1/2 + \varepsilon)II), ((1/2 - \varepsilon)I + (1/2 + \varepsilon)II), (1 - \varepsilon)III + \varepsilon IV, (1 - \varepsilon)III + \varepsilon IV)$, the best response against this strategy for player 1 is II. Similarly, it can easily be seen that I will appear in player 1's best response as well, when the above perturbation is reversed. Moreover, for the retract given by $\Delta(\{I, II\})$ for players 1 and 2 and III for the others is not persistent. Because, when players 1 and 2 choose I, I (or II, II), the persistent retract in the middle game is $(\Delta(\{III, IV\}))^2$. Therefore, there is no persistent retract which includes this equilibrium other than the whole game.

Also note that, there is a persistent retract, neighborhoods around I for player 1 and 2, $\Delta(\{III, IV\})$ for players 3 and 4. For any strategy in this retract, the best response of first and second players are still I. Third and fourth players best response to this tremble will be trivially be in this retract as well. Therefore, there is a persistent retract not including s (other than the whole game), hence, the whole game is not a minimal persistent retract. Hence, s is not a persistent equilibrium.

We claim that s is in the entropic selection set of Nash Equilibria. Note that, there are no pure strategy equilibria in this game. Note further that there is no equilibrium where only one player mixes among his strategies. Therefore, in all equilibria, at least two players randomize among their strategies. At this point recall that if a player randomizes, then his entropy will be at least 2. Hence, in this game, we will encounter entropy vectors with at least two numbers greater than 1. Now, consider s, and $\bar{\varepsilon} > 0$, such that for all $\eta, \eta' \in (0, \bar{\varepsilon})$ we have $\mathbb{E}^{\eta}(s) = \mathbb{E}^{\eta'}(s)$. Then, $\mathbb{E}_i^{\eta}(s) = 2$, $i = 1, 2$. Third and fourth players are playing the strategy III and if player 1 and player 2 make small mistakes (due to the strict dominance in the game on the right) their best response will still be III, therefore $\mathbb{E}_3^{\eta}(s) = 1 = \mathbb{E}_4^{\eta}(s)$, for $\eta > 0$ sufficiently small. Then, we have an entropy vector $(2, 2, 1, 1)$ which is the best possible entropy vector with respect to summation aggregation f. Thus, this equilibrium is in the entropic selection of Nash equilibrium, however is not persistent.

4 Entropic Selection of Undominated Nash equilibrium

In all of the examples in the above, the notion of entropic selection of Nash equilibrium did not involve any weakly dominated Nash equilibrium. However, this property does not hold in general. In the game given in table 9, it can be shown that there are two Nash equilibria, s and s': $s_i(I) = 0$ and $s_i(II) = 1$ for $i = 1, 2, 3$; and, $s'_i(I) = s'_i(II) = 1/2$ for $i = 1, 2$, and $s'_3(I) = 1$ and $s'_3(II) = 0$. Clearly, s involves a weakly dominated strategy (by player 3), and because that for $\varepsilon > 0$ small enough $\mathbb{E}^{\varepsilon}(s) = (1, 1, 2)$ and $\mathbb{E}^{\varepsilon}(s') = (2, 2, 1)$, s is the only point in the entropic selection of Nash equilibria.

This observation is why we continue our analysis with the entropic selection of undominated Nash equilibrium.

It is important to point out that in all the games handled previously, with the exception

3					
I			*II*		
1\2	*I*	*II*	1\2	*I*	*II*
I	$(1,0,1)$	$(0,1,1)$	*I*	$(0,0,0)$	$(0,0,0)$
II	$(0,1,0)$	$(1,0,0)$	*II*	$(0,0,0)$	$(1,0,0)$

Table 9: A 3-player game

of the last one (given in table 9), the entropic selection of Nash equilibrium coincides with the entropic selection of undominated Nash equilibrium. Hence, the entropic selection of undominated Nash equilibrium meets our objectives. Yet, obviously the problem displayed in that example is significant.

When we restrict attention to the entropic selection of undominated Nash equilibrium, on the other hand, we show that existence holds and a further noteworthy aspect is added to the achievements of our notion: If the set of undominated Nash equilibrium contains a pure strategy profile, then it has to be in the entropic selection of undominated Nash equilibrium and there cannot be any other mixed strategy profile that involves at least one player allocating strictly positive probabilities to two or more actions. First, we present this result, and next handle existence.

Proposition 1 *Suppose* $a^* \in \mathcal{N}^d(\Gamma)$ *and* $f : \mathbb{N}^N \to \mathbb{R}$ *is strictly increasing. Then,* $a^* \in \mathcal{E}^f(\mathcal{N}^d(\Gamma))$*, and there is no* $s \in \mathcal{E}^f(\mathcal{N}^d(\Gamma))$ *with* $s_i(a_i), s_i(a_i') > 0$ *for some* $i \in N$ *and* $a_i, a_i' \in A_i$ *with* $a_i \neq a_i'$*.*

Proof. Clearly, for $\varepsilon > 0$ small enough, $\mathbb{E}_i^{\varepsilon}(s^*) = 1$ where $s_i^*(a_i) = 1$ whenever $a_i = a_i^*$ and 0 otherwise. Moreover, because that for any $\varepsilon' > 0$ it must be that (1) $\mathbb{E}_i^{\varepsilon'}(s) > 1$ for some $i \in N$, s as specified in the hypothesis; and, (2) for any $s' \in S$ it must be that $\mathbb{E}_i^{\varepsilon'}(s') \geq 1$, the result follows by construction and the hypothesis that f is strictly increasing. ■

In words, this proposition establishes that for any given strictly increasing aggregation function f, the f–induced entropic selection of undominated Nash equilibrium must contain only pure strategies, whenever there exists a undominated pure strategy Nash equilibrium.

Theorem 3 $\mathcal{E}^f(\mathcal{N}^d(\Gamma))$ *is non–empty.*

Proof. Follows from Theorem 1. ■

Notice that the above theorem only concerns existence, and not compactness. This is because the set of undominated Nash equilibrium may not be compact. This is displayed in the game given in table 10, which also establishes that the set of perfect equilibria is not compact. Action III of player 2 is strictly dominated, hence, $\mathcal{N}^d(\Gamma) =$

1\2	I	II	III	IV
I	(4,0)	(0,0)	(2,1)	(3,0)
II	(2,0)	(2,0)	(2,1)	(2,0)
III	(0,0)	(4,0)	(2,1)	(3,0)

Table 10: A 2-player game with non-compact set of undominated Nash equilibria

$\{((s_1^I, s_1^{II}, s_1^{III}), (s_2^I, s_2^{II}, s_2^{III}, s_2^{IV})) : \sum_k s_i^k = 1, \text{ and } s_i^k \in [0,1], k = I, II, ...\} \backslash ((0,1,0),(0,0,1,0))$, because $s' = ((0,1,0),(0,0,1,0))$ is weakly dominated by $((1/2,0,1/2),(0,0,1,0))$.

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