An Analysis of Dynamic Bankruptcy Problems

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September 4, 2009

1 Introduction

The *bankruptcy problem* involves the allocation of a single resource in a situation where the amount available is insufficient to satisfy the claims of all agents simultaneously. This is very old problem which even appears in the Talmud (e.g., see Dagan, 1994).[1] The available quantity of the good to be divided is usually called the *estate*. The agents are also referred to as *creditors*, whereas the term *claims* is used to describe the agents' entitlements, demands or needs, depending upon the problem at hand.

The literature contains many examples of rules that prescribe how the resource should be allocated among the claimants for every possible bankruptcy problem (e.g. see Thomson, 2007). A solution to a bankruptcy problem is to be interpreted as the application of an allocation rule that gives a sensible distribution of the estate as a function of the agents' claims.

Problems of this type arise in many real life situations. The canonical example would be that of a bankrupt firm that is to be liquidated. A bankruptcy judge is to allocate the remaining *assets* E of a bankrupt firm among its creditors, N. Each agent i has *credited* c_i to the bankrupt firm and now, claims this amount. For example, see O'Neill (1982) and the following literature. For a detailed review of the extensive literature on taxation and bankruptcy problems, see Thomson (2003 and 2007).

[1]The Talmud is a record of rabbinic discussion pertaining to jewish law, ethics, customs, and history. It is a central text of mainstream judaism.

Another example would be the division of an estate among several heirs, particularly when the estate cannot meet all the deceased's commitments.

Taxation is also a good example of resource allocation problems. A public authority is to collect an amount E of *tax* from a society N. Each agent i has *income* c_i. This is a central and very old problem in public finance. For example, see Edgeworth (1898) and the following literature. Young (1987) proposes a class of "parametric solutions" to this problem

Many economic situations as discussed with examples above can be modelled as a problem of how to divide a resource among agents who have claims on it in a single period. However, for some situations it is necessary to model them as a *dynamic bankruptcy problem.* Dynamic bankruptcy problems consider cases where an estate is distributed in several periods and the agents have claims on the estate for all periods. We study such problems.

In this paper we focus on problems where there are two periods in which an estate is to be allocated between the two claimants. Agents have claims on the estate both in the first and the second periods. To motivate the problem, we provide the following examples.

Consider many state universities and several periods for the allocation problem. The state tries to grant an allowance for those universities. The claims mentioned above are interpreted as the universities' claims for their spendings in the given periods. Total endowments for all periods show the state's maximum budget to be allocated in all periods. It is reasonable since in the development plans of countries, they state budget constraints for several years (e.g., five years development plans in Turkey). Quite naturally, we consider the case where the total claims of the universities exceed the government budget for each period.

Another example would be that of allocation of an inheritance. In some cases, the total asset can consist of different kind of goods and commodities in terms of their liquidities. When those different type of assets are allocated in different times, the problem turns into a dynamic bankruptcy problem. For such an example, the numbers of periods is same as the total number of asset types. All heirs have their claims on all of asset types.

In our problem, we have to allocate the first period endowment and the second period endowment, namely E_1 and E_2, respectively, between two agents who have claims in both periods. Our first contribution is to provide a formulation of these kind of dynamic bankruptcy problems. We also evaluate some commonly used allocation rules for the static problem in our dynamic setting. Additionally, we provide new rules that can be applied to the dynamic bankruptcy setting. The static bankruptcy literature consists of many axioms according to which allocation rules are evaluated (e.g. see Thomson, 2007). In this study, we adapt some of the main axioms used in static bankruptcy literature to the dynamic bankruptcy domain. We then check the proposed allocation rules with respect to these axioms.

Our first main axiom is *Pareto optimality.* We characterize *Pareto optimal* allocations in the dynamic setting when the agents have linear utilities. The idea here is similar to the 2-person allocation on the domain of exchange economies. We consider the estate to be allocated in the first period and the second periods as different goods and hence transform our problem to a classical economy which can be represented with an *Edgeworth box.*

We then characterize *Pareto optimal* and *strategy-proof* allocations. We find that a *strategy-proof* and *Pareto optimal* allocation rule in such a setup must be a member of a *Generalized Dictatorial Family* which we introduce.

In **Section 2**, we present a review of the literature on both the static bankruptcy problems and the strategy-proofness of solutions on the domain of exchange economies. The model is presented in **Section 3** in detail. In **Section 4**, we give examples of some well-known static allocation rules that we adapt for dynamic bankruptcy problems[2] and we check if they satisfy the axioms we defined in Section 2. A special class of rules named **Generalized Dictatorial rules** is defined in **Section 5**. Our two characterization results are presented in **Section 6.** In **Section 7** we conclude.

2 Literature Review

The formalization of static bankruptcy problems were introduced by O'Neill (1982) and studied later by Aumann and Maschler (1985). O'Neill (1982) suggested that the static bankruptcy problem could be analyzed by a game-theoretic methodology. He proposed a formal presentation of the problem and gave an axiomatic treatment of rules that appear in rabbinical sources. In addition he proposed a method of transforming the static bankrupcty problem into a transferrable utility (TU) game. By applying the Shapley value to this game, O'Neill generated a rule that generalizes the Contested Garment principle[3] to any n-creditor

[2] The allocation rules we consider are commonly used allocation rules in the static bankruptcy literature. Since our problem requires two stage allocation for two periods seperately, we analyze the implications of those static allocation rules being repeatedly used in each period.

[3] The Contested Garment principle is provided in the Talmud through examples. A famous Mishna states: "Two hold a garment; one claims it all, the other claims half. Then one is awarded $\frac{3}{4}$, the other $\frac{1}{4}$."

bankruptcy problem.

Aumann and Maschler (1985) revisited the Contested Garment principle from the Talmud and defined a *consistency* property for the solutions. Then, they showed that each bankruptcy problem has a unique *consistent* solution. They converted the static bankruptcy problem to a coalitional game by an algorithm called the "*coalitional procedure*". They showed that the consistent solution of a static bankruptcy problem is the nucleolus of the corresponding coalitional game. In short, they applied the nucleus to the TU game mentioned above and generated a new bankruptcy rule. They showed that their rule recommends the same allocations recommended in Talmud some two thousands years ago to the three particular cases he considered. Their proposal is now referred to as the Talmudic rule.

Young (1987) analyzes solutions to static taxation problems which are the motivating example in his paper. However, his results hold for the static bankruptcy problem as well. He first precisely defines the *consistency* axiom and *parametric tax schedules* which are the form of many historical tax proposals. His first result shows that a continuous, symmetric method is consistent if and only if it is representable by a parametric tax schedule. Another result shows the equivalence between consistency and minimizing an additive loss function in the presence of continuity and symmetry. He also discusses the connection between these results and Lensberg's (1983) work in bargaining theory. He shows how the methods of proofs he developed throughout the paper can be used to construct an objective function for the Talmudic bankruptcy method.

Chun (1988a) considers why the Proportional solution is the most widely used. To that purpose, he adopts the axiomatic approach; he suggests a set of axioms which a desirable solution should satisfy and he shows that the proportional solution is the only solution to satisfy these axioms. His main axioms are *no advantageous reallocation* and *additivity.* A solution satisfies *no advantageous reallocation* if no subgroup of claimants ever benefits by transferring parts of their claims between themselves. A solution satisfies *additivity* if it yields the same allocation whether the total estate is divided at once or in several steps. Chun (1988a) characterizes the class of allocation rules satisfying *additivity,continuity* on the estate and the claims and *Pareto optimality.*

Chun (1988b) introduces a new bargaining solution which is commonly used later: the *Equal Losses rule.* This solution equalizes across agents the losses from the ideal point. In the bankruptcy literature we can interpret the ideal point as claims point. Two characterizations of the solution are presented by formulating axioms specifying how bargaining solutions should respond to changes in the feasible set and the ideal point. This paper is in bargaining literature but the rule he suggests in this paper is also used in bankruptcy literature.

Chun (1999) investigates the logical relations between various axioms in the context of static bankruptcy. Those axioms are: *population-monotonicity, resource-monotonicity, consistency, converse consistency, agreement*, and *separability.* In most axiomatic models, these axioms are not logically related. However, he shows that they are equivalent on the class of bankruptcy problems under minor additional requirements.

Dagan (1994) presents axiomatic characterizations of two bankruptcy rules discussed in the Jewish legal literature; the Constrained Equal Awards rule and the Contested Garment rule (which is defined only for two-creditor problems). A major property in his characterization is independence of irrelevant claims which requires that if an individual claim exceeds the total to be allocated the excess claim should be considered irrelevant. He shows that the Constrained Equal Award rule is the unique rule that satisfies *independence of irrelevant claims*, *composition*, and *equal treatment.* He also shows that The Contested Garment principle is the unique two-creditor rule that satisfies *self-duality* and *independence of irrelevant claims.* His last result is that the Contested Garment principle is the unique two-creditor rule that satisfies v - *separability*, *independence of irrelevant claims* and *equal treatment.*

Thomson (2003) gives a survey of the axiomatic and game-theoretic analyses of static bankruptcy and taxation problems. This essay is organized as an introduction to the literature devoted to the formal analysis of such problems. He presents the rules that are commonly used in practise or discussed in theoretical work. He shows how many can be obtained by applying solution concepts developed in cooperative game theory for bargaining games and for coalitional games. Thomson formulates properties of rules, first when the population of agents is fixed, then it may vary, compare the rules on the basis of these properties, and search for rules satisfying the greatest number of properties together. He models the resolution of conflicting claims as strategic games, and extends the model to handle surplus sharing and situations in which the feasible set is specified in utility space. He identifies well-behaved taxation rules is formally identical to identifying rules to reconcile conflicting claims, and all of the results he presents can be reinterpreted in that context.

Moulin (2001) also gives a survey on the equitable division of a joint cost or a jointly produced output among agents with different shares or types of output commodities which is a central theme of the theory of cooperative games with transferable utility. Another rationing model Mouling reviews in the survey is the static bankruptcy problem. Moulin's survey reviews the normative literature on these two models, and emphasizes their deep structural link via the *additivity* axiom for cost sharing. This principle requires that the individual cost shares depend additively upon the cost function. He explains that an additive cost sharing method can be written as the integral of a rationing method, and this representation defines a linear isomorphism between additive cost sharing methods and rationing methods.

Herrero and Villar (2001) provide a comparative analysis of some classical solutions to bankruptcy problems from an axiomatic viewpoint. They show that there are only three rules on the bankruptcy domain satisfying *equal treatment of equals*, *scale invariance*, *composition*, *path-independence*, and *consistency*: *Constrained Equal Award rule, Constrained Equal Loses rule,* and *Proportional rule. They* also use the *exemption, exclusion, independence of claims truncation,* and *composition from minimal rights* properties in letter sections and characterize these three rules and also *Talmud rule.* All these results illuminate on the kind of problems for which each solution might be better. They claim that the constrained equal awards rule seems appropriate for those problems in which individuals are the primary concern, whereas their claims only represent maximal aspirations. The constrained equal-losses rule is a sensible rationing scheme for those problems in which claims represent real entities of an absolute nature. The proportional rule lies somewhere in between since it gives priority neither to smaller nor larger claims. Hence claims and agents are treated on an equal foot.

The only paper regarding dynamic bankruptcy problems by Inarra and Skonhoft (2008). This paper considers *Total Allowable Catch (TAC)*, as regulating scheme. According to Inerra and Skonhoft, any overexploited fishery subject to a *TAC-regulating scheme* in which claims are based on historical catches can be modeled as a bankruptcy problem. *TAC-regulating scheme*, quotas have been introduced in most fisheries. A typical *TAC-regulating scheme* implies that in stage one the regulating authority sets a *TAC* for the actual fish stock for a given fishing period. In the next stage, the *TAC* is distributed among, different vessel groups or fisherman who claim to have fishing rights inherited in historical catches. How the total quota should be distributed among the various harvesters involved is, however, far from clear as the sum of the claims generally exceeds what is available, that is, the *TAC*. Therefore, a bankruptcy problem exists. The *TAC* in their cases corresponds to total endowment in each period in this paper.

In this paper we use a similar dynamic bankruptcy setup as Inarra and Skonhoft (2008) but there are significant differences between our works. First of all, Inarra and Skonhoft does not provide an axiomatic treatment of the dynamic bankruptcy problems. In Inarra and Skonhoft (2008), the endowment is stochastic and also there is dependence between the endowments of the first and the second period although in our paper endowments are deterministic Claims of an individual in different periods are also correlated in their paper. In our setup, endowments to be allocated in two periods are known. In this paper, there is no dependence between the endowment of the period one and the endowment of the period two. In Inarra and Skonhoft, first, the allocation of period one is done and after the first period's endowment is allocated then the endowment and claims are determined for the second period by *TAC-regulating scheme* and *quotas*. This is not the case in our paper. We set the endowments and the claims for both period in the beginning. The dynamic allocation rules we recommend gives the allocation for both of the periods by applying them to the problem once.

Apart from the bankruptcy literature, there is a huge literature on *strategy-proof* and *efficient* solutions to the allocation problems on the domains of exchange economies. Gibbard (1973) and Satterthwaite (1975) shows that strategy-proofness is a very demanding property in the sense that it is essentially equivalent to dictatorship on unrestricted domain of preferences.

Zhou (1991) considers 2-agent exchange economies in which agents have strictly convex and monotonic preferences. He shows that any *strategy-proof* and *efficient* solution on that domain is *dictatorial*.

Barberà and Jackson (1995) replace *efficiency* with *individual rationality* and some minor auxiliary conditions. On their domain which is the same as Zhou's (1991), they characterize *fixed-price trading* solutions: agents trade only in certain fixed proportions from their endowments.

On a domain with two goods and convex, strictly monotonic preferences, Sprumont (1995) drops *efficiency* and requires a *continuity* condition with respect to preferences. He characterizes a class of solutions in which a fixed agent receives his most preferred bundle from a predetermined set. While these domains are smaller than the one of Gibbard (1973) and Satterhwaite (1975) they still appear to be rich enough to give *strategy-proofness* the strength it has on the unrestricted domain of social choice.

Schummer (1996) investigates two-agent exchange economies in which agents have homothetic preferences. Homothetic preferences are commonly used in consumer and producer theory, international trade theory, and the theory of the aggregation of preferences. Cobb-Douglass and CES preferences are examples of the homothetic preferences. On that domain, Schummer shows that any *strategy-proof* and *efficient* solution is *dictatorial.* Schummer also shows that on the domain of two-agent exchange economies in which agents have linear, strictly monotonic preferences, any *strategy-proof* and *efficient* solution is *dictatorial.* In this paper, we reach similar results as Schummer (1996) althought there are important differences. Schummer uses homothetic preferences while we use linear preferences. The domains in this paper and in Schummer's work are quite different. We use the dynamic bankruptcy problem which includes individual claims although Schummer's domain is 2-agent exchange economy. The similarity between the two papers comes from the fact that our problems closely related to classical economies.

3 Model

In this section we formulate the dynamic bankruptcy problem as below.

Let $N = \{1, 2\}$ be the two agents and let $T = \{1, 2\}$ be the two periods. For $i \in N$ and $t \in T$, let $c_i^t \in \mathbb{R}_+$ be the claim of agent $i \in N$ at period $t \in T$. In each period $t \in T$, let $E^t > 0$ be the total endowment to be allocated at time $t \in T$. For each agent $i \in N$, let $\delta_i \in [0, 1]$ be the discount factor of agent $i \in N$. We will denote claims as $c = (c_1^1, c_1^2, c_2^1, c_2^2) \in \mathbb{R}_+^4$. For notational convenience, let $c_i = [c_i^1, c_i^2] \in \mathbb{R}_+^2$ be the claims of agent $i \in N$ in periods 1 and 2 respectively and let $c^t = \begin{bmatrix} c_1^t \\ c_2^t \end{bmatrix} \in \mathbb{R}_+^2$ be the claims of player 1 and 2 respectively in period $t \in T$. The total endowment vector will be denoted as $E = (E^1, E^2) \in \mathbb{R}_{++}^2$. We will denote the agents' discount factors as $\delta = (\delta_1, \delta_2) \in [0, 1]^2$.

Note that, for all $i \in N$,$t \in T$, $c_i^t \geq 0$, $E^t > 0$ and $\delta_i \in [0, 1]$. We will assume that $c_1^1 + c_2^1 \geq E^1$ and $c_1^2 + c_2^2 \geq E^2$. This is a standard assumption in the literature. In almost all papers on the bankruptcy problem, authors analyze the case where total claims exceed the endowment is considered. The simple reason is that when the claims add up to a smaller amount than the estate each agent can get his claim and no interesting allocation problem arises.

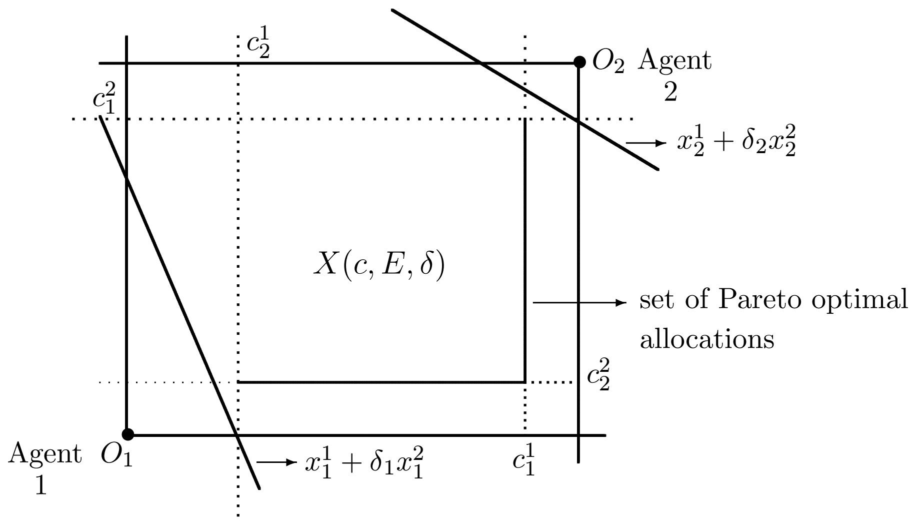
We will also assume that $c_i^t \leq E^t$ for all $i \in N$ and $t \in T$. This assumption simplifies the model and is valid for all of the rules which satisfy the propery knowns as *"truncation invariance"* in the literature. There are papers which also analyze the case in which there are some agents whose individual claim exceed the available estate. This assumption states that scaling down this unfeasible claim to the estate should not affect the outcome. Consider a bankruptcy problem in which the claim of some individual agent is larger than the estate. How a rule should treat his demand? One of the principles that appears in the Talmud says that one should not consider any claim that is larger than the estate. That is, replacing c by E if $c > E$ should not affect the recommendation. The well-known allocation rules such as Constrained Equal Awards rule and Talmud rule are *truncation invariant* although Proportional rule and Constrained Equal Awards rule are not. It is possible to create a new *truncation invariant* rule from a rule which is not *truncation invariant.*

Definition 1 *The triple* $(c, E, \delta) \in \mathcal{B}$ *is called a dynamic bankruptcy problem where* $\mathcal{B}=\{(c, E, \delta) \mid E^t > 0, 0 \leqq c_i^t \leqq E^t$ *for all* $i \in N,\ t \in T,\ \delta \in [0,1]^2$ *and* $c_1^t + c_2^t \geq E^t\}$

The set of feasible allocations for the problem (c, E, δ) is $X(c, E, \delta) = \{x \in \mathbb{R}_+^4 \mid 0 \leq x_i^t \leq c_i^t$ for all $i \in N$; $t \in T$ and $x_1^t + x_2^t = E^t$ for $t \in T\}$.

Definition 2 *An allocation rule for dynamic bankruptcy problems is a function* $F : \mathcal{B} \rightarrow \mathbb{R}_+^4$ *such that for all* $t \in T$, $x_1^t + x_2^t = E^t$ *and for all* $i \in N$, $t \in T$, $x_i^t \leq c_i^t$, $F_1^t(c, E, \delta) + F_2^t(c, E, \delta) = E^t$. *We will use* $\mathcal{F}$ *to denote the set of all such rules F.*

In this paper we use the utility function of agent i$\in N$ as $U_i(x) = x_i^1 + \delta_i x_i^2$. The model can be extended by defining the utility function of agent i as $U_i = u_i(x_i^1) + \delta_i u_i(x_i^2)$ where $u_i(x_i^t)$ is the utility of agent i$\in N$ gained in period t$\in T$. It can also be extended to more general preferences such as Cobb-Douglass.



The above figure illustrates the dynamic bankruptcy problem in the form of Edgeworth box economy. We consider the endowments of the period 1 and period 2 as different goods. Agents' claims and feasible set of solutions are also shown in the figure. Linear utilities of both agents are represented for both agents. For the case $\delta_1 < \delta_2$, the set of Pareto optimal allocations can also be seen from the figure.

3.1 Axioms

In this section we define some significant properties for the dynamic bankruptcy problems. We focus on *Pareto optimality, strategy-proofness, equal treatment* and *no-envy.* There is a large inventory of axioms that have been used in the literature. However, we just consider the properties appeared as the most fundamental axioms in the related literature.

Our first axiom is *Pareto optimality.* Given $(c, E, \delta) \in \mathcal{B}$ an allocation $x \in X(c, E, \delta)$ is *Pareto optimal* if there is no an allocation $x' \in X(c, E, \delta)$ such that for all $i \in N$, $U_i(x_i') \geq U_i(x_i)$ and there exists $j \in N$ such that $U_j(x_j') > U_j(x_j)$. An allocation is *Pareto optimal* if there are no alternative allocations at which one agent can be strictly better off while the other agent is not worse.

A rule $F \in \mathcal{F}$ is *Pareto optimal* if for all (c, E, δ), $F(c, E, \delta)$ is a *Pareto optimal allocation.* A Pareto optimal rule gives a Pareto optimal allocation for all the problems $(c, E, \delta) \in \mathcal{B}$.

Another important property in the standart model is *equal treatment of equals.* This property is a weak form of the *anonymity* property which requires that the identity of agents should not matter. Two different versions of the equal treatment of equal axiom are defined below for the dynamic case.

A rule $F \in \mathcal{F}$ satisfies *equal treatment of equals in allocation* if for all $(c, E, \delta) \in \mathcal{B}$ such that $c_1 = c_2$ and $\delta_1 = \delta_2$, we have $x_1 = x_2$. *Equal treatment of equals in allocation* axiom is one of the fairness properties in the literature and simply says that if both agents have the same claims and discount factors, then their resulting allocations must be the same.

A rule $F \in \mathcal{F}$ satisfies *equal treatment of equals in utility* if for all $(c, E, \delta) \in \mathcal{B}$ such that $c_1 = c_2$ and $\delta_1 = \delta_2$, we have $U_1(x_1) = U_2(x_2)$. This is the modified version of *equal treatment of equals* axiom which says if the both agents have the same claims in both periods and the same discount factors, then they have to obtain the same utility but not necessarily the same allocation.

Equal treatment of equals in allocation implies *equal treatment of equals in utility* but the reverse is not necessarily true.

Envy-freeness and *hierarchical envy-freeness* are also examples of the standard fairness properties in the allocation literature. They can be described for a dynamic bankruptcy problem as follows;

A rule $F \in \mathcal{F}$ is *envy-free* if for all $(c, E, \delta) \in \mathcal{B}$ and $i \in N$, $U_i(x_i) \geq U_i(x_j)$. An allocation rule F is *envy-free* if each agent prefers his share to the share of another agent.

A rule $F \in \mathcal{F}$ is *hierarchical envy-free* if for all (c, E, δ) such that $c_i \geqq c_j$, $U_i(x_i) \geq U_i(x_j)$, that is, agent i prefers his share to agent j's share whenever the condition $c_i \geqq c_j$ is satisfied. An allocation rule F satisfies *hierarchical no-envy* if each agent prefers his share to the share of another agent with a smaller claim. (e.g., see Kıbrıs, 2003)

A rule $F \in \mathcal{F}$ is strategy-proof if for all $(c, E, \delta) \in \mathcal{B}$ for all $i \in N$, for all $\delta_i^{'} \in [0, 1]$ for $x = F(c, E, \delta)$ and $x^{'} = F(c, E, \delta_j, \delta_i^{'})$, $U_i(x_i) \geq U_i(x_i^{'})$. *Strategy-proofness* axiom gives incentives to the agents to make truthful declaration of their discount factors. It says that an agent cannot get higher utility by declaring his discount factor different that what it is.

4 Allocation Rules

The followings are the most common allocation rules in static bankruptcy problems.

In solving bankruptcy problems, the most common practice in most countries is to make awards *proportional* to claims. In fact, *proportionality* has a long documented history as the primary method of handling simple allocation problems of the kind considered here. The **Proportional rule** allocates the endowment proportional to the characteristic values: for each $i \in N$, $PRO_i(c,E) = \frac{c_i}{\sum_N c_j}E$. In the taxation literature, this rule is called a *Linear Tax.* The proportional rule satisfies a number of appealing properties.

The **Constrained Equal Award rule** allocates the endowment equally, subject to no agent receiving more than his characteristic value: for each $i \in N$, $CEA_i(c,E) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \min\{c_i, \lambda\} = E$. In the single-peaked allocation literature, this rule is called the *Uniform rule,* and in the taxation literature, it is called the *Leveling Tax.*

The **Constrained Equal Losses rule** equalizes the losses agents incur, subject to no agent receiving a negative share: for each $i \in N$, $CEL_i(c,E) = \max\{0, c_i - \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \max\{0, c_i - \lambda\} = E$. In the single-peaked allocation literature, this rule is called the *Equal Distance rule,*, and in the taxation literature, it is called the *Head Tax.*

The **Talmud rule** (Aumann and Maschler, 1985) assigns equal gains until each agent receives half his characteristic value and then uses the equal losses idea: $TAL(c,E) = CEA\left(\frac{1}{2}c, \min\left\{E, \frac{1}{2}\sum_N c_i\right\}\right) + CEL\left(\frac{1}{2}c, \max\left\{0, E - \frac{1}{2}\sum_N c_i\right\}\right).$

Without a dynamic theory of bankruptcy, the only available option is the use of the proposals of the rules from the static literature repeatedly in every period. Below we present and analyze such applications of the above rules to the dynamic setting. As it can be seen below, those rules provided by applying the same static bankruptcy rules in every period do not satisfy basic properties we defined in Section 4, although they satisfy them on static problems. For example, the Proportional rule satisfies the *Pareto optimality* while the dynamic rule created by applying the Proportional rule in both periods does not satisfies it.

Now we suggest the following rules for the dynamic bankruptcy problem and check if they satisfy the axioms we defined in Subsection 3.1 or not.

Definition 3 *Proportional rule in both periods. Hence, the allocation is as follows;*
In period 1; $DPRO_1^1(c,E,\delta) = \frac{c_1^1}{c_1^1+c_2^1}E^1$ *and* $DPRO_2^1(c,E,\delta) = \frac{c_2^1}{c_1^1+c_2^1}E^1$
In period 2; $DPRO_1^2(c,E,\delta) = \frac{c_1^2}{c_1^2+c_2^2}E^2$ *and* $DPRO_2^2(c,E,\delta) = \frac{c_2^2}{c_1^2+c_2^2}E^2$

The dynamic rule created by applying the Proportional rule in both periods does not satisfy *Pareto optimality.* Consider the problem such that claim vector is $c = (8,8,8,8)$, discount factor vector $\delta = (0.6, 0.8)$ and endowment vector is $E = (10,10)$. The resulting allocation according to dynamic rule defined above is $DPRO(c,E,\delta) = (5,5,5,5)$ and the resulting utilities are 8 and 9, respectively. However, the allocation $x = (7, 2, 3, 8)$ gives higher utility to both agents.

It is trivial to see that the dynamic rule created by applying the proportional rule in both periods satisfies both *equal treatment of equals in allocation* and *in utilities*. The rule also satisfy *strategy-proofness* since the resulting allocation is independent of discount factors of the agents.

In order to check envy-freeness consider the following example; $c = (5, 5, 10, 10)$ and $E = (12, 12)$. Hence, $DPRO(c, E) = x = (4, 4, 8, 8)$. So, envy-freenes is violated in this example for all $\delta \in [0, 1]^2$. *Hierarchical envy-freeness* is satisfied for the dynamic rule since if $c_i \geqq c_j$ then $x_i \geqq x_j$ by definition of the Proportional rule.

Definition 4 *Constrained equal award rule in both periods. Hence, the allocation is as follows;*

In period 1 ; $DCEA_1^1 = \min\{c_1^1, \lambda_1\}$ *and* $DCEA_2^1 = \min\{c_2^1, \lambda_1\}$ *such that* $\min\{c_1^1, \lambda_1\} + \min\{c_2^1, \lambda_1\} = E^1$

In period 2; $DCEA_1^2 = \min\{c_1^2, \lambda_2\}$ *and* $DCEA_2^2 = \min\{c_2^2, \lambda_2\}$ *such that* $\min\{c_1^2, \lambda_2\} + \min\{c_2^2, \lambda_1\} = E^2$

The dynamic rule created by applying the Constrained Equal Awards rule in both periods does not satisfy *Pareto optimality* either. It can be seen above example in which $c = (8, 8, 8, 8)$, $\delta = (0.6, 0.8)$ and $E = (10, 10)$ since it gives the same allocation as the dynamic rule defined above.

The dynamic rule satisfies *strategy-proofness* since it also yields an allocation which is independent of discount factors. It satisfy both *equal treatment of equals in allocation* and *in utilities* by the definition of CEA.

For the *envy-freeness*, again, consider the example such that $c = (5, 5, 10, 10)$ and $E = (12, 12)$. Hence, $DCEA(c, E) = (5, 5, 7, 7)$. So, *envy-freeness* is not satisfied. By the definition of CEA rule, *hierarchical envy freeness* is satisfied for the dynamic rule that uses the CEA rule in both periods since if $c_i \geqq c_j$ then $x_i \geqq x_j$.

Definition 5 *Constrained Equal Losses Rule in both periods. Hence the allocation is as follows;*

In period 1; $DCEL_1^1 = \max\{c_1^1 - \lambda_1, 0\}$ *and* $DCEL_2^1 = \max\{c_2^1 - \lambda_1, 0\}$ *such that* $\max\{c_1^1 - \lambda_1, 0\} + \max\{c_2^1 - \lambda_1, 0\} = E^1$

In period 2; $DCEL_1^2 = \max\{c_1^2 - \lambda_2, 0\}$ *and* $DCEL_2^2 = \max\{c_2^2 - \lambda_2, 0\}$ *such that* $\max\{c_1^2 - \lambda_2, 0\} + \max\{c_2^2 - \lambda_2, 0\} = E^2$

The dynamic rule created by applying the Constrained Equal Losses rule in both periods does not satisfy *Pareto optimality.* It can be seen from the same example given above since it gives the same allocation as the dynamic rule created by applying Proportional rule in both periods.

It satisfy strategy-proofnees since it also yields an allocation which is independent of discount factors. It also satisfy both *equal treatment of equals in allocation and in utilities* by the definition of CEL.

Consider the same example used for *envy-freeness* in the above examples. Let $c = (5, 5, 10, 10)$ and $E = (12, 12)$. Hence, $DCEL(c, E) = (3.5, 3.5, 8.5, 8.5)$. Hence, *envy-freeness* is not satisfies for also for the dynamic rule uses the CEL in both periods. On the other hand, *hierarcical envy-freeness* is satisfied by the definition of CEL rule.

Definition 6 *Dictatorial Rules*

Two dictatorial rules $D[1], D[2] \in \mathcal{F}$ are defined as follows;

$D[1](c, E, \delta) = \begin{bmatrix} c_1^1 & c_1^2 \\ E^1 - c_1^1 & E^2 - c_1^2 \end{bmatrix}$ gives to Player 1 as much as he claims in both periods.

$D[2](c, E, \delta) = \begin{bmatrix} E^1 - c_2^1 & E^2 - c_2^2 \\ c_2^1 & c_2^2 \end{bmatrix}$ gives to Player 2 as much as he claims in both periods.

We generalize dictatorial rules as follows;

Definition 7 *Generalized Dictatorial Rules*

Let define $\boldsymbol{\pi}$ *as follows;* $\boldsymbol{\pi} : \{(c, E) \mid E \geqq 0, 0 \leqq c_i \leqq E$ *for all* $i \in N, c_1^t + c_2^t \geq E^t$ *for all* $t \in T\} \longrightarrow N$*. So,* $\boldsymbol{\pi}$ *is a function that maps* (c, E) *to an agent* i*, that is,* $\pi(c, E) \in N$*. The rule* $D[\pi] \in \mathcal{F}$ *is defined as* $D[\pi](c, E, \delta) = D[\pi(c, E)](c, E, \delta)$*. We call the set* $\mathcal{D} = \{D[\pi] \mid \pi \in \boldsymbol{\pi}\}$ *as "**Generalized Dictatorial Rules**.*

Generalized Dictatorial rules satisfy *Pareto optimality.* Since it gives as much as he claims to one of the agents and in order to make the other agent better off, the dictator must be made worse off. Generalized Dictatorial rules also satisfy *strategy-proofness* since the resulting allocation is independent of the discount factors of the agents.

It is obvious that Generalized Dictatorial rules do not satisfy *equal treatment of equals both in allocation and utilities*. Since those rules favor one of the agent.

To check *envy-freeness* let us consider the example such that $c = (5, 5, 10, 10)$, $\delta = (0.5, 0.5)$ and $E = (12, 12)$. Suppose that $\pi(c, E) = 2 \in N$. Hence, the resulting allocation is $D[2](c, E, \delta) = (2, 2, 10, 10)$. Agents' utilities are 3 and 15, respectively. Therefore, Generalized Dictatorial rules do not satisfy *envy-freeness*. They also do not satisfy *hierarchical envy-freeness* since if $c_i \geq c_j$, then it is possible to have a resulting allocation such that $x_i < x_j$.

5 Results

In this section, we will focus on to characterize Pareto optimal and strategy-proof allocation rules on the domain of dynamic bankruptcy. We first characterize Pareto optimal allocations on our domain. Then, we combine strategy-proofness with Pareto optimality and search for the rules which satisy both of them.

Theorem 1 *An allocation rule F is Pareto optimal if and only if F satisfies the following;*

(i) for each $(c, E, \delta) \in \mathcal{B}$ such that $\delta \in (0, 1]^2$ if $\delta_i < \delta_j$ then $F_i^1(c, E, \delta) = c_i^1$ or $F_j^2(c, E, \delta) = c_j^2$

(ii) for each $(c, E, \delta) \in \mathcal{B}$ such that $\delta \in (0, 1]^2$ if $\delta_i = \delta_j$ then $F(c, E, \delta) \in X(c, E, \delta)$

Proof. *We first prove for (i)*

($\Rightarrow$) Let $x = F(c, E, \delta)$. Suppose that $x_i^1 \neq c_i^1$ and $x_j^2 \neq c_j^2$. For all such allocation x, there is a sufficiently small $\varepsilon > 0$ and an allocation $x' \in X(c, E, \delta)$ such that $x_i'^1 = x_i^1 + \epsilon\frac{\delta_1+\delta_2}{2}$, $x_j'^1 = x_j^1 - \epsilon\frac{\delta_1+\delta_2}{2}$, $x_i'^2 = x_i^2 - \epsilon$ and $x_j'^2 = x_j^2 + \epsilon$. The new allocation x' makes both of the players better off. Thus, x is not Pareto Optimal.

($\Leftarrow$) Let $(c, E, \delta) \in \mathcal{B}$ such that $\delta_i < \delta_j$ and $x = F(c, E, \delta)$.

Case 1: *$x_i^1 = c_i^1$.*

If $x_i^1 = c_i^1$ then $x_j^1 = E^1 - c_i^1$. Then players' utilities are $u_i(x) = c_i^1 + \delta_i x_i^2$ and $u_j(x) = E^1 - c_i^1 + \delta_j(E^2 - x_i^2)$, respectively. Let $y \in X(c, E, \delta)$. Suppose $y_i^1 + \delta_i y_i^2 > c_i^1 + \delta_i x_i^2$. Then,

$\delta_i < \frac{y_i^1 - c_i^1}{x_i^2 - y_i^2}$. *Since* $\delta_i > 0$ *and* $y_i^1 - c_i^1 \leq 0$ *then* $x_i^2 - y_i^2 < 0$.

$$\begin{aligned} u_j(y) - u_j(x) &= (E^1 - y_i^1) + \delta_j(E^2 - y_i^2) - (E^1 - c_i^1) - \delta_j(E^2 - x_i^2) \\ &= c_i^1 - y_i^1 + \delta_j(x_i^2 - y_i^2) < \delta_i(y_i^2 - x_i^2) + \delta_j(x_i^2 - y_i^2) \\ &= (y_i^2 - x_i^2)(\delta_i - \delta_j) < 0. \end{aligned}$$

So, player j *is worse off.*

Now suppose that player j *is strictly better off at the allocation* y. *So,* $E^1 - y_i^1 + \delta_j(E^2 - y_i^2) > E^1 - c_i^1 + \delta_j(E^2 - x_i^2)$ *and hence,* $c_i^1 - y_i^1 < \delta_j(y_i^2 - x_i^2)$. *Since* $\delta_j > 0$ *and* $c_i^1 - y_i^1 \geq 0$ *then* $y_i^2 - x_i^2 > 0$.

$$\begin{aligned} u_i(y) - u_i(x) &= y_i^1 + \delta_i y_i^2 - c_i^1 - \delta_i x_i^2 \\ &= y_i^1 - c_i^1 + \delta_i(y_i^2 - x_i^2) < \delta_j(x_i^2 - y_i^2) + \delta_i(y_i^2 - x_i^2) \\ &= (y_i^2 - x_i^2)(\delta_i - \delta_j) < 0. \end{aligned}$$

So, player i *worse off.*

Case 2: $x_j^2 = c_j^2$.

If $x_j^2 = c_j^2$, *then* $x_i^2 = E^2 - c_j^2$. *Then players' utilities are* $u_i(x) = E^1 - x_j^1 + \delta_i(E^2 - c_j^2)$ *and* $u_j(x) = x_j^1 + \delta_j c_j^2$, *respectively. Let* $y \in X(c, E, \delta)$. *Suppose that* $y_j^1 + \delta_j y_j^2 > x_j^1 + \delta_j c_j^2$. *Then,* $x_j^1 - y_j^1 < \delta_j(y_j^2 - c_j^2)$.

$$\begin{aligned} u_i(y) - u_i(x) &= E^1 - y_j^1 + \delta_i(E^2 - y_j^2) - (E^1 - x_j^1) - \delta_i(E^2 - c_j^2) \\ &= x_j^1 - y_j^1 + \delta_i(c_j^2 - y_j^2) < \delta_j(y_j^2 - c_j^2) + \delta_i(c_j^2 - y_j^2) \\ &= (c_j^2 - y_j^2)(\delta_1 - \delta_2) < 0. \end{aligned}$$

So, player i *is worse off.*

Now suppose that player i *is strictly better off at the allocation* y. *So,* $E^1 - y_j^1 + \delta_i(E^2 - y_j^2) > E^1 - x_j^1 + \delta_i(E^2 - c_j^2)$ *and hence,* $y_j^1 - x_j^1 < \delta_i(c_j^2 - y_j^2)$.

$$\begin{aligned} u_j(y) - u_j(x) &= y_j^1 + \delta_j y_j^2 - x_j^1 - \delta_j c_j^2 \\ &= y_j^1 - x_j^1 + \delta_j(y_j^2 - c_j^2) < \delta_i(c_j^2 - y_j^2) + \delta_j(y_j^2 - c_j^2) \\ &= (c_j^2 - y_j^2)(\delta_i - \delta_j) < 0. \end{aligned}$$

So, player j is worse off.

Now, we prove for (ii)

($\Rightarrow$) *It is trivial from our assumption on the model that $x_i^t \leq c_i^t$ for all $i \in N$ and for all $t \in T$.*

($\Leftarrow$) *Let $F(c, E, \delta) = x \in X(c, E, \delta)$ and $\delta_i = \delta_j$. Suppose that x is not Pareto optimal. Then, there exists an allocation $y \in X(c, E, \delta)$ such that one agent can be strictly better off while the other agent is not worse. Without loss of generality, suppose that $U_i(y) > U_i(x)$. Therefore, $y_i^1 + \delta_i y_i^2 > x_i^1 + \delta_i x_i^2$. For the other agent, $U_j(y) \geq U_j(x)$. Since $\delta_i = \delta_j$,*

$$\begin{aligned} E^1 - y_i^1 + \delta_i(E^2 - y_i^2) &\geq E^1 - x_i^1 + \delta_i(E^2 - x_i^2) \\ x_i^1 + \delta_i x_i^2 &\geq y_i^1 + \delta_i y_i^2 \end{aligned}$$

It is a contradiction. Hence, $x \in X(c, E, \delta)$ is Pareto optimal. ■

After now, we consider the allocation rules which are both Pareto optimal and strategy-proof. First lemma considers two different dynamic bankruptcy problems. Only difference between those two problems is the discount factors of one agent who has smaller discount factor in those two problems. Lemma 1 simply states that if the agent who has smaller discount factor in both problems has different discount factors, then, the Pareto optimal and strategy-proof allocation rules give the same resulting allocation for both problems.

Lemma 1 *Let $F \in \mathcal{F}$ be a Pareto optimal and strategy-proof allocation rule and let (c, E, δ),(c, E, δ') $\in \mathcal{B}$ be such that $\delta \in (0,1]^2, \delta_j = \delta_j'$, $\delta_i < \delta_j$ and $\delta_i' < \delta_j$. Then, $F(c, E, \delta) = F(c, E, \delta')$.*

Proof. *Let $x = F(c, E, \delta)$ and $y = F(c, E, \delta')$. Since x is Pareto optimal, by Theorem 1, either $x_j^2 = c_j^2$ or $x_i^1 = c_i^1$.*

Case 1: *$x_j^2 = c_j^2$*

Now $x = \begin{bmatrix} x_i^1 & E^2 - c_j^2 \\ E^1 - x_i^1 & c_j^2 \end{bmatrix}$ and $y = \begin{bmatrix} y_i^1 & y_i^2 \\ y_j^1 & y_j^2 \end{bmatrix}$. Since F is a Pareto optimal rule $y_i^1 = c_i^1$ or $y_j^2 = c_j^2$. By strategy-proofness of F, $x_i^1 + \delta_i x_i^2 \geq y_i^1 + \delta_i y_i^2$. For y to be a Pareto optimal allocation, $y_j^2 = c_j^2$. If $y_i^1 = c_i^1$, then player, since $c_i^1 + \delta_i' y_1^2 > x_1^1 + \delta_i(E^2 - c_2^2)$ player i announces his discount factor as δ_i' rather than δ_i. Thus, the rule can not be strategy-proof. Hence, $y = \begin{bmatrix} y_i^1 & E^2 - c_j^2 \\ E^1 - y_i^1 & c_j^2 \end{bmatrix}$. Again by strategy-proofness, $x_i^1 + \delta_i(E^2 - c_j^2) \geq$

$y_i^1 + \delta_i(E^2 - c_j^2) \Longrightarrow x_i^1 \geq y_i^1$. *Also,* $y_i^1 + \delta_i^{'}(E^2 - c_j^2) \geq x_i^1 + \delta_i^{'}(E^2 - c_j^2) \Longrightarrow y_i^1 \geq x_i^1$. *Then,* $x_i^1 = y_i^1$. *Hence,* $x = y$.

Case 2: $x_i^1 = c_i^1$

Now $x = \begin{bmatrix} c_i^1 & x_i^2 \\ E^1 - c_i^1 & E^2 - x_i^2 \end{bmatrix}$ *and* $y = \begin{bmatrix} y_i^1 & y_i^2 \\ E^1 - y_i^1 & E^2 - y_i^2 \end{bmatrix}$. *By Pareto optimality,* $y_i^1 = c_i^1$ *or* $y_j^2 = c_j^2$. *In both cases, by strategy-proofness,* $c_i^1 + \delta_i x_i^2 \geq c_i^1 + \delta_i y_i^2$ *and* $c_i^1 + \delta_i^{'} y_i^2 \geq c_i^1 + \delta_i^{'} x_i^2$. *Then,* $x_i^2 = y_i^2$. *Hence,* $x = y$. ■

Lemma 2 *Let F be a Pareto optimal and strategy-proof allocation rule and let* (c, E, δ),$(c, E, \delta^{'}) \in \mathcal{B}$ *be such that* $\delta \in (0, 1]^2, \delta_i = \delta_i^{'}$, $\delta_i < \delta_j$ *and* $\delta_i < \delta_j^{'}$. *Then,* $F(c, E, \delta) = F(c, E, \delta^{'})$.

Note that Lemma 2 is the symmetric case of the Lemma 1.

Lemma 3 can be regarded as a more general version of Lemma 1 and Lemma 2. In Lemma 3 there are again two different bakruptcy problems in which both agents have different discount factor between those two problems. Only common feature between the two problems is that the order between the discount factors is preserved. In such a case, the Pareto optimal and strategy-proof allocation rules yield the same resulting allocation for those two problems.

Lemma 3 *Let F be a Pareto optimal and strategy-proof allocation rule and let* (c, E, δ),$(c, E, \delta^{'}) \in \mathcal{B}$ *be such that* $\delta \in (0, 1]^2, \delta_i < \delta_j$ *and* $\delta_i^{'} < \delta_j^{'}$. *Then,* $F(c, E, \delta) = F(c, E, \delta^{'})$.

Proof. There are several cases to consider;

Case 1: $\delta_i < \delta_i^{'} < \delta_j < \delta_j^{'}$

By Lemma 1, $F(c, E, (\delta_i, \delta_j)) = F(c, E, (\delta_i^{'}, \delta_j))$. Using Lemma 3 yields. $F(c, E, (\delta_i^{'}, \delta_j)) = F(c, E, (\delta_i^{'}, \delta_j^{'}))$.

Case 2: $\delta_i < \delta_i^{'} < \delta_j^{'} < \delta_j$

By Lemma 1, $F(c, E, (\delta_i, \delta_j^{'})) = F(c, E, (\delta_i^{'}, \delta_j^{'}))$. Using Lemma 3 yields. $F(c, E, (\delta_i^{'}, \delta_j^{'})) = F(c, E, (\delta_i^{'}, \delta_j))$.

Case 3: $\delta_i < \delta_j < \delta_i^{'} < \delta_j^{'}$

By Lemma 1, $F(c, E, (\delta_i, \delta_j^{'})) = F(c, E, (\delta_i^{'}, \delta_j^{'}))$. Using Lemma 3 yields. $F(c, E, (\delta_i^{'}, \delta_j^{'})) = F(c, E, (\delta_i^{'}, \delta_j))$.

Case 4: $\delta_i^{'} < \delta_i < \delta_j^{'} < \delta_j$

By Lemma 2, $F(c, E, (\delta_i, \delta_j^{'})) = F(c, E, (\delta_i, \delta_j))$. Using Lemma 3 yields $F(c, E, (\delta_i, \delta_j)) = F(c, E, (\delta_i^{'}, \delta_j))$

Case 5: $\delta_i^{'} < \delta_i < \delta_j < \delta_j^{'}$

By Lemma 2, $F(c, E, (\delta_i, \delta_j)) = F(c, E, (\delta_i^{'}, \delta_j))$. Using Lemma 3 yields $F(c, E, (\delta_i^{'}, \delta_j)) = F(c, E, (\delta_i^{'}, \delta_j^{'}))$

Case 6: $\delta_i^{'} < \delta_j^{'} < \delta_i < \delta_j$

By Lemma 2, $F(c, E, (\delta_i^{'}, \delta_j^{'})) = F(c, E, (\delta_i^{'}, \delta_j))$. Using Lemma 3 yields $F(c, E, (\delta_i^{'}, \delta_j)) = F(c, E, (\delta_i, \delta_j))$ ■

The following theorem is the main result of this paper. It characterizes the Pareto optimal and strategy-proof allocation rules. In the domain of dynamic bankruptcy problem a rule is Pareto optimal and strategy-proff if and only if it is a dictatorial rule.

Theorem 2 *If $\delta \in (0, 1]^2$ the allocation rule $F \in \mathcal{F}$ is Pareto optimal and strategy-proof if and only if F is in the Dictatorial Family, $F \in \mathcal{D}$.*

Proof. ($\Rightarrow$) By the definition of the dictatorial rule and Theorem 1, F is Pareto optimal. Suppose $(c, E) \in \mathcal{B}$. Let $\pi(c, E) = i \in N$ for all $(\delta_i, \delta_j) \in (0, 1]^2$. Suppose that $D\,[i]\,(c, E, \delta) = x$. By the definition of $D[i]$, $x = x^{'}$ where $x^{'} = D[i](c, E, \delta_i, \delta_j^{'})$ such that $\delta_j \neq \delta_j^{'}$. Hence, dictatorial rules are strategy-proof.

($\Leftarrow$) Suppose $F \notin \mathcal{D}$. Assume that $\delta_i < \delta_j$ By Pareto optimality of F let $F(c, E, \delta_i, \delta_j) = x$ such that $x_i^1 = c_i^1$ or $x_i^2 = c_i^2$. If $\delta_i < \delta_i^{'} < \delta_j$,by Lemma 1, $F(c, E, \delta_i^{'}, \delta_j) = x$ Suppose that $\delta_i < \delta_i^{''} = \delta_j$. Now, all allocations having the same utility with the allocation x is Pareto optimal and strategy-proof. All $z \in X$ such that $z_i^1 + \delta_i z_i^2 = x_i^1 + \delta_i x_i^2$ are Pareto optimal and strategy proof. Call the allocation z such that $z_i^2 = c_i^2$ or $z_j^1 = c_i^1$ as y.

Now suppose that $\delta_i < \overline{\delta^{'}}_j$ and hence $F(c, E, \delta_i, \overline{\delta^{'}}_j) = x$ by Lemma 1.If $\delta_i < \overline{\delta_i} < \overline{\delta_j}$then $F(c, E, \overline{\delta_i}, \overline{\delta_j}) = x$. Now suppose that $\delta_i < \overline{\delta_i^{'}} < \delta_j$ then all the allocation having the same utility with the allocation x is Pareto optimal and strategy proof. All $z \in X$ such that $z_i^1 + \overline{\delta_i^{'}} z_i^2 = x_i^1 + \overline{\delta_i^{'}} x_i^2$ are Pareto optimal. Call the allocation z such that $z_i^2 = c_i^2$ or $z_j^1 = c_i^1$ as $\overline{y}$. Assume that $y \neq \overline{y}$. Suppose that $\overline{\delta_i^{'}} > \overline{\delta_j}$ Then by Lemma 1, $F(c, E, \overline{\delta_i^{'}}, \overline{\delta_j}) = \overline{y}$. However, by Lemma 2, $y \neq \overline{y}$. It is a contradiction. ■

6 Conclusion

In this paper, we considered a two-period distribution problem between two agents in which in both periods, both agents have claims on the endowment of that period. In the first section we gave a brief history of the literature on the static bankruptcy problem. In the static bankruptcy literature, characterization of rules according to some significant axioms have been made. We characterize some of the allocation rules in the two-period, two-agent bankruptcy problems according to those axioms. In addition to the bankruptcy literature, our results are also connected with the literature that looks for *strategy-proof* and *Pareto optimal* solutions to allocation problems on the domain of exchange economies.

One of the most important contributions of this paper is that we construct a model for the dynamic bankruptcy problem which is introduced by Inarra and Skonhoft (2008). Our model is an extension of the static bankruptcy problem. Therefore, we borrow some of the more prominent standard axioms for solutions to the static bankruptcy problems and adopt them for our model. We redefine them for the dynamic environment.

In this paper we first analyze the well-known allocation rules in the static bankruptcy literature. We adapt them to our dynamic setting. By applying the same static rule in all periods of our dynamic problem, we check if these dynamic rules satisfy the axioms we defined.

We next define a new class of allocation rules for the dynamic problem: Generalized Dictatorial rules. We also check which of the axioms are satisfied by that new class of rules.

Our first result, Theorem 1, characterizes Pareto optimal allocations for the dynamic bankruptcy problem. Theorem 1 states that, in a dynamic bankruptcy problem, a Pareto optimal allocation rule determines awards according to the discount factors of the agents. If the agents have different discount factors, then a Pareto optimal allocation rule fully awards the agent who has smaller discount factor in the first period and fully awards the agent who has higher discount factor in the second period. If the agents have the same discount factor then any allocation in the feasible set is Pareto optimal.

Our last result characterizes Generalized Dictatorial rules which are defined in Section 4. It states that, in the dynamic setup, a rule is Pareto optimal and strategy-proof if and only if it is a member of the Generalized Dictatorial rules family. It means that, for each (c, E), a Pareto optimal and strategy-proof allocation rule determines an agent and then gives him as much as he claims for all claims δ.

In the paper we just focused on the characterization of the Pareto optimal and strategy-proof allocation rules. Other characterizations are left for future research. Also we only consider linear preferences in this work. Non-linear utilities are also left for future research.

Non-cooperative game theoretic versions of the same problem can also be a good research question to work on.

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