On Fraction Order Modeling and Control of Dynamical Systems ⋆

Islam S. M. Khalil * **A. Teoman Naskali** * **Asif Sabanovic** *

* *Faculty of Engineering and Natural Science, Sabanci University, 34956 Turkey (e-mail: kahalil,teoman@su.sabanciuniv.edu, asif@sabanciuniv.edu).*

Abstract: This paper demonstrates the feasibility of modeling any dynamical system using a set of fractional order differential equations, including distributed and lumped systems. Fractional order differentiators and integrators are the basic elements of these equations representing the real model of the dynamical system, which in turn implies the necessity of using fractional order controllers instead of controllers with integer order. This paper proves that fractional order differential equations can be used to model any dynamical system whether it is continuous or lumped.

Keywords: Fraction calculus, Continuous and lumped systems, Fraction order control, Laplace transform.

1. INTRODUCTION

Dynamics of any lumped system is descried by a set of differential equations that can be obtained by the newtonian approach based on the entire system forces computations or the variational approach that is based on system energies computations. On the other hand, continuous systems such as beams and flexible manipulators are modeled using a set of partial differential equations. And as its well known that these ordinary differential equations can be represented as a set of linear equations in the laplace domain. Surprisingly enough, acting by the laplace operator on a set partial differential equations turns them into ordinary differential equations, and if it was applied twice on the same set of partial differential equations, we end up with another set of fraction order differential equations. It turns out that, in the laplace domain any dynamical system whether it is lumped or continuous is composed of a set of fraction order differentiators and integrators, that describe the exact systems dynamics. In other words, if the dynamical system is described by a set of partial differential equations, fraction order transfer function has to be obtained, on the other hand, if the system dynamics are described by ordinary differential equations, we may end up with a fractional order transfer function when we represent these equations in the laplace domain. However, dynamical systems can be classified into fractional or integer order systems. Simply, fractional order systems are systems described by fractional order transfer functions.

Ma and Hori (2007) pointed out that integer order model is equivalent to the fraction order model in the low frequency range, while at the high frequency range integer order model doesn't describe the system dynamics at all. Moreover, the control system or the compensator have to be with a fraction order as it was presented in Podlunbny et al. (1997).

Fractional order controllers show better transient response results compared with integer order controllers when they are applied on fractional order systems Podlubny (1999). Minimum over shoot, less sensitivity to system parameters variations and controller parameters are shown in Podlunbny (1999) when fraction-order control(FOC) was applied to a fraction order system.

The idea of using fractional-order controllers belongs to Oustaloup, who developed the so called *Command Robuste d'Ordre Non Entire* (CRONE) Outaloup (1995). Fraction order PID controllers were investigated in Podlunbny et al. (1997) and instead of jumping in the P-I-D plane by using integer order differentiator and integrator. The fraction order PID controller makes it possible to move continuously in the P-I-D plane. The FOC concept was restricted due to the unfamiliar idea of taking fractional order and the computational efficiency of computing the fraction order differentiators and integrators, that is basically based on approximations and memory length, that indicates how this approximation is close to the fraction ordered operator Chen and Moore (2002).

This paper proves that dynamical systems whether they are lumped or continuous can be modeled using fractional order differential equations. In other words, using integer order differentiators and integrators is nothing but a special case of the general fraction order operators. The paper is organized as follows, in section 2, fraction order transfer function is derived for both lumped and continuous dynamical systems. In section 3, the fraction order differentiator or integrator is approximated by a discretization process. A comparison between integer and fractional order controller for a plant with a fractional order transfer function is included in section 4.

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2. FRACTION ORDER TRANSFER FUNCTION

2.1 Continuous Systems

Flexible beam A flexible beam is modeled by the following partial differential equation

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^4} = f(x,t) \quad (1)$$

where E, I, ρ and A are respectively the elasticity modulus, moment of inertia, material density and beams cross section. $f(x,t)$ is the external forcing function. Taking the laplace transform of (1) we get

$$EI\frac{d^4 w(x,s)}{dx^4} + \rho A s^2 w(x,s) = f(x,s) \quad (2)$$

where (2) is an ordinary differential equation, solving for the homogenous solution of $w(x,s)$ and making the following definition

$$c \triangleq \frac{\rho A}{EI} \ .$$

The characteristic equation of (2) is

$$\lambda^4 + cs^2 = 0 \quad (3)$$

and the roots are

$$\begin{aligned} \lambda_1 &= i^{1/2}\ c^{1/4}s^{1/2} \quad (4)\\ \lambda_2 &= -i^{1/2}\ c^{1/4}s^{1/2}\\ \lambda_3 &= i^{3/2}\ c^{1/4}s^{1/2}\\ \lambda_3 &= -i^{3/2}\ c^{1/4}s^{1/2} \ . \end{aligned}$$

The homogenous solution describing the beams transient response is

$$w(x,s) = c_1 e^{\lambda_1 s} + c_2 e^{\lambda_2 s} + c_3 e^{\lambda_3 s} + c_4 e^{\lambda_4 s} \quad (5)$$

where $c_{1,2,3,4}$ are constants that depend on beam's boundary conditions. The previous equation indicates that the transient response of the beam is governed by a fraction order differentiator $s^{3/2}$, which in turn implies that flexible beam can be described precisely using a fraction order transfer function.

Rotor attached to a flexible shaft the partial differential equation describing this system in Manabe (2002)

$$\frac{I_1}{l}\frac{\partial^2\theta(x,t)}{\partial t^2} - kl\frac{\partial^2\theta(x,t)}{\partial x^2} = 0 \quad (6)$$

where I_1,l are the rotor inertia and the shaft length, k is the torsional stiffness of the shaft. Taking the laplace transform of (6) we get

$$\frac{I_1}{l}s^2\theta(x,s) - kl\frac{\partial^2\theta(x,s)}{\partial x^2} = 0 \quad (7)$$

and the fraction order transfer function between the input torque τ and the rotor angular position θ is

$$\frac{\tau(x,s)}{\theta(x,s)} = I_1 s^2 + kl\mu s\ tanh(\mu l s) \quad (8)$$

where

$$\mu^2 \triangleq \frac{I_1}{kl}$$

that is nothing but a fraction order transfer function, and the results obtained from (5) and (8) were expected as they depend on fraction order differentiator or integrators, because of the continuous nature of both systems.

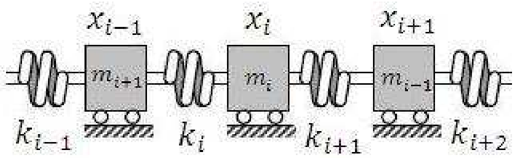


Fig. 1. Lumped flexible system

2.2 Lumped Systems

For the lumped mass spring system shown in Fig.1, the motion of each mass can be modeled by an ordinary differential equation, and the entire system can be modeled by a set of coupled linear or nonlinear differential equations. The motion of each mass is related to its neighbor mass by the following equation W.J.O'Connar (2007b).

$$X_{i+1}(s) = G(s)X_i(s) \quad (9)$$

where $G(s)$ is the transfer function relating any particular mass with its neighbor mass. And the equation of motion for the i^{th} mass

$$m\ddot{x}_i = k(x_{i-1} - 2x_i + x_{i+1}) \ . \quad (10)$$

Taking laplace transform we get a quadratic equation in $G(s)$

$$G^2(s) - (ms^2 + 2k)G(s) + k = 0 \quad (11)$$

solving the quadratic equation we get

$$G_{1,2}(s) = 1 + \frac{1}{2}\frac{s^2}{2\omega_n^2} \pm \sqrt{\frac{s^2}{2\omega_n^2}(1 + \frac{s^2}{2\omega_n^2})} \quad (12)$$

where

$$\omega_n^2 = \frac{k}{m}$$

and the general solution for the motion of each mass is a result of the superposition of two components W.J.O'Connar (2007a)

$$X_i(s) = \alpha_i(s)G_1(s) + \beta_i(s)G_2(s) \ . \quad (13)$$

Where $\alpha_i(s)$ and $\beta_i(s)$ are arbitrary, and the transfer function (12) has a non-integer order and its also nonrational. On one hand this transfer function is hard to work with, but on the other hand it can be approximated by a second order transfer function that will not represent a problem in case of feedback control as Occonar pointed out in W.J.O'Connar (2007b). In any event, the point here is to prove that the exact model that describes the dynamics of these systems is with fractional order, that is not easy to work with in the sense of mathematical computation.

3. FRACTION ORDER OPERATOR DISCRETIZATION

3.1 Generating Function Method

As the fraction order transfer function is based on a set of fraction order differentiators s^r and integrators s^{-r}, where r is any arbitrary real number not necessarily integer, they can be approximated using the following generating function

$$s = \omega(z^{-1}) \quad (14)$$

$$(\omega(z^{-1}))^{\pm r} = (\frac{2}{T})^r \frac{A_n(z^{-1}, r)}{A_n(z^{-1}, -r)} \tag{15}$$
$$= (\frac{2}{T})^r \lim_{\Delta \to 0} \frac{A_n(z^{-1}, r)}{A_n(z^{-1}, -r)}$$

where n represent the memory length of the generating function, the larger n we select the closer approximation to the exact operator we get with more computations.

$$A_o(z^{-1}, r) = 1 \tag{16}$$
$$A_n(z^{-1,r}) = A_{n-1}(z^{-1}, r) - c_n z^n A_{n-1}(z, r)$$

$$c = \begin{cases} r/n, & \text{n is odd;} \\ 0, & \text{n is even.} \end{cases} \tag{17}$$

Assuming that the fraction order differentiator $s^{0.5}$ is required to be approximated using the previous generating function we get the following. Maione (2006).

For n=1 we get the following discrete approximation

$$A_1(z^{-1}, 0.5) = 1 - 0.5z^{-1} \tag{18}$$
$$A_1(z^{-1}, -0.5) = 1 + 0.5z^{-1}$$

$$G_1(z) = \frac{44.72z - 22.36}{z + 0.5} . \tag{19}$$

For n=3 we get the following discrete approximation

$$G_3(z) = \frac{44.72z^3 - 22.36z^2 + 3.727z - 7.454}{z^3 + 0.5z^2 + 0.0833z + 0.1667} . \tag{20}$$

For n=7 we get the following discrete approximation

$$G_7(z) = \frac{\begin{array}{c} 44.72z^7 - 22.36z^6 \\ +4.792z^5 - 7.986z^4 + 2.795z^3 \\ -4.792z^2 + 1.597z - 3.194 \end{array}}{\begin{array}{c} z^7 + 0.5z^6 \\ +0.107z^5 + 0.1786z^4 + 0.0625z^3 \\ +0.107z^2 + 0.0357z + 0.07143 \end{array}} \tag{21}$$

For n=9 we get the following discrete approximation

$$G_9(z) = \frac{\begin{array}{c} 44.72z^9 - 22.36z^8 + 4.969z^7 - 8.075z^6 \\ +3.061z^5 - 4.947z^4 + 2.041z^3 - 3.461z^2 \\ +1.242z - 2.485 \end{array}}{\begin{array}{c} z^9 + 0.5z^8 + 0.111z^7 + 0.1806z^6 \\ +0.0684z^5 + 0.1106z^4 + 0.0456z^3 \\ +0.0773z^2 + 0.02778z + 0.05556 \end{array}} \tag{22}$$

Fig.2 shows the frequency responses of that discrete approximations of the fractional order operator, it turns out that increasing the memory length n makes the approximation closer the exact fraction differentiator, Fig.2 doesn't show the actual frequency response of the fraction order operator $s^{0.5}$, but as the generating function describing this operator is an infinite series, the length of the memory can be adjusted when the high order terms starts to have no pioneer impact on the frequency response. Therefore a generating function with memory length $n = 9$ is suitable approximation for the fraction order differentiator $s^{0.5}$.

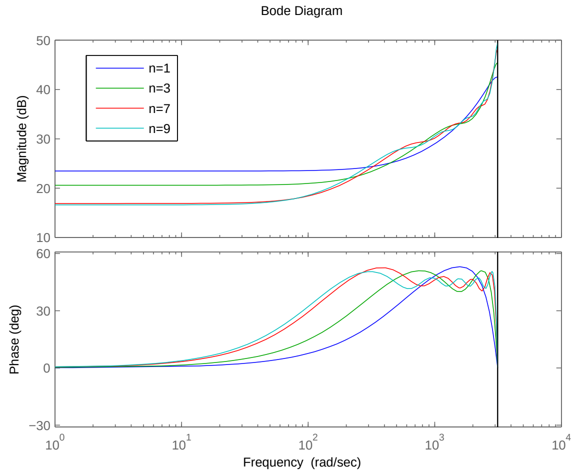


Fig. 2. Frequency response of the discrete approximation of the operator $s^{0.5}$

3.2 Integer Order Control

For the position control of an inertial mass as shown in Fig.3, the closed loop transfer function between the reference input and the actual inertial mass position is

$$G(s) = \frac{ks^r}{Js^2 + ks^r} . \tag{23}$$

For $r = 0$ the closed loop transfer function will have two poles on the imaginary axis that are

$$s_{1,2} = \pm i\sqrt{\frac{k}{J}}$$

which in turn implies that the response of the system will be oscillatory. On the other hand, setting $r = 1$ means that we are using a pure derivative controller, that is supposed to act on the time derivative of the error not the error it self.

The previous analysis shows that using integer order differentiators or integrator is equivalent to jumping on PID plane and just using four point on the PID plane. Fig.4-a shows the PID plane when integer order controller is used, where entire plane is reduced into 4 possible combinations and the rest of the plane is not used.

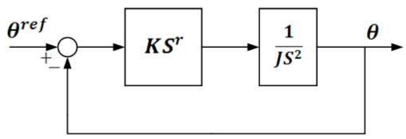
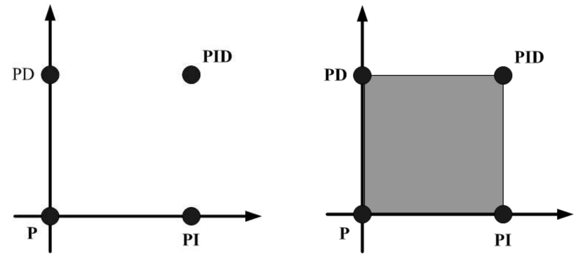


Fig. 3. Position control of an inertial mass

3.3 Fraction Order Control

Fraction order control allows moving in a continuous fashion in the PID plane, and instead of using finite points on the plane or jumping from a point to another the entire plane can be used and infinitely many combinations of differentiators and integrators can be used. Fig4-b shows the entire PID plane that can be used if the controller has a fraction order Podlubny et al. (1997).



(a) Integer order PID controller

(b) Fraction order PID controller

Fig. 4. Controller DOF region for fraction and Integer order PID controller

If an intger order controller is used for the inertial system shown in Fig.4, the system will be oscillatory or unstable as the only two choices are the pure proportional controller or the pure derivative. Assuming that the order r is a fraction we can achieve better tradeoff between stability and robustness, assuming that r=0.5 we get a controller with the following structure

$$G_c(s) = Ks^{0.5} \ . \tag{24}$$

The discrete approximation of the controller is

$$G_c(z) = K\frac{\begin{array}{c}44.72z^9 - 22.36z^8 + 4.969z^7 - 8.075z^6 \\ +3.061z^5 - 4.947z^4 + 2.041z^3 - 3.461z^2 \\ +1.242z - 2.485\end{array}}{\begin{array}{c}z^9 + 0.5z^8 + 0.111z^7 + 0.1806z^6 \\ +0.0684z^5 + 0.1106z^4 + 0.0456z^3 \\ +0.0773z^2 + 0.02778z + 0.05556\end{array}} \tag{25}$$

where k, is the fraction order controller gain, Fig.5 shows the response of the inertial system (22) for the fractional order controller (24) for different gain values. The purpose here is not to achieve a specific transient response, but to show that using fraction order control we can achieve a better tradeoff between stability and robustness, that can not be achieved in the case of integer order controller as jumping between the pure proportional controller to pure derivative, makes the system oscillatory or uncontrollable at all since the derivative actions acts on the time derivative of the error not the error itself. Indeed, the plant here is not described by a fraction order transfer function so that we use a fraction order controller that is more suitable, but the purpose here is to show the better tradeoff that can be achieved when we continuously move in the PID-plane instead of jumping between four finite points representing the possible combinations of the classical PID-controller.

4. $PI^\lambda D^\mu$-CONTROLLER

The transfer function of the fraction order PID controller is, Podlunbny (1999)

$$G_c(s) = \frac{U(s)}{E(s)} = K_p + K_i S^{-\lambda} + K_D S^{\mu} \tag{26}$$

where λ and μ are any arbitrary real numbers, selecting $\lambda = 1$ and $\mu = 1$ turns the controller into classical PID controller, to show the difference between using a classical PID-controller and a fractional order one we assume that we have a plant that is governed by the following discrete transfer function

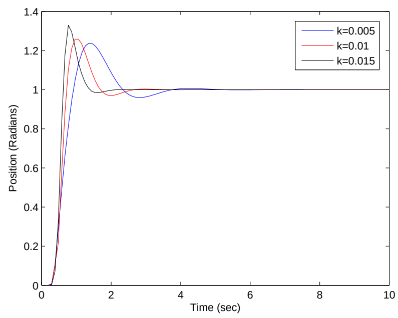


Fig. 5. Fraction order control of an inertial mass

$$G_p(z) = \frac{z^3 + 0.9z^2 + 0.27z + 0.3}{468.6z^3 - 420z^2 + 126.5z - 139.9} \tag{27}$$

selecting the following position on the continous PID plane

$$\mu = 0.5 \tag{28}$$
$$\lambda = 0.2$$

that represent some location on the continous PID plane rather than the classical PID locations that are shown in Fig.4, and the controller transfer function becomes

$$G_c(s) = \frac{U(s)}{E(s)} = K_p + K_i S^{-0.2} + K_D S^{0.5} \ . \tag{29}$$

This requires a discrete approximation for the fraction order differentiators$s^{0.5}$ and integrators$s^{-0.2}$. The discrete approximation of integrator $s^{-0.2}$ is

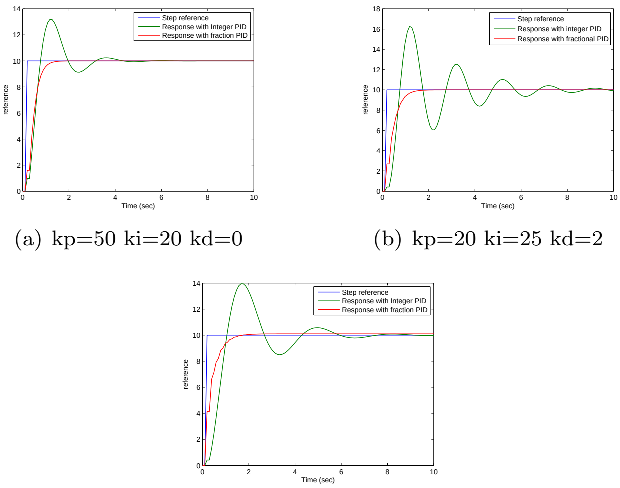
$$G_{s^{-0.2}}(z) = \frac{0.218z^3 + 0.043z^2 + 0.0029z + 0.014}{z^3 - 0.2z^2 + 0.0134z - 0.066}$$

while the approximation of the fraction order differentiator was previously computed in (22).

Applying both integer and fraction order PID controller to the discrete plant (27) using a variety of controller gains is shown in Fig.6. The fraction order controller shows better transient response characteristics in the sense of having less overshoot and higher response. Indeed, the reason behind selecting these particular $PI^\lambda D^\mu$ points on the PID plane is not explained or even investigated in this work, but the purpose is to show that using fraction order controllers makes all the PID plane available and reachable by the controller, unlike the integer controller where controller does finite jumps between the integer order differentiator and integrators.

5. CONCLUSION

Majority of dynamical systems are described by a fraction order transfer functions, not only distributed systems such as flexible shafts and beams, but also lumped systems. This fact doesn't implies that the real transfer function should be derived or worked with, as approximations for those function are quite enough especially in feedback control, In other words the for the lumped masses system the exact transfer function (12) can be approximated by a second order transfer function that shows similar response. Using fraction order controller makes all the PID plane reachable in the sense of building controllers with any required set of fraction order differentiators and integrators, not necessarily the four points on the PID



(a) kp=50 ki=20 kd=0

(b) kp=20 ki=25 kd=2

(c) kp=20 ki=10 kd=5

Fig. 6. Fraction and integer order control of a plant described by a fraction order transfer function

plane in the case of integer order control. In other words, using fraction order controllers increase the flexibility of the control process, with the ability to achieve better tradeoff between stability and robustness as it was shown in inertial mass control example. This work shows that fraction order controllers allows the continuous move in the PID plane that could not be achieved in the case of integer order controllers, but on the other hand, why and how to select these particular point on the PID plane was not investigated in this work. But considering the controller with a fraction order keeps the entire PID-plane usable and reachable, giving the controller much more flexibility over the integer order ones.

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