Critical Percolation Phase and Thermal BKT Transition in a Scale-Free Network with Short-Range and Long-Range Random Bonds

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Percolation in a scale-free hierarchical network is solved exactly by renormalization-group theory, in terms of the different probabilities of short-range and long-range bonds. A phase of critical percolation, with algebraic (Berezinskii-Kosterlitz-Thouless) geometric order, occurs in the phase diagram, in addition to the ordinary (compact) percolating phase and the non-percolating phase. It is found that no connection exists between, on the one hand, the onset of this geometric BKT behavior and, on the other hand, the onsets of the highly clustered small-world character of the network and of the thermal BKT transition of the Ising model on this network. Nevertheless, both geometric and thermal BKT behaviors have inverted characters, occurring where disorder is expected, namely at low bond probability and high temperature, respectively. This may be a general property of long-range networks.

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Scale-free networks are of high current interest [1–5], due to their ubiquitous occurrence in physical, biological, social, and information systems and due to their distinctive geometric and thermal properties. The geometric properties reflect the connectivity of the points of the network. The thermal properties reflect the interactions, along the geometric lines of connectivity, between degrees of freedom located at the points of the network. These interacting degrees of freedom could be voters influencing each other, persons communicating a disease, etc., and can be represented by model systems. Among issues most recently addressed have been the occurrence of true or algebraic [6, 7] order in the geometric or thermal long-range correlations, and the connection between these geometric and thermal characteristics. In Ising magnetic systems on a one-dimensional inhomogeneous lattice [8–11] and on an inhomogeneous growing network [12], a Berezinskii-Kosterlitz-Thouless (BKT) phase in which the thermal correlations between the spins decay algebraically with distance was found. In growing networks [13–20], geometric algebraic correlations were seen with the exponential (non-power-law) scaling of the size of the giant component above the percolation threshold. The connection between geometric and thermal properties was investigated with an Ising magnetic system on a hierarchical lattice that can be continuously tuned from non-small world to highly clustered small world via increase of the occurrence of quenched-random long-range bonds [21]. Whereas in the non-small-world regime a standard second-order phase transition was found, when the small-world regime is entered, an inverted BKT transition was found, with a high-temperature algebraically ordered phase and a low-temperature phase with true long-

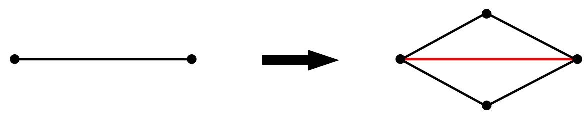

FIG. 1: (Color online) The scale-free random network is constructed by the repeated imbedding of the graph as shown in this figure. The four edges surrounding the graph here will be imbedded at the next phase of the construction. Such surrounding edges of the innermost graphs of the created infinite network are called the innermost edges. Along the innermost edges, a bond occurs with probability q. Along each of the other edges, a bond occurs with probability p. The latter are the long-range random bonds. Different realizations are illustrated in Fig.2.

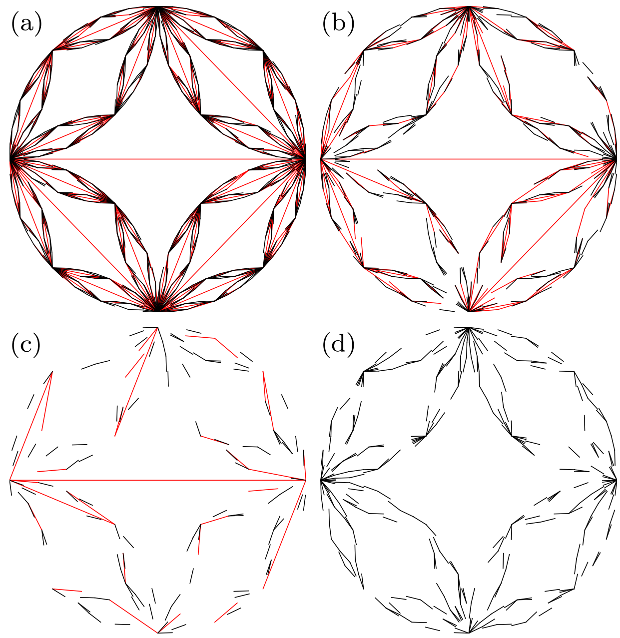

FIG. 2: (Color online) Different realizations of the random network: (a) In the compact percolating phase, with $q = p = 0.8$. (b) In the compact percolating phase, with $q = p = 0.4$. (c) In the algebraic percolating phase, with $q = p = 0.1$. (d) In the non-percolating phase, with $q = 0.3, p = 0$.

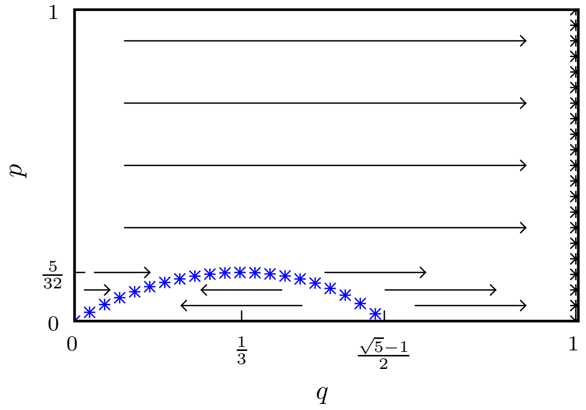

FIG. 3: (Color online) Renormalization-group flow diagram of percolation on the network with short-range and long-range random bonds.

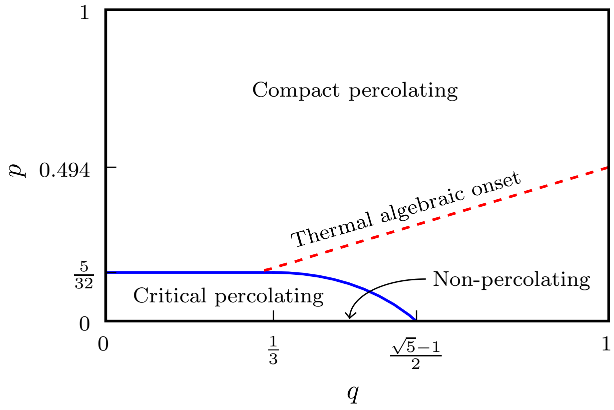

FIG. 4: (Color online) Geometric phase diagram of the network with short-range and long-range random bonds, exhibiting compact percolating, critical percolating, and non-percolating phases. The dashed line indicates the onset of high-temperature algebraic order in an Ising magnetic model on this network. It is thus seen that this thermal onset has no signature in the geometric correlations.

range order but delayed short-range order. Algebraic order in the thermal correlations has also been found in a community network.[22] In the current work, the geometric percolation property of the quenched-random long-range bonds is studied, aiming to relate the geometric properties to the algebraic thermal properties. From an exact renormalization-group solution, surprising results are found both for the geometric properties in themselves and in their would-be relation to the thermal properties.

The solved infinite network is constructed on a very commonly used hierarchical lattice [23–25] with the addition of long-range random bonds, as indicated in Fig. 1. The lattice formed by the innermost edges in the construction explained in Fig. 1 is indeed one of the most commonly used two-dimensional hierarchical lattices. In our study, on each of these edges, a bond occurs with probability q. To this hierarchical lattice, all further-neighbor edges are added between vertices of the same level. On each of these further-neighbor edges, a bond occurs with probability p, thus completing the random network studied here. Note that, due to the scale-free nature of this network, phase transition behaviors as a function of p must be identical along the lines $q = 0$ and $q = p$, which is indeed reflected in the results below.

Hierarchical lattices provide exact renormalization-group solutions to network [21, 22, 26–32] and other diverse complex problems, as seen in recent works [33–46]. The percolation problem presented by the random network defined above is also readily solved by renormalization-group theory. The recursion relation is obtained by replacing graphs at the innermost level of the random network by equivalent, renormalized, nearest-neighbor bonds, which thereby occur with renormalized short-range bond probability

$$q' = 1 - (1 - q^2)^2(1 - p)\,. \qquad (1)$$

This equation is derived as the probability $1 - q'$ of not having any path across the unit, each $(1 - q^2)$ factor being the probability of one sequence of short-range bonds being missing and $(1 - p)$ being the probability of the long-range bond being missing. The long-range bond probability p does not get renormalized, similarly to the thermal long-range interaction in the Ising model lodged on this network [21]. The renormalization-group flow of Eq.(1) has fixed points at $q = 1, p$ arbitrary and at $q = p = 0$. These fixed points are stable under the renormalization-group flows and respectively correspond to the sinks of the ordinary percolating and non-percolating phases. Another continuum of fixed points is obtained from the solution of

$$(1 - p)(q^3 + q^2 - q - 1) + 1 = 0\,. \qquad (2)$$

This equation gives a continuously varying line of fixed points in the region $0 \leq q \leq (\sqrt{5} - 1)/2, 0 \leq p \leq 5/32$. As seen in the flow diagram given in Fig. 3, this fixed line starts at $(q, p) = (0, 0)$, continues to $(1/3, 5/32)$, and terminates at $((\sqrt{5}-1)/2, 0)$. The renormalization-group eigenvalue along this fixed line is

$$\frac{dq'}{dq} = \frac{4q^*}{1 + q^*}\,, \qquad (3)$$

where the fixed point values q^* are determined by p as the solutions of Eq.(2). Thus, the fixed line is stable in its low-q segment and unstable in its high-q segment. Such reversal of stability along a fixed line, at $(q, p) = (1/3, 5/32)$ here, has also been seen in the BKT transition of the two-dimensional XY model [7] and in the Potts critical-tricritical fixed line in one [47], two [48], and three [49, 50] dimensions. The renormalization-group flows along the entire q direction, at any of the fixed p values in $0 < p \leq 5/32$, are as seen for the thermal behavior of antiferromagnetic Potts models [52, 53, 55].

As seen in Fig. 3, for $p > 5/32$, renormalization-group flows from all initial conditions are to the sink $q^* = 1$.

This basin of attraction is, therefore, an ordinary (compact) percolating geometric phase. For $p \leq 5/32$, the higher values of q flow to the sink $q^* = 1$, thereby also being in the ordinary (compact) percolating geometrical phase. For $0 < p \leq 5/32$, the low values of q flow to the stable critical fixed point at finite $0 < q^* \leq 1/3$, thereby being in a critical percolating phase. The infinite cluster in this phase is not compact at the largest length scales, but occurs with the bond probability of q^*. For $p = 0$, the low-q phase is the ordinary non-percolating geometric phase, with sink $q^* = 0$.

The horizontal portion, in Fig. 4, of the phase boundary between the compact and critical percolating phases is controlled by the fixed point at $(q, p) = (1/3, 5/32)$ with a marginal direction. The non-horizontal portion of the phase boundary between the compact and critical percolating phases is controlled by the unstable fixed line segment between $(q, p) = (1/3, 5/32)$ and $((\sqrt{5}-1)/2, 0)$, and has continuously varying critical exponents as a function of the long-range bond probability p. In an interesting contrast, Kaufman and Kardar [54] have found, for percolation on the Cayley tree with added long-range equivalent-neighbor bonds, continuously varying critical exponents as a function of the nearest-neighbor bond probability, between compact percolating and non-percolating phases. The emergent phase diagram of our current model is given in Fig. 4.

One of the motivations of our study was to relate the geometric and thermal properties of this scale-free network. The Ising magnetic system located on this network, with Hamiltonian

$$-\beta\mathcal{H} = \sum_{\langle ij \rangle} J s_i s_j \,, \qquad (4)$$

where $s_i = \pm 1$ at each site i, $\langle ij \rangle$ indicates summation over all pairs of sites connected by a short-range or long-range bond, and the interaction $J > 0$ is ferromagnetic, has an inverted BKT transition, with a high-temperature algebraically ordered phase, in the compact percolation phase in the region above the dashed line in Fig. 4. In the region below the dashed line in the compact percolation phase, the Ising transition is an ordinary second-order phase transition. In the critical percolation phase, the Ising model has no thermal phase transition.

The rightmost point of the dashed line, $(q, p) = (1, 0.494)$, was calculated in Ref.[21]. It was also seen that this point separates the non-small-world geometric regime at low p and the highly clustered, small-world geometric regime at high p. The flows of q onto $q^* = 1$ given in Eq.(1) dictate that, for all q in the currently studied infinite network, highly clustered small-world behavior occurs for $p \gtrsim 0.494$ and non-small-world behavior occurs for $p \lesssim 0.494$.

The rest of the dashed line in Fig. 4, with the leftmost point at $(q, p) = (0.31, 5/32)$, has been currently calculated using the renormalization-group recursion relation for the quenched probability distribution $\mathcal{Q}(J)$ for the interactions on the innermost level,

$$\mathcal{Q}^{(n)}(J'_{i'j'}) = \int \left(\prod_{ij}^{i'j'} dJ_{ij}\, \mathcal{Q}^{(n-1)}(J_{ij}) \right) dJ_{i'j'} \mathcal{P}^{(0)}(J_{i'j'}) \times \delta(J'_{i'j'} - R(\{J_{ij}\}, J_{i'j'}))\,, \qquad (5)$$

where (n) indicates the distribution after n renormalization-group transformations, $\mathcal{P}^{(0)}(J)$ is the initial (double-delta function) and conserved quenched probability distribution for the interactions on higher levels than innermost, and $R(\{J_{ij}\}, J_{i'j'})$ is the local interaction recursion relation,

$$R(\{J_{ij}\}, J_{i'j'}) = \frac{1}{2} \ln \left[\frac{\cosh(J_{i'k} + J_{kj'})}{\cosh(J_{i'k} - J_{kj'})} \right] + \frac{1}{2} \ln \left[\frac{\cosh(J_{i'l} + J_{lj'})}{\cosh(J_{i'l} - J_{lj'})} \right] + J_{i'j'}\,. \qquad (6)$$

Thus, it is seen that, although the geometric correlations of this network show an interesting critical percolating phase, no quantitative connection exists between the onset of geometric BKT behavior on the one hand, and the onsets of thermal BKT behavior and small-world character on the other hand. Qualitatively speaking however, note that an algebraically ordered geometric phase at low bond probability is akin to an algebraically ordered thermal phase at high temperature, both of which are rendered possible on the network. Thus, inverted algebraic order where disorder is expected may be a commonly encountered property, both geometrically and thermally, for long-range random networks. Finally, we note that to-date all renormalization-group calculations exploring thermal behavior on scale-free networks have been done using discrete Ising or Potts degrees of freedom. This is because of the compounded technical burden introduced in position-space renormalization-group calculations by continuum XY or Heisenberg degrees of freedom, for example requiring the analysis of the global flows of the order of a dozen Fourier components of the renormalized potentials.[55] However, in view of the rich BKT and other collective phenomena inherent to these continuum-spin models, such large undertakings may well be worth considering.

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