KOLMOGOROV PROBLEM ON WIDTHS ASYMPTOTICS AND PLURIPOTENTIAL THEORY

by

ÖZCAN YAZICI

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Master of Science

Sabanci University

Spring 2008

KOLMOGOROV PROBLEM ON WIDTHS ASYMPTOTICS AND PLURIPOTENTIAL THEORY

APPROVED BY

Prof. Dr. Vyacheslav Zakharyuta ....
(Thesis Supervisor)

Prof. Dr. Aydın Aytuna ....

Prof. Dr. Albert Erkip ....

Dr. Uğur Gül ....

Assist. Prof. Mert Çağlar ....

DATE OF APPROVAL: July 23, 2008

KOLMOGOROV PROBLEM ON WIDTHS ASYMPTOTICS AND PLURIPOTENTIAL THEORY

Özcan Yazıcı

Mathematics, Master of Science Thesis, 2008

Thesis Supervisor: Prof. Dr. Vyacheslav Zakharyuta

Keywords: Widths asymptotics, pluripotential theory, spaces of analytic functions, extendible bases, Hilbert scales, pluricomplex Green function.

Abstract

Given a compact set K in an open set D on a Stein manifold Ω of dimension n, the set A_K^D of all restriction of functions to K, analytic in D with absolute value bounded by 1 is a compact subset of $C(K)$. The problem on the strict asymptotics for Kolmogorov diameters (widths):

$$\ln d_i(A_K^D) \sim -\sigma i^{1/n}, i \longrightarrow \infty$$

was stated by Kolmogorov in an equivalent formulation for ϵ-entropy of that set. For $n = 1$, this problem is solved by efforts of many authors (Erokhin, Babenko, Zahariuta, Levin-Tikhomirov, Widom, Nguyen, Skiba - Zahariuta, Fisher - Miccheli, et al) with $\sigma = 1/\tau$ where $\tau(K, D) := 1/2\pi \int \Delta w$ (Δw is a positive measure supported on K).
For $n > 1$ Zakharyuta conjectured that for "good" pairs (K, D) such an asymptotics holds with $\sigma = 2\pi \left(n!/C(K, D)\right)^{1/n}$ where $C(K, D)$ is the pluricapacity of the pairs (K, D) introduced by Bedford-Taylor [3]. In [31, 35] Zakharyuta reduced this problem to a problem of pluripotential theory about approximating $w(K, D; z) - 1$ on any compact subset of $D \setminus K$ by pluricomplex Green functions on D. The latter problem which is known as Zakharyuta's conjecture has been solved by Nivoche [23] and Poletsky [25]. In this thesis we give the detailed proofs of Zakharyuta's reduction of Kolmogorov problem to his conjecture and the Nivoche-Poletsky result.

KOLMOGOROV ÇAPININ ASİMTOTU VE ÇOKLU POTANSİYEL TEORİSİ

Özcan Yazıcı

Matematik, Yüksek Lisans Tezi, 2008

Tez Danışmanı: Vyacheslav Zakharyuta

Anahtar Kelimeler: Kolmogorov çapı, çoklu potansiyel teorisi, analitik fonksiyonlar uzayı, çoklu karmaşık Green fonksiyonu.

Özet

n boyutlu Ω Stein manifoldu üzerindeki D açık kümesinde bir K kompakt seti verilsin. D üzerinde analitik ve boyu 1 ile sınırlı fonksiyonların K ya indirgenmesiyle elde edilen fonksiyonların oluşturduğu A_K^D kümesi $C(K)$ nın kompakt bir alt kümesidir. Kolmogorov çapının asimtotu ile ilgili problem aşağıdaki gibidir:

$$\ln d_i(A_K^D) \sim -\sigma i^{1/n}, i \longrightarrow \infty.$$

Problem tek boyutta bir çok matematikçinin (Erokhin, Babenko, Zahariuta, Levin-Tikhomirov, Widom, Nguyen, Skiba - Zahariuta, Fisher - Miccheli) çabasıyla çözülmüştür. $n > 1$ için, Zakharyuta yukardaki asimtotun $\sigma = 2\pi\,(n!/C(K,D))^{1/n}$ için geçerli olacağını iddia etmiştir. [31, 35] te, Zakharyuta bu problemi tamamen çoklu potansiyel teorik bir problem olan $w(D, K; z) - 1$ fonksiyonunun $D \setminus K$ içindeki kompakt setler üzerinde çoklu karmaşık Green fonksiyonlarıyla yaklaşımına indirgemiştir. Bu son problem Nivoche [23] ve Poletsky [25] tarafından çözülmüştür. Bu tezde Zakharyuta' nın problemi indirgemesinin ve Nivoche-Poletski sonucunun ayrıntılı ıspatı verilmiştir.

to my family and Esen

Acknowledgements

First of all, I would like to thank my supervisor Prof. Vyacheslav Zakharyuta for his motivation, guidance and encouragement throughout this thesis. His understanding and great vision has helped me to find my way throughout my studies at Sabanci University. I would like to thank my friends in the Mathematics Program. I am thankful to Esen Aksoy for her love, care, friendship and everything we shared. Finally, many thanks goes to my family for their love, and endless support.

This work is supported by TÜBİTAK.

Contents

CHAPTER 1

INTRODUCTION

Complexity of the binary search algorithm that distinguishes a definite element in a finite set S of $N(S)$ elements is equal to $[\log_2 N(S)]+1$. For infinite case, consider a compact set K in a metric space X. Let $\cup_i K_i$ be a covering of K such that diameter of $K_i \leq 2\epsilon$, $\forall i \in \mathbb{N}$. In $1950's$, Kolmogorov introduced the concept of approximate specification of an element $x \in K$ by finding K_i that contains x. Let $N_\epsilon(K, X)$ denote the smallest cardinality of such covering $\{K_i : \text{diameter of } K_i \leq 2\epsilon\}$. Then the *$\epsilon$-entropy* of the set K is defined by

$$H_\epsilon(K, X) = \ln N_\epsilon(K, X).$$

Note that in the information theory ϵ-entropy $\log_2 N_\epsilon(K, X)$ is asymptotically equivalent to $H_\epsilon(K, X)/\ln 2$ as $\epsilon \to 0$. Let K be a compact set in an open set D on a Stein manifold Ω, $H^\infty(D)$ be the Banach space of all bounded and analytic functions in D with the uniform norm, and A_K^D be the compact subset of $C(K)$ consisting of all restrictions of functions analytic in D and satisfy the inequality $||f||_D \leq 1$, endowed with the sup norm on K. Kolmogorov stated the problem of finding strict asymptotics for ϵ-entropy of A_K^D

$$H_\epsilon(A_K^D) \sim \tau \left(\ln \frac{1}{\epsilon}\right)^{n+1} \text{ as } \epsilon \to 0, \textit{i.e.} \lim_{\epsilon \to 0} H_\epsilon(A_K^D)/\ln(1/\epsilon) = \tau, \tag{1.1}$$

with some constant τ. For a subset A in a Banach space X the *Kolmogorov diameters (or widths)* of A with respect to the unit ball $\mathbb{B}_X$ of X are the numbers

$$d_i(A, \mathbb{B}_X) := \inf_{L \in \mathcal{L}_i} \sup_{x \in A} \inf_{y \in L} ||x - y||_X, \tag{1.2}$$

where $\mathcal{L}_i$ is the set of all i-dimensional subspaces of X, $i = 0, 1, ...$ We shall write $d_i(X, Y)$ instead of $d_i(\mathbb{B}_Y, \mathbb{B}_X)$ for a pair of normed spaces (X, Y) with a linear continuous imbedding $Y \hookrightarrow X$.

From the result of Mityagin [21] and Levin-Tikhomirov [18], we know that the problem about strict asymptotics (1.1) is equivalent to the following problem related with Kolmogorov diameters of the set A_K^D

$$\ln d_i(A_K^D) \sim -\sigma i^{1/n}, i \to \infty, \tag{1.3}$$

where $\sigma = \left(\frac{2}{(n+1)\tau}\right)^{1/n}$ and $d_i(A_K^D) := d_i(H^\infty(D), AC(K))$, where $AC(K)$ is the the completion of the set of all traces of functions, analytic on K, in the space $C(K)$.

In one dimensional case, for *good* pairs (K, D) Kolmogorov conjectured that the constant τ is equal to

$$\tau = \tau(K, D) := \frac{1}{2\pi} \int \Delta w,$$

where

$$w(z) := \limsup_{\zeta \to z} \sup\{u(\zeta) : u \text{ is subharmonic in } D,\ u|_K \leq 0,\ u < 1 \text{ in } D\}, \tag{1.4}$$

and Δw is a positive measure whose support is contained in K. This problem is solved (see [18, 2, 7, 8, 9, 36]): Let K be a non-polar compact subset of an open set D on an open one-dimensional Riemann surface Ω, $K = \widehat{K}_D$ and $D \Subset \Omega$ with ∂D consisted of a countable set of compact connected components at least one of which has more than one point. Then

$$-\ln d_i(A_K^D) \sim \frac{i}{\tau(K, D)}, i \to \infty.$$

Proof of this result based on a property of one dimensional multipole Green function :

$$g_D(P, z) = \sum_{k=1}^{N} c_k g_D(p_k, z)$$

where P is the finite set $\{(p_k, c_k), 1 \leq k \leq N\}$. But we do not have such an equality in multidimensional case (see (3.6) for the definition of pluricomplex Green function).

In [35] Zakharyuta conjectured that for *good* pairs (K, D) on a Stein manifold Ω of dimension $n \geq 2$, the asymptotics (1.3) holds with $\sigma = 2\pi \left(\frac{n!}{C(K,D)}\right)^{1/n}$, where $C(K, D)$ is the pluricapacity of K in D ([3],[26]) defined by

$$C(K, D) := \int_D (dd^c w(z))^n = \int_K (dd^c w(z))^n, \tag{1.5}$$

where,

$$w(z) = w(D, K; z) := \limsup_{\zeta \to z} \sup\{u(\zeta) : u \in \mathcal{PSH}(D), u|_K \leq 0, u < 1 \text{ in } D\} \tag{1.6}$$

is so called *relative extremal function* for K in D. Let Ω be a Stein manifold. Then we say (K, D) is a *pluriregular pair* on Ω if the following conditions are satisfied:

- K is a compact subset of open set D such that $K = \widehat{K}_D$ and K intersects with every connected component of D.

- $w(D, K; z) = 0$ on K and $\lim_{j \longrightarrow \infty} w(D, K; z_j) = 1$ for any discrete sequence $\{z_j\}$ in D.

In [35] Zakharyuta reduced the Kolmogorov problem on asymptotics (1.3) to the certain problem of pluripotential theory about approximating of $w(D, K; z) - 1$ by a sequence of multipole Green functions with finite set of logarithmic singularities. This problem is known as *Zakharyuta conjecture* and solved recently by Nivoche [23] and Poletsky [25]. In Chapter 5 we give the proof in detail. In Chapter 4 we give Zakharyuta's reduction of Kolmogorov problem to his conjecture. Zakharyuta used the theory of Hilbert scales, interpolation properties of analytic functions and functionals, method of extendable bases and extremal plurisubharmonic functions with isolated singularities to show that in order to solve Kolmogorov problem, it is enough to solve his conjecture. To modify Kolmogorov problem, Zakharyuta used Kolmogorov diameters $d_i(X_1, X_0)$ of an *admissible* couple of Banach spaces (X_1, X_0) for (K, D). The concept of admissiblity is given below.

Definition 1.0.1. *We say that a couple of Banach spaces (X_0, X_1) satisfying the dense linear continuous imbeddings $X_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow X_0$ is admissible for*

(K, D) *if for any other couple of Banach spaces* (E_0, E_1) *satisfying the dense linear continuous imbeddings*

$$X_1 \hookrightarrow E_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow E_0 \hookrightarrow X_0,$$

we have $\ln d_i(E_1, E_0) \sim \ln d_i(X_1, X_0),\ i \to \infty$.

Let (X_1, X_0) and (Y_1, Y_0) be two admissible pairs for (K, D). Consider a couple of Banach spaces (E_1, E_0) satisfying

$$X_1 \hookrightarrow E_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow E_0 \hookrightarrow X_0 \text{ and}$$

$$Y_1 \hookrightarrow E_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow E_0 \hookrightarrow Y_0.$$

Then by admissibility of the pairs (X_1, X_0) and (Y_1, Y_0),

$$\ln d_i(X_1, X_0) \sim \ln d_i(E_1, E_0) \sim \ln d_i(Y_1, Y_0).$$

Therefore the asymptotic class of $\ln d_i(X_1, X_0)$ is a characteristic of the pair (K, D) rather than a characteristic of (X_1, X_0).
By Corollary 4.2.1 in this text, for any pluriregular pair (K, D) there exists an admissible couple (X_0, X_1). The following theorem was proved by Zakharyuta with the assumption that *Zakharyuta Conjecture* is true. We give the details in Chapter 4.

Theorem 1.0.1. *Let* (K, D) *be a pluriregular pair on a Stein manifold* Ω. *Assume that Zakharyuta Conjecture is true. Then the strict asymptotics*

$$\ln d_i(X_1, X_0) \sim -2\pi \left(\frac{n!i}{C(K, D)} \right)^{1/n},\ i \to \infty$$

hold for any couple of Banach spaces (X_0, X_1) *admissible for* (K, D).

As a consequence we have an answer to the question about asymptotics (1.3) with some additional conditions on K and D.

Corollary 1.0.1. *Assume that Zakharyuta Conjecture is true. Let* (K, D) *be a pluriregular pair such that* $(AC(K), H^\infty(D))$ *is admissible for* (K, D). *Then the following strict asymptotics*

$$\ln d_i(A_K^D) \sim -2\pi \left(\frac{n!i}{C(K,D)} \right)^{1/n}, \ i \to \infty$$

holds.

CHAPTER 2

SPACES TO BE CONSIDERED

2.1 Stein Manifolds

A Hausdorff topological space Ω is called a *manifold of dimension* n if any point in Ω has a neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 2.1.1. *A manifold Ω (of dimension $2n$) is called a complex analytic manifold (of complex dimension n) if there is a given family $\mathcal{F}$ of homeomorphisms κ, called complex analytic coordinate systems, of open sets $\Omega_\kappa \subset \Omega$ on open sets $\tilde{\Omega}_\kappa \subset \mathbb{C}^n$ such that*

(i) *If κ and $\kappa' \in \mathcal{F}$, then the mapping*

$$\kappa'\kappa^{-1} : \kappa(\Omega_\kappa \cap \Omega_{\kappa'}) \longrightarrow \kappa'(\Omega_\kappa \cap \Omega_{\kappa'})$$

defines an analytic mapping,

(ii) $\cup_{\kappa\in\mathcal{F}}\Omega_\kappa = \Omega$,

(iii) *If κ_0 is a homeomorphism of an open set $\Omega_{\kappa_0} \subset \Omega$ onto an open set in $\mathbb{C}^n$ and the mapping $\kappa\kappa_0^{-1} : \kappa_0(\Omega_\kappa \cap \Omega_{\kappa_0}) \longrightarrow \kappa(\Omega_\kappa \cap \Omega_{\kappa_0})$ and its inverse are analytic for every $\kappa \in \mathcal{F}$, then $\kappa_0 \in \mathcal{F}$.*

We say that n complex valued functions $(f_1, ..., f_n)$ defined in a neighborhood of a point $z \in \Omega$ are a *local coordinate system* at z if they define a coordinate system of a neighborhood of z into $\mathbb{C}^n$.

Definition 2.1.2. *A closed subset V of a complex analytic manifold Ω of dimension n is called an analytic submanifold of dimension m if for each $v \in V$, there exist a neighborhood U of v and local coordinates $f_1, ..., f_n$ such that $U \cap V = \{z \in U : f_{m+1}(z) = ... = f_n(z) = 0\}$.*

Ω is called *countable at infinity* if there exist a family of compact subsets $\{K_i : i \in \mathbb{N}\}$ such that each compact subset of Ω is contained in some K_i.

Definition 2.1.3. *A complex analytic manifold Ω of dimension n which is countable at infinity is called Stein manifold if*

(i) *Ω is holomorphically convex, that is*

$$\widehat{K} = \widehat{K}_\Omega := \{z \in \Omega : |f(z)| \le \sup_K |f| \text{ for all } f \in A(\Omega)\}$$

is a compact subset of Ω for every compact subset K of Ω.

(ii) *For given two points z_1, z_2 with $z_1 \neq z_2$, there exists a function $f \in A(\Omega)$ such that $f(z_1) \neq f(z_2)$.*

(iii) *For every $z \in \Omega$, there exist n analytic functions on Ω $f_1, ..., f_n$ which form a coordinate system at z.*

Due to the following theorem, a Stein manifold can be represented as a submanifold of $\mathbb{C}^N$ where N is sufficiently large.

Theorem 2.1.1. *([12]) Any Stein manifold of dimension n is isomorphic to an analytic submanifold of $\mathbb{C}^{2n+1}$.*

2.2 Spaces of Analytic Functions

Let Ω be a complex manifold. $A(\Omega)$ is the space of all analytic functions on Ω with the topology of uniform convergence on compact subsets of Ω, i.e. with the locally convex topology generated by seminorms

$$|x|_K := \max\{|x(z)| : z \in K\} \tag{2.1}$$

where K is any compact subset of Ω. If Ω is countable at infinity, then $A(\Omega)$ is Fréchet space whose topology is given by the sequence of seminorms $\{|x|_{K_s}\}_{s=1}^{\infty}$ where $K_s \subset K_{s+1}$ and $\cup_s K_s = \Omega$.

Let E be an arbitrary subset of Ω. By $\mathcal{G}(E) = \mathcal{G}_{\Omega}(E)$, we denote the collection of all open neighborhoods of E in Ω. For $D_f, D_g \in \mathcal{G}(E)$, the functions $f \in A(D_f)$ and $g \in A(D_g)$ are said to equivalent ($f \sim g$) if there exist a $D \in \mathcal{G}(E)$ such that $D \subset D_f \cap D_g$ and $f \equiv g$ on D. A germ of analytic functions is an eqivalence class obtained by the relation $\sim$. If x is a germ on E and $f \in x$ then we say that f represents the germ x. We denote by $A(E)$ the locally convex space of all germs on E endowed with the inductive limit topology

$$A(E) = \lim \operatorname{ind}_{D \in \mathcal{G}(E)} A(D)$$

that is, the finest topology on $A(E)$ for which all natural mappings

$$J_{D,E} : A(D) \longrightarrow A(E) \; , D \in \mathcal{G}(E)$$

are continuous.

Let K be a compact set in Ω and $J : A(K) \longrightarrow C(K)$ be the natural restriction homomorphism. We denote by $AC(K)$, the Banach space obtained by the completion of $J(A(K))$ in $C(K)$ according to the norms (2.1).

We shall say that a locally convex space X is imbedded in a locally convex space Y if there exist an injective linear continuous mapping $i : X \longrightarrow Y$. We denote this imbedding by $X \hookrightarrow Y$. If this imbedding is dense, that is, $i(X)$ is a dense set in Y, then the conjugate mapping $i^* := Y^* \longrightarrow X^*$ is also a linear continuous injection. Thus any linear functional $y^* \in Y^*$ can be identified with its image $y' := i^*(y^*)$. Similarly we write $Y^* = Y' := i^*(Y^*) \hookrightarrow X^*$. If X is reflexive, this imbedding is also dense. In particular for a pluriregular pair (K, D) we write

$$A(D) = J_{D,K}(A(D)) \hookrightarrow A(K);$$

$$A(K)^* = A(K)' := J_{D,K}^*(A(K)^*) \hookrightarrow A(D)^*.$$

The elements of $A(D)^*$ are called *analytic functionals* on D. For $F \subset D$ the non-bounded seminorm is given by

$$|x'|_F^* := \sup\{|x'(x)| : x \in A(D), |x|_F \leq 1\} \tag{2.2}$$

on $A(D)^*$.

2.3 GKS- Duality

The result of Grothendieck-Köthe-Silva (see [10, 13, 15, 28]) allows us to realize for any set $E \subset \overline{\mathbb{C}}$ the space $A(E)^*$ as the space of analytic functions $A(E^*)$ where $E^* := \overline{\mathbb{C}} \setminus E$ with the assumption that all germs of $A(E)$ are equal to zero at the point ∞ if $\infty \in E$.

Theorem 2.3.1. *For any set $E \subset \overline{\mathbb{C}}$ there exists an isomorphism $\gamma : A(E)^* \to A(E^*)$ such that the following formula holds*

$$x^*(x) = \int_\Gamma x'(\zeta)x(\zeta)d\zeta, \ x \in A(E),$$

where $x' = \gamma(x^*)$, $\Gamma = \Gamma(x, x')$ is a rectifiable contour separating the singularities of the analytic germs x and x^*.

In several complex variables, there is no similar universal representation of $A(E)^*$ as a space of analytic functions. However, for polydisks in $\mathbb{C}^n$ we have the following proposition:

Proposition 2.3.1. *Let $U^n(r)$ be a polydisk around zero with polyradius $r = (r_\nu)$,*

$$U^n(r)^* := \{z = (z_\nu) \in \overline{\mathbb{C}}^n : |z_\nu| > r_\nu, \nu = 1, ..., n\}.$$

Then there exist a natural isomorphism $J : A(U^n(r))^* \to A(U^n(r)^*)$ such that for $x' = J(x^*)$, we have

$$x^*(x) = \int_\Gamma x(\zeta)x'(\zeta)\frac{d\zeta}{\zeta_1 \ldots \zeta_n}$$

where

$$\Gamma = \Gamma(x^*) = \{z = (z_\nu) \in \overline{\mathbb{C}}^n : |z_\nu| = \lambda r_\nu, \ \nu = 1, ..., n\}, \ \lambda = \lambda(x^*) < 1.$$

2.4 The Dual Form of Cartan Theorem

Let M be a closed analytic submanifold of Stein manifold Ω. Then according to Cartan theorem the restriction operator

$$R : A(\Omega) \to A(M) : Rx = x|M,\ x \in A(\Omega),$$

is a surjection. The adjoint operator $R^* : A(M)^* \to A(\Omega)^*$ maps any functional $\varphi \in A(M)^*$ to $\psi = \varphi \circ R \in A(\Omega)^*$. Using the theorem about dual relation between endomorphisms and monomorphisms we get the following dual version of Cartan theorem:

Proposition 2.4.1. *The adjoint operator $R^* : A(M)^* \to A(\Omega)^*$ of the restriction operator $R : A(\Omega) \to A(M)$ is an isomorphic embedding.*

2.5 Scales and Diameters

Definition 2.5.1. *A family of Banach spaces X_α, $\alpha_0 \le \alpha \le \beta_0$ is called a scale of Banach spaces if for arbitrary $\alpha_0 \le \alpha < \beta \le \alpha_1$, the following conditions are met:*

(i) $X_\beta \hookrightarrow X_\alpha$,

(ii) $||x||_{X_\gamma} \le C(\alpha,\beta,\gamma)(||x||_{X_\alpha})^{1-\tau(\gamma)}(||x||_{X_\beta})^{\tau(\gamma)}$ *with* $\tau(\gamma) = \frac{\gamma-\alpha}{\beta-\alpha}$, $\alpha < \gamma < \beta$

for any $x \in X_\beta$.

A scale of Banach spaces X_α, $\alpha_0 \le \alpha \le \beta_0$, is called *normal* if $C(\alpha,\beta,\gamma) \equiv 1$.

Definition 2.5.2. *A normal scale of Banach spaces X_α, $\alpha_0 \le \alpha \le \beta_0$, is called continuous normal scale if the function $\theta_x(\alpha) = ||x||_{X_\alpha}$, $x \in X_{\beta_0}$, is continuous on $[\alpha_0, \beta_0]$.*

Definition 2.5.3. *A continuous normal scale X_α, $0 \le \alpha \le 1$, is said to be regular if the norm in the dual spaces X'_α satisfies the following:*

$$||x^*||_{X'_\gamma} \le ||x^*||_{X'_\alpha}^{(1-\tau(\gamma))^{-1}} ||x^*||_{X'_\beta}^{\tau(\gamma)^{-1}}$$

for $\alpha < \gamma < \beta$ and $x^* \in X'_\alpha$, where τ is the same function as in Definition 2.5.1.

For further information about scales, we send the reader to the monograph [16]. Let $H_\alpha = H_0^{1-\alpha} H_1^\alpha$, $\alpha \in (-\infty, \infty)$, be a *Hilbert scale* generated by Hilbert spaces with dense imbedding $H_1 \hookrightarrow H_0$. If this imbedding is compact, i.e. every set bounded in the norm of H_1 is relatively compact in H_0, then there is a *common orthogonal basis* $\{e_i\}$ for H_0 and H_1, normalized in H_0 and enumerated by non-decreasing of norms in H_1:

$$||e_i||_{H_0} = 1, \ \mu_i = \mu_i(H_0, H_1) := ||e_i||_{H_1} \nearrow \infty. \tag{2.3}$$

The scale $\{H_\alpha\}$ is determined by the norms

$$(||x||_{H_\alpha})^2 := \sum_{i \in \mathbb{N}} |\zeta_i|^2 \mu_i^{2\alpha}, \ x = \sum_{i \in \mathbb{N}} \zeta_i e_i, \tag{2.4}$$

so that H_α consists of $x \in H_0$ with norm (2.4) if $\alpha \geq 0$, otherwise H_α is the completion of H_0 by the norm (2.4).

For a given a pair of normed spaces $X_1 \hookrightarrow X_0$ with a linear continuous imbedding, an equivalent definition of the Kolmogorov diameters can be given by following numbers:

$$d_i(X_1, X_0) = \inf\{\inf\{\lambda > 0 : \mathbb{B}_{X_1} \subset \lambda \mathbb{B}_{X_0} + L\} : L \in \mathcal{L}_i\}, \tag{2.5}$$

where $\mathcal{L}_i$ is the set of all i dimensional subspaces of X_0.

For a pair of Hilbert spaces H_1, H_0 which satisfies the conditions in (2.3), we have the following simple expression for the diameters ([21]):

$$d_i(H_1, H_0) = \frac{1}{\mu_{i+1}(H_0, H_1)}, \ i \in \mathbb{N}. \tag{2.6}$$

Now we will give a simple expression for the diameters of Hilbert scale $H_\alpha = H_0^{1-\alpha} H_1^\alpha$. For $\alpha_0 < \alpha_1$, due to (2.4), $\tilde{e}_i := \frac{e_i}{||e_i||_{H_1}^{\alpha_0}}$ is a common basis for H_{α_1} and H_{α_0} with

$$||\tilde{e}_i||_{H_{\alpha_0}} = 1 \ \text{ and } \tilde{\mu}_i(H_{\alpha_0}, H_{\alpha_1}) := ||\tilde{e}_i||_{H_{\alpha_1}} = \mu_i(H_0, H_1)^{\alpha_1 - \alpha_0}.$$

Hence we have

$$d_i(H_{\alpha_1}, H_{\alpha_0}) = \frac{1}{\tilde{\mu}_{i+1}(H_{\alpha_0}, H_{\alpha_1})} = \frac{1}{\mu_{i+1}(H_0, H_1)^{\alpha_1 - \alpha_0}} = d_i(H_1, H_0)^{\alpha_1 - \alpha_0}. \tag{2.7}$$

Proposition 2.5.1. *Let $X_1 \hookrightarrow Y_1 \hookrightarrow Y_0 \hookrightarrow X_0$ be a quadruple of Banach spaces with dense imbeddings. Then*

$$d_i(X_1, X_0) \prec d_i(Y_1, Y_0).$$

Proof. Since $X_1 \hookrightarrow Y_1$ and $Y_0 \hookrightarrow X_0$, there exist C_1, C_0 such that $\mathbb{B}_{X_1} \subset C_1\mathbb{B}_{Y_1}$ and $\mathbb{B}_{Y_0} \subset C_0\mathbb{B}_{X_0}$. For any L in $\mathcal{L}_i$ let

$$d_L := \inf\{\lambda > 0 : \mathbb{B}_{Y_1} \subset \lambda\mathbb{B}_{Y_0} + L\}.$$

Then for any $\epsilon > 0$

$$\mathbb{B}_{X_1} \subset C_1\mathbb{B}_{Y_1} \subset C_1(d_L + \epsilon)\mathbb{B}_{Y_0} + L \subset C_0C_1(d_L + \epsilon)\mathbb{B}_{X_0} + L.$$

By taking infimum over $\mathcal{L}_i$, and since $\epsilon > 0$ is arbitrary we obtain

$$d_i(X_1, X_0) \leq C_0C_1d_i(Y_1, Y_0).$$

Since the constants C_0, C_1 do not depend on i, we have

$$d_i(X_1, X_0) \prec d_i(Y_1, Y_0).$$

□

CHAPTER 3

SOME INFORMATION ON PLURIPOTENTIAL THEORY

In this chapter, we first present some fundamental properties of plurisubharmonic functions. Then we define the Monge-Ampère operator and the relative extremal function. For further study of plurisubharmonic functions the reader can consult [14].

3.1 Plurisubharmonic Functions

Let $z = (z_1, ..., z_n) \in \mathbb{C}^n$. The two norms on $\mathbb{C}^n$ that we shall be using are the *Euclidean norm*

$$||z|| = (z_1\bar{z}_1 + ... + z_n\bar{z}_n)^{1/2}$$

and the *maximum norm*

$$|z| = \max\{|z_1|, ..., |z_n|\}.$$

Note that, this norms are equivalent and $|z| \leq ||z|| \leq \sqrt{n}|z|$.
Let $a \in \mathbb{C}^n$ and $r > 0$. The *open polydisc*, with center at a, and radius r, is the set $\{z \in \mathbb{C}^n : |z - a| < r\}$.
Let Ω be an open subset of $\mathbb{C}^n$, and let $u : \Omega \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. The function u is said to be *plurisubharmonic* if for each $a \in \Omega$ and $b \in \mathbb{C}^n$, the function $\lambda \longmapsto u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in \Omega\}$. We denote by $\mathcal{PSH}(\Omega)$, the set of all plurisubharmonic functions in Ω.
The following theorem can be taken as an equivalent definition of plurisubharmonic

functions.

Theorem 3.1.1. *Let $u : \Omega \longrightarrow [-\infty, \infty)$ be upper semicontinuous and not identically $-\infty$ on any connected component of $\Omega \subset \mathbb{C}^n$. Then $u \in \mathcal{PSH}(\Omega)$ if and only if for each $a \in \Omega$ and $b \in \mathbb{C}^n$ such that*

$$\{a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega,$$

we have

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{it}b)dt. \tag{3.1}$$

It should be noted that plurisubharmonicity is a local property.
Let $\Omega \subset \mathbb{C}^n$ be open. If $\Omega \neq \mathbb{C}^n$, define

$$\Omega_\epsilon := \{z \in \Omega : \quad dist(z, \partial\Omega) > \epsilon\}$$

for $\epsilon > 0$. If $\Omega = \mathbb{C}^n$, we set $\Omega_\epsilon = \mathbb{C}^n$. The following theorem is known as *main approximation theorem* for plurisubharmonic functions.

Theorem 3.1.2. *Let Ω be an open subset of $\mathbb{C}^n$, and let $u \in \mathcal{PSH}(\Omega)$. Then $\exists u_\epsilon \in \mathcal{C}^\infty \cap \mathcal{PSH}(\Omega_\epsilon)$ such that u_ϵ decreases with decreasing ϵ, and $\lim_{\epsilon \to 0} u_\epsilon(z) = u(z)$ for each $z \in \Omega$.*

Theorem 3.1.3. *Let Ω be an open subset of $\mathbb{C}^n$*

(i) *If $u, v \in \mathcal{PSH}(\Omega)$ then $\max(u, v) \in \mathcal{PSH}(\Omega)$.*

(ii) *The family $\mathcal{PSH}(\Omega)$ is a convex cone, i.e. if α, β are non-negative numbers and $u, v \in \mathcal{PSH}(\Omega)$, then $\alpha u + \beta v \in \mathcal{PSH}(\Omega)$.*

(iii) *If Ω is connected and $\{u_j\}_{j \in \mathbb{N}} \subset \mathcal{PSH}(\Omega)$ is a decreasing sequence then $u = \lim_{j \to \infty} u_j \in \mathcal{PSH}(\Omega)$ or $u \equiv -\infty$.*

(iv) *Let $\{u_\alpha\}_{\alpha \in A} \subset \mathcal{PSH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha \in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization u^* is plurisubharmonic in Ω where*

$$u^*(y) = \limsup_{\substack{z \to y \\ z \in \Omega}} u(z) \quad y \in \bar{\Omega}$$

Proposition 3.1.1. *Let Ω be a domain in $\mathbb{C}^n$. Let $V \subset \Omega$ be an open subset. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(V)$, and*

$$\limsup_{z \to y} v(z) \leq u(y), \; y \in \partial V \cap \Omega, \tag{3.2}$$

then

$$w = \begin{cases} \max\{u, v\} & \text{in } V \\ u & \text{in } \Omega \setminus V \end{cases}$$

is plurisubharmonic in Ω.

Proof. The boundary condition (3.2) on v ensures that w is upper semicontinuous on Ω. By Theorem 3.1.3 (i) w satisfies the local submean inequality (3.1) at each $z \in V$, and it does also so when $z \in \Omega \setminus V$ since $w \geq u$ on Ω. □

3.2 The Complex Monge-Ampère Operator

3.2.1 Maximal Plurisubharmonic Functions

Definition 3.2.1. *A function $u \in \mathcal{PSH}(\Omega)$ is called maximal if for every relatively compact open subset G of Ω, and for each upper semicontinuous function v on $\bar{G}$ such that $v \in \mathcal{PSH}(G)$ and $v \leq u$ on ∂G, we have $v \leq u$ in G.*

We will use $\mathcal{MPSH}(\Omega)$ to denote the family of all maximal plurisubharmonic functions on Ω.

The differential operators d and d^c are defined by

$$d = \partial + \overline{\partial}, \; d^c = \mathrm{i}(\bar{\partial} - \partial)$$

where

$$\partial = \sum_j \frac{\partial}{\partial z_j} dz_j \text{ and } \overline{\partial} = \sum_j \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j.$$

Note that

$$dd^c = 2\mathrm{i}\partial\bar{\partial}$$

and if $u \in \mathcal{C}^2(\Omega)$, then

$$dd^c u = 2\mathrm{i} \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k.$$

Then *the Monge-Ampère operator* for $\mathcal{C}^2$- functions is defined as

$$(dd^c)^n = \underbrace{dd^c \wedge dd^c ... \wedge dd^c}_{n-times}.$$

Note that if $u \in \mathcal{C}^2(\Omega)$, then

$$(dd^c u)^n = 4^n n! \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] dV,$$

where $dV = (\frac{i}{2})^n dz_1 \wedge d\bar{z}_1 ... dz_n \wedge \bar{z}_n$.

The following theorem characterizes the maximality of a function $u \in \mathcal{C}^2 \cap \mathcal{PSH}(\Omega)$ in terms of the Monge-Ampère operator.

Theorem 3.2.1. *Let $u \in \mathcal{C}^2 \cap \mathcal{PSH}(\Omega)$. Then u is maximal if and only if $(dd^c u)^n = 0$ in Ω.*

3.2.2 Currents

By $\Lambda^k(\mathbb{C}^n, \mathbb{C})$, we denote the set of all k- forms. If p, q are positive integers such that $p + q = k$, then by $\Lambda^{p,q}(\mathbb{C}^n, \mathbb{C})$ we shall denote the subspace of $\Lambda^k(\mathbb{C}^n, \mathbb{C})$ generated by

$$\{dz_{\alpha_1} \wedge ... \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} ... \wedge d\bar{z}_{\beta_q} : 1 \leq \alpha_1 < ... < \alpha_p \leq n, 1 \leq \beta_1 < ... < \beta_q \leq n\}.$$

A $2n$ form w is called *positive* if $w = \tau dV$ for some non-negative number τ. A form $w \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ is called *elementary strongly positive* if there are linearly independent $\mathbb{C}$-linear mappings $\varphi_j : \mathbb{C}^n \longrightarrow \mathbb{C}$, $j : 1, ..., p$, such that

$$w = \left(\frac{\mathrm{i}}{2}\right)^p \varphi_1 \wedge \bar{\varphi}_1 \wedge .... \wedge \varphi_p \wedge \bar{\varphi}_p.$$

A form w is called *strongly positive* if it is belongs to the convex cone $SP^{p,p}(\mathbb{C}^n)$ in $\Lambda^{p,p}(\mathbb{C}^n, \mathbb{R})$ generated by elementary strongly positive forms.

Now we are in a position to define the positive forms of degree less than $2n$.

Definition 3.2.2. *A form $w \in \Lambda^{p,p}(\mathbb{C}^n, \mathbb{C})$ is called positive if for any $\varphi \in SP^{n-p,n-p}(\mathbb{C}^n)$, the $2n$-form $w \wedge \varphi$ is positive.*

Let Ω be a domain in $\mathbb{C}^n$, and $\mathcal{C}_0(\Omega, \mathbb{C})$ be the family of all continuous functions u on Ω such that supp$u$ is a compact subset of Ω.
A *Radon measure* on Ω is a continuous $\mathbb{C}$-linear functional on $\mathcal{C}_0(\Omega, \mathbb{C})$. By $\mathcal{D}_0^{p,q}(\Omega)$ (respectively, $\mathcal{D}^{p,q}(\Omega)$) we denote the set of all differential forms of bidegree (p, q) whose coefficients belong to $\mathcal{C}_0(\Omega, \mathbb{C})$ (respectively, $\mathcal{C}_0^\infty(\Omega, \mathbb{C})$). *i.e.*

$$\mathcal{D}_0^{p,q}(\Omega) = \mathcal{C}_0(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C}))$$

and

$$\mathcal{D}^{p,q}(\Omega) = \mathcal{C}_0^\infty(\Omega, \Lambda^{p,q}(\mathbb{C}^n, \mathbb{C})).$$

The elements of $\mathcal{D}^{p,q}(\Omega)$ are known as *test forms*. Let $\mathcal{D}^{p,q}(\Omega)$ be equipped with Schwartz' topology. Any continuous linear functional on the space $\mathcal{D}^{p,q}(\Omega)$ is called *a current* of bidegree $(n-p, n-q)$. The family of such currents will be denoted by $(\mathcal{D}^{p,q})'(\Omega)$
A current T is called *positive of degree* p if it is a (p, p) current such that for each $w \in \mathcal{C}_0^\infty(\Omega, SP^{n-p,n-p}(\mathbb{C}^n))$ we have $T(w) \geq 0$.
For $u \in \mathcal{PSH}(\Omega)$ and $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$ we can define a current $dd^c u$ by

$$dd^c u(\varphi) := \int_\Omega u dd^c(\varphi)$$

Theorem 3.2.2. *If $u \in \mathcal{PSH}(\Omega)$, then $dd^c u$ is a positive $(1,1)$- current.*

3.2.3 Generalized Complex Monge-Ampère Operator

In this part we will extend the definition of the Monge-Ampère operator so that it can be applied to locally bounded plurisubharmonic functions.

Lemma 3.2.1. *([14]) Let Ω be an open neighborhood of a compact set $K \subset \mathbb{C}^n$. Then there exist a constant $C > 0$ and a compact set $L \subset \Omega \setminus K$, such that for all $u_1, ...u_n \in \mathcal{C}^2(\Omega) \cap \mathcal{PSH}(\Omega)$, we have*

$$\int_K dd^c u_1 \wedge ... \wedge dd^c u_n \leq C||u_1||_L...||u_n||_L. \tag{3.3}$$

Bedford and Taylor (1976) constructed an inductive definition of the Monge-Ampère operator acting on locally bounded plurisubharmonic function by using the fact that $dd^c u$ is a positive $(1,1)$ current and inequality (3.3) with integration by part formula.

Let $u^1, ..., u^k \in L^\infty_{loc}(\Omega) \cap \mathcal{PSH}(\Omega)$. If $1 \leq k \leq n$, then $dd^c u^1 \wedge ... \wedge dd^c u^k$ can be defined inductively as a positive (k,k)-current like this:

$$dd^c u^1 \wedge ... \wedge dd^c u^k(\varphi) := \int_\Omega dd^c u^1 \wedge ... \wedge dd^c u^k \wedge \varphi = \int_\Omega u^k dd^c u^1 \wedge ... \wedge dd^c u^{k-1} \wedge dd^c \varphi,$$

where φ is a test form of bidegree $(n-k, n-k)$.

The operator $(dd^c)^n$, acting on locally bounded plurisubharmonic functions, is called the *generalized complex Monge-Ampère operator.* Note that for a locally bounded plurisubharmonic function u in Ω, $(dd^c u)^n$ is a positive distribution. Therefore it is a (Radon) measure on Ω.

Theorem 3.2.3 (Convergence theorem). *Let $\Omega \subset \mathbb{C}^n$ be a domain, $k \leq n$, $\{u^j_m\}_j \in \mathcal{PSH}(\Omega)$ and let $u^j_m \downarrow u_m \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ pointwise in Ω for $m \leq k$ as $j \longrightarrow \infty$. Then*

$$dd^c u^j_1 \wedge ... \wedge dd^c u^j_k \longrightarrow dd^c u_1 \wedge ... \wedge dd^c u_k \tag{3.4}$$

as $j \to \infty$, in the sense of weak*- convergence of currents, that is,

$$dd^c u^j_1 \wedge ... \wedge dd^c u^j_k(\varphi) \longrightarrow dd^c u_1 \wedge ... \wedge dd^c u_k(\varphi), \forall \varphi \in \mathcal{D}^{n-k,n-k}(\Omega).$$

Theorem 3.2.4 (Comparison theorem). *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$ and for each $\zeta \in \partial\Omega$,*

$$\liminf_{\substack{z \to \zeta \\ z \in \Omega}} (u(z) - v(z)) \geq 0.$$

Then

$$\int_{u<v} (dd^c v)^n \leq \int_{u<v} (dd^c u)^n$$

Theorem 3.2.5. *Let Ω be a domain in $\mathbb{C}^n$ and $u \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega)$. Then u is maximal if and only if $(dd^c u)^n = 0$.*

3.3 The Relative Extremal Functions

Let Ω be a domain in $\mathbb{C}^n$ and E is a subset of Ω. The *relative extremal function* for E in Ω is defined as

$$u_{E,\Omega}(z) = \sup\{v(z) : v \in \mathcal{PSH}(\Omega), v|_E \leq -1, v \leq 0\} \tag{3.5}$$

Note that the function $(u_{E,\Omega})^*$ is plurisubharmonic in Ω and $(u_{E,\Omega})^* = w - 1$ where $w := w(\Omega, E; z)$ is defined in (1.6). The next proposition follows directly from the definition of the relative extremal function.

Proposition 3.3.1. *If $E_1 \subset E_2 \subset \Omega_1 \subset \Omega_2$, then*

$$u_{E_1,\Omega_1} \geq u_{E_2,\Omega_1} \geq u_{E_2,\Omega_2}$$

A domain Ω in $\mathbb{C}^n$ is called *pluriregular* (or *hyperconvex*) if there exist a continuous plurisubharmonic function $\phi : \Omega \to (-\infty, 0)$ such that $\lim_{\zeta \to \partial D} \varphi(\zeta) = 0$. If Ω is open and K is a non pluripolar relatively compact subset of Ω, then Ω is pluriregular if and only if $\lim_{z \to \delta} u_{K,D} = 0$ for each $\delta \in \partial\Omega$.

Theorem 3.3.1. *If Ω is pluriregular and $K \subset \Omega$ is compact, then $u^*_{K,\Omega}$ is maximal in $\Omega \setminus K$, i.e.*

$$(dd^c u^*_{K,\Omega})^n = 0 \quad in \quad \Omega \setminus K$$

Proposition 3.3.2. *If Ω is a pluriregular domain containing a compact set K with $u^*_{K,\Omega} = -1$ on K, then $u_{K,\Omega}$ is a continuous function.*

Proposition 3.3.3. *Let Ω be a connected open set containing E. Then $u^*_{E,\Omega} \equiv 0$ if and only if E is pluripolar.*

Proposition 3.3.4. *Let Ω be a pluriregular domain in $\mathbb{C}^n$ containing a compact set K. Suppose that Ω_j is an increasing sequence of open sets such that $\Omega = \cup_{j=1}^{\infty} \Omega_j$ and $K \subset \Omega_1$. Then for each $z \in \Omega$,*

$$\lim_{j \to \infty} u_{K,\Omega_j}(z) = u_{K,\Omega}(z).$$

Let Ω be a domain in $\mathbb{C}^n$ and P be a finite set

$$\{(p_j, c_j) : p_j \in \Omega,\ c_j > 0,\ 1 \leq j \leq N\}$$

where $p_j \in \Omega$ and c_j are positive weights. Then

$$\begin{aligned} g_\Omega(P,z) = \sup\{v(z) \quad &: \quad v \text{ psh on } \Omega,\ \forall j = 1,...,N \\ v \quad &\leq 0, v(z) \leq c_j \log||z - p_j|| + O(1)\} \end{aligned} \tag{3.6}$$

is called *pluricomplex Green function on Ω with logarithmic poles p_j of weight c_j*. If Ω is pluriregular domain, then $g_\Omega(P,z)$ is the unique solution to the following problem:

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega \setminus \{p_1,...,p_N\}) \quad , u|\Omega < 0 \\ (dd^c u)^n = 0 \quad \text{on} \quad \Omega \setminus \{p_1,...,p_N\}, \\ u(z) = c_j \log||z - p_j|| + O(1) \quad as \quad z \longrightarrow p_j \\ u(z) \longrightarrow 0 \quad as \quad z \longrightarrow \zeta \in \partial\Omega \end{cases}$$

We have the following equality for the capacity of Green function $g := g_\Omega(P,z)$:

$$(dd^c g)^n = (2\pi)^n \sum_{j=1}^{N} c_j^n \delta_{p_j} \quad in \quad \Omega$$

where δ_{p_j} denotes the Dirac measure at p_j. If $g > c$ on $\partial\Omega$ then,

$$\int_\Omega (dd^c \max\{g, c\})^n = (2\pi)^n \sum_{j=1}^{m} c_j^n. \tag{3.7}$$

CHAPTER 4

KOLMOGOROV PROBLEM ON WIDTHS ASYMPTOTICS

4.1 Interpolation of Analytic Functions and Functionals

Definition 4.1.1. *An open set $D \Subset \Omega$ is said to be strongly pluriregular if there exist a function u which is continuous and plurisubharmonic on some pseudoconvex open set G such that $D \Subset G$ and $D = \{z \in G : u(z) < 0\}$.*

For analytic functions, we have the following interpolational estimates (see [33])

$$|f|_{D_\alpha} \leq (|f|_K)^{1-\alpha}(|f|_D)^\alpha, 0 < \alpha < 1, f \in H^\infty(D), \tag{4.1}$$

where (K, D) is a pluriregular pair and $D_\alpha := \{z \in D : w(D, K; z) < \alpha\}$.
In one variable, since any analytic functional can be represented as an analytic function in the complement of K (GKS duality), the inequalities (4.1) can be used to estimate analytic functionals. However in higher dimension ($n \geq 2$), we do not have GKS duality. Instead, we have the following interpolational estimates:

Theorem 4.1.1. *([33, 35]) Let (K, D) be a pluriregular pair on a Stein manifold Ω and D be strongly pluriregular. Then for each $\epsilon > 0$ and $0 < \alpha < 1$, we have*

$$|x'|^*_{D_\alpha} \leq M\ (|x'|^*_K)^{1-\alpha+\epsilon}(|x'|^*_D)^{\alpha-\epsilon},\ x' \in AC(K)' \hookrightarrow A(D)^* \tag{4.2}$$

with some constant $M = M(\alpha, \epsilon)$.

At first, we will consider the special case when D and K are analytic polyhedrons in Ω defined by the same collection of analytic functions. Here we consider the Stein manifold Ω as a closed complex manifold in $\mathbb{C}^N$ for some big enough N. Let us consider a family of polydisks in $\mathbb{C}^N$ given by:

$$V_\alpha := \{z = (z_\nu) \in \mathbb{C}^N : |z_\nu| < r_\nu(\alpha),\ \nu = 1, ..., N\},$$

where $r_\nu(\alpha) = R_\nu^\alpha r_\nu^{1-\alpha},\ r_\nu < R_\nu,\ \nu = 1, ..., N,\ \alpha \in \mathbb{R}$. Now we consider two families of analytic polyhedrons in Ω:

$$\Delta_\alpha := V_\alpha \cap \Omega,\ \tilde{\Delta}_\alpha := \overline{V}_\alpha \cap \Omega,\ \alpha \in [-\sigma, 1+\sigma], \sigma > 0$$

and let Φ_α and $\tilde{\Phi}_\alpha$ be polyhedrons that are the connected components of Δ_α and $\tilde{\Delta}_\alpha$ respectively which have non-empty intersection with $\tilde{\Phi}_{-\sigma} = \bar{\Delta}_{-\sigma}$.

Proposition 4.1.1. *For every $\epsilon > 0$ and $\alpha \in (0, 1)$ there is a constant $C = C(\alpha, \epsilon)$ such that*

$$|x^*|^*_{\tilde{\Phi}_\alpha} \leq C \left(|x^*|^*_{\tilde{\Phi}_0}\right)^{1-\alpha+\epsilon} \left(|x^*|^*_{\tilde{\Phi}_1}\right)^{\alpha-\epsilon}, \quad x^* \in AC(\tilde{\Phi}_0)^* \tag{4.3}$$

Proof. All polyhedrons Φ_α are closed submanifolds of polyhedrons V_α, and $\tilde{\Phi}_\alpha = \cap_{\beta>\alpha}\Phi_\beta$. Let $\alpha_0 < \alpha_1$ and both are contained in $[-\sigma, 1+\sigma],\ \sigma > 0$. For any $\alpha \in [\alpha_0, \alpha_1]$, since $\tilde{\Phi}_{\alpha_0} \subset \tilde{\Phi}_\alpha \subset \tilde{\Phi}_{\alpha_1}$ we have natural imbeddings

$$A(\tilde{\Phi}_{\alpha_0}) \hookleftarrow A(\tilde{\Phi}_\alpha) \hookleftarrow A(\tilde{\Phi}_{\alpha_1}).$$

Hence, taking the duals

$$A(\tilde{\Phi}_{\alpha_0})^* \hookrightarrow A(\tilde{\Phi}_\alpha)^* \hookrightarrow A(\tilde{\Phi}_{\alpha_1})^*.$$

Since $\overline{V}^*_{\alpha_0} \supset \overline{V}^*_\alpha \supset \overline{V}^*_{\alpha_1}$ natural imbeddings

$$A(\overline{V}^*_{\alpha_0}) \hookrightarrow A(\overline{V}^*_\alpha) \hookrightarrow A(\overline{V}^*_{\alpha_1})$$

hold. Using Proposition 2.4.1 we get the isomorphic imbeddings $T_\alpha : A(\tilde{\Phi}_\alpha)^* \to A(\overline{V}_\alpha)^*\ \forall \alpha \in [\alpha_0, \alpha_1]$. Proposition 2.3.1 implies that there are onto isomorphisms $S_\alpha : A(\overline{V}_\alpha)^* \to A(\overline{V}^*_\alpha)$. Therefore we obtain the following diagram:

$$\begin{array}{ccc} A(\tilde{\Phi}_{\alpha_1})^* & \xrightarrow{S_{\alpha_1}\circ T_{\alpha_1}} & A(\overline{V}^*_{\alpha_1}) \\ \uparrow & & \uparrow \\ A(\tilde{\Phi}_\alpha)^* & \xrightarrow{S_\alpha\circ T_\alpha} & A(\overline{V}^*_\alpha) \\ \uparrow & & \uparrow \\ A(\tilde{\Phi}_{\alpha_0})^* & \xrightarrow{S_{\alpha_0}\circ T_{\alpha_0}} & A(\overline{V}^*_{\alpha_0}) \end{array}$$

Since $A(\tilde{\Phi}_\alpha)^* \hookrightarrow A(\overline{V}^*_\alpha)$ for any $\epsilon : 0 < \epsilon < \sigma$ $\exists \delta = \delta(\epsilon)$ $\exists C = C(\alpha, \epsilon)$ such that

$$|x^*|^*_{\tilde{\Phi}_{\alpha+\epsilon}} \leq C(\alpha, \epsilon)|x'|_{\overline{V}^*_{\alpha+\delta}} \tag{4.4}$$

$$C(\alpha, \epsilon)|x^*|^*_{\tilde{\Phi}_{\alpha+\delta}} \geq |x'|_{\overline{V}^*_{\alpha+\epsilon}} \tag{4.5}$$

where $x' := S_\alpha \circ T_\alpha(x^*)$. Since δ only depends on ϵ for $\beta = \alpha + \epsilon$ we obtain that

$$|x^*|^*_{\tilde{\Phi}_\beta} \leq C(\alpha, \epsilon)|x'|_{\overline{V}^*_{\beta-\epsilon}} \tag{4.6}$$

$$C(\alpha, \epsilon)|x^*|^*_{\tilde{\Phi}_\beta} \geq |x'|_{\overline{V}^*_{\beta+\epsilon}}. \tag{4.7}$$

Now for any ϵ: $0 < \epsilon < \sigma$ we obtain that

$$\begin{aligned} |x^*|^*_{\tilde{\Phi}_\alpha} &\leq C(\alpha, \epsilon)|x'|_{\overline{V}^*_{\alpha-\epsilon}} \leq C(\alpha, \epsilon)(|x'|_{\overline{V}^*_{1+\epsilon}})^{\alpha-2\epsilon}(|x'|_{\overline{V}^*_\epsilon})^{1-\alpha+2\epsilon} \\ &\leq M(\alpha, \epsilon)|x^*|^*_{\tilde{\Phi}_0})^{1-\alpha+2\epsilon}(|x^*|^*_{\tilde{\Phi}_1})^{\alpha-2\epsilon}. \end{aligned}$$

□

Proof. (of Theorem 4.1.1) Let $\epsilon > 0$ and $\delta < \epsilon$ be fixed positive number. By Lelong-Bremermann Lemma we can construct a function

$$v(z) := \max\{\alpha_j \ln|f_j(z)| : j = 1, ..., m\},\ \alpha_j > 0,\ f_j \in A(\Omega)$$

such that

$$v(z) < w(D, K; z) < v(z) + \delta,\ z \in D.$$

The sublevel domains $\Phi_\alpha := \{v(z) < \alpha\}$ of $v(z)$ are analytic polyhedrons that simultaneously approximate the corresponding level domains D_α:

$$\Phi_{\alpha-\delta} \subset D_\alpha \subset \Phi_\alpha$$

for all $\alpha \in [0, 1]$. Hence the following natural imbeddings hold:

$$A(\Phi_{\alpha-\delta})^* \hookrightarrow A(D_\alpha)^* \hookrightarrow A(\Phi_\alpha)^*,\ \forall \alpha \in [0, 1]. \tag{4.8}$$

Then for any $x^* \in AC(K)^*$ using (4.8) and Proposition 4.1.1 with $\epsilon - \delta$ instead ϵ we obtain that

$$|x^*|^*_{D_\alpha} \leq M(\alpha)|x^*|^*_{\Phi_{\alpha-\delta}} \leq C(\alpha,\epsilon)\left(|x^*|^*_{\tilde{\Phi}_0}\right)^{1-\alpha+\epsilon}\left(|x^*|^*_{\tilde{\Phi}_1}\right)^{\alpha-\epsilon} \leq N(\alpha,\epsilon)(|x^*|^*_K)^{1-\alpha+\epsilon}(|x^*|^*_D)^{\alpha-\epsilon}. \quad (4.9)$$

□

Theorem 4.1.2. *Let D be a strongly pluriregular open set on a Stein manifold Ω and $K \subset D$ be a compact set such that (K,D) is a pluriregular pair. Let H_0 and H_1 be a pair of Hilbert spaces with the continuous imbeddings:*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K) \quad (4.10)$$

$$A(\bar{D}) \hookrightarrow H_1 \hookrightarrow A(D). \quad (4.11)$$

Then we have the following continuous imbeddings:

$$A(K_\alpha) \hookrightarrow H_\alpha \hookrightarrow A(D_\alpha),\ 0 < \alpha < 1. \quad (4.12)$$

Proof. Let $\{e_i\}_{i\in\mathbb{N}}$ be a common orthogonal basis for H_1 and H_0 as in (2.3). Since $H_0 \hookrightarrow AC(K)$ there is a constant B such that $|e_i|_K \leq B||e_i||_{H_0} = B$, $\forall i \in \mathbb{N}$. Since $H_1 \hookrightarrow A(D)$, for any $q < 1$, $|e_i|_{D_q} \leq C||e_i||_{H_1} = C\mu_i$ for some constant $C = C(q)$. For any $\alpha < 1$ and $\epsilon > 0$, we choose q with $\beta := \alpha/q < \alpha + \epsilon$. The relation (4.1) implies that

$$|e_i|_{D_{q\beta}} \leq B^{1-\beta}C^\beta(|e_i|_K)^{1-\beta}(|e_i|_{D_q})^\beta.$$

Thus we obtain

$$|e_i|_{D_\alpha} \leq N\mu_i^{\alpha+\epsilon},\ \forall i \in \mathbb{N} \quad (4.13)$$

with a constant $N = N(\alpha,\epsilon)$.

Let $\{e_i'\}_{i\in\mathbb{N}}$ be the biorthogonal system for $\{e_i\}_{i\in\mathbb{N}}$. Since $H_0^* \hookrightarrow A(K)^*$, $\forall\delta > 0$ there is a constant $M = M(\delta)$ such that

$$|e_i'|^*_{D_\delta} \leq M|e_i'|_{H_0^*} = M. \quad (4.14)$$

The continuous imbedding $H_1^* \hookrightarrow A(\bar{D})^*$ implies that $\forall \gamma > 1, \exists P = P(\gamma)$

$$|e_i'|^*_{D_\gamma} \leq P||e_i'||_{H_1^*} = P\mu_i^{-1} \ \forall i\mathbb{N}. \tag{4.15}$$

Then using Theorem 4.1.1 with (4.14) and (4.15) we obtain

$$|e_i'|^*_{D_{\gamma\alpha}} \leq L(\alpha,\epsilon)|e_i'|^*_{D_\delta}{}^{1-\alpha+\epsilon}|e_i'|_{D_\gamma}{}^{\alpha-\epsilon} \leq LM^{1-\alpha+\epsilon}P^{\alpha-\epsilon}\mu_i^{-\alpha+\epsilon}$$

where

$$D_{\gamma\alpha} := \{z \in D_\gamma : w(D_\gamma, D_\delta; z) < \alpha\}.$$

As $\gamma \downarrow 1$, $\gamma\alpha \downarrow \alpha$. Thus

$$|e_i'|^*_{D_\alpha} \leq S(\alpha,\epsilon)\mu_i{}^{-\alpha+2\epsilon}. \tag{4.16}$$

Let $x \in H_\alpha$. For any β with $\beta + \epsilon < \alpha$, using (4.13) with Schwartz's inequality we obtain that

$$\begin{aligned} |x|_{D_\beta} &\leq \sum_i \mu_i^\alpha |e_i'(x)||e_i|_{D_\beta}\mu_i^{-\alpha} \qquad (4.17)\\ &= \left(\sum_i |e_i'(x)|^2\mu_i^{2\alpha}\right)^{1/2} N^2 \left(\sum_i \mu_i^{-2\alpha}\mu_i^{2(\beta+\epsilon)}\right)^{1/2}. \end{aligned}$$

Since the imbedding $H_1 \hookrightarrow H_0$ is nuclear

$$\sum_i \mu_i^{-2\alpha}\mu_i^{2(\beta+\epsilon)} < \infty.$$

Thus we get $|x|_{D_\beta} \leq C||x||_{H_\alpha}$ for some constant C, that is $H_\alpha \hookrightarrow A(D_\alpha)$.
By definiton of dual norm we have that

$$|e_i'(x)| \leq |e_i'|^*_{D_{\alpha+2\epsilon}}|x|_{D_{\alpha+2\epsilon}} \tag{4.18}$$

for any $\alpha < 1$ and $\epsilon > 0$. For any $x \in A(K_\alpha)$, using (4.16) and (4.18) we have the following estimate:

$$\begin{aligned} ||x||_{H_\alpha} = \left(\sum_i |e_i'(x)|^2\mu_i^{2\alpha}\right)^{1/2} &\leq \left(\sum_i (|e_i'|^*_{D_{\alpha+2\epsilon}}|x|_{D_{\alpha+2\epsilon}})^2\mu_i^{2\alpha}\right)^{1/2} \qquad (4.19)\\ &\leq S|x|_{D_{\alpha+2\epsilon}}\left(\sum_i \mu_i^{2(-\alpha-\epsilon)}\mu_i^{2\alpha}\right)^{1/2}\\ &\leq R(\alpha,\epsilon)|x|_{D_{\alpha+2\epsilon}} \end{aligned}$$

for some constant $R(\alpha,\epsilon)$. Thus $A(K_\alpha) \hookrightarrow H_\alpha$. □

4.2 Adherent Spaces

Let E be a Fréchet space with the topology defined by seminorms $\{||x||_p, p \in \mathbb{N}\}$ and

$$||x^*||_p^* := \sup\{|x^*(x)| : x \in E,\ ||x||_p \leq 1\},\ x^* \in E^*,\ p \in \mathbb{N}, \tag{4.20}$$

be the system of non-bounded polar norms.

Definition 4.2.1. *A Banach space $X \hookrightarrow E$ is said to be adherent to E if for each $p \in \mathbb{N}$ and any $\delta > 0$ there exist $q \in \mathbb{N}$ and $C > 0$ such that*

$$||x^*||_q^* \leq C(||x^*||^*)^{1-\delta}(||x^*||_p^*)^{\delta},\ x^* \in E^*,$$

where $||x^*||^*$ is the norm in X^* defined by

$$||x^*||^* = \sup\{|x^*(x)| : x \in \mathbb{B}_X\}.$$

Theorem 4.2.1. *([20]) A Banach space X is adherent to E if and only if for any neighborhood V of zero in E and any $\delta > 0$ there exist $p \in \mathbb{N}$ and $C > 0$ such that*

$$U_p \subset t^{\delta}\mathbb{B}_X + \frac{C}{t^{1-\delta}}V,\ t > 0 \tag{4.21}$$

where $U_P := \{x \in E : ||x||_p \leq 1\}$.

Remark: A Banach space $X_1 \hookrightarrow A(D)$ is adherent to $A(D)$ if and only if for any Banach space $X_0 \hookleftarrow A(D)$, $A(D) \hookrightarrow X_\alpha$ where $(X_\alpha)_{\alpha \in [0,1]}$ is any normal regular Banach scale connecting X_1 and X_0.

Definition 4.2.2. *A Fréchet space E belongs to the class $\mathcal{D}_2$ if for every $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ there is $C > 0$ such that:*

$$(||x^*||_q^*)^2 \leq C||x^*||_p^*\ ||x^*||_r^*,\ x^* \in E^*. \tag{4.22}$$

Proposition 4.2.1. *Let Ω be a Stein manifold having finite set of connected components. Then the followings are equivalent:*

(i) *Ω is pluriregular,*

(ii) $A(\Omega) \in \mathcal{D}_2$,

(iii) *there exists a Hilbert space* $H \hookrightarrow A(\Omega)$ *adherent to* $A(\Omega)$.

Proof. The implications $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (i)$ are due to [33, 34]. For the proof Zakharyuta used Hadamard type inequalities for analytic functionals. (See, Theorem 4.1.1); $(ii) \Rightarrow (iii)$: By Vogt ([30], Lemma 4) there is a Banach space $X \hookrightarrow A(\Omega)$ adherent to $A(\Omega)$. By a result of Pietsch [24], since $A(\Omega)$ is nuclear, there is a Hilbert space H such that $X \hookrightarrow H \hookrightarrow A(\Omega)$. Therefore H is also adherent to $A(\Omega)$. □

Remark: In [1] with the assumption (i), using L^2-estimates of Hörmander for $\bar{\partial}$-operator (see, [12]) Aytuna constructed an adherent Hilbert space for $A(\Omega)$ as a weighted L^2 space.

Proposition 4.2.2. *Let* D *be a strongly pluriregular domain on a Stein manifold. Then any Banach space* X *satisfying the dense imbeddings* $A(\overline{D}) \hookrightarrow X \hookrightarrow A(D)$ *is adherent to* $A(D)$*; in particular,* $H^\infty(D)$ *is adherent to* $A(D)$.

Proof. Let $K \subset D$ be a compact set making up a pluriregular pair with D. Let X be a Banach space as above and H_0 be a Hilbert space satisfying the following imbedding:

$$A(D) \hookrightarrow H_0 \hookrightarrow AC(K).$$

Since $A(D)$ is a nuclear space there is a Hilbert space H_1 such that $A(\bar{D}) \hookrightarrow H_1 \hookrightarrow X \hookrightarrow A(D)$. Let H_α be any Banach scale connecting H_1 and H_0. By Theorem 4.1.2 the imbeddinds $A(K_\alpha) \hookrightarrow H_\alpha$ hold for $\alpha \in (0,1)$. Thus $A(D) \hookrightarrow H_\alpha$, that is; H_1 is adherent to $A(D)$. Consequently X is adherent to $A(D)$. □

Definition 4.2.3. *Let* K *be a compact set in* Ω *and a Banach space* X *satisfy the dense imbedding* $A(K) \hookrightarrow X$. *Then we say that* X *is *-adherent to* $A(K)$ *if the dual space* $X^* \hookrightarrow A(K)^*$ *is adherent to* $A(K)^*$ *in the usual sense.*

A compact set $K \subset \Omega$ is said to be *Runge set on* Ω if $A(\Omega)$ is dense in $A(K)$.

Proposition 4.2.3. *([32, 33]) Let K be a Runge set on Stein manifold Ω and each connected component of Ω has non-empty intersection with K. Then the following statements are equivalent:*

(i) *K is pluriregular; that is, there is some open neighborhood $D \Subset \Omega$ of K such that $w(D, K, z) \equiv 0$ on K.*

(ii) *$A(K)^* \in \mathcal{D}_2$;*

(iii) *There is a Hilbert space $H \hookleftarrow A(K)$ *-adherent to $A(K)$;*

(iv) *$AC(K)$ is *-adherent to $A(K)$.*

Example 4.2.1. *Let K be a compact set in Ω and (K, Ω) be a pluriregular pair. From the implication (i) $\Rightarrow$ (iv) any Hilbert space H, satisfying the dense imbeddings $A(K) \hookrightarrow H \hookrightarrow AC(K)$, *-is adherent to $A(K)$. More explicitly, as an example of Hilbert space $H \hookleftarrow A(K)$ *-adherent to $A(K)$ we consider $H = AL_2(K, \mu)$ obtained as a completion of $A(K)$ by the norm*

$$||x|| := \left(\int_K |x|^2 d\mu \right)^{1/2}$$

with $\mu := (dd^c w)^n$, $w = w(D, K; z)$.

Definition 4.2.4. *Let (K, Ω) be a pluriregular pair. A couple of Banach spaces (X_0, X_1) is said to be adherent to $(A(K), A(\Omega))$ if*

$$X_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow X_0$$

*and X_1 is adherent to $A(\Omega)$, X_0 is *-adherent to $A(K)$.*

As a direct consequence of Proposition 4.2.1 and Proposition 4.2.3, we have the following corollary:

Corollary 4.2.1. *Given any pluriregular pair (K, Ω), there exist a couple of Hilbert spaces (H_0, H_1) adherent to $(A(K), A(\Omega))$.*

Theorem 4.2.2. *([37])Let D be a Stein manifold and (K, D) be a pluriregular pair. Let (X_0, X_1) be a couple of Banach spaces adherent to the couple $(A(K), A(D))$ such that the imbedding $X_1 \hookrightarrow X_0$ is normal and $\mathbb{B}_{X_1}$ is closed in X_0. Let X_α, $0 \le \alpha \le 1$, be any regular normal scale of Banach spaces connecting the spaces X_0 and X_1. Then the following continuous imbeddings hold*

$$A(K_\alpha) \hookrightarrow X_\alpha \hookrightarrow A(D_\alpha),\ 0 < \alpha < 1, \tag{4.23}$$

where $K_\alpha = \{z \in D : w(z) \le \alpha\}$ and $D_\alpha = \{z \in D : w(z) < \alpha\}$ are sublevel domains of the function $w(z) = w(D, K; z)$ which was defined in (1.6).

Theorem 4.2.3. *Let $X_1 \hookrightarrow A(\Omega)$ be a Banach space and $X_0 \hookleftarrow A(K)$ be a Hilbert space such that (X_1, X_0) forms a couple adherent to $(A(K), A(\Omega))$. Then (X_1, X_0) is admissible for (K, Ω).*

Proof. Let (Y_1, Y_0) be a couple of Banach spaces satisfying the following imbeddings

$$X_1 \hookrightarrow Y_1 \hookrightarrow A(\Omega) \hookrightarrow A(K) \hookrightarrow Y_0 \hookrightarrow X_0.$$

Then we need to show that

$$\ln d_i(X_1, X_0) \sim \ln d_i(Y_1, Y_0) \text{ as } i \to \infty. \tag{4.24}$$

Since $A(\Omega)$ is nuclear, by the result of Pietsch [24], there is a Hilbert space H_1 satisfying the continuous imbeddings $X_1 \hookrightarrow H_1 \hookrightarrow A(\Omega)$. Clearly, H_1 is adherent to $A(\Omega)$. Consider the Hilbert scale $H_\alpha = (X_0)^{1-\alpha}(X_1)^\alpha$ which satisfies the imbeddings (4.23). Then the system of norms $\{||x||_{H_\alpha}, 0 < \alpha < 1\}$ defines the original topology of the spaces $A(\Omega)$. Since X_1 is adherent to $A(\Omega)$, applying (4.21) with $V = \mathbb{B}_{X_0} \cap A(\Omega)$, $U_p = \mathbb{B}_{\alpha_p} \cap A(\Omega)$ and any $\delta > 0$, there exist $\alpha = \alpha(\delta) < 1$ and $C = C(\delta) > 0$ such that

$$\mathbb{B}_{H_\alpha} \subset \left(\frac{1}{\lambda}\right)^\delta \mathbb{B}_{X_1} + C\lambda^{1-\delta}\mathbb{B}_{X_0},\ \lambda > 0. \tag{4.25}$$

Choose any $\lambda = \lambda(i) > 0$ such that

$$d_i(X_1, X_0) < \lambda \le 2d_i(X_1, X_0). \tag{4.26}$$

Then due to the definition of diameters, there is $L \in \mathcal{L}_i$ such that

$$\mathbb{B}_{X_1} \subset \lambda \mathbb{B}_{X_0} + L \tag{4.27}$$

Combining (4.25), (4.26), (4.27), we obtain

$$\mathbb{B}_{H_\alpha} \subset (1+C)(2d_i(X_1, X_0))^{1-\delta}\mathbb{B}_{X_0} + L. \tag{4.28}$$

Let $\alpha > \beta > 0$. Then by (2.5.1) and (4.28) we get

$$\begin{aligned} d_i(X_1, X_0) &\prec d_i(Y_1, Y_0) \prec d_i(H_\alpha, H_\beta) = d_i(H_1, X_0)^{\alpha-\beta} \\ &= d_i(H_\alpha, X_0)^{\frac{\alpha-\beta}{\alpha}} \prec d_i(X_1, X_0)^{\frac{(\alpha-\beta)(1-\delta)}{\alpha}}. \end{aligned}$$

Since δ and β are arbitrary, we obtain the result (4.24). □

Corollary 4.2.2. *Let $X_1 \hookrightarrow A(\Omega)$ be a Banach space adherent to $A(\Omega)$. Then $(AC(K), X_1)$ is admissible for (K, Ω).*

Proof. Consider the Hilbert space $H = AL^2(K, \mu)$ (defined as in Example 4.2.1) which is *-adherent to $A(K)$. Since $A(K) \hookrightarrow AC(K) \hookrightarrow H$, preceding theorem implies that $(AC(K), X_1)$ is admissible for (K, Ω). □

4.3 Maximal Plurisubharmonic Functions with Isolated Singularities

Let Ω be a pluriregular Stein manifold. By $G(\Omega)$, we denote the set of all plurisubharmonic functions u on Ω satisfying the following conditions:

- $\lim_{z \to \partial\Omega} u(z) = 0$.
- there is a finite set $\Lambda = \Lambda(u) \subset \Omega$ such that $u \in MP(\Omega \setminus \Lambda)$ and $u(z) \equiv -\infty$ on Λ.

By $G_\Lambda(\Omega)$, we denote the set of all functions $u \in G(\Omega)$ with a fixed set Λ.

Definition 4.3.1. *For an open neighborhood U of ζ, we say that two functions ϕ and $\psi \in \mathcal{PSH}(U) \cap \mathcal{MPSH}(U \setminus \{\zeta\})$ with $\phi(\zeta) = \psi(\zeta) = -\infty$ generate the same singularity in ζ if*

$$\phi \sim \psi := \lim_{z \to \zeta} \frac{\phi(z)}{\psi(z)} = 1. \tag{4.29}$$

The equivalence class $\sigma = [\phi]$ generated by ϕ, is called *standart singularity at the point* ζ. A standart singularity σ at the point ζ is called *continuous* if there is a representative $\phi \in \sigma$ which is continuous in some punctured neighborhood of ζ.

Theorem 4.3.1. *([31, 35]) Given a pluriregular Stein manifold* Ω *, finite set* $\Lambda = \{\zeta_\mu : \mu = 1, ..., m\} \subset \Omega$ *and continuous standart singularities* $\sigma_\mu = [\phi_\mu]$ *at the points* ζ_μ*, there exist the unique function* $g \in G_\Lambda(\Omega)$ *having the singularities* σ_μ *at the points* ζ_μ*. This function is continuous in* $\Omega \setminus \Lambda$ *and defined by*

$$g(z) := \sup\{u(z) : u \in P(\Omega, \Lambda, (\sigma_\mu))\} \tag{4.30}$$

where $P(\Omega, \Lambda, (\sigma_\mu))$ *is the class of all negative plurisubharmonic functions* u *in* Ω *such that there is a constant* c *provided* $u(z) \leq \phi_\mu(z) + c$ *in some neighborhood of* ζ_μ*,* $\mu = 1, ..., m$*.*

Given any point $\zeta_\mu \in \Lambda$, we have a *logarithmic singularity* $\sigma_\mu = [\phi_\mu]$ defined by the function $\phi_\mu(z) := \alpha_\mu \ln |t_\mu(z) - t_\mu(\zeta_\mu)|$ where $\alpha_\mu > 0$ and $t_\mu(z)$ are local coordinates in ζ_μ. Then the function (4.30) is called *multipolar Green function* and denoted by $g_\Omega(\Lambda, \alpha, z)$ where $\alpha = (\alpha_\mu) \in \mathbb{R}^m_+$. Note that in Chapter 3 (3.6) we have defined $g_\Omega(\Lambda, \alpha, z)$ as $g_\Omega(P, z)$ for domains in $\mathbb{C}^n$.

Definition 4.3.2. *Let* $u \in \mathcal{PSH}(\Omega)$ *and* $G \Subset \Omega$*. Then MP-balayage of the function* u *is defined by*

$$s(G, u; z) := \limsup_{\zeta \to z} \sup\{v(\zeta) : v \in P(\Omega, G; u)\}, \tag{4.31}$$

where $P(\Omega, G; u)$ *is the class of all plurisubharmonic function on* Ω *with* $v \leq u$ *on* $\Omega \setminus G$*.*

Proposition 4.3.1. *Let* u *be a continuous plurisubharmonic function on* Ω *and* $G \Subset D \Subset \Omega$*. Then*

$$\int_D (dd^c u(z))^n = \int_D (dd^c s(G, u; z))^n. \tag{4.32}$$

Proof. First we will show that the statement is true for functions $u \in C^2(\Omega')$ where $D \Subset \Omega' \Subset \Omega$. By Stokes' theorem

$$\int_D (dd^c u)^n = \int_{\partial D} d^c u \wedge (dd^c u)^{n-1}.$$

Since $u(z) \equiv s(z) := s(G, u; z)$ in $\Omega \setminus G \supset \partial D$, we obtain 4.32.
In general, take a decreasing sequence $u_k \in C^2(\overline{\Omega'}) \cap \mathcal{PSH}(\Omega'), D \Subset \Omega' \Subset \Omega$ with $|u - u_k|_{\overline{\Omega'}} \to 0$. On ∂G, since $s_k(z) := s(G, u_k; z) \equiv u_k(z)$ and $s(z) \equiv u(z)$ we have

$$s(z) - |u - u_k|_{\partial G} \leq s_k(z) \leq s(z) + |u - u_k|_{\partial G}. \tag{4.33}$$

By the maximality of s and s_k in G, we have the relation (4.33) in G, that is, $|s - s_k|_{\Omega'} \to 0$. Then by Theorem 3.2.3,

$$\int_D (dd^c u(z))^n = \lim_k \int_D (dd^c u_k(z))^n = \lim_k \int_D (dd^c s_k(z))^n = \int_D (dd^c s(z))^n.$$

□

Definition 4.3.3. *Let $\sigma = [\phi]$ be a standart singularity at $\zeta \in \Omega$. Then the charge of σ is defined by*

$$\nu_\zeta(\sigma) = \nu_\zeta\{\phi\} := \left(\frac{1}{2\pi}\right)^n \int_\Omega (dd^c s(\Delta_\lambda, \phi; z))^n, \lambda > \delta, \tag{4.34}$$

where $\Delta_\lambda := \{z \in \Delta : \phi(z) < -\lambda\}$, $\Delta = \Delta(\phi) \Subset \Omega$ is an open neighborhood of ζ provided $\phi \in \mathcal{PSH}(\Delta) \cap \mathcal{MPSH}(\Delta \setminus \{\zeta\})$ and $\delta > 0$ is such that $\Delta_\delta \Subset \Omega$.

Proposition 4.3.2. *The charge of a singularity σ is well-defined, that is; $\nu_\zeta(\sigma)$ does not depend on a choice of λ or $\phi \in \sigma$.*

Proof. By Proposition 4.3.1, $\nu_\zeta(\sigma)$ does not depend on $\lambda > \delta$. Therefore it is enough to show that $\nu_\zeta(\sigma)$ is independent of a choice of ϕ. Let ψ and ϕ be two representative function in the class σ. Then for each $\epsilon > 0$, there is γ such that

$$\Delta_{(1+\epsilon)\lambda} \subset \Delta'_\lambda \subset \Delta_{(1-\epsilon)\lambda}, \lambda \geq \gamma, \tag{4.35}$$

where Δ'_λ are sublevel domains for the function ψ. Since

$$s(\Delta_{\lambda_0}, \phi; z) = (\lambda_0 - \lambda_1) w(\Delta_{\lambda_1}, \overline{\Delta_{\lambda_0}}; z) - \lambda_0, \tag{4.36}$$

$$\nu_\zeta\{\phi\} = (\lambda_0 - \lambda_1)^n C(\overline{\Delta_{\lambda_0}}, \Delta_{\Lambda_1}), \tag{4.37}$$

where $\gamma < \lambda_1 < \lambda_0$.

Using the relations (4.37) (with $\lambda_0 = 2(1+\epsilon)\lambda$ and $\lambda_1 = (1-\epsilon)\lambda$), (4.35) and monotonicity of the capacity we obtain

$$\begin{aligned}\frac{\nu_\zeta\{\phi\}}{(1+3\epsilon)^n\lambda^n} &= C(\overline{\Delta_{2(1+\epsilon)\lambda}}, \Delta_{(1-\epsilon)\lambda}) \leq C(\overline{\Delta'_{2\lambda}}, \Delta'_\lambda) = \frac{\nu_\zeta\{\psi\}}{\lambda^n} \\ &\leq C(\overline{\Delta_{2(1-\epsilon)\lambda}}, \Delta_{(1+\epsilon)\lambda}) = \frac{\nu_\zeta\{\phi\}}{(1-3\epsilon)^n\lambda^n}. \end{aligned} \tag{4.38}$$

Since $\epsilon > 0$ is arbitrary, we have the equality $\nu_\zeta\{\phi\} = \nu_\zeta\{\psi\}$. □

Definition 4.3.4. *Let $g \in G_\Lambda(\Omega)$ where $\Lambda = \{\zeta_\mu : \mu = 1, ..., m\}$. Then the charge of g is defined by*

$$\nu\{g\} := \left(\frac{1}{2\pi}\right)^n \int_\Omega (dd^c s(\Omega_\lambda, g; z))^n = \sum_{\mu=1}^m \nu_{\zeta_\mu}([g]), \tag{4.39}$$

where

$$\Omega_\lambda := \{z \in \Omega : g(z) < -\lambda\}, \lambda > 0.$$

Proposition 4.3.3. *The charge of the multipole Green plurifunction $g(z) = g_\Omega(\Lambda, \alpha; z)$ is equal to*

$$\nu\{g\} = \sum_{\mu=1}^m (\alpha_\mu)^n. \tag{4.40}$$

Proof. In the definition of $g_\Omega(\Lambda, \alpha; z)$, we choose local coordinates t_μ such that $t_\mu(\zeta_\mu) = 0$. Then in a neighborhood of ζ_μ, $[g] = [\alpha_\mu \ln|t_\mu|]$ for $\mu = 1, ..., m$. Therefore, using (4.3.3), (4.3.4), and the Jensen equality ([14], Example 6.5.6)

$$\left(\frac{1}{2\pi}\right)^n \int_{\partial\mathbb{B}(0,r)} d^c(\ln|z|) \wedge (dd^c(\ln|z|))^{n-1} = 1,$$

we get the result. □

4.4 Estimates of Analytic Functions and Functionals with Given Zeros and Poles

Definition 4.4.1. *Let Ω be a pluriregular Stein manifold of dimension n, $F = \{\zeta_\mu : \mu = 1, ..., m\} \subset \Omega$ and $\sigma = (s_\mu) \in \mathbb{Z}_+^m$. We say that a functional $f' \in A(F)'$ is* *discrete rational functional having the poles of order at least s_μ at the point ζ_μ if $f'(f) = 0$ for all $f \in A_0((F,\sigma),\Omega)$ where $A_0((F,\sigma),\Omega)$ is the set of all functions in $A(\Omega)$ vanishing on F and having zero of order $\le s_\mu$ at the point ζ_μ. By $A_0^\perp(F,\sigma)$, we denote the set of all such functionals.*

Theorem 4.4.1. *For any $f \in A_0((F,\sigma),\Omega)$, we have*

$$|f(z)| \le |f|_\Omega \exp s(\alpha) g_\Omega(F,\alpha;z),\ z \in \Omega, \tag{4.41}$$

where Ω, F, σ are defined as in Definition 4.4.1 and $s(\alpha) := \inf\{s_\mu/\alpha_\mu : \mu = 1,...,m\}$.

Proof. It is clear that the plurisubharmonic function $u(z) := \frac{\ln|f(z)| - \ln|f|_\Omega}{s(\alpha)}$ belongs to the class $P(\Omega, F, (\sigma_\mu))$ with the logarithmic singularities σ_μ defined by the function $\phi_\mu(z) := \alpha_\mu \ln|t(\zeta_\mu) - t(z)|$, where t are local coordinates in a neighborhood of ζ_μ, $\mu = 1,...,m$. By definition, we have the inequality $u(z) \le g_\Omega(F,\alpha;z)$ in Ω which is equivalent to (4.41). □

Theorem 4.4.2. *Let D be a strongly pluriregular open set on a Stein manifold Ω, f' be a discrete rational functional with the pole set $F = \{\zeta_\mu : \mu = 1,...,m\} \subset D$, $\sigma = (s_\mu) \in \mathbb{Z}_+^m$ be the corresponding set of multiplicities. Let*

$$\Phi_\lambda = \Phi_\lambda(F,\alpha) := \{z \in D : g_D(F,\alpha;z) < -\alpha\}, 0 < \alpha < \infty,$$

where $\alpha = (\alpha_\mu), \alpha_\mu > 0$. Then for each $\delta > 0$ the following estimates

$$|f'|^*_{\Phi_\lambda} \le C|f'|^*_D \exp(\lambda+\delta)s(\alpha),\ 0 < \lambda < \infty, \tag{4.42}$$

hold with some constant $C = C(\lambda,\delta)$ and $s(\alpha) := \max\{s_\mu/\alpha_\mu : \mu = 1,...,m\}$.

Proof. Let $t^{(\mu)} = (t_i^{(\mu)}) : \Delta^{(\mu)} \to \mathbb{U}^n$ be local coordinates for each $\zeta_\mu \in F$ with mutually disjoint neighborhoods $\Delta^{(\mu)}$ of ζ_μ. Let $\Delta = \cup_{\mu=1}^m \Delta_\mu$. Then

$$g_\Delta(F,\alpha;z) = \sup\{\ln|t_j^{(\mu)}(z)| : j = 1,...,n\},\ z \in \Delta^{(\mu)}.$$

Let Δ_τ denote the sublevel domains

$$\Delta_\tau := \{z \in \Delta : g_\Delta(F,\alpha;z) < -\tau\} = \cup_{\mu=1}^m \Delta_\tau^{(\mu)}, \tag{4.43}$$

where

$$\Delta_\tau^{(\mu)} = \{z \in \Delta^{(\mu)} : |t_j^{(\mu)}(z)| < \exp(-\tau/\alpha_\mu)\}, 0 < \tau < \infty.$$

We choose a common basis for $A(\Delta_\tau)$ and $A(F)$

$$f_{k,\mu}(z) = \begin{cases} t^{(\mu)}(z)^k := t_1^{(\mu)}(z)^{k_1}...t_n^{(\mu)}(z)^{k_n}, & z \in \Delta^\mu \\ 0 & z \in \Delta \setminus \Delta^{(\mu)}, \end{cases} \tag{4.44}$$

with $k = (k_1, ..., k_n) \in \mathbb{Z}_+^n$ and $\mu = 1, ..., m$. Its biorthogonal system can be given by

$$f'_{k,\mu}(f) := \left(\frac{1}{2\pi i}\right)^n \int_S \frac{f(\nu_\mu(t))}{t^{k+I}} dt, \ f \in A(\Delta_\tau), \ 0 < \tau < \infty, \tag{4.45}$$

where $\nu_\mu : \mathbb{U}^n \to \Delta^{(\mu)}$ is the inverse mapping of $t^{(\mu)}$, $\mu = 1, ..., m$, $k = (k_1, ..., k_n)$, $I = (1, ..., 1)$ and $S = S_{\tau,\mu}$ is the Shilov boundary of polydisc $\mathbb{U}_r^n$, where $r = r(\lambda) := \exp(-\lambda - \tau/\alpha_\mu)$, $0 < \lambda < \infty$. Any functional $f' \in A_0^\perp(F, \sigma)$ can be written

$$f' = \sum_{\mu=1}^m \sum_{|k| \le s_\mu} f'(f_{k,\mu}) f'_{k,\mu}. \tag{4.46}$$

Then using expression (4.46) with

$$|f_{k,\mu}|_{\Delta_\tau} = \exp\left(-\frac{\tau|k|}{\alpha_\mu}\right); |f'_{k,\mu}|^*_{\Delta_\tau} = \exp\left(\frac{\tau|k|}{\alpha_\mu}\right)$$

we obtain

$$\begin{aligned} |f'|^*_{\Delta_\tau} &\le |f'|^*_{\Delta_\tau} \sum_{\mu=1}^m (\alpha_\mu)^n (s_\mu/\alpha_\mu)^n \exp \tau s_\mu/\alpha_\mu \\ &\le |f'|^*_{\Delta_\tau} \sum_{\mu=1}^m (\alpha_\mu)^n s(\alpha)^n \exp \tau s(\alpha). \end{aligned}$$

with $0 < \tau < \infty$, $s(\alpha) := \max\{s_\mu/\alpha_\mu : \mu = 1, ..., m\}$. Then we have the estimates

$$|f'|^*_{\Delta_\tau} \le M(\epsilon)|f'|^*_\Delta \exp(\tau + \epsilon)s(\alpha) \tag{4.47}$$

with $\epsilon > 0$, $M(\epsilon) = \sum_{\mu=1}^m (\alpha_\mu)^n \sup\{r^n \exp(-\epsilon r) : r > 0\}$. Choose γ so that $\Phi_\gamma \Subset \Delta_\tau$. Then we obtain from (4.47) that

$$|f'|^*_{\Phi_\tau} \le M(\epsilon)|f'|^*_{\Phi_\gamma} \exp(\tau + \epsilon)s(\alpha), 0 < \tau < \infty. \tag{4.48}$$

Let $F_\tau := \{z \in D : g_D(F, \alpha; z) \leq -\tau\}$. Then we have the following relation

$$D_\alpha^{(\tau)} := \{z \in D : w(D, F_\tau; z) < \alpha\} = \Phi_{(1-\alpha)\tau}, 0 < \alpha < 1,\ 0 < \alpha < 1. \quad (4.49)$$

comes from

$$w(D, F_\tau; z) = \frac{1}{\tau} g_D(F, \alpha; z) + 1,\ z \in D \setminus F_\tau,\ 0 < \tau < \infty.$$

Thus applying Theorem 4.1.1 with

$$K = F_\tau,\ \alpha = 1 - \lambda/\tau, 0 < \lambda < \tau,\ \epsilon > 0,$$

we obtain

$$|f'|_{\Phi_\lambda}^* \leq N(\tau, \lambda, \epsilon)(|f'|_{F_\tau}^*)^{\frac{\lambda}{\tau}+\epsilon}(|f'|_D^*)^{1-\frac{\lambda}{\tau}-\epsilon}. \quad (4.50)$$

Since the relation (4.42) is homogeneous, we can assume that $|f'|_D^* = 1$. Therefore combining (4.48) and (4.50) we obtain

$$\begin{aligned} |f'|_{\Phi_\lambda}^* &\leq N\left(M(\epsilon)|f'|_{\Phi_\gamma}^* \exp(\tau + \epsilon)s\right)^{\frac{\lambda}{\tau}+\epsilon} \quad (4.51) \\ &\leq N'\left(|f'|_{\Phi_\gamma}^*\right)^{\frac{\lambda}{\tau}+\epsilon} \exp(\lambda + \epsilon')s \end{aligned}$$

with some constant $N' = N'(\tau, \lambda, \epsilon)$ and $\epsilon' = \tau\epsilon + \epsilon + \epsilon^2,\ 0 < \lambda < \tau$. In particular, for $\tau = 4\gamma$, $\lambda = \gamma$ and $\epsilon = 1/4$ we obtain

$$|f'|_{\Phi_\gamma}^* \leq (N')^2 \exp(2\gamma + 1)2s. \quad (4.52)$$

For given δ and λ we choose τ and ϵ so that

$$2(2\gamma + 1)(\lambda/\tau + \epsilon) < \delta/2 \text{ and } \tau\epsilon + \epsilon + \epsilon^2 < \delta/2. \quad (4.53)$$

Then (4.51) and (4.52) with the parameters satisfying the conditions (4.53) imply (4.42) in the case $|f'|_D^* = 1$.

□

Corollary 4.4.1. *Let Ω, F and σ be as in Theorem 4.4.2. Let $H \hookrightarrow A(\Omega)$ be a Hilbert space adherent to $A(\Omega)$. Then for each $f' \in A_0^\perp(F, \sigma)$ and any $\delta > 0$, we have the following:*

$$|f'|_{\Omega_\lambda}^* \leq C(\lambda, \delta)||f'||_{H^*} \exp(\lambda + \delta)s(\alpha),\ 0 < \lambda < \infty$$

where $s(\alpha) := \sup\{\frac{s_\mu}{\alpha_\mu} : \mu = 1, ..., k\}$.

4.5 Extendible Bases

Theorem 4.5.1. *([31, 35]) Let Ω be a pluriregular Stein manifold, $F = \{\zeta_\mu : \mu = 1, ..., m\}$ be a finite set in Ω, of dimension n, having no connected component disjoint with the set F; $\alpha = (\alpha_\mu), \alpha_\mu > 0$. Then there exists a common basis $\{\varphi_i(z)\}_{i\in\mathbb{N}}$ in the spaces*

$$A(\Omega), A(F), A(\Omega_\lambda), A(F_\lambda),\ 0 < \lambda < \infty,\ \textit{where} \tag{4.54}$$

$$\Omega_\lambda := \{z \in \Omega : g_\Omega(F, \alpha; z) < -\lambda\},\ F_\lambda := \{z \in \Omega : g_\Omega(F, \alpha; z) \leq -\lambda\}$$

with $0 < \lambda < \infty$. Also the estimates

$$\frac{1}{C} \exp \sigma_n(-\lambda - \epsilon) i^{1/n} \leq |\varphi_i(z)|_{F_\lambda} \leq C \exp \sigma_n(-\lambda + \epsilon) i^{1/n}, i \in \mathbb{N} \tag{4.55}$$

hold with some constant

$$C = C(\lambda, \epsilon),\ \textit{and}\ \sigma_n = \left(\frac{n!}{\sum_{\mu=1}^{m} (\alpha_\mu)^n}\right)^{1/n}.$$

By the way, we have the following formula for Green multipole function:

$$\limsup_{\zeta \to z} \limsup_{i \to \infty} \frac{\ln |\varphi_i(\zeta)|}{i^{1/n}} = \sigma_n g_\Omega(F, \alpha; z),\ z \in \Omega \setminus F. \tag{4.56}$$

Proof. Let $\{f'_{k,\mu}, k \in \mathbb{Z}^n_+, \mu = 1, ..., m\} \subset A(F)'$ be a basis for $A(F)'$ as in (4.45). We enumarate

$$e'_i = f'_{k(i),\mu(i)},\ i \in \mathbb{N} \tag{4.57}$$

such that $s_\alpha(i) := \frac{|k(i)|}{|\alpha_{\mu(i)}|}$ is non-decreasing.
We take any Hilbert space H adherent to $A(\Omega)$ and orthonormalize the sequence (4.57) in the space $H^* \supset A(\Omega)^*$. Then we obtain the system

$$\varphi'_i = \sum_{j \leq i} t_{ij} e'_j,\ i \in \mathbb{N}.$$

We will show that the biorthogonal system $\{\varphi_i\}_{i\in\mathbb{N}} \subset H \subset A(\Omega) \subset A(F)$ is a required basis for the spaces (4.54). Let $f \in A(F)$ and $\varphi'_i(f) = 0,\ \forall i \in \mathbb{N}$. Then $e'_i(f) = 0,\ \forall i \in \mathbb{N}$. Thus $f \equiv 0$ in a neighborhood of F. Therefore $\{\varphi_i\}_{i\in\mathbb{N}}$ is total and hence complete in the spaces (4.54), due to the reflexivity of all the spaces there. Let $< f, \varphi_i >_H = \varphi_i'(f) = 0$ for all $i \in \mathbb{N}$. Then $f \equiv 0$. Thus $\{\varphi_i\}_{i\in\mathbb{N}}$ is also complete in H.

By definition, $\varphi_i' \in A_0^{\perp}(F, \sigma)$ with $\sigma = [\alpha_\mu s_\alpha(i)] + 1$. Since $|\varphi_i'|_H^* = 1$, Corollary 4.4.1 implies the following inequality:

$$|\varphi_i'|_{\Omega_\lambda}^* \leq C \exp(\lambda + \delta) s_\alpha(i), \; i \in \mathbb{N}, \; 0 < \lambda < \infty \tag{4.58}$$

with some constant $C = C(\lambda, \delta), \delta > 0$. Since $H \hookrightarrow A(\Omega)$, for any $\delta > 0$

$$|\varphi_i|_{\Omega_\delta} \leq L |\varphi_i|_H = L$$

for some constant $L = L(\delta)$. Note that $\varphi_i \in A_0((F, \sigma), \Omega_\delta)$ with $\sigma = ([\alpha_\mu s_\alpha(i)])$ and $g_{\Omega_\delta}(F, \alpha; z) = g_\Omega(F, \alpha; z) + \delta$. Thus using Theorem 4.4.1 we obtain

$$|\varphi_i|_{\Omega_\lambda} \leq N \exp s_\alpha(i)(-\lambda + \delta), \; i \in \mathbb{N}, \; 0 < \delta < \lambda < \infty \tag{4.59}$$

for some constant $N = N(\lambda, \delta)$. From (4.58) and (4.59) for any function f in any space in (4.54), $f(z) = \sum_i \varphi_i'(f)\varphi_i(z)$ converges in the topology of that space. Using (4.58) we get

$$\frac{1}{|\phi_i|_{\Omega_\lambda}} = \frac{|\varphi_i'(\varphi_i)|}{|\varphi_i|_{\Omega_\lambda}} \leq |\varphi_i'|_{\Omega_\lambda}^* \leq C \exp(\lambda + \delta) s_\alpha(i), \; i \in \mathbb{N}, \; 0 < \lambda < \infty. \tag{4.60}$$

The strict asymptotics

$$s_\alpha(i) \sim \sigma_n i^{1/n}, \; i \to \infty \tag{4.61}$$

follows from

$$|\{i : s_\alpha(i) < t\}| \sim \sum_{\mu=1}^{k} \frac{(\alpha_\mu t)^n}{n!}, \; t \to \infty.$$

Now (4.59), (4.60) with (4.61) imply the result (4.55).

□

4.6 Solution of Kolmogorov Problem on Widths Asymptotics

Here we give a detailed proof of Theorem 1.0.1. This proof is due to Zakharyuta [37] and based on the following recent result of Nivoche and Poletsky.

Proposition 4.6.1. *Let (K, D) be a pluriregular pair on a Stein manifold Ω. Then there exist a sequence of multipole Green functions (with finite numbers of poles) converging to $w(D, K; z) - 1$ uniformly on any compact subset of $D \setminus K$.*

A detailed proof of Proposition 4.6.1 is given in next chapter.

Proof. (of Theorem 1.0.1) Take any pair of Hilbert spaces (H_0, H_1) adherent to $(A(K), A(D))$. By Theorem 4.2.3, (H_0, H_1) is admissible for (K, D). Since the strict asymptotics is independent of the choice of admissible pair of Banach spaces, it is enough to prove that

$$\lim_{i\to\infty} \frac{\ln d_i(H_1, H_0)}{i^{1/n}} = -2\pi \left(\frac{n!}{C(K, D)}\right)^{1/n}. \tag{4.62}$$

By Theorem 4.2.2 the Hilbert scale $H_\alpha := H_0^{1-\alpha} H_1^\alpha$ connecting H_1 and H_0 satisfies the following imbeddings:

$$A(K_\alpha) \hookrightarrow H_\alpha \hookrightarrow A(\Omega_\alpha) \tag{4.63}$$

where K_α and D_α are defined as in Theorem 4.2.2. Take two sequences of real numbers $\epsilon_j \downarrow 0$ and $\delta_j \downarrow 0$ such that

$$\epsilon_{j+1} < \epsilon_j - 2\delta_j. \tag{4.64}$$

By Proposition 4.6.1, for each j, there exists a multipole Green function $g_j(z) := g_\Omega(F^{(j)}, \alpha^{(j)}; z)$ with a finite set of poles $F^{(j)} = \{\zeta_{j,i} : i = 1, ..., m_j\} \subset D$ and a vector $\alpha^{(j)} = (\alpha_{j,i}) \in \mathbb{R}_+^{m_j}$ such that

$$|g_j(z) - w(D, K; z) + 1| < \delta_j, \; z \in K_{1-\epsilon_j} \setminus D_{\epsilon_j}. \tag{4.65}$$

Consider the function

$$w_j(z) := s\left(\theta_j, \frac{g_j + 1 - \epsilon_j}{1 - \epsilon_j}; z\right) = \begin{cases} \frac{g_j(z)+1-\epsilon_j}{1-\epsilon_j} & \text{if } z \in D \setminus \theta_j \\ 0 & \text{if } z \in \theta_j \end{cases} \tag{4.66}$$

where $\theta_j := \{z \in D : g_j(z) \leq -1 + \epsilon_j\}$. Clearly $w_j(z) = w(D, \theta_j; z)$ in D. Since $\theta_{j+1} \subset \theta_j$, $w_j(z) = w(D, \theta_j; z)$ is non-decreasing. Using (4.64) and (4.65) we get

$$K_{\epsilon_{j+1}-\delta_{j+1}} \subset \theta_{j+1} \subset K_{\epsilon_j-\delta_j}.$$

Thus

$$w(D, K_{\epsilon_j-\delta_j}; z) \leq w_{j+1}(z) \leq w(D, K_{\epsilon_{j+1}-\delta_{j+1}}; z), j \in \mathbb{N}.$$

So we proved that

$$w_j(z) \uparrow w(D, K; z),\ z \in D. \tag{4.67}$$

Using the fact that Monge-Ampère operator continuous on increasing sequence of locally bounded functions (see Theorem 3.6.1 in [14]) and (4.40) we have

$$(2\pi)^n \sum_{i=1}^{m_j} (\alpha_{j,i})^n = C(\theta_j, D) := \int_D (dd^c w_j(z))^n \downarrow \int_D (dd^c w(z))^n =: C(K, D). \tag{4.68}$$

Thus for a given $\epsilon > 0$, there is j such that

$$C(K, D) \leq C(\theta_j, D) \leq (1 + \epsilon)C(K, D),\ \epsilon_j < \epsilon. \tag{4.69}$$

Using Theorem 4.5.1 with $H = H_1, F = F^{(j)}, \alpha = \alpha^{(j)}$, we choose a basis $\{\phi_i\}$ orthonormal in H_1. Consider the Hilbert space G consist of all $x = \sum_{i\in\mathbb{N}} \zeta_i \phi_i$ with the norm

$$||x||_G := \left(\sum_{i\in\mathbb{N}} |\zeta_i|^2 \exp 2\sigma_n(\epsilon_j - 1) i^{1/n} \right)^{1/2} < \infty \tag{4.70}$$

where

$$\sigma_n = \left(\frac{n!}{\sum_{i=1}^{m_j} (\alpha_{j,i})^n} \right)^{1/n} = 2\pi \left(\frac{n!}{C(\theta_j, D)} \right)^{1/n}. \tag{4.71}$$

Then by (2.6)

$$d_{i-1}(H_1, G) = \exp \sigma_n(-1 + \epsilon_j) i^{1/n}. \tag{4.72}$$

Using the estimate (4.69) and (4.71) we have

$$(1 - \epsilon)2\pi \left(\frac{n!}{(1 + \epsilon)C(K, D)} \right)^{1/n} \leq -\frac{\ln d_{i-1}(H_1, G)}{i^{1/n}} \leq 2\pi \left(\frac{n!}{C(K, D)} \right)^{1/n}. \tag{4.73}$$

Due to (4.55) and (4.63), we have the following imbeddings:

$$H_1 \hookrightarrow H_{\epsilon_j+\delta_j} \hookrightarrow A(D_{\epsilon_j+\delta_j}) \hookrightarrow A(\theta_j) \hookrightarrow G \hookrightarrow A(D_j) \hookrightarrow A(K) \hookrightarrow H_0.$$

By Proposition 2.5.1, there is $C > 0$ such that

$$\frac{1}{C} d_i(H_1, H_0) \leq d_i(H_1, G) \leq C d_i(H_1, H_{\epsilon_j+\delta_j}). \tag{4.74}$$

From the inequalities in (4.73), (4.74) with the relation

$$d_i(H_1, H_{\epsilon_j+\delta_j}) = d_i(H_1, H_0)^{1-\epsilon_j-\delta_j}$$

we get

$$-\liminf_{i\to\infty} \frac{\ln d_i(H_1, H_0)}{i^{1/n}} \geq (1-\epsilon) 2\pi \left(\frac{n!}{(1+\epsilon) C(K, D)} \right)^{1/n}$$

and

$$-\limsup_{i\to\infty} \frac{\ln d_i(H_1, H_0)}{i^{1/n}} \leq \frac{2\pi}{1-2\epsilon} \left(\frac{n!}{C(K, D)} \right)^{1/n}.$$

Since ϵ is arbitrary we obtain (4.62). □

Due to Proposition 4.2.2 and Corollary 4.2.2 we have the following sufficient condition for strict asymptotics (1.3):

Corollary 4.6.1. *Let D be strongly pluriregular domain. Then the strict asymptotics (1.3) holds for any compact set K making up a pluriregular pair with D.*

CHAPTER 5

PROOF OF THE CONJECTURE OF ZAHARIUTA

Zakharyuta's Conjecture: *Given a pluriregular pair (K, D) on a Stein manifold Ω of dimension n, the relative extremal function $w(D, K; z) - 1$ can be approximated uniformly on any compact subset of $D \setminus K$ by pluricomplex Green functions on D.*

Nivoche [23] and Poletsky [25] solved this conjecture recently. We will follow the proof of Poletsky with slight differences and improvements.

5.1 Approximation of Condensers by Holomorphic Functions

Definition 5.1.1. *Let D be an open set in Ω. An open set $P := \{z \in D : |f_j(z)| < 1 \text{ for } j = 1, ..., N\}$, where $P \subset\subset D$ and $f = (f_j)_{j=1}^N \in A(D)^N$, is called an analytic polyhedron of type N. We say that $(D, f_1, ..., f_N)$ is a frame for P. An analytic polyhedron of type $n = dim\Omega$ is called a special analytic polyhedron.*
Let P be an open set in Ω such that ∂P is compact. If there are neighborhood $U \subset \Omega$ of ∂P and $(f_j)_{j=1}^N \in A(U)^N$ such that $U \cap P = \{z \in U : |f_i(z)| < 1,\ 1 \leq i \leq N\}$, then P is called polyhedral region with a frame $(U, f_1, ..., f_N)$. If $N > n$ we say that P is unreduced. P is called prepared if P is unreduced and $(f_2 f_1^{-1}, ..., f_N f_1^{-1})$ is light on $U \setminus \{f_1 = 0\}$, that is, all its level sets consist of just isolated points of $U \setminus \{f_1 = 0\}$.

Let D be a strongly pluriregular domain on a Stein manifold Ω. For all integer $j \geq 1$ we define

$$D^j := \{z \in \Omega : \phi(z) < \frac{1}{j}\}$$

A *pluriregular condenser* $K = (K_1, ..., K_m, \sigma_1, ..., \sigma_m)$ in D is a system of pluriregular compact sets $K_m \subset K_{m-1} \subset ... \subset K_1 \subset D \subset \bar{D} = K_0$ and the numbers $\sigma_m < \sigma_{m-1} < ... < \sigma_1 < \sigma_0 = 0$ such that there is a continuous plurisubharmonic function $w(z)$ on D with 0 boundary values, $K_i = \{z \in D : w \leq \sigma_i\}$ and w is maximal on int $(K_{i-1}) \setminus K_i$ for all i, $1 \leq i \leq m$. Let $D_r = \{z \in D : w(z) < r\}$. Suppose that $f = (f_k)_{k=1}^N \in A(D^j)$ and $p > 0$ is an integer. Now define

$$v(z) = \sup_{1 \leq k \leq N} \frac{1}{p} \log |f_k(z)|.$$

We say that $f = (f_k)_{k=1}^N \in A(D^j)$ *approximates* K for $\epsilon > 0$ with p if there exists $\tau : 0 < \tau < \epsilon$ such that $\forall\, i: \ 1 \leq i \leq m$,

1. $\sigma_i + 2\tau < 0$,

2. $v(z) < w(z)$ on $\overline{D}$,

3. F_i, the union of all connected components of the set $\{v \leq \sigma_i + \tau\}$ intersecting K_i, must satisfy the condition $F_i \subset D_{\sigma_i + 2\tau}$.

This is the slight modification of the notion of approximation given by Poletsky [25].

Denote by G_i the interior of F_i. Then G_i is an analytic polyhedron.

Existence of approximation of pluriregular condenser is given in the following lemma.

Lemma 5.1.1. *Let K be a pluriregular condenser in a strongly pluriregular domain D. For any sufficiently small $\epsilon > 0$ and integer j there exist $p \in \mathbb{N}^*$ and $(f_k)_{k=1}^N \in A(D^j)$ that approximate K for ϵ.*

Proof. We assume that $\sigma_1 + 2\epsilon < 0$. We take $\tau : 0 < \tau < \epsilon$, $\delta : 0 < \delta < \tau/2$ and $a > 0$ such that $a\phi < w$ on $\bar{D}_{-\delta}$. Then

$$w'(z) := \begin{cases} \max\{w, a\phi\} - \delta & for\ z \in \bar{D}, \\ a\phi - \delta & for\ z \in D^j \setminus D \end{cases} \tag{5.1}$$

is plurisubharmonic on D^j. By Lelong-Bremermann Lemma [5](see [27] for proof), there exist a positive integer p and $(f_k)_{k=1}^N \in A(D^j)$ such that

$$w'(z) < v(z) := \sup_{1\le k\le N} \frac{1}{p}\log|f_k(z)| < w'(z) + \delta$$

on D^{2j}. Since $w' = w - \delta$ on $D_{-\delta}$, $w - \delta < v < w$ on $D_{-\delta}$. On ∂D $v < 0$ and on $\partial D_{-\delta}$ $v < -\delta$. By maximality of w in $D \setminus D_{-\delta}$, we have that $v < w$ on $D \setminus D_{-\delta}$. Since $v > -2\delta$ on $D \setminus D_{-\delta}$, we obtain that

$$w(z) - \tau < v(z) < w(z) \tag{5.2}$$

on $\bar{D}$. By inequality 5.2, G_i, the set of all connected components of $\{v < \sigma_i + \tau\}$ that intersect K_i, belongs to $D_{\sigma_i+2\tau}$. Therefore p and $(f_k)_{k=1}^N$ approximate K for ϵ. □

Next lemma shows that approximation is stable under a small shifting of approximating function $f = (f_k)_{k=1}^N$.

Lemma 5.1.2. *([25]) Suppose that an integer p and $(f_k)_{k=1}^N \in A(D^j)^N$ approximate K for $\epsilon > 0$. Then $\exists\delta > 0$ such that for any analytic functions $(h_k)_{k=1}^N \in A(D^j)^N$ with $||h_k||_{\bar{D}} < \delta$, $(g_k = f_k + h_k)_{k=1}^N$, approximates K with the same ϵ and p.*

Proof. Choose $a > 0$ such that $|f_k| < e^{pw} - a$ on $\overline{D}$ $\forall k : 1 \le k \le N$. If $\delta < a$, then $|g_k| < e^{pw}$. We take δ so small that $1 - \delta e^{-p(\sigma_i+\tau)} = e^{-bp}$ for all $1 \le i \le m$ where $0 < b < \tau$. If $z \in \partial G_i$, then $|f_k(z)| = e^{p(\sigma_i+\tau)}$ for some k. Then

$$|g_k(z)| > e^{p(\sigma_i+\tau)}(1 - \delta e^{-p(\sigma_i+\tau)}) = e^{p(\sigma_i+\tau')},$$

where $\tau' = \tau - b$. Thus

$$v'(z) := \sup_{1\le k\le N} \frac{1}{p}\log|g_k(z)| > \sigma_i + \tau'.$$

Let G_i' be the interior of the union F_i' of connected components of the set $\{z \in D^j : v'(z) \le \sigma_i + \tau'\}$ that intersect K_i. Let F' be one of the connected components of the set F_i'. Then $F' \cap K_i \ne \varnothing$. Therefore F' intersects a connected components

G of the set G_i. Since $v' > \sigma_i + \tau'$ on ∂G_i, $F' \subset G$. Thus $F'_i \subset G_i \subset F_i$. This implies that $G'_i \subset G_i$.
There is a positive $\tau'' < \tau$ such that $F_i \subset D_{\sigma_i + 2\tau''}$. We choose $\delta > 0$ so small that $\tau'' < \tau'$. Then $F'_i \subset D_{\sigma_i + 2\tau'}$. Thus $(g_k)_{k=1}^N$, approximates K with the same ϵ and p.

□

The following theorem shows that any pluriregular condenser can be approximated by n functions.

Theorem 5.1.1. *For any sufficiently small $\epsilon > 0$, there exist $p \in \mathbb{N}^*$ and n analytic functions $f_1, ..., f_n$ on D^j that approximate K for ϵ. Moreover $f = (f_k)_{k=1}^n$ can be choosen such that isolated zeros f in D are simple.*

The following Lemma will be needed in the proof of Theorem 5.1.1.

Lemma 5.1.3. *Let P be a polyhedral region with a frame $(U, f_1, ..., f_k)$. Let $V \subset U$ be a relatively compact neighborhood of ∂P such that $P \setminus V$ is a non-empty compact subset of P and $||f_i||_{\partial V \cap P} = r^{-1}$, $1 \leq i \leq k$ for some $r > 1$. Let Q^N be the union of the components of*

$$\{z \in \overline{V} : r^N |f_i^N - f_1^N| < 1,\ 2 \leq i \leq k\}$$

which intersects $P \setminus V$. Then $R^N = Q^N \cup (P \setminus V)$ is a polyhedral region with frame $(P \cap V, (rf_2)^N - (rf_1)^N, ..., (rf_k)^N - (rf_1)^N)$ if N is sufficiently large.

Lemma 5.1.3 is stated in [11] as Lemma $7B2$ without saying that $P \setminus V$ is non-empty. In this case, if we take $V \supset P$ as a neighborhood of ∂P, then $\partial V \cap P$ will be empty, and the number r in the lemma has no role anymore.
Note that for the proof of Lemma 5.1.3 is same with the proof of Lemma $7B2$ in [11].
***Proof.*(of Theorem** 5.1.1) Take $\epsilon > 0$ such that $\sigma_i + 2\epsilon < \sigma_{i-1}$ $\forall i = 1, ..., m$. Let $N > n$ be the minimal number of holomorphic functions $(f_k)_{k=1}^N \in A(D)^N$ that approximates K for ϵ. Note that existence of such an approximation is guaranteed

by Lemma 5.1.1. Let δ be as in Lemma 5.1.2. Since the set of $N-1$ tuples in $A(D^j)^{N-1}$ which give light maps on $D^j \setminus \{f_1 = 0\}$ is dense in $A(D^j)^{N-1}$ by Lemma $7B1$ (or by Theorem $5D4$ in [11]), there exist $h_2, ..., h_N \in A(D^j)^{N-1}$ such that $||h_k||_{\bar{D}^{2j}} < \delta$ for $2 \leq k \leq N$ and $(f_2 f_1^{-1} + h_2, ..., f_N f_1^{-1} + h_N)$ is light on $D^{2j} \setminus \{f_1 = 0\}$. Since $|f_1|_{\bar{D}_{\sigma_1}} < 1$, by Lemma 5.1.2, $g_k = f_k + h_k f_1$ for $2 \leq k \leq N$ and $g_1 = f_1$ approximate K for ϵ with same p. Now G_i, interior of the union of all connected components of $\{z \in D : v(z) := \sup_{1 \leq k \leq N} \frac{1}{p} \log |g_k(z)| \leq \sigma_i + \tau\}$ that intersect K_i, are prepared analytic polyhedra. We want to show that for some $q \in \mathbb{Z}$ big enough, $g_k^q - g_1^q$, $2 \leq k \leq N$, also approximate K for ϵ with some p'.
Since the functions g_k are continuous and $|g_k|_{\bar{D}} < 1$, we can find $a < 1$ and $j_1 > j$ such that $|g_k|_{D^{j_1}} < a$ for all k: $1 \leq k \leq N$. Therefore $|g_k^q - g_1^q|_{\bar{D}^{j_1}} < 1$ when $q \geq q_0$ for some q_0. $v(z) < w(z) \leq \sigma_i$, for $z \in K_i$. Thus we can choose a $\gamma > 0$ such that $v(z)|_{K_i} \leq \sigma_i - \gamma$ and $v(z)|_D \leq -\gamma$. Then $v < w - \gamma$ on $\bar{D}$. Now we have

$$|g_k^q - g_1^q| \leq |g_k^q| + |g_1^q| \leq 2e^{pq(w-\gamma)} \text{ on } \bar{D}.$$

Let us choose $q_1 > q_0$ such that for any $q > q_1$, we have $\frac{\ln 2}{pq} < \gamma$. Then we obtain

$$v' := \sup_{2 \leq k \leq N} \frac{1}{p'} \log |g_k^q - g_1^q| \leq w - \gamma + \frac{\ln 2}{pq} < w$$

on $\bar{D}$ where $p' = pq$.
Let us show that $\overline{D}_{\sigma_i+\tau} \subset G_i$. Since $\sigma_i + \tau < \sigma_{i-1}$, $\overline{D}_{\sigma_i+\tau}$ belongs to the interior K_{i-1}° of K_{i-1}. $\overline{G}_i \subset D_{\sigma_i+2\tau} \subset K_{i-1}^\circ$ because $\sigma_i + 2\tau < \sigma_{i-1}$. Since $G_i \supset K_i$, w is maximal on $K_{i-1}^\circ \setminus \overline{G}_i$. The boundary of $K_{i-1}^\circ \setminus \overline{G}_i$ consists of the boundary of K_{i-1}°, where $w = \sigma_{i-1} > \sigma_i + \tau$, and the boundary of G_i, where $w > v = \sigma_i + \tau$. By the maximality of w, $w > \sigma_i + \tau$ on $K_{i-1}^\circ \setminus \overline{G}_i$. Hence $\overline{D}_{\sigma_i+\tau} \subset \overline{G}_i$. Since $w > v = \sigma_i + \tau$ on ∂G_i, $\bar{D}_{\sigma_i+\tau} \subset G_i$.
Thus we can take $\tau' < \tau$ such that

$$\bar{D}_{\sigma_i+\tau'} \subset G_i \subset \bar{G}_i \subset D_{\sigma_i+2\tau'}$$

$i = 1, ..., m$. Let us take an open set $U_i \subset\subset D_{\sigma_i+2\tau'}$ such that G_i is a prepared analytic polyhedron with the frame $(U_i, g_{1i}, ..., g_{Ni})$, where $g_{ki} = e^{-p(\sigma_i+\tau)} g_k$, that is, $G_i = \{z \in U_i : |g_{ki}(z)| < 1, 1 \leq k \leq N\} \subset\subset U_i$. Take an open set U'_i such that $\overline{G_i} \subset U'_i \subset\subset U_i$. Consider $V_i := U'_i \setminus \overline{D}_{\sigma_i+\tau'}$ which is a neighborhood of ∂G_i. $V_i \subset\subset U_i$. Since $\partial V_i \cap G_i = \partial D_{\sigma_i+\tau'}$, $|w|_{\partial V_i \cap G_i} = \sigma_i + \tau'$. Thus $|g_{ki}| < e^{p(\tau'-\tau)}$ on $\partial V_i \cap G_i$. Put $r_i = e^{-p(\tau'-\tau)}$, then $|g_{ik}| < r_i^{-1} < 1$ on $\partial V_i \cap G_i$. So by Lemma 5.1.3 $\exists q_2 > q_1$ such that $\forall q > q_2$ the union R^i_q of $\overline{D}_{\sigma_i+\tau'}$ and all connected components of the set $\{z \in \overline{V_i} : r^q |g^q_{ki} - g^q_{1i}| < 1,\ 2 \leq k \leq N\}$ intersecting $\overline{D}_{\sigma_i+\tau'}$ is a polyhedral region with the frame

$$(V_i \cap G_i, r_i^q(g^q_{2i} - g^q_{1i}), ..., r_i^q g^q_{Ni} - g^q_{1i}). \tag{5.3}$$

Choose $\tau'' < \tau'$ such that $\overline{U'_i} \subset D_{\sigma_i+2\tau''}$.

Let F' be a connected component of $\{v' \leq \sigma_i + \tau''\}$ intersecting K_i. If $z_0 \in F' \cap K_i$, then $z_0 \in \bar{D}_{\sigma_i+\tau'}$ since $K_i \subset \overline{D}_{\sigma_i+\tau'}$. Thus $z_0 \in R$ which is one of the connected component of R^i_q. So $F' \cap K_i \subset R$. By (5.3), ∂R is contained in V_i. Therefore if $z_1 \in \partial R$, then $r_i^q |g^q_{ki}(z_1) - g^q_{Ni}(z_1)| = 1$ for some k. So $|g^q_k(z_1) - g^q_N(z_1)| e^{-pq(\tau-\tau')} e^{-pq(\sigma_i+\tau)} = |g^q_k(z_1) - g^q_N(z_1)| e^{-pq(\sigma_i+\tau')} = 1 \Rightarrow |g^q_k(z_1) - g^q_N(z_1)| = e^{pq(\sigma_i+\tau')} > e^{pq(\sigma_i+\tau'')}$. So $v'(z_1) > \sigma_i + \tau''$ i.e $\partial R \cap F' = \varnothing$. Since $F' \cap K_i \subset R$ and F' is connected, $F' \subset R \subset R^i_q$. Since $R^i_q \subset U'_i \subset\subset D_{\sigma_i+2\tau''}$, $F' \subset D_{\sigma_i+2\tau''}$ i.e, the functions $g^q_k - g^q_1$, $2 \leq k \leq N$ and $p' = pq - \mu$ approximate K for ϵ. But this contradicts to minimality of N.

Suppose that $f = (f_1, ..., f_n)$ has non-simple isolated zeros. By Lemma 5.1.2, our approximation is stable for some $\delta > 0$. Sard's theorem states that the set $\{y \in f(D) : Df(x) = 0 \text{ for some } x \in f^{-1}(y)\}$ has measure 0. So there exist a point $c = (c_1, ..., c_n) \in \mathbb{C}^n$ such that Df is different from 0 at all preimages of c and $|c_k| < \delta$ for all $k : 1, ..., n$. Let $g_k = f_k - c_k$. Now $g_1, ..., g_n$ and p also approximate K for same ϵ and isolated zeros of $g = (g_1, ..., g_n)$ are simple. □

5.2 Approximation of the Relative Extremal Function by Multipole Green functions

Lemma 5.2.1. *Let K be a pluriregular condenser in a strongly pluriregular domain $D \subset \Omega$. Then there are positive numbers $\delta_j \downarrow 0$, $\epsilon_j \downarrow 0$, Green functions g_j on D,* *numbers $\sigma'_{ij} < \sigma_i$ converging to σ_i for all $i = 1, ..., m$ and open sets V_{ij} and W_{ij} ($W_{mj} \neq \emptyset$) such that*

$$D_{\sigma'_{ij}} \subset\subset W_{ij} \subset\subset D_{\sigma_i} \subset\subset V_{ij} \subset\subset D_{\sigma_i+2\epsilon_j}$$

$g_j > \sigma_i$ on ∂V_{ij}, $g_j > \sigma'_{ij}$ on ∂W_{ij}, the poles of g_j are contained in the union of sets $Z_{ij} := V_{ij} \setminus \overline{W}_{ij}$, $1 \leq i \leq m$, and

$$\int_{Z_{ij}} (dd^c w)^n - \delta_j \leq \int_{Z_{ij}} (dd^c g_j)^n \leq \int_{Z_{ij}} (dd^c w)^n + \delta_j.$$

Proof. For each $1 \leq i \leq m$ let us take a sequence of numbers σ'_{ij} satisfying $\sigma_{i+1} < \sigma'_{ij} < \sigma_i$ and $\sigma'_{ij} \uparrow \sigma_i$. For each j we consider the pluriregular condenser

$$K^j := \{K_{1j}, ..., K_{2m\,j}, \sigma_{1j}, ..., \sigma_{2m\,j}\}$$

where $K_{2i-1\,j} = K_i$, $K_{2i\,j} = \overline{D}_{\sigma'_{ij}}$, $\sigma_{2i-1\,j} = \sigma_i$, $\sigma_{2i\,j} = \sigma'_{ij}$. Note that $w(D, K; z) = w(D, K^j; z)$ for all j. Let $(f_{kj})_{k=1}^n$ be a sequence of holomorphic functions that approximates K^j for $\epsilon_j < 1/j$. Assume that the system $f_{1j} = ... = f_{nj} = 0$ have simple roots and ϵ_j satify the following:

$$\alpha_j := \frac{\sigma_{ij} - \sigma_{i-1\,j} + \epsilon_j}{\sigma_{ij} - \sigma_{i-1\,j} + \epsilon_j + \epsilon_j^2} < \left(1 + \frac{1}{j}\right)^{1/n}$$

for all $1 \leq i \leq 2m$. Then there exist $\tau_j < \epsilon_j$ satisfying $\sigma_{ij} + 2\tau_j < \sigma_{i-1\,j}$ and

$$K_{ij} \subset G_{ij} \subset\subset D_{\sigma_{ij}+2\tau_j}$$

where G_{ij} is the interior of union of all connected components of $\{v_j \leq \sigma_{ij} + \tau_j\}$ that intersect K_{ij} with $v_j := \sup_{1\leq k\leq n} 1/p_j \log |f_{kj}|$. Let $w_{ij} := (1-\tau_j)(w - \sigma_{ij} - \tau_j - \tau_j^2)$ and $v_{ij} := v_j - \sigma_{ij} - \tau_j$. On ∂G_{ij} $v_{ij} = 0$ and $w_{ij} < 0$. Since $w \leq \sigma_{ij}$ on K_{ij} and $v_j < w$ on D, $w_{ij} > v_{ij}$ on K_{ij}. Therefore $H_{ij} := \{v_{ij} < w_{ij}\} \cap G_{ij} \supset K_{ij}$. By Comparison Principle,

$$\int_{G_{ij}} (dd^c v_j)^n \geq \int_{H_{ij}} (dd^c v_{ij})^n \geq \int_{H_{ij}} (dd^c w_{ij})^n.$$

We assume that $(1 - \tau_j)^n > 1 - \epsilon_j$. Since w_{ij} is maximal on $G_{ij} \setminus K_{ij}$ we obtain

$$\begin{aligned}\int_{G_{ij}}(dd^c v_j)^n \geq \int_{G_{ij}}(dd^c w_{ij})^n &= (1-\tau_j)^n \int_{G_{ij}}(dd^c w)^n \\ &> (1-\epsilon_j)\int_{G_{ij}}(dd^c w)^n. \end{aligned} \tag{5.4}$$

Let P_{ij} be the set of poles of v_j that lie in G_{ij}. We consider the Green function g_{ij} on $D_{\sigma_{i-1\,j}}$ with poles in P_{ij} of weight $1/p_j$ and $g_{ij}=0$ on $\partial D_{\sigma_{i-1\,j}}$. By definition of g_{ij} we have that $g_{ij} > v_j - \sigma_{i-1\,j}$ on $\overline{D}_{\sigma_{i-1\,j}}$.
Let w'_{ij} be the restriction of $\alpha_j(w-\sigma_{i-1\,j})$ to $D_{\sigma_{i-1\,j}}$ and $v'_{ij} := \max\{g_{ij}, \sigma_{ij} - \sigma_{i-1\,j} + \tau_j\}$. Since $w'_{ij} < \sigma_{ij} - \sigma_{i-1\,j} + \tau_j$ on G_{ij}, $w'_{ij} < v'_{ij}$ on G_{ij}. On $\partial D_{\sigma_{i-1\,j}}$ $g_{ij} = w'_{ij} = 0$. On ∂G_{ij} $g_{ij} > \sigma_{ij} - \sigma_{i-1\,j} + \tau_j > w'_{ij}$. Since g_{ij} is maximal on $D_{\sigma_{i-1\,j}} \setminus \overline{G}_{ij}$ we see that $v'_{ij} \geq g_{ij} > w'_{ij}$ on $D_{\sigma_{i-1\,j}}$. By the Comparison Principle,

$$\int_{D_{\sigma_{i-1\,j}}}(dd^c v'_{ij})^n \leq (\alpha_j)^n \int_{D_{\sigma_{i-1\,j}}}\int (dd^c w)^n.$$

By

$$\int_{D_{\sigma_{i-1\,j}}}(dd^c v'_{ij})^n = \int_{D_{\sigma_{i-1\,j}}}(dd^c g_{ij})^n.$$

Since g_{ij} and w are maximal on $D_{\sigma_{i-1\,j}} \setminus \overline{G}_{ij}$,

$$\int_{G_{ij}}(dd^c v_j)^n = \int_{G_{ij}}(dd^c g_{ij})^n \leq (\alpha_j)^n \int_{G_{ij}}(dd^c w)^n. \tag{5.5}$$

Then by (5.4), (5.5) and the equality

$$\int_{G_{2i\,j}}(dd^c w)^n = \int_{G_{2i+1\,j}}(dd^c w)^n$$

we obtain that

$$\int_{G_{2i\,j}}(dd^c v_j)^n - \int_{G_{2i+1\,j}}(dd^c v_j)^n \leq \frac{2}{j}\int_{G_{2i+1\,j}}(dd^c w)^n. \tag{5.6}$$

Let $V_{ij} := G_{2i-1\,j}$ and $W_{ij} := G_{2i\,j}$ for all $i = 1, ..., m$. For each j we introduce the Green function g_j on D with poles of weight $1/p_j$ at those poles of v_j that lie in the union of the sets $G_{2m-1\,j}$ and $G_{2i-1\,j} \setminus \overline{G}_{2i\,j}$, $1 \leq i \leq m-1$. By definiton of g_j and inequality (5.5),

$$\int_{V_{ij}}(dd^c g_j)^n \leq \int_{V_{ij}}(dd^c v_j)^n \leq \left(1+\frac{1}{j}\right)\int_{V_{ij}}(dd^c w)^n. \tag{5.7}$$

Since the set of poles of g_j in V_{ij} is equal to the set of poles of v_j that lie in the set

$$V_{ij} \setminus \cup_{k=i}^{m-1}(G_{2k\,j} \setminus \overline{G}_{2k+1\,j})$$

we have

$$\int_{V_{ij}} (dd^c g_j)^n = \int_{V_{ij}} (dd^c v_j)^n - \sum_{k=i}^{m-1} \int_{G_{2k\,j}\setminus\overline{G}_{2k+1\,j}} (dd^c v_j)^n.$$

By (5.4) and (5.6),

$$\int_{V_{ij}} (dd^c g_j)^n \geq \left(1 - \frac{2m}{j}\right) \int_{V_{ij}} (dd^c w)^n. \tag{5.8}$$

Let $\gamma_j := 2m/j$ and $\delta_j := 2\gamma_j \int_D (dd^c w)^n$. By inequalities (5.7) and (5.8),

$$\begin{aligned}
\int_{Z_{ij}} (dd^c g_j)^n &= \int_{V_{ij}} (dd^c g_j)^n - \int_{V_{i+1\,j}} (dd^c g_j)^n \\
&\leq (1+\gamma_j) \int_{V_{ij}} (dd^c w)^n - (1-\gamma_j) \int_{V_{i+1\,j}} (dd^c w)^n \leq \int_{Z_{ij}} (dd^c w)^n + \delta_j.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{Z_{ij}} (dd^c g_j)^n &\geq (1-\gamma_j) \int_{V_{ij}} (dd^c w)^n - (1+\gamma_j) \int_{V_{i+1\,j}} (dd^c w)^n \\
&\geq \int_{Z_{ij}} (dd^c w)^n - \delta_j.
\end{aligned}$$

By definiton, $g_j > v_j$ on D. Thus $g_j > \sigma_i$ on ∂V_{ij} and $g_j > \sigma'_{ij}$ on ∂W_{ij}. Now the proof is complete. □

Lemma 5.2.2. *Let $(g_j)_j$ be a sequence of Green function on D that satisfies the conditions of Lemma 5.2.1. Then $(g_j)_j$ converges uniformly to $w(z)$ on every compact set in $D_{\sigma_{i-1}} \setminus K_i$, $i = 1, ..., m$.*

Proof. Let g_{ij} be the restriction of g_j to the open set Z_{ij}. Since $g_j > \sigma_i > \sigma'_{ij}$ on ∂V_{ij} and $g_j > \sigma'_{ij}$ on ∂W_{ij}, $\{g_{ij} < \sigma'_{ij} - \delta_j\} \subset\subset Z_{ij}$. Let $g'_{ij} := \max\{g_{ij}, \sigma'_{ij} - \delta_j\}$. Then the function g'_j defined as g'_{ij} on Z_{ij} for all $i = 1, ..., m$ and g_j on $D \setminus \cup_{i=1}^m Z_{ij}$ is plurisubharmonic in D. Let

$$c_j := \min\left\{\frac{\sigma_i + 2\epsilon_j}{\sigma'_{ij} - \delta_j} : 1 \leq i \leq m\right\}.$$

Note that $c_j \to 1$. Let $h_j := c_j g'_j$. On Z_{ij}, $h_j \geq \sigma_i + 2\epsilon_j > w$ for all $i = 1, ..., m$. By the maximality of h_j outside the union of the Z_{ij}, $h_j > w$ on D. Thus integrating by parts we have,

$$\begin{aligned}\int_D (-w)(dd^c w)^n &\geq \int_D (-h_j)(dd^c w)^n = \int_D (-w) dd^c h_j \wedge (dd^c w)^{n-1} \\ &\geq \int_D (-h_j) dd^c h_j \wedge (dd^c w)^{n-1} = ... \geq \int_D (-h_j)(dd^c h_j)^n.\end{aligned}$$

Then we obtain

$$\begin{aligned}0 \leq \int_D (h_j - w)(dd^c w)^n &\leq \int_D h_j (dd^c h_j)^n - \int_D w (dd^c w)^n \\ &= \sum_{i=1}^{m} \left(\int_{Z_{ij}} h_j (dd^c h_j)^n - \int_{Z_{ij}} w (dd^c w)^n \right). \quad (5.9)\end{aligned}$$

The support of $(dd^c w)$ lies on ∂K_i, where $w = \sigma_i$. The support of $(dd^c h_j)$ lies where $g'_j < \sigma'_{ij}$. Thus by (5.9)

$$0 \leq \int_D (h_j - w)(dd^c w)^n \leq \sum_{i=1}^{m} \left(c_j^n \sigma'_{ij} \int_{Z_{ij}} (dd^c g'_j)^n - \sigma_i \int_{Z_{ij}} (dd^c w)^n \right).$$

By (3.7) and Lemma 5.2.1,

$$\int_{Z_{ij}} (dd^c g'_j)^n = \int_{Z_{ij}} (dd^c g_j)^n \geq \int_{Z_{ij}} (dd^c w)^n - \delta_j.$$

For any $a > 0$,

$$\begin{aligned}0 \leq a \int_{\{h_j - w > a\}} (dd^c w)^n &\leq \int_D (h_j - w)(dd^c w)^n \\ &\leq \sum_{i=1}^{m} \left((c_j^n \sigma'_{ij} - \sigma_i) \int_{Z_{ij}} (dd^c w)^n - \delta_j c_j^n \sigma'_{ij} \right).\end{aligned}$$

Thus

$$\lim_{j \to \infty} \int_{\{h_j - w > a\}} (dd^c w)^n = 0. \quad (5.10)$$

For any $\delta > 0$ we choose $\epsilon > 0$ such that $\epsilon |z|^2 < \delta/2$ on D and denote $u_j := h_j + \epsilon |z|^2 - \delta$. Note that $(dd^c(\epsilon |z|^2 - \delta))^n = \epsilon^n k_n dV$, where k_n depends only on

n and dV is the volume form. Let $E_j := \{z \in D : w < u_j\}$. Since $u_j < -\delta/2$ on ∂D and $h_j - w > \delta/2$ on E_j, E_j is relatively compact in D and contained in the set $\{h_j - w > \delta/2\}$. By the subadditivity of the complex Monge-Ampère operator and the Comparison Principle we have

$$\begin{aligned}\epsilon^n k_n m(E_j) &\leq \int_{E_j} (dd^c h_j)^n + \int_{E_j} (dd^c \epsilon|z|^2 - \delta)^n \\ &\leq \int_{E_j} (dd^c u_j)^n \leq \int_{E_j} (dd^c w)^n \leq \int_{\{h_j - w > \delta/2\}} (dd^c w)^n.\end{aligned}$$

Thus by (5.10),

$$\lim_{j\to\infty} m(E_j) = 0.$$

Let $F_j := \{w < h_j - \delta\} \subset E_j$. There exists $r > 0$ such that $|w(z) - w(z')| < \delta$ whenever $|z - z'| < r$. Take j_0 such that $m(E_j) < \delta m(B(z, r))$ $\forall j > j_0$. If $B = B(z_0, r) \subset D$, then

$$\begin{aligned}w(z_0) &\leq h_j(z_0) \leq \frac{1}{m(B)} \int_B h_j dV \leq \frac{1}{m(B)} \int_{B\setminus F_j} h_j dV \\ &\leq \frac{1}{m(B)} \left(\int_B (w + \delta) dV - \int_{B\cap F_j} (w + \delta) dV \right) \leq w(z_0) + (2 - \sigma_m)\delta.\end{aligned}$$

Thus h_j converge to w uniformly on D. Consequently on every compact set in $D_{i-1} \setminus K_i$, g_j converge uniformly to w for all $i = 1, ..., m$. □

Now we can prove Proposition 4.6.1 which is the positive solution of Zakharyuta Conjecture. Proposition 4.6.1 can be considered as a particular case of Lemma 5.2.2. Only we need to extend the notion of approximation from strongly pluriregular domain to pluriregular domain. Here we follow Nivoche [23] for the proof.

Proof. (of Proposition 4.6.1) For any δ sufficiently small, $D(-\delta)$ is a strongly pluriregular domain containing a compact set K. By definition, we have on $D(-\delta)$

$$u_{K,D(-\delta)} = \frac{u_{K,D} + \delta}{1 - \delta}.$$

Therefore, for any $\epsilon > 0$, there exist $\delta_0 < \delta$ such that,

$$u_{K,D(-\delta_0)} \leq \frac{u_{K,D}}{(1 + \epsilon)^{\frac{1}{3}}}.$$

By Proposition 3.3.1, for any $\delta' < \inf\{\delta_0, \delta^2\}$, we have

$$(1+\epsilon)^{1/3} u_{K,D(-\delta')} \leq u_{K,D} \leq u_{K,D(-\delta')} \text{ on } \overline{D(-\delta)}. \tag{5.11}$$

Now Lemma 5.2.2 for the couple $(K, D(-\delta'))$ with $m = 1$ and $u_{K,D}$ instead of w implies that there exist pluricomplex Green function g on $D(-\delta')$ such that on each compact set in $\overline{D(-\delta')} \setminus D(-1+\delta'(1-\delta'))$,

$$(1+\epsilon)^{1/3} g \leq u_{K,D(-\delta')} \leq (1-\epsilon)^{1/3} g. \tag{5.12}$$

By combining the equation (5.12) with the equation (5.11), we get

$$(1+\epsilon)^{2/3} g \leq u_{K,D} \leq (1-\epsilon)^{2/3} g \text{ on } \overline{D(-\delta')} \setminus D(-1+\delta'(1-\delta')). \tag{5.13}$$

Next step is to replace g by a pluricomplex Green function on D. Let g' be the pluricomplex Green function with the same poles as g but with weights all multiplied by the constant $(\delta-\delta')/\delta$. Since $D(-\delta') \subset D$,

$$g' \leq \frac{(\delta-\delta')}{\delta} g \text{ on } D(-\delta'). \tag{5.14}$$

By (5.13), on $\partial D(-\delta)$ we have,

$$\frac{u_{K,D}(z)}{(1-\epsilon)^{2/3}} = \frac{-\delta'}{(1-\epsilon)^{2/3}} \leq \frac{(\delta-\delta')}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}}.$$

On $\partial D(-\delta')$, since $g(z) = 0$ and $u_{K,D}(z) = -\delta'$,

$$\frac{u_{K,D}(z)}{(1-\epsilon)^{2/3}} = \frac{-\delta'}{(1-\epsilon)^{2/3}} = \frac{(\delta-\delta')}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}}.$$

Let v denote the following negative psh function:

$$v(z) = \begin{cases} \frac{(\delta-\delta')}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}} & \text{on } \overline{D(-\delta)} \\ \max\{\frac{(\delta-\delta')}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}}, \frac{u_{K,D}(z)}{(1-\epsilon)^{2/3}}\} & \text{on } D(-\delta') \setminus \overline{D(-\delta)} \\ \frac{u_{K,D}(z)}{(1-\epsilon)^{2/3}} & \text{on } D \setminus D(-\delta') \end{cases}$$

By definition of g', $v \leq g'$ on D. In particular on $\overline{D(-\delta)}$,

$$g' \geq \frac{(\delta-\delta')}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}}. \tag{5.15}$$

By combining the inequalities (5.13), (5.14) and (5.15),
on $\overline{D(-\delta)} \setminus D(-1+\delta'(1-\delta'))$ we have,

$$\frac{\delta(1+\epsilon)^{2/3}}{\delta-\delta'}.g'(z) \leq u_{K,D}(z) \leq \frac{\delta(1-\epsilon)^{2/3}}{\delta-\delta'}.\left(g' + \frac{\delta'}{(1-\epsilon)^{2/3}}\right).$$

Since $u_{K,D}$ and g' are maximal in $D \setminus D(-\delta)$, we have the same inequalities on $\bar{D}\setminus D(-\delta)$. Since $\delta'(1-\delta') < \delta$, we have also the same inequalities on $\bar{D}\setminus D(-1+\delta)$. By choosing δ' sufficiently small such that $\frac{\delta}{\delta-\delta'} \leq (1+\epsilon)^{1/3}$ and $\frac{\delta\delta'}{\delta-\delta'} + \frac{\delta(1-\epsilon)^{2/3}}{\delta-\delta'}g' \leq (1-\epsilon)g'$ on $D \setminus D(-1+\delta)$, we obtain the following inequality on $D \setminus D(-1+\delta)$,

$$(1+\epsilon)g'(z) < u_{K,D}(z) < (1-\epsilon)g'(z)$$

which completes the proof. □

Bibliography

[1] Aytuna A., *On Stein manifolds M for which $O(M)$ is isomorphic to $O(\Delta^n)$ as Fréchet spaces,* Manuscripta Math. 62 (1988), 297-315.

[2] Babenko K. I., *On ϵ-entropy of a class of analytic functions*, Nauchn. Doklady Vyssh. Shkoly, 2 (1958), 9-16.

[3] Bedford E., Taylor B.A., *A new capacity for plurisubharmonic functions*, Acta Math. 149, 1-40 (1982)

[4] Bishop E., *Mappings of partially analytic spaces*, Am. J. Math. 83, 209-242 (1961)

[5] Bremermann M.J., *On the conjecture of the equivalence of the plurisubharmonic functions and Hartogs functions,* Math. Ann., vol.131, 76-86, (1956)

[6] Demailly J.P., *Mesures de Monge-Ampère et mesures pluriharmoniques.*

[7] Erokhin V. D., *On asymptotics of ϵ-entropy of analytic functions*, Dok- lady AN SSSR 120 (1958), 949-952.

[8] Erokhin V. D., *On the best approximation of functions analytically extendable from a given continuum into a given domain*, Uspehi Matem. Nauk 14 (1968), 91-132

[9] Fisher S.D., Micchelli C.A., *The n-width of sets of analytic functions*, Duke Math. J. 47 (1980), 789-801.

[10] Grothendieck A., *Sur Certain Escapes de Fonctions Holomorphes*, J. Reine und Angrew. Math., 192 (1953), 35-64, 77-95.

[11] Gunning R.C., Rossi H., *Analytic Functions of Several Complex Variables,* Prentice-Hall, Inc., (1974). MR 31:4927(1st ed)

[12] Hörmander L., *An Introduction to Complex Analysis in Several Variables*, North Holland Publishing Company, (1973)

[13] Khavin V. P., *Spaces of Analytic Functions*, Itogi Naugi i Tecniki, VINITI AN SSSR, Mat. Anal. 1964 (1966), 76-164. Engl. transl. Progress in Math., Vol. 1, Plenum Press, New York (1968), 75-167.

[14] Klimek M., *Pluripotential theory*, volume 6 of London Math. Soc.Monographs (N.S.). New York: Oxford Univ. Press (1991)

[15] Köthe G., *Dualität in der Funktionentheorie*, J. Reine und Angrew. Math. 191 (1953), 30-39.

[16] Krein S.G., Petunin Y.I., Semenov E. M., *Interpolation of Linear Operators*, Translation of Mathematical Monographs 54, AMS, Providence-Rhode Island, 1980. Math. Z. 194, 519-564 (1987)

[17] Lelong P., Rashkovskii A., *Local indicators for plurisubharmonic functions*, J. Math. Pures Appl., IX. Sėr. 78, 233-247 (1999)

[18] Levin A.L., Tikhomirov V. M., *On theorem of V. D. Erokhin*, Russian Math. Surveys 23 (1968), 119-13

[19] Lojasiewicz S., *Introduction to Complex Analytic Geometry*, Basel: Birkhäuser (1991)

[20] Meise R., Vogt D., *Introduction to Functional Analysis, Clarendon Press, Oxford*, (1997).

[21] Mityagin B. S., *Approximative dimension and bases in nuclear spaces*, Russian Math. Survey 16 (1963), 59-127.

[22] Nivoche S., Poletsky E.A., *Multipoled Green functions,* Preprint (2001)

[23] Nivoche S., *Proof of the conjecture of Zahariuta concerning a problem of Kolmogorov on the ϵ-entropy*, Invent. Math. 158 (2004), 413-450.

[24] Pietsch A., *Nuclear Locally Convex Spaces*, Springer, Berlin and Newyork (1972)

[25] Poletsky E., *Approximation of plurisubharmonic functions by multipole Green functions*, Trans. Amer. Math. Soc. 335 (2003), 1579-1591.

[26] Sadullaev A., *Plurisubharmonic measures and capacities on complex manifolds* Russ. Math. Surv. 36, 61-119 (1981)

[27] Sibony N., *Prolongement des fonctions holomorphes bornées et métrique de Caratheodory*, Invent. Math. 29 (1975), 205-230.

[28] Silva J. *Sebastiao e, Analytic Functions in Functional Analysis*, Portug. Math. 9 (1950) 1-130.

[29] Skiba N.I., Zakharyuta V.P., *Estimates of the n-widths of certain classes of functions that are analytic on Riemann surfaces*, Math. Notes 19, 525-532 (1976)

[30] Vogt D., *Eine Charakterisierung der Potenzreihenträume von endlichen Typ und ihre Folgerungen,* Manuscripta Math. 37 (1982), 269-301.

[31] Zahariuta V.P., *Spaces of analytic functions and complex potential theory*, Linear Topol. Spaces Complex Anal. 1, 74-146 (1994)

[32] Zahariuta V.P., *Extremal plurisubharmonic functions, orthogonal polynomials and the Bernstein-Walsh theorem for functions of several complex variables*, Ann. Pol. Math. 33, 137-148 (1976/77)

[33] Zahariuta V.P., *Extremal plurisubharmonic functions, Hilbert scales, and the isomorphism of spaces of analytic functions of several variables*, I, II. Teor.

Funkcii, Funkcional. Anal. i Priložen. 19 (1974), 133-157; ibid. 21 (1974), 65-83.

[34] Zahariuta V.P., *Isomorphism of spaces of analytic functions* , Sov. Math. Dokl. 22 (1980), 631-634.

[35] Zahariuta V.P., *Spaces of analytic functions and maximal plurisubharmonic functions*, Dr. Sci. Thesis, Rostov State University, Rostov-na-Donu, (1984), 281pp.

[36] Zahariuta V.P., Skiba N., *Estimates of n-diameters of some classes of functions analytic on Riemann surfaces*, Mat. zametki 19 (1976), 899-911.

[37] Zahariuta V.P.,*Kolmogorov problem on widths asymptotics and pluripotential theory*, to appear in Proceeding of International Conference on Functional Analysis and Complex Analysis, Sabanci University 2007.