ON ISOMORPHISMS OF SPACES OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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To Mom

&

Dad

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Abstract

In this thesis, we discuss results on isomorphisms of spaces of analytic functions of several complex variables in terms of pluripotential theoretic considerations. More specifically, we present the following result:

Theorem 1 *Let Ω be a Stein manifold of dimension n. Then,*

$$A(\Omega) \simeq A(U^n)$$

if and only if Ω is pluriregular and consists of at most finite number of connected components.

The problem of isomorphic classification of spaces of analytic functions is also closely related to the problem of existence and construction of bases in such spaces. The essential tools we use in our approach are Hilbert methods and the interpolation properties of spaces of analytic functions which give us estimates of dual norms and help us to obtain extendable bases for pluriregular pairs.

ÇOKLU KARMAŞIK DEĞİŞKENLİ ANALİTİK FONKSİYON UZAYLARINDA EŞBİÇİMLİLİK ÜZERİNE

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Özet

Bu tezde çoklu karmaşık değişkenli analitik fonksiyon uzaylarında eşbiçimliliğin karmaşık potansiyel teorisi yardımıyla karakterize edilmesi konusundaki gelişmelere değinilmekte ve n boyutlu bir Stein manifoldu üzerindeki analitik fonksiyon uzayının $\mathbb{C}^n$'in birim diski üzerindeki analitik fonksiyon uzayına eşbiçimli olması için gerekli ve yeterli şartlara dair sonuca yer verilmektedir.

Analitik fonksiyon uzayları arasındaki eşbiçimlilik problemi bu uzaylarda tabanların varlığı ve bu tabanların oluşturulması problemiyle de yakın ilişki içerisindedir. Analitik fonksiyon uzaylarının interpolasyon özellikleri ve Hilbert teknikleri bu uzaylar için taban oluşturmada ve eşbiçimliliklerin belirlenmesinde önemli yardımlar sağlamaktadır.

Contents

CHAPTER 1

SPACES OF ANALYTIC FUNCTIONS

1.1. Stein Manifolds

A Hausdorff topological space Ω is called a *manifold* (of dimension n) if any point in Ω has a neighbourhood which is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 1.1.1 *A manifold Ω is called a complex analytic manifold if there is a given family F of homeomorphisms κ of open sets $\Omega_\kappa \subset \Omega$ on open sets $\tilde{\Omega}_\kappa \subset \mathbb{C}^n$ such that*

(i) For κ, $\kappa' \in F$, the mapping

$$\kappa'\kappa^{-1} : \kappa(\Omega_\kappa \cap \Omega_{\kappa'}) \to \kappa'(\Omega_\kappa \cap \Omega_{\kappa'})$$

between open sets in $\mathbb{C}^n$ is analytic,

(ii)

$$\bigcup_{\kappa \in F} \Omega_\kappa = \Omega,$$

(iii) For a homeomorphism κ_0 of an open set $\Omega_0 \subset \Omega$ on an open set in $\mathbb{C}^n$ where the mapping

$$\kappa\kappa_0^{-1} : \kappa_0(\Omega_0 \cap \Omega_\kappa) \to \kappa(\Omega_0 \cap \Omega_\kappa)$$

and its inverse are analytic for every $\kappa \in F$, we have $\kappa_0 \in F$.

If F satisfies (i) and (ii) only, then F can be extended in only one way to a family F' satisfying (i), (ii) and (iii), which is the set of all mappings satisfying (iii) relative to F. So, if we drop the last condition, then we can find different families defining the same complex analytic structure. Such a family is called *a complete set of complex analytic coordinate systems.* We also say that n complex valued functions $(z_1, \ldots, z_n)$ defined in a neighbourhood of a point $w \in \Omega$ are *a local coordinate system at* w if they define a mapping of a neighbourhood of w into $\mathbb{C}^n$ which is a complex analytic coordinate system.

We may define analyticity in several complex variables simply via Cauchy-Riemann conditions:

Definition 1.1.2 *$u \in C^1(\Omega)$, where Ω is an open set in $\mathbb{C}^n$, is called analytic in Ω if $\bar{\partial} u = 0$ (i.e. it satisfies the Cauchy-Riemann equations).*

Then, the concept of analyticity can be extended to functions on complex manifolds as follows:

Definition 1.1.3 *Let Ω_1 and Ω_2 be complex analytic manifolds. Then a mapping $f : \Omega_1 \to \Omega_2$ is called analytic if $\kappa_2 \circ f \circ \kappa_1^{-1}$ is analytic for all coordinate systems κ_1 in Ω_1 and κ_2 in Ω_2.*

Every open subset of a complex analytic manifold Ω has a structure of a complex analytic manifold, so the concept of an analytic function on an open subset is also well defined. By the definition of a complex analytic manifold, analytic functions exist locally. Here is a class of complex manifolds where we can obtain globally defined analytic functions:

Definition 1.1.4 *An n-dimensional complex analytic manifold Ω is called a* Stein manifold *if*

(i) Ω is countable at infinity, i.e. if there exists a countable number of compact subsets $\{K_i : i \in \mathbb{N}\}$ such that every compact subset of Ω is contained in some K_i.

(ii) $\widehat{K} := \{z \in \Omega \ : \ |f(z)| \leq \sup_K |f| \quad \forall f \in A(\Omega)\}$ is a compact subset of Ω for any compact subset K of Ω.

(iii) for any different points z_1 *and* z_2 *in* Ω*, there exists* $f \in A(\Omega)$ *such that* $f(z_1) \neq f(z_2)$.

(iv) for any $z \in \Omega$ *there exist n functions* $f_1, \cdots, f_n \in A(\Omega)$ *which form a coordinate system at* z.

Every domain of holomorphy in $\mathbb{C}^n$ is a Stein manifold since an open set Ω in $\mathbb{C}^n$ is a domain of holomorphy if and only if K is relatively compact in Ω implies $\widehat{K}$ is relatively compact in Ω. Also, any submanifold of a Stein manifold is a Stein manifold itself. For the definition of submanifold and the proof of this statement, one can look at [8].

1.2. Spaces of Analytic Functions

The space of all analytic functions on a complex manifold Ω, with the topology of uniform convergence on compact subsets of Ω, will be denoted by $A(\Omega)$. More precisely, the topology on $A(\Omega)$ is the locally convex topology generated by the seminorms

$$|f|_K = \max_{z \in K}\{|f(z)|\} \tag{1.1}$$

where K is any compact set in Ω. If Ω is countable at infinity, e.g. Ω is a Stein manifold, then the topology on $A(\Omega)$ can be defined by some countable sequence of seminorms of the form (1.1). Also, completeness follows from the fact that for any sequence in $A(\Omega)$ converging locally uniformly to a function $f : \Omega \to \mathbb{C}$, we have $f \in A(\Omega)$. Hence, $A(\Omega)$ becomes a Fréchet space when Ω is a Stein manifold.

For any arbitrary subset E of Ω, we can also construct a locally convex space by using germs of analytic functions on E as follows:

Let $\mathcal{N}(E)$ denote the collection of all open neighbourhoods of E in Ω. We define an equivalence relation by stating that two functions $f \in A(D_f)$ and $g \in A(D_g)$, where $D_f, D_g \in \mathcal{N}(E)$, are equivalent if there exists $D \in \mathcal{N}(E)$ such that $D \subset D_f \cap D_g$ and $f(z) = g(z)$ for every $z \in D$. An equivalence class of this relation is called *a germ of analytic functions* (or, briefly *a germ*).

If E is a non-empty open set in Ω, then if any two functions $f, g \in A(E)$ are in the same germ, we have $f \equiv g$ on E. Since E is a non-empty open set in Ω and $f - g \equiv 0$ on E, we should have $f - g \equiv 0$ on Ω. So, $f \equiv g$ on any neighbourhood of E, which implies that any germ on E consists of a unique analytic function on E.

Let us denote by $A(E)$, all analytic germs on E equipped with the inductive limit topology

$$A(E) = \lim \operatorname{ind}_{D \in \mathcal{N}(E)} A(D), \tag{1.2}$$

i.e. the finest topology on $A(E)$ for which the natural restriction mappings from $A(D)$ to $A(E)$, where $D \in \mathcal{N}(E)$, are continuous. Then, $A(E)$ is also a locally convex space.

If K is a compact set in Ω, then we can represent $A(K)$ as the countable inductive limit

$$A(K) = \lim \operatorname{ind}_{n \to \infty} A(D_n)$$

where D_n is any countable basis of $\mathcal{N}(K)$. Without loss of generality, we may choose D_n such that D_{n+1} is relatively compact in D_n for every n and no D_n contains a connected component disjoint from K. So, in this setting, $x_n \to x$ in $A(K)$ if there exists a neighbourhood $D \in \mathcal{N}(K)$ such that $x_n \in A(D)$ for every n, $x \in A(D)$ and (x_n) converges uniformly to x on any compact subset of D.

Let K be a compact set in Ω and $J : A(K) \to C(K)$ be the natural restriction homomorphism. We denote by $AC(K)$, the Banach space obtained from the completion of $J(A(K))$ in $C(K)$ with respect to the norm (1.1).

CHAPTER 2

ON DUALITY

2.1. Analytic Functionals

Let Ω be a Stein manifold. An element of the dual space $A(\Omega)^*$ of $A(\Omega)$, i.e. the space of all linear continuous functionals on $A(\Omega)$, is called *an analytic functional.* If E is an arbitrary subset of a connected Stein manifold Ω, then

$$j^* : A(E)^* \to A(\Omega)^* \tag{2.1}$$

which maps any $x^* \in A(E)^*$ to its restriction on $A(\Omega)^*$, is linear and continuous.

If E is a Runge set in Ω, i.e. $A(\Omega)$ is dense in $A(E)$, then j^* becomes a dense imbedding. This result is obtained from the following proposition:

Proposition 2.1.1 *Let X_0 and X_1 be separable locally convex spaces with a linear continuous dense imbedding $j : X_1 \to X_0$ where X_1 is reflexive. Then, the adjoint operator $j^* : X_0^* \to X_1^*$, where $j^*(x^*) = x^* \circ j$ for every $x^* \in X_0^*$, is also a linear continuous dense imbedding.*

Proof: Linearity and continuity of j^* follows simply from the definition of j^*.

To show that j^* is injective, let $j^*(x_1^*) = j^*(x_2^*)$ where $x_1^*, x_2^* \in X_0^*$. Then $x_1^* \circ j = x_2^* \circ j$ and, since j is a dense imbedding, we have $x_1^* = x_2^*$ on the dense image set $j(X_1) \subset X_0$. Any continuous function is determined by its values on a dense subset, so $x_1^* = x_2^*$ on X_0. Thus, j^* is injective.

To show that j^* is a dense mapping, we need to use the reflexivity of X_1. Since X_1 is reflexive, there exists an isometric isomorphism

$$\lambda : X_1 \to X_1^{**}$$
$$\lambda(x)(u) = u(x), \quad u \in X_1^*$$

and $\lambda(X_1) = X_1^{**}$. Now, fix $v \in X_1^{**}$ such that $v(y^*) = 0$ for every $y^* \in j^*(X_0^*)$. Since X_1 is reflexive, there exists $x \in X_1$ such that $\lambda(x) = v$. So, for all $y^* \in j^*(X_0^*)$, $v(y^*) = 0$ implies $\lambda(x)(y^*) = 0$, which implies $y^*(x) = 0$. Hence $x^*(j(x)) = 0$ for every $x^* \in X_0^*$. Since X_0^* separates points of X_0, we have $j(x) = 0$, which implies that $x = 0$ since j is injective. This implies that $v = \lambda(x) = 0$.

So, we have shown that for any $v \in X_1^{**}$ such that $v(y^*) = 0$ for all $y^* \in L$, we get $v \equiv 0$. Hence, by a corollary to Hahn-Banach theorem, L must be a dense subset of X_1^*. Thus, j^* is a dense imbedding. □

For a Runge set $E \subset \Omega$, let $A'(E)$ denote the image of $A(E)^*$ under j^* as in (2.1). Then, for Runge subsets E, F in Ω where $E \subset F$ and $A(F)$ is dense in $A(E)$, we have the natural imbeddings $A'(F) \subset A'(E) \subset A'(\Omega)$.

2.2. GKS-Duality

The result of Grothendieck, Köthe and Silva, called GKS-duality, which we will state now, supplies us with the machinery in one-dimensional case so that for any set $E \in \overline{\mathbb{C}}$, we can realize the dual space $A(E)^*$ as the space of analytic functions $A(E^*)$, where $E^* = \overline{\mathbb{C}} \backslash E$, with the assumption that all germs of $A(E)$ are equal to zero at the point ∞ if $\infty \in E$.

Theorem 2.2.1 *For any set $E \subset \overline{\mathbb{C}}$ there exists an isomorphism*

$$\gamma : A(E)^* \to A(E^*)$$

such that

$$x^*(x) = \int_\Gamma x'(\zeta)x(\zeta)d\zeta, \quad x \in A(E),$$

where $x' = \gamma(x^*)$ and $\Gamma = \Gamma(x, x')$ is a rectifiable contour separating the singularities of the analytic germs x and x^*.

However, in several complex variables, there is no similar universal representation of $A(E)^*$ as a space of analytic functions. Here is a simple special case that will be necessary for us:

Proposition 2.2.2 *Let U_r be a polydisc in $\mathbb{C}^n$ around zero with the polyradius $r = (r_1, \ldots, r_n)$ and define U_r^* as*

$$U_r^* := \{z = (z_\nu) \in \bar{\mathbb{C}}^n : |z_\nu| > r_\nu, \quad \nu = 1, \ldots, n\}$$

Then there exists a natural isomorphism $J : A(U_r)^* \to A(U_r^*)$ such that for $x' = J(x^*)$, we have

$$x^*(x) = \int_{\Gamma_\lambda} x(\zeta)x'(\zeta)\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_\mathrm{n},$$

where

$$\Gamma_\lambda = \Gamma_\lambda(x^*) = \{\zeta = (\zeta_\nu) \in \bar{\mathbb{C}}^n : |\zeta_\nu| = \lambda r_\nu, \quad \nu = 1, \cdots, n\}, \quad \lambda = \lambda(x^*) < 1$$

Proof: From the Cauchy integral formula, we know that any $x \in A(U_r)$ can be represented as the integral

$$x(z) = \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{x(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_\mathrm{n} \tag{2.2}$$

where $\Gamma_\lambda = \{\zeta = (\zeta_\nu) \in \mathbb{C}^n : |\zeta_\nu| = \lambda r_\nu, \nu = 1, \ldots, n\}$ for $0 < \lambda < 1$ and $z = (z_1, \ldots, z_n) \in A(U_r)$. Let

$$u_\zeta(z) := \left(\frac{1}{2\pi i}\right)^n \frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}$$

Then, for any $\zeta \in \Gamma_\lambda$, we have $u_\zeta \in A(U_{\lambda r})$, where $U_{\lambda r} = \lambda U_r = \{z = (z_\nu) \in \bar{\mathbb{C}}^n : |z_\nu| < \lambda r_\nu, \quad \nu = 1, \ldots, n\}$. Also , if we denote by $v_z(\zeta) := u_\zeta(z)$, a function of ζ where $z \in U_{\lambda r}$ is fixed, then $v_z(\zeta) \in A(U_{\lambda r}^*)$.

Now, let $x^* \in A(U_r)^*$. Then there exist $\gamma = \gamma(x^*)$ in $(0, 1)$ and a positive constant $C = C(x^*)$ so that $|x^*(x)| \leq |x|_{U_{\gamma r}}$ for any $x \in A(U_r)$.

Let us take $\lambda = \lambda(x^*)$ such that $\gamma < \lambda < 1$. Then

$$\begin{aligned} x(z) &= \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_\lambda} \frac{x(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_\mathrm{n} \\ &= \int_{\Gamma_\lambda} x(\zeta_1, \ldots, \zeta_n) u_\zeta(z_1, \ldots, z_n) \mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_\mathrm{n} \end{aligned}$$

Now, if we take a sequence of partitions $\{\Delta_j^{(m)}\}_{j=1}^m$ of Γ_λ with $\zeta_j^{(m)} \in \Delta_j^{(m)}$ and $\mu_j^{(m)}$ is the measure of $\Delta_j^{(m)}$ for every $j = 1, \ldots, m$ such that $\delta^{(m)} := \max_j\{\mu_j^{(m)}\} \to 0$ as $m \to \infty$, then, by the continuity of $x(\zeta)u_\zeta(z)$, for any fixed z, we have

$$x(z) = \lim_{m\to\infty} \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)}$$

Let $\epsilon > 0$, then by the additivity of integral we have

$$\int_{\Gamma_\lambda} x(\zeta) u_\zeta(z) \mathrm{d}\zeta = \sum_{\mathrm{j}=1}^{\mathrm{m}} \int_{\Delta_\mathrm{j}^{(\mathrm{m})}} \mathrm{x}(\zeta)\mathrm{u}_\zeta(\mathrm{z})\mathrm{d}\zeta,$$

and

$$\begin{aligned} &\left| \int_{\Gamma_\lambda} x(\zeta) u_\zeta(z) d\zeta - \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)} \right| \\ = &\left| \sum_{j=1}^m \int_{\Delta_j^{(m)}} x(\zeta) u_\zeta(z) \mathrm{d}\zeta - \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{x}(\zeta_\mathrm{j}^{(\mathrm{m})}) \mathrm{u}_{\zeta_\mathrm{j}^{(\mathrm{m})}}(\mathrm{z}) \mu_\mathrm{j}^{(\mathrm{m})} \right| \\ \leq &\sum_{j=1}^m \left| \int_{\Delta_j^{(m)}} \left(x(\zeta) u_\zeta(z) - x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \right) \mathrm{d}\zeta \right| \end{aligned}$$

Since $v_\zeta(z)$ is uniformly continuous on $\overline{U}_{\gamma r}$, for the given ϵ, there exists $\delta > 0$ such that $\left|v_{\zeta^{(1)}}(z) - v_{\zeta^{(2)}}(z)\right| < \epsilon$ whenever $\left|\zeta^{(1)} - \zeta^{(2)}\right| < \delta$. So, if we choose a partition $\Delta^{(m_0)}$ such that $\delta^{(m_0)} < \delta$, then we have

$$\sum_{j=1}^m \left| \int_{\Delta_j^{(m)}} \left(x(\zeta) u_\zeta(z) - x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \right) \mathrm{d}\zeta \right| < \epsilon \sum_{j=1}^m \mu_j^{(m)}$$

whenever $m \geq m_0$. Hence, the partial sums converge to $x(z)$ uniformly on $A(\overline{U}_{\gamma r})$. Thus, if we apply x^* to both sides, we get

$$\begin{aligned} x^*(x) &= x^*(\lim_{m\to\infty} \sum_{j=1}^m x(\zeta_j^{(m)}) u_{\zeta_j^{(m)}}(z) \mu_j^{(m)}) \\ &= \lim_{m\to\infty} \sum_{j=1}^m x(\zeta_j^{(m)}) x^*(u_{\zeta_j^{(m)}}(z)) \mu_j^{(m)} \end{aligned}$$

By using the similar arguments as above, we can show that this limit tends to $\int_{\Gamma_\lambda} x(\zeta)x'(\zeta)\mathrm{d}\zeta$, where $x'(\zeta) = x^*(u_\zeta)$. Thus, $x^*(x) = \int_{\Gamma_\lambda} x(\zeta)x'(\zeta)\mathrm{d}\zeta$.

To show that x' is analytic on $U^*_{\lambda r}$, let us look at the partial derivatives

$$\frac{\partial}{\partial \zeta_\nu} u_\zeta(z) = \lim_{h\to 0} \frac{u_{\zeta+he_\nu}(z) - u_\zeta(z)}{h}, \quad \zeta \in U^*_{\lambda r} \tag{2.3}$$

where e_ν are the unit vectors in the respective coordinates ζ_ν, $\nu = 1, \ldots, n$. Since we have uniform continuity on $\overline{U}_{\gamma r}$,

$$\begin{aligned} x^*\left(\lim_{h\to 0} \frac{u_{\zeta+he_\nu(z)} - u_\zeta(z)}{h}\right) &= \lim_{h\to 0} x^*\left(\frac{u_{\zeta+he_\nu} - u_\zeta}{h}\right) \\ &= \lim_{h\to 0} \frac{x^*(u_{\zeta+he_\nu}) - x^*(u_\zeta)}{h} \\ &= \frac{\partial}{\partial \zeta_\nu}(x^*(u_\zeta)) \\ &= \frac{\partial}{\partial \zeta_\nu}(x'(\zeta)) \end{aligned}$$

Hence, x' is analytic in each variable ζ_ν, and so x' is analytic on $U^*_{\lambda r}$. □

2.3. Dual Form Of Cartan Theorem

Let M be a closed analytic submanifold of a Stein manifold Ω. Then, according to Cartan theorem, the operator

$$R : A(\Omega) \to A(M) : Rx = x|_M, \quad x \in A(\Omega)$$

is a surjection. If we consider the adjoint operator $R^* : A(M)^* \to A(\Omega)^*$ which maps any functional $\phi \in A(M)^*$ to $\psi = \phi \circ R \in A(\Omega)^*$, we get a dual form of Cartan theorem:

Proposition 2.3.1 *The adjoint operator $R^* : A(M)^* \to A(\Omega)^*$ of the restriction operator $R : A(\Omega) \to A(M)$ is an isomorphic imbedding.*

The proof can be found in [7].

CHAPTER 3

SOME TOPICS OF PLURIPOTENTIAL THEORY

3.1. Maximal plurisubharmonic functions

Let $PSH(D)$ denote the class of all plurisubharmonic functions in a domain D. For a detailed information on plurisubharmonic functions and pluripotential theory, one can refer to [9].

Definition 3.1.1 *A function $u \in PSH(D)$ is called* maximal *in D if for any relatively compact open subset G of D and any function $v \in PSH(D)$, upper semicontinuous on $\overline{G}$ such that $v \leq u$ on ∂G, we have $v \leq u$ in G.*

Let us denote the class of all maximal plurisubharmonic functions on D by $MPSH(D)$. Due to Bremermann ([6]), we know that if $u \in MPSH(D) \cap C^2(D)$, then u is maximal if and only if

$$(dd^c u)^n = 0, \tag{3.1}$$

where $d = \partial + \overline{\partial},\ d^c = i(\overline{\partial} - \partial)$. The equation (3.1) is called the *homogeneous Monge-Ampère equation.*

Bedford and Taylor, in [3] and [4], had shown that the Monge-Ampère operator

$$u \to (dd^c u)^n$$

can be defined as a continuous operator from the space $L^\infty_{loc}(D) \cap PSH(D)$ equipped with the topology induced by $L^\infty_{loc}(D)$, to the space $M(D)$ of non-negative Borel measures on D equipped with the weak convergence topology, by continuously extending the corresponding differential operator acting on $PSH(D) \cap C^2(D)$. In light of these results, maximal plurisubharmonic functions can be characterized as follows:

Proposition 3.1.1 *A function $u \in PSH(D) \cap L^\infty(D)$ is maximal if and only if u satisfies the homogeneous Monge-Ampère equation, $(dd^c u)^n = 0$, in the generalized sense.*

3.2. Green Pluripotential

Let E be a subset of a Stein manifold Ω. The following extremal function is called the *Green pluripotential* (or MP-measure):

$$\omega(z) = \omega(\Omega, E, z) = \limsup_{\zeta \to z} \omega^0(\Omega, E, z), \quad z \in \Omega, \tag{3.2}$$

where

$$\omega^0(z) = \omega^0(\Omega, E, z) = \sup\{u(z) : U \in P(E, \Omega)\},$$

$$P(E, \Omega) = \{u \in PSH(\Omega) : u|_E \leq 0, \quad u(z) < 1, \quad z \in \Omega\}.$$

$\omega(z)$ is an analogue to the general solution of the Dirichlet problem for the class of harmonic functions in the domain $\Omega = \Omega_1 \backslash \Omega_0 \subset \mathbb{R}^n$ with the boundary conditions $u|_{\partial\Omega_1} \equiv 1, u|_{\partial\Omega_0} \equiv 0$.

Here are some elementary properties of (3.2):

1. $\omega \in PSH(\Omega)$,

2. if $\Omega_1 \subset \Omega,\ E_1 \subset E$, then $\omega(\Omega, E, z) \leq \omega(\Omega_1, E_1, z)$ for all $z \in \Omega_1$,

3. if Ω' is a connected component of Ω and $\Omega' \cap E = \emptyset$, then $\omega(\Omega, E, z) \equiv 1$ in Ω'.

Bedford and Taylor had proven that $\omega(z)$ satisfies the homogeneous Monge-Ampère equation on $\Omega\backslash E$, hence $\omega(z)$ is maximal in $\Omega\backslash E$.

With the help of $\omega(z)$, we can state an analogue of the Two Constant Theorem for analytic functions of several complex variables:

Proposition 3.2.2 *Let K be a compact subset of a Stein manifold Ω and f be a bounded analytic function in Ω. Then,*

$$|f(z)| \leq (|f|_\Omega)^{\omega(z)}(|f|_K)^{1-\omega(z)}, \quad z \in \Omega. \tag{3.3}$$

Proof: Since $f(z)$ is a bounded analytic function in Ω, $\ln|f(z)|$ is plurisubharmonic in Ω. Also, by the definiton of the norms $|f|_\Omega$ and $|f|_K$, we have $ln|f| \leq |f|_\Omega$ on Ω and $ln|f| \leq |f|_K$ on K. Hence we can consider the function

$$\frac{\ln|f(z)| - \ln|f|_K}{\ln|f|_\Omega - \ln|f|_K}$$

which is plurisubharmonic in $\Omega\backslash K$ and

$$\frac{\ln|f(z)| - \ln|f|_K}{\ln|f|_\Omega - \ln|f|_K} \leq \omega(z),$$

by the maximality of $\omega(z)$. Thus we have,

$$\ln|f(z)| \leq \omega(z)\ln|f|_\Omega + (1-\omega(z))\ln|f|_K$$

which implies

$$|f(z)| \leq |f|_\Omega^{\omega(z)}|f|_K^{(1-\omega(z))}.$$

□

3.3. Pluriregularity

Definition 3.3.1 *A Stein manifold Ω is called* pluriregular *(or* strongly pseudoconvex*) if there exists a plurisubharmonic function $u \in PSH(\Omega)$ such that $u(z) < 0$ for every $z \in \Omega$ and $u(z_j) \to 0$ for every sequence $\{z_j\} \subset \Omega$ without limit points in Ω.*

Definition 3.3.2 *A compact set K in a Stein manifold Ω is called* pluriregular *on Ω if $\omega(D, K, z) = 0$ on K for some open neighbourhood D relatively compact in Ω. K is called* strongly pluriregular *on Ω if for any open neighbourhood D of K, relatively compact in Ω, we have $\omega(\widetilde{D}, K, z) \equiv 0$ on K, where $\widetilde{D}$ is the envelope of holomorphy of D.*

Pluriregularity of a compact set K in Ω is equivalent to the pluriregularity of its envelope of holomorphy $\widehat{K}_\Omega$. Also, K is strongly pluriregular if and only if for every pseudoconvex covering domain Ω_1 over Ω such that $\Omega_1 \supset K$, the envelope of holomorphy $\widehat{K}_{\Omega_1}$ is pluriregular on Ω_1. For the definition of envelope of holomorphy and some of its properties, one can look at [8], chapter 5.4.

Definition 3.3.3 *A pair (K, Ω) is called pluriregular if K is a pluriregular holomorphically convex compact set on the pluriregular Stein manifold Ω and every connected component of Ω has a nonempty intersection with K.*

Given a pluriregular pair (K, Ω), the Green pluripotential $\omega(z)$ has the following useful properties:

Theorem 3.3.1 *If (K, Ω) is a pluriregular pair, then $\omega(z) = \omega(\Omega, K, z)$ is continuous in Ω and satisfies*

(i) $\omega(z) = 0, \quad z \in K,$

(ii) $0 < \omega(z) < 1, \quad z \in \Omega \backslash K,$

(iii) $\lim_{z \to \partial\Omega} \omega(z) = 1.$

The proof can be found in [16].

A set $\Omega \in \mathbb{C}^n$ is called *circled* if for every $a = (a_1, \cdots, a_n) \in \Omega$, the torus

$$\{z \in \mathbb{C}^n \ : \ z = (a_1 e^{i\theta_1}, \cdots, a_n e^{i\theta_n}, \quad 0 \leq \theta_j \leq 2\pi\}$$

lies in Ω. An open circled set in $\mathbb{C}^n$ is called a *circular* (*Reinhardt*) domain. A circular domain Ω is called *complete* if for every $a \in \Omega$, we have

$$\{z \in \mathbb{C}^n \ : \ |z_j| < |a_j|, \quad j = 1, \cdots, n\} \subset \Omega.$$

Let D be a complete circular domain in $\mathbb{C}^n$ and let us define the function

$$\psi_D(z) := \ln \inf\{t > 0 : \frac{z}{t} \in D\}, \quad z \in \mathbb{C}^n \tag{3.4}$$

$\psi_D(z)$ is upper semicontinuous in $\mathbb{C}^n$ and it is known that it characterizes the pseudoconvexity of D. So, a complete circular domain D is pluriregular if and only if $\psi_D(z)$ is plurisubharmonic in $\mathbb{C}^n$ and continuous in $\mathbb{C}^n\backslash\{0\}$.

CHAPTER 4

INTERPOLATION PROPERTIES OF SPACES OF ANALYTIC FUNCTIONS

4.1. Interpolation Estimates of the Norms of Analytic Functionals

In this section, we consider the two constant theorem in the case of analytic functionals. In the implicit form, it was considered in [16] as a result about Hilbert scales of analytic functions. Here, we give a proof based on the dual version of Cartan Theorem (theorem 2.3.1), without using Hilbert space techniques.

Theorem 4.1.1 *Let (K, D) be a pluriregular pair on a Stein Manifold Ω, where D is a strongly pluriregular open set on Ω. Then, for any $\epsilon > 0$ and $\alpha \in (0, 1)$, there exists a constant $C = C(\alpha, \epsilon)$ such that for any $x^* \in AC(K)^*$ the following estimate holds*

$$|x^*|^*_{D_\alpha} \leq C(|x^*|^*_K)^{1-\alpha+\epsilon}(|x^*|^*_D)^{\alpha-\epsilon},$$

where

$$D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\}, \quad 0 < \alpha < 1 \tag{4.1}$$

Let us first consider a special case when D and K are analytic polyhedrons in Ω defined by the same collection of analytic functions. We have a one-parameter family of polydiscs in $\mathbb{C}^N$:

$$V_\alpha = \{z = (z_\nu) \in \mathbb{C}^N : |z_\nu| < r_\nu(\alpha), \quad \nu = 1, \ldots, N\}$$

where

$$r_\nu(\alpha) = R_\nu^\alpha r_\nu^{1-\alpha}, \quad r_\nu < R_\nu, \quad \nu = 1, \dots, N, \quad \alpha \in \mathbb{R}$$

For $\alpha \in [0,1]$, let $\Delta_\alpha := V_\alpha \cap \Omega$ and $\tilde{\Delta}_\alpha := \overline{V}_\alpha \cap \Omega$ be two families of analytic polyhedrons in Ω and let Φ_α and $\tilde{\Phi}_\alpha$ be polyhedrons that are the connected components of Δ_α and $\tilde{\Delta}_\alpha$ respectively, which have nonempty intersections with $\tilde{\Phi}_0 = \tilde{\Delta}_0$. Then, we have the following:

Proposition 4.1.2 *For every $\epsilon > 0$ and $\alpha \in (0,1)$, there exists a constant $C = C(\alpha, \epsilon)$ such that*

$$|x^*|^*_{\tilde{\Phi}_\alpha} \le C \left(|x^*|^*_{\tilde{\Phi}_0} \right)^{1-\alpha+\epsilon} \left(|x^*|^*_{\tilde{\Phi}_1} \right)^{\alpha-\epsilon}, \quad x^* \in AC(\tilde{\Phi}_0)^* \tag{4.2}$$

Proof: We know that all polyhedrons Φ_α are closed submanifolds of polyhedrons V_α, and so are $\tilde{\Phi}_\alpha$, since $\tilde{\Phi}_\alpha = \bigcap_{\beta > \alpha} \Phi_\beta$. So, for $\alpha \in (0,1)$, we have

$$\tilde{\Phi}_0 \subset \Phi_\alpha \subset \tilde{\Phi}_\alpha \subset \Phi_1,$$

which implies that

$$A(\tilde{\Phi}_0) \supset A(\Phi_\alpha) \supset A(\tilde{\Phi}_\alpha) \supset A(\Phi_1).$$

Hence, taking the duals, we have the inclusions

$$A(\tilde{\Phi}_0)^* \subset A(\Phi_\alpha)^* \subset A(\tilde{\Phi}_\alpha)^* \subset A(\Phi_1)^*.$$

Thus, we get the natural imbeddings

$$A(\tilde{\Phi}_0)^* \to A(\Phi_\alpha)^* \to A(\tilde{\Phi}_\alpha)^* \to A(\Phi_1)^*.$$

Similarly, we obtain the natural imbeddings

$$A(\overline{V}_0)^* \to A(V_\alpha)^* \to A(\overline{V}_\alpha)^* \to A(V_1)^*.$$

Also, if we consider the polydiscs

$$V_\alpha^* = \{z = (z_\nu) \in \mathbb{C}^N : |z_\nu| > r_\nu(\alpha), \quad \nu = 1, \dots, N\},$$

then we have the inclusions $\overline{V}_0^* \supset V_\alpha^* \supset \overline{V}_\alpha^* \supset V_1^*$. Hence, we have the natural imbeddings

$$A(\overline{V}_0^*) \to A(V_\alpha^*) \to A(\overline{V}_\alpha^*) \to A(V_1^*).$$

Now, according to the dual form of Cartan theorem (theorem 2.3.1), for any $\alpha \in [0,1]$, we get the isomorphic imbeddings $T_\alpha : A(\Phi_\alpha)^* \to A(V_\alpha)^*$ and $\tilde{T}_\alpha : A(\tilde{\Phi}_\alpha)^* \to A(\overline{V}_\alpha)^*$. Also, by theorem 2.2.2, we have the isomorphisms $S_\alpha : A(V_\alpha)^* \to A(V_\alpha^*)$ and $\tilde{S}_\alpha : A(\overline{V}_\alpha)^* \to A(\overline{V}_\alpha^*)$. Hence, we obtain the following diagram:

$$\begin{array}{ccccc} A(\Phi_1)^* & \to & A(V_1)^* & \to & A(V_1^*) \\ \uparrow & & \uparrow & & \uparrow \\ A(\tilde{\Phi}_\alpha)^* & \to & A(\overline{V}_\alpha)^* & \to & A(\overline{V}_\alpha^*) \\ \uparrow & & \uparrow & & \uparrow \\ A(\Phi_\alpha)^* & \to & A(V_\alpha)^* & \to & A(V_\alpha^*) \\ \uparrow & & \uparrow & & \uparrow \\ A(\tilde{\Phi}_0)^* & \to & A(\overline{V}_0)^* & \to & A(\overline{V}_0^*) \end{array}$$

So, if we apply the two constant theorem (theorem 3.2.2) for any given $x \in A(V_\alpha^*)$, $0 < \alpha < 1$, then we have

$$|x|_{V_\alpha^*} \leq \left(|x|_{V_0^*}\right)^{1-\alpha} \left(|x|_{V_1^*}\right)^{\alpha}.$$

Furthermore, since we have the continuous imbeddings as shown in the diagram, given $\epsilon > 0$ and $0 < \alpha < 1$, there exists a constant $C = C(\alpha, \epsilon)$ such that

$$|x^*|^*_{\tilde{\Phi}_\alpha} \leq C(\alpha,\epsilon)|x|_{V^*_{\alpha-\epsilon}}.$$

So, combining these two inequalities, for any $x^* \in A(\tilde{\Phi}_\alpha^*)$, we get

$$|x^*|^*_{\tilde{\Phi}_\alpha} \leq C\left(|x^*|^*_{\tilde{\Phi}_0}\right)^{1-\alpha+\epsilon}\left(|x^*|^*_{\tilde{\Phi}_1}\right)^{\alpha-\epsilon}.$$

□

The main case can be obtained with the help of Lelong-Bremermann theorem about the uniform approximation of an arbitrary continuous plurisubharmonic function, which can be found in [5], by constructing the following plurisubharmonic function:

$$v(z) = \max\{\alpha_j \ln|f_j(z)| \ : \ j = 1, \ldots, n\},$$

where $\alpha_j > 0$ and $f_j \in A(\Omega)$, and considering the fact that level domains of $v(z)$ are analytic polyhedrons that simultaneously approximate the corresponding level domains D_α. Plurisubharmonicity of $v(z)$ is shown in [16].

4.2. Hilbert Scales

Hilbert scales are a special class of analytic scales, so let us start by introducing the notion of analytic scale of spaces, as done in [10]. Let M be a normed space in which a family of linear operators $T(z)$ acts satisfying the following conditions:

(i) $T(z)x$ is an entire function of the complex variable z.

(ii) $\|T(z)x\|_M$ is a bounded function on every straight line parallel to the imaginary axis.

(iii) $T(0)x = x$.

(iv) $\sup_{\mu,\nu} \|T(\alpha + i\mu)T(\beta + i\nu)x\| \leq \sup_\tau \|T(\alpha + \beta + i\tau)x\|_M$.

(v) $T(i\mu)\frac{T(z+\Delta z)x - T(z)x}{\Delta z} \to T(i\mu)\,(T(z)x)'$ uniformly in μ as $\Delta z \to 0$.

The family E_α, $-\infty < \alpha < \infty$, of Banach spaces where E_α are the completions of M with respect to the respective norms

$$\|x\|_\alpha = \sup_{-\infty<\tau<\infty} \|T(\alpha + i\tau)x\|_M$$

is called an *analytic scale of spaces.*

Now, let H_0 be a complex Hilbert space and let j be an unbounded positive definite selfadjoint operator in H_0 with a domain D_j such that

$$\|x\|_{H_0} \leq \|jx\|_{H_0} \quad x \in D_j.$$

Let us denote by E_λ the spectral resolution of the identity corresponding to j, and consider the set M of all elements representable in the form $x = \int_1^N \lambda \mathrm{dE}_\lambda \mathrm{x}$ for some $N < \infty$. This set M is dense in H_0 and on this set we can define the operators $T(z) = j^z$ where

$$j^z x = \int_1^N \lambda^z \mathrm{dE}_\lambda \mathrm{x}.$$

This family of operators satisfies the conditions necessary to construct an analytic scale from them. So, on M, we introduce the norms

$$\|x\|_\alpha = \sup_{-\infty<\tau<\infty} \|j^{\alpha+i\tau}x\|_{H_0} = \|j^\alpha x\|_{H_0}.$$

The completions H_α of M with respect to these norms are Hilbert Spaces and they form an analytic scale which is called a *Hilbert scale*.

Now let $H^\alpha = (H_0)^{1-\alpha}(H_1)^\alpha$, $\alpha \in \mathbb{R}$, be a Hilbert scale generated by a pair $H_1 \subset H_0$ of Hilbert spaces with continuous imbedding. If this imbedding is compact, i.e. every set bounded in the norm of H_1 is relatively compact in H_0, then we have the following proposition:

Proposition 4.2.1 *Let $H_1 \hookrightarrow H_0$ be a linear dense compact imbedding, then there exists a common orthogonal basis $\{e_k\}$ in every space H^α such that $\|e_k\|_{H_0} = 1$ and $\mu_k := \|e_k\|_{H_1} \nearrow \infty$.*

Proof: Given the pair of Hilbert spaces H_0 and H_1, let us consider the restriction operator $R : H_1 \to H_0$, which is a linear dense compact imbedding. Then the adjoint operator $R^* : H_0 \to H_1$ can be defined as

$$\langle Rx, y\rangle_{H_0} = \langle x, R^* y\rangle_{H_1} \quad x \in H_1, \quad y \in H_0.$$

Now, let $S := R^* R$, then S is a self adjoint operator and given $x, y \in H_1$, we have

$$\begin{aligned}
\langle x, y\rangle_{H_0} &= \langle Rx, Ry\rangle_{H_0} && \text{by the definition of } R,\\
&= \langle x, R^* Ry\rangle_{H_1} \\
&= \langle x, Sy\rangle_{H_1} && \text{by the definition of } S,\\
&= \langle Sx, y\rangle_{H_1} && \text{since } S \text{ is selfadjoint.}
\end{aligned}$$

From the above equality we get that $\langle Sx, x\rangle_{H_1} = 0$ if and only if $x = 0$. Hence, S is strictly positively defined. Also, S is compact since it is the composition of a continuous and a compact operator. Therefore, there exists a complete orthonormal sequence of eigenvectors $\{g_k\}$ with respective strictly positive eigenvalues λ_k, where $\lambda_k \to 0$. Without loss of generality we can take $\lambda_k \searrow 0$. Then, for any m, n we have

$$\langle g_m, g_n\rangle_{H_0} = \langle S g_m, g_n\rangle_{H_1} = \langle \lambda_m g_m, g_n\rangle_{H_1} = \lambda_m \langle g_m, g_n\rangle_{H_1} = \lambda_m \delta_{m,n}$$

Hence, we have $\|g_k\|_{H_0} = \sqrt{\lambda_k}$ and $\|g_k\|_{H_1} = 1$ where $\{g_k\}$is a common orthogonal basis in H_0 and H_1. Setting $e_k = \frac{1}{\lambda_k} g_k$, we get the common orthogonal basis with the properties $\|e_k\|_{H_0} = 1$ and $\mu_k := \|e_k\|_{H_1} \nearrow \infty$, where $\mu_k = \frac{1}{\sqrt{\lambda_k}}$.

Using this basis, the Hilbert scale is determined by the norms:

$$\|x\|_{H_\alpha} = \left(\sum_{k=1}^{\infty} |\xi_k|^2 \mu_k^{2\alpha} \right)^{\frac{1}{2}}, \qquad x = \sum_{k=1}^{\infty} \xi_k e_k.$$

□

Theorem 4.2.2 *Let D be a strongly pluriregular open set on the Stein manifold Ω and a compact set $K \subset D$ be pluriregular on D. Given a pair of Hilbert spaces H_0, H_1 with the continuous imbeddings*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K),$$

$$A(\overline{D}) \hookrightarrow H_1 \hookrightarrow A(D),$$

the following continuous imbeddings hold:

$$A(K_\alpha) \subset H^\alpha \subset A(D_\alpha), \qquad 0 < \alpha < 1.$$

Sketch of proof: Using the common orthogonal basis from the previous proposition, $\{e_k\} \subset H_1 \subset A(D)$, and we obtain the estimate

$$|e_k|_{D_\alpha} \le C(\alpha, \epsilon) \mu_k^{\alpha+\epsilon}$$

by using the projective limit topology properties of the spaces considered and applying the two constant theorem for analytic functions. Also, applying the interpolational estimates as in theorem 4.1.1, we get

$$|e_k^*|_{D_\alpha}^* \le C(\alpha, \epsilon) \mu_k^{-\alpha+\epsilon}.$$

Furthermore, we use the fact that the imbedding $H_1 \subset H_0$ is nuclear and so for any $\delta > 0$ we get

$$\sum_{k=1}^{\infty} \mu_k^{-\delta} < \infty$$

Thus we can obtain the norm estimates which provides us with the continuous imbeddings $A(K_\alpha) \subset H^\alpha \subset A(D_\alpha)$.

The following generalization of the previous theorem can be stated by means of Vogt's idea, which will be presented in the next section. This statement was also proved in [16] under the assumption of existence of a basis in $A(D)$:

Theorem 4.2.3 *Let (K, Ω) be a pluriregular pair. Then there exists a Hilbert space $H_1 \subset A(\Omega)$ such that for any Hilbert space H_0 satisfying*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K),$$

the following continuous imbeddings hold:

$$A(K_\alpha) \subset H^\alpha \subset A(D_\alpha), \qquad 0 < \alpha < 1.$$

The proof can be found in [17].

The conditions on H_0 can also be weakened by considering the H-space $AL^2(E, \mu)$, which is the completion of the set $A(E)$ in the space $L^2(E, \mu)$ with respect to the norm

$$\|x\| = \left(\int_E |x(z)|^2 d\mu \right)^{1/2}$$

where μ is some Borel measure on an unrestricted Borel set $E \subset \Omega$. Let us state this fact formally in the next theorem:

Theorem 4.2.4 *Let E be a relatively compact Borel set on a pluriregular Stein manifold Ω not pluripolar in any connected component of Ω, and μ be a Borel measure supported on E satisfying*

$$\int_A (dd^c \omega(\Omega, K, z))^n \leq \left(\ln \frac{1}{\mu A} \right)^{-n(1+\delta)}$$

for any Borel set $A \subset E$ with a sufficiently small measure $\mu A \leq \epsilon(\delta)$. Then there exists a Hilbert space $H_1 \subset A(\Omega)$ such that for any Hilbert space H_0 satisfying the continuous imbeddings

$$A(\overline{E}) \subset H_0 \subset AL^2(E, \mu)$$

the following continuous imbeddings hold:

$$A(E_\alpha) \subset H^\alpha \subset A(\Omega_\alpha), \qquad 0 < \alpha < 1,$$

where

$$\Omega_\alpha = \{z \in \Omega : \omega(\Omega, E, z) < \alpha\},$$

$$E_\alpha = \{z \in \Omega : \omega^0(\Omega, \overline{E}, z) \leq \alpha\}.$$

4.3. Dragilev Classes

Let X be an F-space with topology defined by the system of norms

$$\{\|x\|_p,\ p \in \mathbb{N}\}.$$

Then, in the strong dual space X^*, we have the system of dual unbounded norms

$$\|x^*\|_p^* := \sup\{|x^*(x)| : x \in U_p\}, \qquad x^* \in X^*, \quad p \in \mathbb{N},$$

where

$$U_p := \{x \in X : \|x\|_p \leq 1\}.$$

Also, let us introduce another notation

$$U_p^0 := \{x^* \in X^* : \|x^*\|_p^* \leq 1\}, \quad p \in \mathbb{N}.$$

Definition 4.3.1 *A Fréchet space X belongs*

(i) to the class $\mathcal{D}_1$ if

$$\exists p\ \forall q\ \exists r\ \exists C \mid \|x\|_q^2 \leq C\|x\|_p\|x\|_r, \quad x \in X,$$

(ii) to the class $\mathcal{D}_2$ if

$$\forall p\ \exists q\ \forall r\ \exists C \mid \left(\|x^*\|_q^*\right)^2 \leq C\|x^*\|_p^*\|x^*\|_r^*, \quad x^* \in X^*.$$

The inequalities in conditions (i) and (ii) are equivalent to the following relations respectively:

$$U_q^0 \subset tU_p^0 + \frac{C}{t}U_r, \quad t > 0,$$
$$U_q \subset tU_r + \frac{C}{t}U_p, \quad t > 0.$$

With the help of these relations, the next theorem was proved by Vogt in [13], which was also used in the generalization of existence of a Hilbert space H_1 as in the previous section.

Theorem 4.3.1 *Let X be a Fréchet-Schwarz space. Then, $X \in \mathcal{D}_2$ if and only if there exists a bounded closed absolutely convex set $B \subset X$ such that*

$$\forall p\ \forall \mu : 0 < \mu < 1\ \exists q\ \exists c \mid U_q \subset t^\mu B + \frac{C}{t^{1-\mu}}U_p, \quad t > 0. \tag{4.3}$$

The statement was also proved in [16] under the assumption of existence of an unconditional basis in a Hilbert space (not necessarily Schwarz).

The inclusion in (4.3) can also be written in an equivalent form with the same quantifiers as

$$\|x^*\|_q^* \leq C\left(\|x^*\|^*\right)^{1-\mu}\left(\|x^*\|_p^*\right)^{1-\mu}, \quad x^* \in X^*, \tag{4.4}$$

where

$$\|x^*\|^* := \sup\{|x^*(x)| : x \in B\}, \quad x^* \in X^*.$$

Definition 4.3.2 *A Banach space E continuously imbedded in X is called a* Vogt space*, denoted by $E \in \mathcal{V}(X)$, if (4.4) holds for the norm $\|x^*\|^*$ defined above.*

Now, we can relate the Dragilev classes to the spaces of analytic functions in terms of pluripotential theoretic considerations.

Theorem 4.3.2 *Let Ω be a Stein manifold. Then, $A(\Omega) \in \mathcal{D}_2$ if and only if Ω is pluriregular.*

Sketch of proof: For sufficiency, given p, $\|x\|_p$ is defined on a compact set, say K, and (K, Ω) becomes a pluriregular pair. So, given a strongly pluriregular open set D in Ω and μ such that $0 < \mu < 1$, by theorem 4.1.1, we have the interpolational estimate where for every $\epsilon > 0$ there exists C such that

$$\|x^*\|_{D_\mu}^* \leq C\left(\|x^*\|_K^*\right)^{1-\mu+\epsilon}\left(\|x^*\|_D^*\right)^{\mu-\epsilon}.$$

So, we can apply theorem 4.3.1 to obtain that $A(\Omega) \in \mathcal{D}_2$.

In order to prove necessity, without loss of generality, we may assume that Ω is connected. We can also choose a suitable compact set $K \subset \Omega$, a Hilbert space H_0 such that $A(K) \hookrightarrow H_0 \hookrightarrow AC(K)$, and a space $H_1 \in \mathcal{V}(A(\Omega))$ such that there is a common orthogonal basis $\{e_k\}$ for H_0 and H_1 satisfying $\|e_k\|_{H_0} = 1$ and $\mu_k = \|e_k\|_{H_1} \nearrow \infty$. Then the function

$$u(z) = \overline{\lim}_{\zeta \to z} \overline{\lim}_{k \to \infty} \left(\frac{\ln |e_k(\zeta)|}{\ln \mu_k} - 1\right)$$

is a negative plurisubharmonic function such that $\lim_{z \to \partial\Omega} u(z) = 0$. This fact was proved in [16]. Hence, Ω is pluriregular.

Similarly, one can state the following theorem:

Theorem 4.3.3 *Let K be a compact set on a Stein manifold Ω. Then $A(K)^* \in \mathcal{D}_2$ if and only if K is strongly pluriregular on Ω.*

CHAPTER 5

ISOMORPHISMS OF SPACES OF ANALYTIC FUNCTIONS

5.1. Extendable Basis For a Pluriregular Pair

Theorem 5.1.1 *Let (K, D) be a pluriregular pair. Then, there exists a common basis $\{x_i(z)\}$ in the spaces $A(D)$, $A(K)$, $A(D_\alpha)$, $A(K_\alpha)$, $0 < \alpha < 1$, satisfying the asymptotic estimate*

$$\limsup_{\zeta \to z} \limsup_{i \to \infty} \frac{\ln |x_i(z)|}{a_i} = \omega(D, K, z), \quad z \in D \backslash K \tag{5.1}$$

where

$$K_\alpha = \{z \in D : \omega(D, K, z) \le \alpha\}, \quad D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\}$$

and $\{a_i\}$ is some nondecreasing sequence of positive numbers such that

$$a_i \asymp i^{\frac{1}{\dim D}}, \quad i \to \infty.$$

Let us give a sketch of the proof. We can take a common orthogonal basis $\{x_i(z)\}$ for some pair of Hilbert spaces H_0, H_1 with the continuous imbeddings

$$H_1 \hookrightarrow A(D) \hookrightarrow A(K) \hookrightarrow H_0.$$

According to a method proposed in [14], we can choose H_0 and H_1 so that $H_1 \in \mathcal{V}(A(D))$ and $H_0^* \in \mathcal{V}(A(K)^*)$. By the proposition 2.1.1, the dual space H_0^* is naturally imbedded in $A(K)^*$. The existence of such spaces follows from the theorems in section . The system $\{x_i(z)\}$ is normed and ordered by theorem 4.2.1, so if we take $a_i = \ln \mu_i(H_0, H_1)$ then the conclusion follows from the inequalities

$$|x_i|_{D_\alpha} \leq C(\alpha, \epsilon)\mu_i^{\alpha+\epsilon}, \quad |x_i^*|_{D_\alpha}^* \leq C(\alpha, \epsilon)\mu_i^{-\alpha+\epsilon}, \quad \sum_{k=1}^{\infty} \mu_k^{-\delta} < \infty.$$

How to obtain the asymptotic equality is explained in [16].

There exists simpler sufficient conditions for $H_1 \in \mathcal{V}(A(D))$ and $H_0^* \in \mathcal{V}(A(K)^*)$. For example, it is sufficient for the space H_0 to satisfy the continuous imbeddings

$$A(K) \hookrightarrow H_0 \hookrightarrow AL^2(K, \mu)$$

where μ is a Borel measure satisfying the sonditions of Theorem 4.2.4. Also, if a manifold D is strongly pluriregular on the Stein manifold Ω, then the continuity of the imbeddings

$$A(\overline{D}) \hookrightarrow H_1 \hookrightarrow A(D)$$

imply $H_1 \in \mathcal{V}(A(D))$. However, in the case of arbitrary pluriregular Stein manifolds, we have only the fact of existence of Vogt spaces.

As an application of the previous theorem, we can state the following proposition, which was considered in [14]:

Proposition 5.1.2 *Let D_0, D be pluriregular circular (Reinhardt) domains where $\overline{D}_0 \subset D$. Then, there exists a common basis in $A(D)$and $A(\overline{D}_0)$ consisting of homogeneous polynomials.*

5.2. Isomorphisms of Pluriregular Domains and Compact Sets

By means of extendable bases considered in the previous section, we obtain the following result:

Theorem 5.2.1 *Let Ω be a Stein manifold of dimension n. Then,*

$$A(\Omega) \simeq A(U^n)$$

if and only if Ω is pluriregular and consists of at most finite number of connected components.

Sketch of Proof: Let Ω be a Stein manifold of dimension n satisfying $A(\Omega) \simeq A(U^n)$. Then $A(\Omega) \in \mathcal{D}^2$ since it is isomorphic to $A(U^n)$. So, the necessity follows from the fact that $A(\Omega) \in \mathcal{D}^2$ if and only if Ω is pluriregular.

Conversely, let Ω be pluriregular, consisting of at most finite number of connected components and let K be a pluriregular compact set having a non-empty intersection with every connected component of Ω with the property $K = \hat{K}_\Omega$. Given a common basis $\{x_i(z)\}$, which exists due to theorem 5.1.1, we have the isomorphism $T : A(\Omega) \to A(U^n)$ defined by

$$x_i(z) \to (\exp a_i)\ e_i(z), \quad i \in \mathbb{N},$$

where $\{e_i\}$ is a common orthogonal basis such that

$$\|e_i\|_{H_0} = 1, \quad \mu_i := \|e_i\|_{H_1} \nearrow \infty.$$

It should be pointed out here that the result of Vogt which provides us a suitable Hilbert space H_1 was used.

Under some additional assumptions on Ω, such as strong pluriregularity or existence of a basis in $A(\Omega)$, the sufficiency was also proved in [16].

Similarly, we can obtain the following result:

Theorem 5.2.2 *Let K be a compact set on a Stein manifold Ω. Then,*

$$A(K) \simeq A(U^n)$$

if and only if K has a Runge neighbourhood in Ω and K is strongly pluriregular.

Necessity follows from the fact that $A(K)^* \in \mathcal{D}^2$ if and only if K is strongly pluriregular on Ω. For sufficiency, the isomorphism can be obtained by using the basis from Theorem 5.1.1 for the pluriregular pair $(K, \tilde{D})$, where D is some Runge neighbourhood of K such that its envelope of holomorphy $\tilde{D}$ is pluriregular.

Here is a corollary of Theorem 5.2.1:

Proposition 5.2.3 *Let Ω be a complete pseudoconvex circular domain in $\mathbb{C}^n$ and*

$$\psi_\Omega(z) := \ln \inf\{t > 0 : \frac{z}{t} \in \Omega\}, \quad z \in \mathbb{C}^n.$$

Then, $A(\Omega) \simeq A(U^n)$ if and only if $\psi_\Omega(z)$ is continuous in $\mathbb{C}^n$.

Bibliography

[1] A. Aytuna, *On the linear topological structures of spaces of analytic functions*, Doğa, Turkish J. Math., vol.10, 46-49, 1986

[2] A. Aytuna, *Stein spaces M for which $\mathcal{O}(M)$ is isomorphic to a power series spaces*, Advances in the Theory of Fréchet Spaces, Kluwer Academic Publishers, Dordrecht-Boston-London, 115-154, 1988

[3] E. Bedford, B.A. Taylor, *The Dirichlet complex Monge-Ampère equation*, Invent. Math., vol.37, 1-4, 1976

[4] E. Bedford, B.A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math., vol.149, 1-40, 1982

[5] M.J. Bremermann, *On the conjecture of the equivalence of the plurisubharmonic functions and Hartogs functions*, Math. Ann., vol.131, 76-86, 1956

[6] M.J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and convex domains*, Trans. Amer. Math. Soc., vol.91, 246-276, 1959

[7] G.M. Henkin, B.S. Mitiagin, *Linear problems of complex analysis*, Russian Math. Surveys, vol.26, 99-164, 1971

[8] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North Holland Publishing Company, 1973

[9] M. Klimek, *Pluripotential Theory*, Clarendon Press, 1991

[10] S.G. Krein, Ju.I. Petunin, E.M. Semenov, *Interpolation of Linear Operators*, American Mathematical Society, Providence, Rhode Island, (1982)

[11] R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Graduate Texts in Mathematics, Springer, 1986

[12] W. Rudin, *Functional Analysis, Second Edition*, McGraw-Hill, Inc., 1991

[13] D. Vogt, *Charakterisierung deer Potenzreihenräume von endlichen Typ und ihre Folgerungen*, Manuscr. Math., vol.37 N3, 269-301, 1982

[14] V.P. Zahariuta, *On extandable bases in spaces of analytic functions of one and several variables*, Siberian Math. J., vol.8, 204-216, 1967

[15] V.P. Zahariuta, *Isomorphism of spaces of holomorphic functions of several complex variables*, Func. and Anal. and Appl., vol.5, 326-328, 1972

[16] V.P. Zahariuta, *Extremal plurisubharmonic functions, Hilbert scales, and the isomorphim of spaces of analytic functions of several variables I and II*, Theory of functions, functional analysis and its applications, Kharkov, vol.19, 133-157, 1974; vol.21, 65-83, 1974 (Russian)

[17] V.P. Zahariuta, *Spaces of analytic functions and maximal plurisubharmonic functions*, D. Sc. Dissertation, Rostov-on-Don, 1984 (Russian)