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A higher-order Boussinesq equation in locally non-linear theory of one-dimensional non-local elasticity

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In one space dimension, a non-local elastic model is based on a single integral law, giving the stress when the strain is known at all spatial points. In this study, we first derive a higher-order Boussinesq equation using locally non-linear theory of 1D non-local elasticity and then we are able to show that under certain conditions the Cauchy problem is globally well-posed.

Keywords: higher-order Boussinesq equation; non-local elasticity; global well-posedness; Cauchy problem.

1. Introduction

This article deals with both a derivation of a higher-order Boussinesq (HBq) equation

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = (g(u))_{xx} \tag{1.1}$$

for a 1D motion in an infinite medium with non-linear and non-local elastic properties and global well-posedness of the Cauchy problem concerning this equation.

In one space dimension, a non-local elastic model is based on a single integral law, giving the stress when the strain is known at all spatial points. There is a large literature concerning non-local problems associated with the linear theory of non-local elasticity (see Eringen, 2002, and the references cited therein). However, to our knowledge there are few studies considering the effect of non-linearity for non-local problems resulting from non-local elasticity. Our objective here is to study one of such problems, using a locally non-linear theory of the non-local elastic model based on a single integral law.

We first show that the propagation of longitudinal waves in an infinite elastic medium with non-linear and non-local properties is described by the HBq equation (1.1). To this aim, in Section 2 the basic equations corresponding to the locally non-linear theory of 1D non-local elasticity are considered. The only difference between these equations and the corresponding equations, considered in the literature, of non-local elasticity is in the constitutive equation in which we replace Hooke's law of linear theory of non-local elasticity by a local stress–strain relationship derived from a local strain-energy density. The associated non-local kernel is determined by matching the dispersion curve of linear harmonic waves

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with the dispersion curve available from lattice dynamics. This matching is performed by approximating the fourth-order Taylor series expansion of the lattice dispersion relation about the zero wave number by a rational function. This approximation is equivalent to defining the non-local kernel function as Green's function of a linear fourth-order differential operator with constant coefficients. Converting the integro-partial-differential equation into a partial differential equation leads to the HBq equation (1.1), one primary goal of the present analysis.

Section 3 starts with a summary of results that were given in the literature about the Cauchy problem for Boussinesq-type equations. We first establish local well-posedness of the Cauchy problem in the Sobolev space H^s with any $s > 1/2$ using the contraction mapping principle (Theorem 2). In Theorem 3, we prove a regularity result in the space variable. In our main result, Theorem 4, we are able to extend the global existence to the HBq equation when the local strain-energy function satisfies a positivity condition.

2. Derivation of the HBq equation

2.1 *Background*

The modelling of small-scale effects has become an interesting subject nowadays due to their applications to nanotechnology (see, for instance, Wang *et al.*, 2006, and the references cited therein) and it is expected that non-local continuum mechanics will play a useful role in researching small-scale effects. The classical (or local) theory of elasticity assumes the existence of zero-range internal forces, i.e. the stress at a reference point depends uniquely on the strain at the same point. Consequently, the classical theory does not admit an intrinsic length scale. The applicability of the conventional theory is limited at small scales. This is natural because the discrete structure of the material becomes increasingly important at small scales and a length scale such as the lattice spacing between individual atoms cannot be avoided. From a wave propagation point of view, the linear harmonic waves propagating in an unbounded elastic medium are, contrary to lattice waves, non-dispersive. That is, the sole source of dispersion in the classical theory of elasticity is the existence of the boundaries. In short, the classical theory of elasticity does not include the 'physical' dispersion produced by the internal structure of the medium, whereas it does include the 'geometric' dispersion resulting from the existence of the boundaries. The discrepancy between the lattice and the continuum approach stimulated research to develop theories that admit long-range internal forces and consequently involve the 'physical' dispersion. One of these generalized elasticity theories is called non-local elasticity theory in which the constitutive equation is written as an integral law, i.e. the stress at a reference point is written as a functional of the strain field at every point in the body. Rigorous foundations of the non-local continuum mechanics were established in the last four decades (see Eringen, 2002, and the references cited therein).

2.2 *Non-linear theory of 1D non-local elasticity*

Here, we study propagation of plane longitudinal waves in a non-local elastic, homogeneous, isotropic and locally non-linear medium. We assume that the plane waves propagate in the $X_1 \equiv X$ direction of a rectangular Cartesian reference frame X_1, X_2 and X_3. The only non-vanishing displacement component at time t of a reference point X is $u_1 = u(X, t)$. Assuming that there are no body forces, the equation of motion is

$$\rho_0 u_{tt} = (S(u_X))_X, \tag{2.1}$$

where ρ_0 is the mass density of the medium, $S = S(u_X)$ is the (non-local) stress and the subscripts denote partial derivatives. The constitutive equation for the non-local stress is taken in a form

$$S(X,t) = \int_{-\infty}^{\infty} \alpha(|X-Y|)\sigma(Y,t)\mathrm{d}Y, \quad \sigma(X,t) = W'(\epsilon(X,t)), \tag{2.2}$$

where $\epsilon = u_X$ denotes the 'strain' at time t of a reference point X, σ is the classical (local) stress, W is the local strain-energy function, Y marks a generic point of the medium, with X being the point of observation, α is a kernel function to be specified as the inverse of a linear differential operator below and the symbol $'$ denotes differentiation. In the present 1D model, the kernel has the dimension of 1/length which implies that the present theory introduces a characteristic length scale to the equations, which does not appear in the classical theory. We assume that the reference configuration is a stress-free undistorted configuration: $W(0) = W'(0) = 0$. When the local strain-energy function is assumed to be in the form of $W(\epsilon) = (\lambda + 2\mu)\epsilon^2/2$, where λ and μ are Lame constants, the above equations reduce to those of the linear theory of 1D non-local elasticity (see Eringen, 2002).

2.3 *Kernel function and lattice model*

How to determine the kernel function is still an open question in non-local continuum mechanics. Within the context of the present study, the most important issue is how to choose the kernel function so that the Cauchy problem for the resulting non-linear equations is well-posed in appropriate function spaces. The form of kernel function α is to be determined by matching the linear dispersion relation of non-local elasticity with that of the lattice dynamics. We first assume that the kernel α is the Green's function associated with a constant-coefficient linear partial differential operator $\mathcal{L}$ (Lazar *et al.*, 2006):

$$\mathcal{L}\alpha(|X-Y|) = \delta(X-Y),$$

where δ denotes the Dirac delta function. Since the Green's function inverts the effect of the differential operator, the first equation in (2.2) can be written as

$$\mathcal{L}S(X,t) = \sigma(X,t).$$

Using this result in (2.1), we obtain the equation of motion in the form

$$\rho_0(\mathcal{L}u)_{tt} = (W'(u_X))_X, \tag{2.3}$$

where we use the fact that $\mathcal{L}$ is a differential operator with constant coefficients.

For the linearized equations of the present theory, consider plane harmonic wave solutions of the form $u(X,t) = A\exp(\mathrm{i}(kX - \omega t))$, where A is a constant and ω and k are wave frequency and wave number, respectively. Then, the linearized form of (2.3) yields the linear dispersion relation

$$\frac{\omega^2}{c^2k^2} = \frac{1}{\mathcal{L}(\mathrm{i}k)}, \tag{2.4}$$

where $c = [(\lambda + 2\mu)/\rho_0]^{1/2}$ is the speed of longitudinal waves according to the classical (local) theory of elasticity and $\mathcal{L}(\mathrm{i}k)$ is the Fourier symbol of $\mathcal{L}$. As it is expected, the above equation shows that the non-local theory of elasticity implies the dispersion of waves even in the absence of the boundaries.

We now consider lattice waves propagating in a 1D monatomic chain with interparticle spacing a. When the particles are connected by nearest-neighbour harmonic springs of equal strength, the corresponding linear dispersion relation (Kittel, 1962, p. 143) is

$$\frac{\omega^2}{c^2k^2} = \left(\frac{2}{ka}\right)^2 \sin^2\left(\frac{ka}{2}\right), \tag{2.5}$$

where the phase velocity at $k = 0$ is assumed to be equal to c. From the point of view of this study, an important aspect of (2.5) is that there exists an upper bound for ω.

One approach to determine the differential operator is to equate the right-hand sides of (2.4) and (2.5) in the form

$$\frac{1}{\mathcal{L}(\mathrm{i}k)} = \left(\frac{2}{ka}\right)^2 \sin^2\left(\frac{ka}{2}\right), \tag{2.6}$$

but this would make $\mathcal{L}$ a pseudo-differential operator represented in the Fourier space by $\mathcal{L}(\mathrm{i}k)$. A more practical approach is based on the use of both the polynomial approximations to $\mathcal{L}(\mathrm{i}k)$ (see Eringen, 2002) and the Taylor series expansions of both sides of (2.6) about $k = 0$. Note that the polynomial approximations of $\mathcal{L}(\mathrm{i}k)$ imply an upper bound for ω in (2.4). For the zeroth-order polynomial approximation of $\mathcal{L}(\mathrm{i}k)$, i.e. for the limiting case $\mathcal{L}(\mathrm{i}k) = 1$, we get the equations corresponding to the local (classical) theory of elasticity. For the second-order polynomial approximation, we set $\mathcal{L}(\mathrm{i}k) = 1+\gamma k^2$, where γ is a non-negative constant. This implies that the differential operator $\mathcal{L}$ is in the form $\mathcal{L}(\cdot) = 1 - \gamma (\cdot)_{XX}$. For the choice of $\gamma = a^2/12$, we note that (2.6) is satisfied up to $\mathcal{O}(a^4)$ and that non-locality is incorporated into the equations by the addition of a characteristic length scale, the lattice spacing a. For the fourth-order polynomial approximation, we set $\mathcal{L}(\mathrm{i}k) = 1 + \gamma_1 k^2 + \gamma_2 k^4$, where γ_1 and γ_2 are non-negative constants. This implies that $\mathcal{L}(\cdot) = (\cdot) - \gamma_1(\cdot)_{XX} + \gamma_2(\cdot)_{XXXX}$. For the choice of $\gamma_1 = a^2/12$ and $\gamma_2 = a^4/240$, (2.6) is satisfied up to $\mathcal{O}(a^6)$. For a more detailed discussion about other choices of γ_1 and γ_2, we refer the reader to Eringen (2002) and Lazar *et al.* (2006) where a similar approach was used for the linear theory of 3D non-local elasticity.

2.4 *Scaling and non-dimensionalization*

We henceforth adopt the fourth-order linear partial differential operator given above as the inverse of our integral operator. For convenience, we separate the quadratic part of the strain-energy function; this corresponds to decomposing the derivative of the strain-energy function into its linear and non-linear parts:

$$W(\epsilon) = (\lambda + 2\mu)\left[\frac{1}{2}\epsilon^2 + G(\epsilon)\right],$$

where $G(0) = G'(0) = 0$. This implies that

$$W'(\epsilon) = (\lambda + 2\mu)[\epsilon + g(\epsilon)], \tag{2.7}$$

where

$$G(\epsilon) = \int_0^{\epsilon} g(s)\mathrm{d}s, \quad g(0) = 0. \tag{2.8}$$

Differentiating both sides of (2.3) with respect to X, we obtain the equation of motion expressed in terms of strain:

$$\rho_0(\mathcal{L}\epsilon)_{tt} = (W'(\epsilon))_{XX}$$

or explicitly

$$\epsilon_{tt} - c^2\epsilon_{XX} - \gamma_1\epsilon_{XXtt} + \gamma_2\epsilon_{XXXXtt} = c^2(g(\epsilon))_{XX}. \tag{2.9}$$

Now, we define the dimensionless independent variables

$$x = X/\sqrt{\gamma_1}, \quad \tau = ct/\sqrt{\gamma_1}$$

and from now on, and for simplicity, we use u for ϵ and t for τ. Thus, (2.9) takes the form given in (1.1), with $\beta = \gamma_2/\gamma_1^2 \geqslant 0$.

We point out that (1.1) was derived in Rosenau (1988) for the continuum limit of a dense chain of particles with elastic couplings. Also, the conserved quantities of (1.1), corresponding to conservation of mass, conservation of momentum and conservation of energy, were derived in Rosenau (1988). The same equation was used to model water waves with surface tension in Schneider & Wayne (2001). We refer the reader to Eringen (2002) and Lazar *et al.* (2006) for the derivation of the linearized form of (1.1) within the context of linear theory of 3D non-local elasticity.

3. Cauchy problem

In this section, we investigate the well-posedness of the Cauchy problem

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(u)_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3.2}$$

The global existence of the Cauchy problem for the generalized improved Boussinesq equation for which $\beta = 0$ has been proved in Chen & Wang (1999). Similarly, the global existence of the Cauchy problem for the generalized double dispersion equation where $\beta = 0$ and a linear term u_{xxxx} is included has been proved in Wang & Chen (2006). It is therefore natural to ask how the higher-order dispersive term affects the global existence. In fact, the method presented in Wang & Chen (2006) for the generalized double dispersion equation was extended to the Cauchy problem (3.1–3.2) for the HBq equation in Duruk (2006). Summarizing the results in Duruk (2006), we prove in this section the global well-posedness when the non-linear term satisfies a positivity condition. Similar results also have been derived independently in Wang & Mu (2007).

In what follows, $H^s = H^s(\mathbb{R})$ will denote the L^2 Sobolev space on $\mathbb{R}$. For the H^s-norm, we use the Fourier transform representation $\|u\|_s^2 = \int (1+\xi^2)^s |\hat{u}(\xi)|^2 \, \mathrm{d}\xi$. We use $\|u\|_\infty$ and $\|u\|$ to denote the L^∞- and L^2-norm, respectively.

3.1 *Linear problem*

For the linear version of (3.1), we prove the following theorem.

THEOREM 1 Let $s \in \mathbb{R}, T > 0$, $\varphi \in H^s$, $\psi \in H^s$ and $h \in L^1([0,T]; H^{s-2})$. Then, the Cauchy problem

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = (h(x,t))_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.3}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \tag{3.4}$$

has a unique solution $u \in C^1([0,T], H^s)$ satisfying the estimate

$$\|u(t)\|_s + \|u_t(t)\|_s \leqslant m(1+T)\left(\|\varphi\|_s + \|\psi\|_s + \int_0^t \|h(\tau)\|_{s-2}\, \mathrm{d}\tau\right) \tag{3.5}$$

for $0 \leqslant t \leqslant T$, with some constant $m \geqslant 2$. Moreover, if $h \in C([0,T]; H^{s-2})$, then $u \in C^2([0,T], H^s)$.

Proof. Taking Fourier transform with respect to the space variable in (3.3) gives

$$\lambda^2(\xi)\hat{u}_{tt} + \xi^2\hat{u} = -\xi^2\hat{h},$$

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi), \quad \hat{u}_t(\xi,0) = \hat{\psi}(\xi),$$

with $\lambda^2(\xi) = 1 + \xi^2 + \beta\xi^4$. This in turn yields the solution formula

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi)\cos\left(\frac{t\xi}{\lambda(\xi)}\right) + \hat{\psi}(\xi)\frac{\lambda(\xi)}{\xi}\sin\left(\frac{t\xi}{\lambda(\xi)}\right) - \int_0^t \sin\left(\frac{(t-\tau)\xi}{\lambda(\xi)}\right)\frac{\xi}{\lambda(\xi)}\hat{h}(\xi,\tau)\mathrm{d}\tau.$$

Differentiating in t and using $|\sin w| \leqslant |w|$, we obtain the estimate (3.5) from which the proof follows. □

3.2 *Local results for the non-linear problem*

In this subsection, we prove local well-posedness of the non-linear problem (3.1–3.2) with a fixed-point technique for data in H^s with $s > \frac{1}{2}$. We utilize the following lemmas in Wang & Chen (2006).

LEMMA 1 Let $f \in C^{[s]+1}(\mathbb{R})$, $s \geqslant 0$, with $f(0) = 0$. Then, for any $M > 0$ there is some constant $K_1(M)$ such that for all $u \in H^s \cap L^\infty$ with $\|u\|_\infty \leqslant M$, we have

$$\|f(u)\|_s \leqslant K_1(M)\|u\|_s.$$

LEMMA 2 Let $f \in C^{[s]+1}(\mathbb{R})$, $s \geqslant 0$. Then, for any $M > 0$ there is some constant $K_2(M)$ such that for all $u, v \in H^s \cap L^\infty$ with $\|u\|_\infty \leqslant M$, $\|v\|_\infty \leqslant M$ and $\|u\|_s \leqslant M$, $\|v\|_s \leqslant M$, we have

$$\|f(u) - f(v)\|_s \leqslant K(M)\|u - v\|_s.$$

REMARK 1 Although Lemmas 1 and 2 look quite similar, easy examples show that the extra bounds on the H^s-norm in Lemma 2 are necessary. The proof for Lemma 1 can be found in Wang & Chen (2006) and many other sources, but Wang & Chen (2006) incorrectly states Lemma 2 without the H^s bounds. The proof of Lemma 2 as we state is quite easy along the lines of the proof for Lemma 1.

THEOREM 2 Let $s > 1/2$, $\varphi \in H^s$, $\psi \in H^s$ and $g \in C^k(\mathbb{R})$ with $g(0) = 0$ and $k = \max\{[s-1], 1\}$. Then, there is some $T > 0$ such that the non-linear Cauchy problem is well-posed with solution $u \in C^2([0,T], H^s)$ satisfying

$$\max_{t\in[0,T]}(\|u(t)\|_s + \|u_t(t)\|_s) \leqslant 2m(\|\phi\|_s + \|\psi\|_s).$$

Proof. Set $\|\varphi\|_s + \|\psi\|_s = A$ and let

$$X(T) = \left\{u \in C^1([0,T], H^s)\colon \|u\|_{X(T)} = \max_{t\in[0,T]}(\|u(t)\|_s + \|u_t(t)\|_s) \leqslant 2mA\right\},$$

where $T > 0$ is to be determined later.

For $\omega \in (T)$, we consider the problem

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = g(\omega)_{xx}, \tag{3.6}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3.7}$$

We see that for $g(\omega(x,t)) = h(x,t)$, this problem reduces to the linearized problem in Theorem 1 in Section 3.1, hence it has a unique solution $u(x,t)$. We define $\mathcal{S}(\omega) = u(x,t)$. Clearly, $\mathcal{S}$ denotes the map which carries ω into the unique solution of (3.6) and (3.7). Our aim is again to show that for appropriately chosen T and A, $\mathcal{S}$ has a unique fixed point in $X(T)$.

The estimate (3.5) implies that

$$\|u(t)\|_s + \|u_t(t)\|_s \leqslant m(1+T)\left(\|\varphi\|_s + \|\psi\|_s + \int_0^t \|g(\omega(\tau))\|_{s-2}\,\mathrm{d}\tau\right). \tag{3.8}$$

So,

$$\begin{aligned}\|\mathcal{S}(\omega)\|_{X(T)} &= \max_{t\in[0,T]}(\|u(t)\|_s + \|u_t(t)\|_s)\\ &\leqslant m(1+T)\left(A + T\left(\max_{t\in[0,T]}\|g(\omega(t))\|_{s-2}\right)\right).\end{aligned}$$

Since $\|w(t)\|_\infty \leqslant \mathrm{d}\|w(t)\|_s \leqslant 2m\,\mathrm{d}A$, Lemma 1 holds:

$$\|g(\omega(t))\|_{s-2} \leqslant K_1\|\omega(t)\|_{s-2} \leqslant K_1\|\omega\|_{X(T)},$$

where $K_1 = K_1(2m\,\mathrm{d}A)$ is a constant dependent on A. Then,

$$\begin{aligned}\|\mathcal{S}(\omega)\|_{X(T)} &\leqslant m(1+T)(A + TK_1\|\omega\|_{X(T)})\\ &\leqslant mA(1+T)(1+TK_1 2m).\end{aligned}$$

For sufficiently small T, $(1+T)(1+TK_1 2m) \leqslant 2$ so we have $\|\mathcal{S}(\omega)\|_{X(T)} \leqslant 2mA$, in other words $\mathcal{S}(w) \in X(T)$.

Now, let $\omega, \bar{\omega} \in X(T)$ and $u = S(\omega)$, $\bar{u} = S(\bar{\omega})$. Set $V = u - \bar{u}$ and $W = \omega - \bar{\omega}$. Then, V satisfies

$$V_{tt} - V_{xx} - V_{xxtt} + \beta V_{xxxxtt} = (g(\omega) - g(\bar{\omega}))_{xx},$$

$$V(x,0) = V_t(x,0) = 0.$$

Hence, by (3.5) and Lemma 2, there is some constant K_2 depending on A so that

$$\begin{aligned}\|V(t)\|_s + \|V_t(t)\|_s &\leqslant m(1+T)\int_0^t \|g(\omega(\tau)) - g(\bar{\omega}(\tau))\|_{s-2}\,\mathrm{d}\tau\\ &\leqslant m(1+T)TK_2 \max_{t\in[0,T]}\|W(t)\|_s.\end{aligned}$$

So,

$$\|V\|_{X(T)} \leqslant m(1+T)TK_2\|W\|_{X(T)}.$$

If we further choose T small enough so that $m(1+T)TK_2 \leqslant \frac{1}{2}$, $\mathcal{S}$ becomes contractive. By the Banach fixed-point theorem, we obtain local existence and uniqueness.

We now look at continuous dependence on the initial data. Let u_1 and u_2 be solutions of (3.1–3.2) with initial data φ_i, ψ_i $(i = 1, 2)$, satisfying $\|u_i\|_s \leqslant A$. Then, again the estimates of Theorem 1 and Lemma 2 yield

$$\|u_1(t) - u_2(t)\|_s \leqslant m(1+T)\left(\|\varphi_1 - \varphi_2\|_s + \|\psi_1 - \psi_2\|_s + \int_0^t \|g(u_1(\tau)) - g(u_2(\tau))\|_{s-2}\,\mathrm{d}\tau\right)$$

and

$$\|g(u_1(\tau)) - g(u_2(\tau))\|_{s-2} \leqslant K_2\|u_1(\tau) - u_2(\tau)\|_s.$$

So,

$$\|u_1(t) - u_2(t)\|_s \leqslant m(1+T)\left(\|\varphi_1 - \varphi_2\|_s + \|\psi_1 - \psi_2\|_s + K_2\int_0^t \|u_1(\tau) - u_2(\tau)\|_s\,\mathrm{d}\tau\right).$$

Gronwall's lemma implies that

$$\|u_1(t) - u_2(t)\|_s \leqslant m(1+T)(\|\varphi_1 - \varphi_2\|_s + \|\psi_1 - \psi_2\|_s)\mathrm{e}^{m(1+T)K_2 t}. \tag{3.9}$$

This completes the proof of the theorem.

Using standard techniques, the solution can be extended to the maximal interval $[0, T_{\max})$, where the maximal time is characterized as follows. If $T_{\max} < \infty$, we have

$$\limsup_{t\to T_{\max}^-}[\|u(t)\|_s + \|u_t(t)\|_s] = \infty. \tag{3.10}$$

We can further characterize blow-up by

$$\limsup_{t\to T_{\max}^-}\|u(t)\|_\infty = \infty. \tag{3.11}$$

Since $s > \frac{1}{2}$, we have $\|u(t)\|_\infty \leqslant \mathrm{d}\|u(t)\|_s$ so if (3.11) holds so thus (3.10). Conversely, if $M = \limsup_{t\to T_{\max}^-}\|u(t)\|_\infty < \infty$, by Lemma 1 and (3.8) we have for $t < T$

$$\|u(t)\|_s + \|u_t(t)\|_s \leqslant m(1+T)\left(\|\varphi\|_s + \|\psi\|_s + K_1(M)\int_0^t \|(\omega(\tau))\|_{s-2}\,\mathrm{d}\tau\right)$$

which implies that $\limsup_{t\to T_{\max}^-}[\|u(t)\|_s + \|u_t(t)\|_s] < \infty$ by Gronwall's Lemma. □

REMARK 2 The condition (3.11) in particular says that $T_{\max}$ does not depend on s for $s > 1/2$. The estimate in Theorem 1 allows us to prove the following result on the x-regularity of the solution.

THEOREM 3 Let $\varphi \in H^s$, $\psi \in H^s$ and $g \in C^k(\mathbb{R})$ with $g(0) = 0$ and $k = \max\{[s-1], 1\}$. Suppose further that for some $1/2 < r < s$ and $T > 0$, we have a solution $u \in C^2([0, T], H^r)$. Then $u \in C^2([0, T], H^s)$.

Proof. Let $r^* = \min(r+2, s)$. Then, (3.8) implies

$$\|u(t)\|_{r^*} + \|u_t(t)\|_{r^*} \leqslant m(1+T)\left(\|\varphi\|_s + \|\psi\|_s + K\int_0^t \|u(\tau)\|_r\,\mathrm{d}\tau\right),$$

so that $u \in C^2\big([0, T], H^{r^*}\big)$. Continuing inductively we prove the theorem. □

3.3 *Global existence*

As we have seen above, looking for the global solution is equivalent to showing that there is no blow-up. We first derive an energy identity. We use the operator $\Lambda^{-\alpha} w = F^{-1}[|\xi|^{-\alpha} F w]$. Then,

$$\Lambda^{-2} u_{tt} + u + u_{tt} - \beta u_{xxtt} = -g(u).$$

Multiplying both sides with u_t and integrating over $\mathbb{R}$ with respect to x, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^{-1} u_t\|^2 + \|u\|^2 + \|u_t\|^2 + \beta\|u_{xt}\|^2 + 2\int_{\mathbb{R}} \left(\int_0^u g(p)\mathrm{d}p \right) \mathrm{d}x \right) = 0.$$

Thus, the following lemma has been proved.

LEMMA 3 Suppose that $g \in C(\mathbb{R})$, $G(u) = \int_0^u g(p)\mathrm{d}p$, $\varphi \in H^1$, $\psi \in H^1$, $\Lambda^{-1}\psi \in H^1$ and $G(\varphi) \in L^1$. Then, for the solution $u(x,t)$ of problem (3.1–3.2), we have the energy identity

$$E(t) = \|\Lambda^{-1} u_t\|^2 + \|u\|^2 + \|u_t\|^2 + \beta\|u_{xt}\|^2 + 2\int_{-\infty}^{\infty} G(u)\mathrm{d}x = E(0) \tag{3.12}$$

for all $t > 0$ for which the solution exists.

THEOREM 4 Assume that $s \geqslant 1$, $g \in C^{s+1}(\mathbb{R})$, $\varphi \in H^s$, $\psi \in H^s$, $\Lambda^{-1}\psi \in H^s$, $G(\varphi) \in L^1$ and $G(u) \geqslant 0$ for all $u \in \mathbb{R}$, then the problem (3.1–3.2) has a unique global solution $u \in C^2([0,\infty), H^s)$.

Proof. By Remark 2 following Theorem 2, it suffices to prove the case $s = 1$. If $G(u) \geqslant 0$, then from (3.12)

$$\|\Lambda^{-1} u_t\|^2 + \|u\|^2 + \|u_t\|^2 + \beta\|u_{xt}\|^2 \leqslant E(0) < \infty.$$

Hence, H^1-norm of u_t, i.e. $\|u_t\|^2 + \|u_{xt}\|^2$, is bounded and does not blow-up in finite time. We need an estimate for $\|u(t)\|_{H^1}$; so we write $u(x,t)$ as an integral equation:

$$u(x,t) = \varphi(x) + \int_0^t u_t(x,\tau)\mathrm{d}\tau.$$

Then,

$$\|u(t)\|_{H^1} \leqslant \|\varphi\|_{H^1} + \int_0^t \|u_t(\tau)\|_{H^1}\,\mathrm{d}\tau \leqslant \|\varphi\|_{H^1} + tE(0).$$

Thus, for any finite $T > 0$,

$$\limsup_{t\to T^-}[\|u(t)\|_{H^1} + \|u_t(t)\|_{H^1}] < \infty.$$

We want to add some concluding remarks. □

REMARK 3 Considering that u in our Cauchy problem (corresponding to $\epsilon(X,t)$ of Section 2) represents, up to scaling, the space derivative of displacement, the artificial looking hypothesis $\Lambda^{-1}\psi \in H^s$ of Theorem 4 is in fact H^s regularity of the initial velocity.

REMARK 4 Following the proof in Wang & Chen (2006), the positivity assumption $G(u) \geqslant 0$ in Theorem 4 can be weakened to $G(u) \geqslant -ku^2$ which is equivalent to $g'(u)$ being bounded from below. This extension covers all odd-degree non-linearities $g(u)$.

REMARK 5 Finally, we want to look at continuous dependence on the initial data. In Theorem 2, we prove this for small t. When the assumptions of Theorem 4 hold, we can extend the result to arbitrary times as follows: the key is noting that the inequality (3.9) holds whenever we have bounds on $\|u(t)\|_\infty$ and $\|u(t)\|_s$ depending on the initial data. For $s = 1$, the proof of Theorem 4 provides such a bound for $\|u(t)\|_1$ and hence for $\|u(t)\|_\infty$ in terms of $E(0)$. For $s > 1$ since we have an H^1 thus an L^∞ bound on $u(t)$, we repeat the proof of the equivalence of two characterizations of $T_{\max}$ ((3.10) and (3.11)) and obtain a bound on $\|u(t)\|_s$.

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