# BASES AND ISOMORPHISMS IN SPACES OF ANALYTIC FUNCTIONS

by

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Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Master of Science

Sabancı University
Fall 2006

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DATE OF APPROVAL: February 14th, 2006

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*sevgili ablama...*

## Acknowledgments

I would like to express my gratitude and deepest regards to my supervisor Prof. Dr. Vyacheslav Zahariuta for his motivation, guidance and encouragement throughout this thesis.

I also would like to thank Kağan Kurşungöz and all my friends their endless support.

**BASES AND ISOMORPHISMS IN SPACES OF ANALYTIC FUNCTIONS**


## Abstract

We will discuss the construction of bases in a space of analytic functions for a given domain and isomorphic classification of spaces of analytic functions. We will focus on results in one dimensional case.

In one dimensional case, we consider the construction of bases in two different ways. Using one of them, we construct interpolational bases for the space of analytic functions on a compactum K and in that part, results of Leja, Walsh, and Zahariuta are used. Then, isomorphic classification follows by the use of Potential Theory. Using the second way, we construct a common basis for the spaces of analytic functions of a regular pair "compact set – domain" by the Hilbert methods that was proposed by Zahariuta. GKS-duality is used for both of the cases.

In multidimensional case, some results about bases and isomorphisms of spaces of analytic functions in several variables that were proved by Zahariuta are represented (see also Aytuna). Since a multidimensional analogue of GKS-duality does not exist, interpolational bases cannot be constructed as in one dimensional case. But the bases constructed by Hilbert methods proves to be applicable for studying the isomorphism of the space of analytic functions on D to the space of analytic functions on the unit circle of n-dimensional complex plane.



Keywords: Hilbert scales, spaces of analytic functions, Green potential, regularity, GKS-duality.

**ANALİTİK FONKSİYON UZAYLARINDA İZOMORFİK SINIFLANDIRMA VE TABAN İNŞASI**


# Özet

Bu tezde belirli bir bölgedeki analitik fonksiyon uzayları için taban inşası ve izomorfik sınıflandırma tartışılacaktır. Tek boyutlu düzlem için bulunan sonuçlar üzerinde yoğunlaşılacaktır.

Tek boyutlu durumda, taban kurulumu için iki farklı yöntem üzerinde durulacaktır. Birini kullanarak, bir tıkız küme K üzerindeki analitik fonksiyon uzayları için enterpolasyon yollu taban inşası yapılacaktır ve bu durumda, Leja, Walsh ve Zahariuta'nın sonuçları kullanılmaktadır. Potansiyel Teori yardımıyla, bulunan tabanları kullanarak izomorfik sınıflandırma yapılacaktır. İkinci yolu kullanarak, regüler bir "tıkız küme-bölge" ikilisi üzerindeki analitik fonksiyon uzayları için Hilbert yöntemleri kullanılarak ortak bir taban kurulacaktır. Bu metod, Zahariuta tarafından bulunmuştur. GKS-düalitesi her iki yöntem için de kullanılmıştır.

Çok boyutlu durumda, çok değişkenli analitik fonksiyon uzaylarında taban inşası ve izomorfik sınıflandırma için Zahariuta tarafından ispat edilen bazı sonuçlar sunulacaktır. GKS-düalitesinin çok boyutlu bir analoğu olmadığı için, tek boyutlu durumda olduğu gibi enterpolasyon yollu tabanlar kurulamaz. Ama, Hilbert yöntemlerini kullanarak inşa edilen tabanlar belirli bir bölge D ve n boyutlu kompleks düzlemdeki birim çember üzerinde tanımlı analitik fonksiyon uzaylarının arasındaki izomorfaları çalışmak için kullanılabilir.



Anahtar Kelimeler: Hilbert skalaları, analitik fonksiyon uzayları, Green potansiyeli, regülerlik, GKS-düalitesi.

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# CHAPTER 1

# INTRODUCTION

Spaces of analytic functions were studied by many mathematicians (e.g., Poincaré, Pincerle, Fantappie, Uryson, Whittaker, Newns, Markushewich, Haplanov, Martineau, Aizenberg, Mityagin, Khavin, Arsove, Boas, Ronkin, Köthe, Grothendieck, Zahariuta, Aytuna et. al.). We will introduce spaces of analytic functions in Chapter 4.

In this thesis, we will discuss the construction of bases in a space of analytic functions for a given domain and isomorphic classification of spaces of analytic functions. We will focus on the one dimensional case.

Many problems of Approximation and Interpolation Theory of analytic functions of one variable require notions and methods of Potential Theory. Therefore, we will use Potential Theory for the results on existence of bases and isomorphisms of spaces of analytic functions. Sufficient information about Potential Theory can be found in Chapter 3. The following result will be discussed in Chapter 5.

**Proposition 1** *Let $K$ be a compactum in $\mathbb{C}$ with a connected complement. If $K$ is regular or polar, there exists a sequence of knots $\{\zeta_1, \zeta_2, \cdots, \zeta_\nu, \cdots\}$ (there may be repetitions) such that the system of Newton interpolation polynomials*

$$p_k(z) = \prod_{\nu=1}^{k} (z - \zeta_\nu), \quad k = 1, 2, \cdots, \quad p_0(z) \equiv 1,$$

*forms a basis in the space $A(K)$.*

The case where $K$ is a regular compactum was proved by J. L. Walsh [32] and F. Leja [17], and the case where $K$ is a polar compactum was proved by V. P.

Zahariuta [34].

After construction of the bases also the following theorems which were proved by Zahariuta [34] will be discussed in Chapter 5.

**Theorem 1** *Let $K$ be a compactum in $\widehat{\mathbb{C}}$. For the spaces $A(K)$ and $\bar{A}_1$ to be isomorphic, it is necessary and sufficient that (a) the compactum $K$ be regular, and (b) the complement $K^* = \widehat{\mathbb{C}} \setminus K$ consist no more than a finite number of connected components.*

**Theorem 2** *Let $K$ be a compactum in $\widehat{\mathbb{C}}$. For the spaces $A(K)$ and $\bar{A}_0$ to be isomorphic, it is necessary and sufficient that $C(K) = 0$.*

**Theorem 3** *Let $K$ be a compactum in $\widehat{\mathbb{C}}$. For the spaces $A(K)$ and $\bar{A}_1 \times \bar{A}_0$ to be isomorphic, it is necessary and sufficient that the compactum $K$ be decomposed into two disjoint non-empty compacta $K^{(1)}$ and $K^{(2)}$, where $K^{(1)}$ is a regular compactum whose complement consists of a finite number of connected components and $C(K^{(2)}) = 0$.*

We will also discuss another method for construction of bases that is based on Hilbert methods which was suggested by Zahariuta [35]. Using that method, a common basis for a regular pair "compact set-domain" $(K, D)$ will be constructed. Information about Hilbert scales is represented in Chapter 2. The following result will be discussed in Chapter 5.

**Theorem 4** *Let $K \subset D$ be a regular pair "compact set-domain". Let $H_0,\ H_1$ be such that the dense continuous imbeddings hold:*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K),$$

$$A(D^*) \hookrightarrow H_1' \hookrightarrow AC(D^*),$$

*where $H_1'$ is a GKS-realization of the dual space $H_1^*$. Then the common orthogonal basis $\{e_k(z)\}$ for $H_0,\ H_1$, normalized in $H_0$ and ordered by non-increasing of its norms in $H_1$ :*

$$\|e_k\|_{H_0} = 1,\ \mu_k = \mu_k(H_0, H_1) := \|e_k\|_{H_1} \nearrow \infty,$$

*is also a common basis in all spaces $A(D), A(D_\alpha), A(K_\delta)$, and $A(K)$ where $D_\alpha, K_\delta$ are the sublevel domains as defined in Lemma 6.*

Notice that in a very particular case (for a 1-connected domain $D$ and continuum $K$) a common basis was constructed by Erokhin [9] by means of refined technique of conformal mappings and by Walsh and Russel [33] in the form of interpolational rational system of functions. Both of the methods are essentially one dimensional and cannot be applied for the case of several variables.

For multidimensional case, we will represent some results about bases and isomorphisms of spaces of analytic functions that were proved by Zahariuta [37], [38] (see also [3]) without detailed proofs. Interpolational bases cannot be constructed as in one dimensional case since there does not exist a multidimensional analogue of GKS-duality. But the bases constructed by Hilbert methods proves to be applicable, as confirmed by Zahariuta [37], [38]. A sketch of the following theorem will be given in Chapter 6.

**Theorem 5** *( [37], [38]) Let $(K, D)$ be a pluriregular pair "compact set-Stein manifold". Then there exists a common basis $\{x_i(z)\}$ in the spaces $A(D)$, $A(K)$, $A(K_\alpha)$, $A(D_\alpha)$, $0 < \alpha < 1$, satisfying the asymptotic estimate*

$$\overline{\lim_{\zeta\to z}}\,\overline{\lim_{i\to\infty}}\frac{\ln\left|x_{i(z)}\right|}{a_i} = \omega(D, K, z),\ \ z \in D \setminus K,$$

*where*

$$K_\alpha = \{z \in D : \omega(D, K, z) \leq \alpha\},\ D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\},\ 0 < \alpha < 1,$$

*and $\{a_i\}$ is a certain non-decreasing sequence of positive numbers such that with $n = \dim D$,*

$$a_i \asymp i^{\frac{1}{n}},\ i \to \infty.$$

Then, using the extendible bases that are constructed, the following theorem was proved by Zahariuta. We will again give a sketch of the proof.

**Theorem 6** *( [37], [38]) Let $\Omega$ be a Stein manifold on dimension $n$. For the isomorphism*

$$A(\Omega) \simeq A(U^n)$$

*it is necessary and sufficient that $\Omega$ is pluriregular and consists of at most finite number of connected components, where $U^n$ is the unit circle in $\mathbb{C}^n$.*

The sufficiency in Theorem 6 was preceded by many results on sufficient condition for isomorphism $A(\Omega) \simeq A(U^n)$, such as: for $n$-circular (Reinhardt) domains Aizenberg-Mityagin [2], Bezdudniy [5], [6], Mityagin [18], Okun [20], Rolewicz [22], for $(p_1, p_2, \cdots, p_n)$-circular domains Aizenberg [1], for convex domains Zahariuta [44], and for strongly pseudoconvex domains Henkin-Mityagin [13].

# CHAPTER 2

# SOME TOPICS OF FUNCTIONAL ANALYSIS

In this chapter, some preliminary concepts about functional analysis will be mentioned. In Section 2.1, locally convex spaces will be introduced [19]. This section is also where we define inductive and projective limit topologies and give definitions of some of the spaces that will be used, like nuclear spaces.

In Section 2.2, Hilbert scales will be introduced ( [16], [18]) and a theorem that constructs a common basis for a pair of Hilbert spaces is discussed [35].

## 2.1 Locally Convex Spaces

Let $X$ be a non-empty set. We will define a *topology* on $X$ as a system $\mathcal{T}$ of subsets of $X$ which has the properties:

1. The union of arbitrarily many open sets is open; $\emptyset$ is open.

2. The intersection of finitely many open sets is open; $X$ is open.

The elements of $\mathcal{T}$ are called open sets. A topological space $(X, \mathcal{T})$ is a set $X$ with a topology $\mathcal{T}$.

A topological space $X$ is called *Hausdorff space* if for each pair $x, y \in X$ with $x \neq y$ there exists disjoint open sets $U_x$ and $U_y$ with $x \in U_x$ and $y \in U_y$. Later on, we will always assume the topological spaces to be Hausdorff.

By a topological vector space, we will mean a $\mathbb{K}$-vector space $E$ with a topology $\mathcal{T}$ for which addition $+ : E \times E \to E$ and a scalar multiplication $\cdot : \mathbb{K} \times E \to E$ are continuous in $\mathcal{T}$. The continuity of the addition means that for each elements

$x,\ y \in E$ and each neighborhood $U_z$ of $z = x + y$ there exist neighborhoods $U_x$ of $x$ and $U_y$ of $y$ such that $U_x + U_y \subset U_z$; where $A + B := \{a + b : a \in A, b \in B\}$ for any two sets $A$ and $B$. The continuity of the scalar multiplication means that for each $\lambda_0 \in \mathbb{K}$, $x_0 \in E$ and every neighborhood $U_{\lambda_0 x_0}$ of $\lambda_0 x_0$ there exist $\varepsilon > 0$ and a neighborhood $U_{x_0}$ such that

$$\{\lambda v : |\lambda - \lambda_0| \leq \varepsilon, v \in U_{x_0}\} \subset U_{\lambda_0 x_0}.$$

A topology $\mathcal{T}$ on a $\mathbb{K}$-vector space $E$ is called a *vector space topology*, if $(E, \mathcal{T})$ is a topological vector space.

**Proposition 2** *The following are direct consequences of the definition of topological vector space $E$ :*

1. *For each $y \in E$, the translation $x \mapsto x + y$ is continuous and therefore a homeomorphism of $E$. In particular the neighborhoods of each $x \in E$ are of the form $x + V := \{x + v : v \in V\}$, where $V$ is a zero neighborhood.*
2. *For each zero neighborhood $U$ in $E$ there exists a zero neighborhood $V$ in $E$ with $V + V \subset U$. In particular $E$ has a zero neighborhood basis consisting of closed zero neighborhoods.*
3. *For each zero neighborhood $U$ in $E$ there exists zero neighborhood $W \subset U$ with*
$$W = \{\lambda w : |\lambda| \leq 1,\ w \in W\}.$$
4. *For every zero neighborhood $U$ in $E$ we have $E = \cup_{n\in\mathbb{N}} nU$.*

**Proof.**

1. Since addition is continuous, for each fixed $y \in E$ and each pair $(x, y) \in E \times E$,
$$(x, y) \longmapsto x + y$$
and its inverse operator
$$(x, y) \longmapsto x - y$$

are continuous. Hence, for each $y \in E$, the translation is a homeomorphism of $E$. Now, let $U$ be a neighborhood of any $x \in E$. Then, for any $x_0 \in U$, there exists $v_0$ such that $x_0 = x + v_0$, where $v_0 \in V$ for some open zero neighborhood $V$. Since translations are continuous, we can therefore write

$$U := x + V : \{x + v : v \in \ V\},$$

where $V$ is a zero neighborhood.

2. We have $0 + 0 = 0$ and we know that the addition is continuous. Hence, by definition, for each neighborhood $U$ of zero in $E$, there exists a neighborhood $V$ of zero in $E$ such that $V + V \subset U$. Let $U,\ V$ be neighborhoods of zero in $E$. Then,

$$\bar{U} \subset U + V = \{u + V : u \in U\} = \cup_{u \in U} (u + V)$$

since any $u \in \bar{U}$ is by definition in some $u + V$ for some zero neighborhood $V$. So, $E$ has a zero neighborhood basis consisting of closed zero neighborhoods.

3. This follows directly from the continuity of the multiplication with $\lambda_0 = 0$, $x_0 = 0$ and $W = \{\lambda v : |\lambda| \leq \varepsilon, v \in V\}$.

4. Let $U$ be a zero neighborhood in $E$. Then, since for each $x \in E$, $\left(\frac{x}{n}\right)$ converges to 0, $x \in nU$ for some $n \in \mathbb{N}$. Hence, $E = \cup_{n \in \mathbb{N}} nU$.

■

**Definition 1** *A locally convex space, $E$, is a topological vector space $E$ in which each point has a neighborhood basis of convex sets.*

A *locally convex topology*, on a $\mathbb{K}$-vector space $E$, is a topology $\mathcal{T}$ on $E$ for which $(E, \mathcal{T})$ is a locally convex space.

**Proposition 3** *For a topological vector space $E$ the following are equivalent:*

1. *$E$ is a locally convex space.*

2. *$E$ has a zero neighborhood basis of convex sets.*

3. *E has a zero neighborhood basis of absolutely convex sets.*

Let $E$ be a locally convex space. A collection $\mathcal{U}$ of zero neighborhoods in $E$ is called a *fundamental system of zero neighborhoods*, if for every zero neighborhood $U$ there exists a $V \in \mathcal{U}$ and an $\varepsilon > 0$ with $\varepsilon V \subset U$.

A family $(\|\cdot\|_\alpha)_{\alpha\in A}$ of continuous semi-norms on $E$ is called a *fundamental system of semi-norms*, if the sets

$$U_\alpha := \{x \in E : \|x\|_\alpha < 1\}, \alpha \in A,$$

form a fundamental system of zero neighborhoods.

**Proposition 4** *Every locally convex space $E$ has a fundamental system of semi-norms. Every fundamental system of semi-norms $(\|\cdot\|_\alpha)_{\alpha\in A}$ has the following properties:*

1. *For every $x \in E$ with $x \neq 0$ there exists an $\alpha \in A$ with $\|x\|_\alpha > 0$.*

2. *For $\alpha, \beta \in A$ there exist $\gamma \in A$ and $C > 0$ with $\max\left(\|\cdot\|_\alpha, \|\cdot\|_\beta\right) \leq C\|\cdot\|_\gamma$.*

**Proposition 5** *Let $E$ be a $\mathbb{K}$-vector space and $(\|\cdot\|_\alpha)_{\alpha\in A}$ be a family of semi-norms on $E$ having properties* 1. *and* 2. *of Proposition* 4. *Then there exists a unique locally convex topology on $E$ for which $(\|\cdot\|_\alpha)_{\alpha\in A}$ is a fundamental system of semi-norms.*

If $(\|\cdot\|_\alpha)_{\alpha\in A}$ is a fundamental system of semi-norms in the locally convex space $E$, then a *net* $(x_\tau)_{\tau\in T}$ *converges* to $x_0 \in E$ if, and only if, $\lim_{\tau\in T}\|x_\tau - x_0\| = 0$, for each $\alpha \in A$, that is for any $\varepsilon > 0$, there exists $\tau_0 \in T$ such that $\|x_\tau - x_0\| < \varepsilon$ whenever $\tau > \tau_0$.

A $\mathbb{K}$-vector space $E$ together with a family of locally convex spaces $(E_i)_{i\in I}$ and linear maps $\pi_i : E \to E_i$, $i \in I$, is called a *projective system*, if for each $x \in E$, $x \neq 0$, there exists an $i \in I$ with $\pi_i(x) \neq 0$. Consider the system of semi-norms

$$\left\{p_M(x) := \max_{i\in M} p_i(\pi_i(x)),\ x \in E,\ p_i \text{ is a continuous semi-norm on } E_i\right\}$$

where $M$ runs through $\mathcal{P}(I)$, the set of all finite subsets of $I$.This system is a fundamental system of semi-norms for a locally convex topology on $E$, which is called the *projective topology* determined by $\{\pi_i : E \to E_i\}_{i\in I}$. We will denote it as:

$$E = limproj_{i\in I}(E_i, \pi_i).$$

**Proposition 6** *Let the locally convex space $E$ have the projective topology of the system $(\pi_i : E \to E_i)_{i\in I}$. Let $F$ be a locally convex space and $T : F \to E$ be a linear map. Then, $T$ is continuous if and only if $\pi_i \circ T$ is continuous for each $i \in I$.*

A $\mathbb{K}$-vector space $E$ together with a family of locally convex spaces $(E_i)_{i\in I}$ and linear maps $\eta_i : E_i \to E$ is called an *inductive system*, if $\cup_{i\in I}\eta_i(E_i) = E$. If a finest locally convex topology for which all the maps $\eta_i$ are continuous exists on $E$, then it is called the *inductive topology* of the system $(\eta_i : E_i \to E)_{i\in I}$. We will denote it as:

$$E = limind_{i\in I}(E_i, \eta_i).$$

**Proposition 7** *Let the locally convex space $E$ have the inductive topology of the system $(\eta_i : E_i \to E)_{i\in I}$. Let $F$ be a locally convex space and $T : E \to F$ be a linear map. Then, $T$ is continuous if and only if $T \circ \eta_i$ is continuous for each $i \in I$.*

We now introduce some special classes of locally convex spaces which will be important for us.

Let $E$ be a locally convex space. If for each absolutely convex zero neighborhood $U$ in $E$ there exists a zero neighborhood $V$ such that for each $\varepsilon > 0$, there exists points $x_1, \cdots, x_n \in V$ such that

$$V \subset \cup_{j=1}^{n}(x_j + \varepsilon U)$$

then $E$ is said to be a *Schwartz space*. For example, the spaces $A(D)$ and $C^\infty(\Omega)$ are Schwartz spaces.

**Definition 2** *Let $E$ be a locally convex space and let $M \subset E$. $M$ is called a barrel if $M$ is absolutely convex, closed, and absorbing. $E$ is said to be barreled if each barrel in $E$ is a zero neighborhood.*

Let $E$ be a locally convex space. If $E$ is a barrelled space in which each bounded set is relatively compact, then it is called a *Montel space*. Note that every Montel space is reflexive. For example, the spaces $A(D)$ and $C^{\infty}(\Omega)$ are Montel spaces.

**Definition 3** *Let $E$ and $F$ be Banach spaces and $A : E \to F$ be a linear map. If there exists sequences $(\lambda_j)_{j\in\mathbb{N}}$ in $E'$ and $(\beta_j)_{j\in\mathbb{N}}$ in $F$ such that $\sum\limits_{j\in\mathbb{N}} \|\lambda_j\| \|\beta_j\| < \infty$, so that*

$$Ax = \sum_{j\in\mathbb{N}} \lambda_j(x)\beta_j \text{ for all } x \in E, \tag{2.1}$$

*then $A$ is called a nuclear operator.* (2.1) *is said to be a nuclear representation of $A$.*

Let $E$ be a locally convex space. Let $p$ be a semi-norm on $E$ and $N_p := \{x \in E : p(x) = 0\}$. A norm is defined on the quotient space $E/N_p$ by $\|x + N_p\|_p := p(x)$. The space $E_p := \left(\widehat{E/N_p}, \|\|_p\right)$ is called the *local Banach space* for the semi-norm $p$. We have $\|\iota^p(x)\|_p = p(x)$, for all $x \in E$, where $\iota^p$ is the canonical map, $\iota^p : E \to E_p$, $\iota^p(x) := x + N_p$. Note that if $p$ and $q$ are semi-norms on $E$ and if $q \geq p$, then the identity map on $E$ induces a continuous linear linking map $\iota^p_q : E_q \to E_p$ between the local Banach spaces determined by the relation $\iota^p_q \circ \iota^p = \iota^q$.

If for each continuous semi-norm $p$ on $E$ there exists a continuous semi-norm $q$ with $q \geq p$, so that $\iota^p_q : E_q \to E_p$ is nuclear, then $E$ is called a *nuclear space*. For example, the spaces $A(D)$ and $C^{\infty}(\Omega)$ are nuclear spaces.

### 2.2 Hilbert Pairs and Scales

**Theorem 7** (*see e.g.* 1001[35]) *Let $H_0$, $H_1$ be a pair of Hilbert spaces with a linear dense compact imbedding $H_1 \hookrightarrow H_0$. Then there exists a system $\{e_k\} \subset H_1$ which is a common orthogonal basis in $H_1$ and $H_0$ such that*

$$\|e_k\|_{H_0} = 1,\ \mu_k = \mu_k(H_0, H_1) := \|e_k\|_{H_1} \nearrow \infty. \tag{2.2}$$

**Proof.** Let $H_0$, $H_1$ be a pair of Hilbert spaces. Define the restriction operator $J : H_1 \to H_0$ as $Jx \equiv x$ for any $x \in H_1$. Then $J$ is a linear dense compact imbedding.

For any $x \in H_1$, $y \in H_0$ the adjoint operator $J^* : H_0 \to H_1$ is defined as

$$\langle Jx, y \rangle_{H_0} = \langle x, J^* y \rangle_{H_1}.$$

Define $A := J^* J$. Then, since $A^* = (J^* J)^* = J^* J = A$, $A$ is self-adjoint. If both $x$ and $y$ are elements of $H_1$, then since $x = Jx$,

$$\begin{aligned} \langle x, y \rangle_{H_0} &= \langle Jx, y \rangle_{H_0} = \langle Jx, Jy \rangle_{H_0} \\ &= \langle x, J^* Jy \rangle_{H_1} = \langle x, Ay \rangle_{H_1} = \langle Ax, y \rangle_{H_1}, \end{aligned}$$

where the last equality follows since $A$ is self-adjoint.

Now, $A$ is compact since it is the superposition of a continuous and a compact operator. Also, since $\langle x, y \rangle_{H_0} = \langle Ax, y \rangle_{H_1}$, for any $x \in H_1$, $\langle Ax, x \rangle_{H_1} \geq 0$ as $\langle Ax, x \rangle_{H_1} = 0$ if and only if $x = 0$.

So, $A$ is a compact, self-adjoint, strictly positively defined operator. Hence there exists a complete orthonormalized sequence of eigenvectors $\{g_k\}$ :

$$Ag_k = \lambda_k g_k, \ k \in \mathbb{N}, \ \lambda_k > 0, \ \lambda_k \to 0.$$

Take $\lambda_k \downarrow 0$. Then,

$$\langle g_k, g_j \rangle_{H_0} = \langle Ag_k, g_j \rangle_{H_1} = \langle \lambda_k g_k, g_j \rangle = \lambda_k \delta_{kj}.$$

So, $\|g_k\|_{H_0} = \sqrt{\lambda_k}$, $\|g_k\|_{H_1} = 1$ and $\{g_k\}$ is a common orthogonal basis in $H_1$ and $H_0$. To renormalize this system, let $e_k := \frac{1}{\lambda_k} g_k$. Then, $\{e_k\}$ is also a common orthogonal basis in $H_1$ and $H_0$ such that

$$\|e_k\|_{H_0} = 1, \ \|e_k\|_{H_1} = \mu_k = \mu_k (H_0, H_1) \nearrow \infty \text{ as } k \to \infty \tag{2.3}$$

where $\mu_k = \frac{1}{\sqrt{\lambda_k}}$. ■

Given a couple of Hilbert spaces $(H_0, H_1)$ with a dense linear continuous imbedding $H_1 \hookrightarrow H_0$ we denote by $H_\alpha = H_0^{1-\alpha} H_1^{\alpha}$, $\alpha \in (-\infty, \infty)$, the *Hilbert scale spanned on* $(H_0, H_1)$ ( [16], [18]). If the imbedding is compact (which is sufficient for all our considerations) this scale can be described especially transparently, since in this case there is a *common orthogonal basis* $\{e_k\}$ for $H_0$ and $H_1$, normalized in

$H_0$ and arranged by non-decreasing of norms in the space $H_1$ as in (2.3). Using this basis the scale is determined by the norms

$$\|x\|_{H_\alpha} := \left({}_{k=1}^{\infty} |\xi_k|^2 \ \mu_k^{2\alpha}\right)^{1/2}, \ x = {}_{k=1}^{\infty} \xi_k \ e_k \tag{2.4}$$

(in the case $\alpha \geq 0$ the space $H_\alpha$ consists of $x \in H_0$ with a finite norm (2.4); for $\alpha < 0$ the space $H_\alpha$ is the completion of $H_0$ by the norm (2.4)).

# CHAPTER 3

# SOME TOPICS OF POTENTIAL AND PLURIPOTENTIAL THEORY

In this chapter, some of the notions about Potential and Pluripotential Theory that will be useful for our considerations will be given ( [12], [27], [23], [28], [36]).

## 3.1 Potential Theory

Since potential theory may be defined as the study of harmonic functions, subharmonic functions, and capacities, we will first define what these are. As a subharmonic function is semi-continuous as part of its definition, first we have to define semi-continuous functions.

### 3.1.1 Semi-continuous Functions

Let $X$ be a topological space. We say that a function $u : X \to [-\infty, \infty)$ is upper semi-continuous if the set $\{x \in X : u(x) < \alpha\}$ is open in $X$ for each $\alpha \in \mathbb{R}$. Also $v : X \to (-\infty, \infty]$ is lower semi-continuous if $-v$ is upper semi-continuous.

Notice that $u$ is continuous if and only if it is both upper and lower semi-continuous.

#### Properties of Semi-continuous Functions

1. A function $u(x)$ that is upper semi-continuous on a compactum $K$ attains its maximum value at $K$.

2. The lower envelope

$$u(x) = \inf_{\alpha} u_{\alpha}(x)$$

of a family $\{u_\alpha(x)\}$ of functions that are upper semi-continuous on a set $A$ is upper semi-continuous in $A$.

3. The limit of a decreasing sequence of upper semi-continuous functions defined on a set $A$ is upper semi-continuous in $A$.

4. If $u(x)$ is upper semi-continuous on a compactum $K$ and $u(x) < \infty$ in $K$, there exists a decreasing sequence of continuous functions that converges to $u(x)$.

### 3.1.2 Subharmonic Functions

Let $U$ be an open subset of $\mathbb{C}$. A function $u : U \to [-\infty, \infty)$ is called subharmonic ($u \in S(U)$) if it is upper semi-continuous and satisfies the local submean inequality, i.e. given $w \in U$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it})dt \qquad (0 \leq r < \rho).$$

Also $v : U \to (-\infty, \infty]$ is superharmonic if $-v$ is subharmonic.

Notice that a function is harmonic if and only if it is both subharmonic and superharmonic function.

**Properties of Subharmonic Functions**

1. If $f$ is analytic on an open set $U$ in $\mathbb{C}$, then $\log|f|$ is subharmonic on $U$.

2. Let $u$ and $v$ be subharmonic functions on an open set $U$ in $\mathbb{C}$. Then:

   (a) $\max(u, v)$ is subharmonic on $U$;

   (b) $\alpha u + \beta v$ is subharmonic on $U$ for all $\alpha$, $\beta \geq 0$.

3. (Maximum Principle) Let $u$ be a subharmonic function on a domain $D$ in $\mathbb{C}$.

   (a) If $u$ attains a global maximum on $D$, then $u$ is constant.

   (b) If $\overline{\lim_{z \to \zeta}} u(z) \leq 0$ for all $\zeta \in \partial D$, then $u \leq 0$ on $D$.

4. Let $U$ be an open subset of $\mathbb{C}$, and let $u : U \to [-\infty, \infty)$ be an upper semicontinuous function. Then the following are equivalent.

   (a) The function $u$ is subharmonic on $U$.

   (b) Whenever $\bar{\Delta}(w, \rho) \subset U$, then for $r < \rho$ and $0 \le t < 2\pi$,

$$u(w + re^{it}) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} u(w + \rho e^{i\theta})\theta d\theta.$$

   (c) Whenever $D$ is a relatively compact subdomain of $U$, and $h$ is a harmonic function on $D$ satisfying

$$\varlimsup_{z \to \zeta} (u - h)(z) \le 0 \qquad (\zeta \in \partial D)$$

   then $u \le h$ on $D$.

5. (Global Submean Inequality) If $u$ is a subharmonic function on an open set $U$ in $\mathbb{C}$, and if $\bar{\Delta}(w, \rho) \subset U$, then

$$u(w) \le \frac{1}{2\pi} \int_0^{2\pi} u(w + \rho e^{i\theta}) d\theta.$$

6. Let $U$ be an open subset of $\mathbb{C}$, and let $u \in C^2(U)$. Then $u$ is subharmonic on $U$ if and only if $\Delta u \ge 0$ on $U$.

7. (Gluing Theorem) Let $u$ be a subharmonic function on an open set $U$ in $\mathbb{C}$, and let $v$ be a subharmonic function on an open subset $V$ of $U$ such that

$$\varlimsup_{z \to \zeta} v(z) \le u(\zeta) \qquad (\zeta \in U \cap \partial V).$$

   Then $\tilde{u}$ is subharmonic on $U$, where

$$\tilde{u} = \begin{cases} \max(u, v) & \text{on } V, \\ u & \text{on } U \setminus V. \end{cases}$$

8. Let $(u_n)_{n \ge 1}$ be subharmonic functions on an open set $U$ in $\mathbb{C}$, and suppose that $u_1 \ge u_2 \ge u_3 \ge \ldots$ on $U$. Then $u := \lim_{n \to \infty} u_n$ is subharmonic on $U$.

9. Let $\{u_k\}_{k\in\mathbb{N}}$ be a family of subharmonic functions that are locally uniformly bounded above in the domain $G$. Then the upper envelope

$$u(z) = \overline{\lim_{z\to z'}} \sup_k u_k(z')$$

is a subharmonic function in $G$.

**Theorem 8** *(Hartogs' Theorem) Let $v_k$ be a sequence of subharmonic functions in $\Omega$ which are uniformly bounded above from above on every compact subset of $\Omega$, and assume that $\overline{\lim\limits_{k\to\infty}} v_k(z) \leq C$ for every $z \in \Omega$. For every $\varepsilon > 0$ and every compactum $K \subset \Omega$, one can then find $k_0$ so that*

$$v_k(z) \leq C + \varepsilon, \qquad z \in K,\ k > k_0.$$

**Theorem 9** *Let $\{u_k\}_{k\in\mathbb{N}}$ be a family of subharmonic functions that are locally uniformly bounded above in the domain $G$. Then the regularized limit superior*

$$v(z) = \overline{\lim_{z'\to z}}\,\overline{\lim_{k\longrightarrow\infty}} u_k(z')$$

*is a subharmonic function in $G$.*

**Proof.** The function $v(z) < +\infty$ and is upper semi-continuous in $G$. $\{u_k\}$ are measurable functions since they are subharmonic. Using the fact that they satisfy the local submean inequality,

$$\begin{aligned}
v(z) &= \overline{\lim_{z'\longrightarrow z}}\,\overline{\lim_{k\longrightarrow\infty}} u_k(z') \leq \overline{\lim_{z'\longrightarrow z}}\,\overline{\lim_{k\longrightarrow\infty}} \frac{1}{2\pi}\int_0^{2\pi} u_k(z'+re^{i\theta})d\theta \\
&\leq \overline{\lim_{z'\longrightarrow z}} \frac{1}{2\pi}\int_0^{2\pi} \overline{\lim_{k\longrightarrow\infty}} u_k(z'+re^{i\theta})d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \overline{\lim_{z'\longrightarrow z}}\,\overline{\lim_{k\longrightarrow\infty}} u_k(z'+re^{i\theta})d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} v(z+re^{i\theta})d\theta,
\end{aligned}$$

where the second inequality follows from Fatou's Lemma. Hence, $v(z)$ is subharmonic. ■

### 3.1.3 Logarithmic Potential

Now we will define logarithmic potentials. They provide an important source of examples of subharmonic functions thereby allow us to construct subharmonic functions with prescribed properties. Also, logarithmic potentials turn out to be almost as general as arbitrary subharmonic functions and for many purposes the two classes are equivalent.

We will define logarithmic potentials only for finite measures of compact support.

Let $\mu$ be a finite Borel measure on $\mathbb{C}$ with compact support. Its logarithmic potential is the function

$$p_{\mu}(z) = \int \log|z-w|\, d\mu(w) \quad (z \in \mathbb{C}).$$

**Properties of Logarithmic Potentials**

1. With the notation above, $p_\mu$ is subharmonic on $\mathbb{C}$, and harmonic on $\mathbb{C}\backslash(\mathrm{supp}\mu)$. Also
$$p_{\mu}(z) = \mu(\mathbb{C}) \log|z| + O(|z|^{-1}) \quad \text{as } z \to \infty.$$

2. (Continuity Principle) Let $\mu$ be a finite Borel measure on $\mathbb{C}$ with compact support $K$.

   (a) If $\zeta_0 \in K$, then $\liminf\limits_{z\to\zeta_0} p_{\mu}(z) = \liminf\limits_{\substack{\zeta\to\zeta_0 \\ \zeta\in K}} p_{\mu}(\zeta)$.

   (b) If further $\lim\limits_{\substack{\zeta\to\zeta_0 \\ \zeta\in K}} p_{\mu}(\zeta) = p_{\mu}(\zeta_0)$, then $\lim\limits_{z\to\zeta_0} p_{\mu}(z) = p_{\mu}(\zeta_0)$.

3. (Minimum Principle) Let $\mu$ be a finite Borel measure on $\mathbb{C}$ with compact support $K$. If $p_\mu \geq M$ on $K$, then $p_\mu \geq M$ on the whole of $\mathbb{C}$.

### 3.1.4 Equilibrium Measures

Let $\mu$ be a finite Borel measure on $\mathbb{C}$ with compact support. Its energy $I(\mu)$ is given by

$$I(\mu) = \int\int \log|z-w|\,d\mu(z)\,d\mu(w) = \int p_\mu(z)\,d\mu(z).$$

Now, let $K$ be a compact subset of $\mathbb{C}$, and denote by $\mathcal{P}(K)$ the collection of all Borel probability measures on $K$. If there exists $\nu \in \mathcal{P}(K)$ such that

$$I(\nu) = \sup_{\mu\in\mathcal{P}(K)} I(\mu),$$

then $\nu$ is called an equilibrium measure for $K$.

**Properties of Equilibrium Measures**

1. If the sequence $(\mu_n)_{n\geq 1}$ in $\mathcal{P}(K)$ is weak*-convergent to $\mu$ in $\mathcal{P}(K)$, i.e.

$$\int_K \phi d\mu_n \longrightarrow \int_K \phi d\mu \text{ for each } \phi \in C(K),$$

   where $C(K)$ is the space of continuous functions with the usual sup-norm, then $\overline{\lim} I(\mu_n) \leq I(\mu)$.

2. Every compactum $K$ in $\mathbb{C}$ has an equilibrium measure.

### 3.1.5 Logarithmic Capacity

The logarithmic capacity of a subset $E$ of $\mathbb{C}$ is given by

$$C(E) := \sup_\mu e^{I(\mu)},$$

where the supremum is taken over all Borel probability measures $\mu$ on $\mathbb{C}$ whose support is a compact subset of $E$. In particular, if $K$ is a compactum with equilibrium measure $\nu$, then

$$C(K) = e^{I(\nu)}.$$

**Properties of Logarithmic Capacity**

1. (a) If $E_1 \subset E_2$ then $C(E_1) \leq C(E_2)$.

(b) If $E \subset \mathbb{C}$ then $C(E) = \sup\{C(K): \text{ compact } K \subset E\}$

(c) If $E \subset \mathbb{C}$ then $C(\alpha E + \beta) = |\alpha| C(E)$ for all $\alpha,\ \beta \in \mathbb{C}$.

(a) If $K$ is a compact subset of $\mathbb{C}$ then $C(K) = C(\partial_\varepsilon K)$.

(b) If $K_1 \supset K_2 \supset K_3 \supset \cdots$ are compact subsets of $\mathbb{C}$ and $K = \cap_n K_n$, then $C(K) = \lim_{n\to\infty} C(K_n)$.

(c) If $B_1 \subset B_2 \subset B_3 \cdots$ are Borel subsets of $\mathbb{C}$ and $B = \cup_n B_n$, then$C(B) = \lim_{n\to\infty} C(B_n)$.

2. Let $(B_n)$ be a (finite or infinite) sequence of Borel subsets of $\mathbb{C}$, let $B = \cup_n B_n$, and let $d > 0$.

(a) If $diam(B) \leq d$, then $C(B) \leq d$ and

$$\frac{1}{\ln(d/C(B))} \leq \sum_n \frac{1}{\ln(d/C(B_n))}.$$

(b) If $dist(B_j, B_k) \geq d$ whenever $j \neq k$, then

$$\frac{1}{\ln^+(d/C(B))} \geq \sum_n \frac{1}{\ln^+(d/C(B_n))}.$$

#### 3.1.6 Transfinite Diameter

**Definition 4** *Let $K$ be a compact subset of $\mathbb{C}$, and let $n \geq 2$. The $n$-th diameter of $K$ is given by*

$$\delta_n(K) := \sup\left\{\prod_{j,k;j<k} |w_j - w_k|^{\frac{2}{n(n-1)}} : w_1, \ldots w_n \in K\right\}.$$

*An $n$-tuple $w_1, \ldots w_n \in K$ for which the supremum is attained is called a Fekete $n$-tuple for $K$.*

As $K$ is compact, there always exists a set of points such that the supremum is attained, but that set is not necessarily unique. The decreasing sequence $\{\delta_n(K)\}_{n=2}^{\infty}$ has a limit that is called the *transfinite diameter*.

**Theorem 10** *(Fekete-Szegö Theorem) Let $K$ be a compact subset of $\mathbb{C}$. Then the sequence $(\delta_n(K))_{n\geq 2}$ is decreasing, and*

$$\lim_{n\to\infty} \delta_n(K) = C(K).$$

For compactum $K$ in $\widehat{\mathbb{C}}$, a *Fekete polynomial* for $K$ of degree $n$ is a polynomial of the form

$$q(z) = \prod_{i=1}^{n} (z - w_i),$$

where $w_1, w_2, \cdots, w_n$ is a Fekete n-tuple for $K$. Then the following useful result is valid, where we define

$$\|q\|_K := \sup\{|q(z)| : z \in K\}.$$

**Lemma 1** *Let $K$ be a compact subset of $\widehat{\mathbb{C}}$.*

1. *If $q$ is a monic polynomial of degree $n \geq 1$, then $\|q\|_K^{\frac{1}{n}} \geq C(K)$.*

2. *If $q$ is a Fekete polynomial of degree $n \geq 2$, then $\|q\|_K^{\frac{1}{n}} \leq \delta_n(K)$.*

The following is another characterization of capacity.

**Proposition 8** *Let $K$ be a compact subset of $\mathbb{C}$, and for each $n \geq 1$ let*

$$m_n(K) = \inf\{\|q\|_K : \ q \textit{ is a monic polynomial of degree } n\}.$$

*Then*

$$\lim_{n\to\infty} m_n(K)^{\frac{1}{n}} = \inf_{n\geq 1} m_n(K)^{\frac{1}{n}} = C(K).$$

A monic polynomial $q$ of degree $n$ for which $\|q\|_K = m_n(K)$ is called a Chebyshev polynomial.

#### 3.1.7 Polar Sets

Polar sets play the role of negligible sets in potential theory.

1. A subset $E$ of $\mathbb{C}$ is called polar if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ for which supp$\mu$ is a compact subset of $E$.

2. A property is said to hold nearly everywhere (n.e.) on a subset $S$ of $\mathbb{C}$ if it holds everywhere on $S\backslash E$, for some Borel polar set $E$.

**Properties of Polar Sets**

1. Let $\mu$ be a finite Borel measure on $\mathbb{C}$ with compact support, and suppose that $I(\mu) > -\infty$. Then $\mu(E) = 0$ for every Borel polar set $E$.

2. Every Borel polar set has Lebesgue measure zero.

3. A countable union of Borel polar set is polar. In particular, every countable subset of $\mathbb{C}$ is polar.

Note that though every countable set is polar, not every polar set is countable.

**Lemma 2** *Let $C(K) > 0$ and let $\Psi(z)$ be a function subharmonic and bounded above in $D \diagdown K$, satisfying the condition*

$$\lim_{z \to z_0} \Psi(z) = -\infty$$

*for each $z_0 \in \partial K^*$. Then $\Psi(z) \equiv -\infty$ in $D \diagdown K$.*

**Proof.** Assume that $\Psi(z) \not\equiv -\infty$ in $D \diagdown K$. Then, extend the function onto $K$ such that:

$$\bar{\Psi}(z) = \begin{cases} \Psi(z), & z \in D \diagdown K, \\ -\infty, & z \in K. \end{cases}$$

Then, $\bar{\Psi}$ is subharmonic in the whole region $D$. Since $\bar{\Psi}(z) = -\infty$ for $z \in K$, $K$ is a polar set and so must have zero capacity. But, that contradicts with the fact that $C(K) > 0$. This contradiction shows that $\Psi(z) \equiv -\infty$ in $D \diagdown K$. ■

### 3.1.8 Solution of The Dirichlet Problem and Regularity

Let $D$ be a subdomain of $\mathbb{C}$, and let $\phi : \partial D \to \mathbb{R}$ be a continuous function. The *Dirichlet problem* is to find a harmonic function $h$ on $D$ such that $\lim\limits_{z \to \zeta} h(z) = \phi(\zeta)$ for all $\zeta \in \partial D$.

For "nice" domains, a solution always exists. Also, it is easily seen that there exists at most one solution to the Dirichlet problem.

We will now introduce the Perron method that can be used to solve the problem.

Let $D$ be a proper subdomain of $\widehat{\mathbb{C}}$, and let $\phi : \partial D \longrightarrow \mathbb{R}$ be a bounded function. The associated *Perron function (the generalized solution of the Dirichlet problem)* $H_D\phi : D \longrightarrow \mathbb{R}$ is defined by

$$H_D\phi = \sup_{u \in \mathcal{U}} u,$$

where $\mathcal{U}$ denotes the family of all subharmonic functions $u$ on $D$ such that $\overline{\lim_{z \longrightarrow \zeta}} u(z) \leq \phi(\zeta)$ for each $\zeta \in \partial D$.

If the Dirichlet problem has a solution, that it should be $H_D\phi$. If $h$ is such a solution, then $h \in \mathcal{U}$ and so $h \leq H_D\phi$. On the other hand, by the maximum principle, if $u \in \mathcal{U}$ then $u \in h$ on $D$ so $H_D\phi \leq h$. Hence, $H_D\phi = h$.

The following result shows that $H_D\phi$ is always a bounded harmonic function.

**Lemma 3** *(Poisson Modification) Let $D$ be a domain in $\mathbb{C}$, let $\Delta$ be an open disc with $\bar{\Delta} \subset D$, and let $u$ be a subharmonic function on $D$ with $u \not\equiv -\infty$. If we define $\tilde{u}$ on $D$ by*

$$u = \begin{cases} P_\Delta u & \text{on } \Delta, \\ u & \text{on } D \backslash \Delta, \end{cases}$$

*then $\tilde{u}$ is subharmonic on $D$, harmonic on $\Delta$, $\tilde{u} \geq u$ on $D$.*

**Theorem 11** *Let $D$ be a proper subdomain of $\mathbb{C}_\infty$, and let $\phi : \partial D \longrightarrow \mathbb{R}$ be a bounded function. Then $H_D\phi$ is harmonic on $D$, and*

$$\sup_D |H_D\phi| \leq \sup_{\partial D} |\phi|.$$

The following notion will be needed so that the Perron function will have the prescribed boundary limits.

Let $D$ be a proper subdomain of $\widehat{\mathbb{C}}$, and let $\zeta_0 \in \partial D$. A *barrier* at $\zeta_0$ is a subharmonic function $b$ defined on $D \cap N$, where $N$ is an open neighborhood of $\zeta_0$, satisfying

$$b < 0 \text{ on } D \cap N \text{ and } \lim_{z \longrightarrow \zeta_0} b(z) = 0.$$

A boundary point at which a barrier exists is called *regular*, otherwise it is *irregular*. If every $\zeta \in \partial D$ is regular, then $D$ is called a *regular domain.*

We say that a *pair* $(K, D)$ *is regular* if $K$ and $D$ are regular, $D$ has no components free from $K$, and $\hat{K}_D = K$, i.e. $K$ has no disjoint holes with $D$.

**Lemma 4** *If $D$ is a proper subdomain of $\widehat{\mathbb{C}}$ and $\phi : \partial D \longrightarrow \mathbb{R}$ is a bounded function, then*

$$H_D\phi \leq -H_D(-\phi) \text{ on } D.$$

**Lemma 5** *(Bouligand's Lemma) Let $\zeta_0$ be a regular boundary point of a domain $D$, and let $N_0$ be an open neighborhood of $\zeta_0$. Then, given $\varepsilon > 0$, there exists a subharmonic function $b_\varepsilon$ on $D$ such that*

$$b_\varepsilon < 0 \text{ on } D,\ b_\varepsilon \leq -1 \text{ on } D \setminus N_0,\ \text{and } \liminf_{z \longrightarrow \zeta_0} b_\varepsilon(z) \geq -\varepsilon.$$

**Theorem 12** *Let $D$ be a proper subdomain of $\widehat{\mathbb{C}}$, and let $\zeta_0$ be a regular boundary point of $D$. If $\phi : \partial D \longrightarrow \mathbb{R}$ is a bounded function which is continuous at $\zeta_0$, then*

$$\lim_{z \longrightarrow \zeta_0} H_D\phi(z) = \phi(\zeta_0).$$

**Green's Functions**

Let $D$ be a proper subdomain of $\widehat{\mathbb{C}}$. A *Green's function* for $D$ is a map $g_D : D \times D \to (-\infty, \infty]$, such that for each $w \in D$:

1. $g_D(\cdot, w)$ is harmonic on $D \setminus \{w\}$, and bounded outside each neighborhood of $w$;
2. $g_D(w, w) = \infty$, and as $z \to w$, $g_D(z, w) = \begin{cases} \ln|z| + O(1), & w = \infty, \\ -\ln|z - w| + O(1), & w \neq \infty; \end{cases}$
3. $g_D(z, w) \to 0$ as $z \to \zeta$, for n.e. $\zeta \in \partial D$.

**Properties of Green's Functions**

1. If $D$ is a domain in $\widehat{\mathbb{C}}$ such that $\partial D$ is non-polar, then there exists a unique Green's function $g_D$ for $D$.
2. Let $D$ be a domain in $\widehat{\mathbb{C}}$ such that $\partial D$ is non-polar. Then

$$g_D(z, w) > 0 \text{ for } z, w \in D.$$

3. Let $D$ be a domain in $\widehat{\mathbb{C}}$ such that $\partial D$ is non-polar, and let $(D_n)_{n\geq 1}$ be subdomains of $D$ such that $D_1 \subset D_2 \subset D_3 \cdots$ and $\cup_n D_n = D$. Then

$$\lim_{n\to\infty} g_{D_n}(z,w) = g_D(z,w) \text{ for } z,w \in D.$$

4. Let $D$ be a domain in $\widehat{\mathbb{C}}$ such that $\partial D$ is non-polar. Then

$$g_D(z,w) = g_D(w,z) \text{ for } z,w \in D.$$

5. Let $D$ be a domain in $\widehat{\mathbb{C}}$ such that $\partial D$ is non-polar, let $w \in D$, and let $\zeta \in \partial D$. Then

$$\lim_{z\to\zeta} g_D(z,w) = 0$$

if and only if $\zeta$ is a regular boundary point of $D$.

**Definition 5** *Let $K \subset D$ be a couple "compact set-open set". The Green potential of that couple is*

$$\omega(z) = \omega(D,K,z) = \overline{\lim_{\zeta\to z}} \sup\{u(\zeta) : u \in S(K,D)\},\ z \in D$$

*where $S(K,D)$ denotes a class of subharmonic functions in $D$ that are nonpositive on $K$ and are bounded above by* 1.

If $z \in D \setminus \hat{K}_D$ then this function coincides with the traditional generalized harmonic measure $\omega\left(z, \partial D, D \setminus \hat{K}_D\right)$, i.e. $\omega(z) = \omega(D,K,z)$ will be the generalized solution of the Dirichlet problem in the region $D \setminus K$ with respect to the function

$$f(z) = \begin{cases} 1 & z \in \partial D, \\ 0 & z \in \partial K^*. \end{cases}.$$

**Remark 1** *The following three cases are possible; we give various equivalent characterizations of each case:*

**(a)** *$\partial K^*$ consists only of regular points $\Longleftrightarrow$ $K$ is a regular compactum $\Longleftrightarrow$ $0 < \omega(z) < 1$ for $z \in D \setminus K$ and $\lim_{z\to z_0} \omega(z) = 0$ for each $z_0 \in \partial K^*$.*

**(b)** *$\partial K^*$ consists only of irregular points $\Longleftrightarrow C(K) = 0 \Longleftrightarrow \omega(z) \equiv 1$ for $z \in D \setminus K$.*

**(c)** *$\partial K^*$ contains both irregular and regular points $\Longleftrightarrow K$ is not regular and $C(K) > 0 \Longleftrightarrow 0 < \omega(z) < 1$ for $z \in D \setminus K$ and there exists a point $z_0 \in \partial K^*$ and a sequence $\{z_n\} \subset D \setminus K,\ z_k \to z_0$ such that $\omega(z_k) \to \alpha_0$.*

**Lemma 6** *Let $\omega(z) = \omega(D, K, z)$ be the Green potential of the "compact set-open set" couple $K \subset D$. Let $D_q$ be a sequence of open sets exhausting $D$ where $D_q := \{z \in D : 0 < \omega(D, K, z) < q\}$, $K_\delta$ be a sequence of compact sets $K_\delta \downarrow K$ such that $K_\delta := \{z \in D : 0 < \omega(D, K, z) \leq \delta\}$. Then,*

$$\omega(D_q, K, z) = \frac{1}{q}\omega(D, K, z),\ z \in D_q,$$

$$\omega(D, K_\delta, z) = \begin{cases} \frac{\omega(D,K,z)-\delta}{1-\delta}, & z \in D \setminus K_{\delta,} \\ 0, & z \in K_{\delta.} \end{cases},$$

$$\omega(D_q, K_\delta, z) = \begin{cases} \frac{\omega(D,K,z)-\delta}{q-\delta}, & z \in D_q \setminus K_{\delta,} \\ 0, & z \in K_{\delta.} \end{cases}.$$

**Theorem 13** *(Hadamard Inequality)Let $D$ be a domain in $\widehat{\mathbb{C}}$, $K$ be a compactum in $D$. If $f \in A(D)$, then*

$$|f|_{D_\alpha} \leq |f|_K^{1-\alpha} |f|_D^\alpha,$$

*where*

$$D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\}, 0 < \alpha < 1.$$

Theorem 13 is also known as the two constant theorem.

**Theorem 14** *(Evans' Lemma) Let $E$ be a compact polar set. Then there exists a Borel probability measure $\mu$ on $E$ such that $p_\mu(z) = -\infty$ for all $z \in E$. Moreover, this measure can be taken discrete, i.e. there is a sequence $\{\zeta_\nu\} \subset E$ and a sequence of nonnegative numbers $\{\alpha_\nu\}$, $\sum\limits_{\nu=1}^{\infty} \alpha_\nu = 1$ such that $\mu = \sum\limits_{\nu=1}^{\infty} \alpha_\nu \mu_{\zeta_\nu}$ where $\mu_\zeta$ means an atomic measure at the point $\zeta$.*

### 3.2 Some Facts of Pluripotential Theory

#### 3.2.1 Plurisubharmonic Functions

Plurisubharmonic functions are the complex analogue of convex functions of several variables and the multidimensional analogue of subharmonic functions.

**Definition 6** *Let $D$ be a domain in $\mathbb{C}^n$. A function $u : D \to [-\infty, \infty]$ is called plurisubharmonic ($u \in P(D)$) if $u$ is upper semi-continuous in $D$ and for any point $z^0 \in D$ and for any complex line $z = l(\zeta) = z^0 + \omega\zeta$, where $\omega \in \mathbb{C}^n$, $\zeta \in \mathbb{C}$, the restriction of $u$ to this line, i.e., the function $u \circ l(\zeta)$, is subharmonic on the open set $\{\zeta \in \mathbb{C} : l(\zeta) \in D\}$.*

**Properties of Plurisubharmonic Functions**

1. For a function $u \in C^2(D)$ to be plurisubharmonic it is necessary and sufficient that at each point $z \in D$ the form $H_z(u, \omega)$ satisfy
$$H_z(u, \omega) = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 u(z)}{\partial z_\mu \partial \bar{z}_\nu}\Big|_z \omega_\mu \bar{\omega}_\nu \geq 0 \text{ for all } \omega \in \mathbb{C}^n.$$

2. If $u$ is a plurisubharmonic function in a domain $D$ and $u$ attains a local maximum at some point $z^0 \in D$, then it is constant in $D$.

3. A function that is plurisubharmonic in some neighborhood of each point $z^0 \in D$ is plurisubharmonic in the domain $D$.

4. If the upper envelope $u(z) = \sup\limits_{\alpha \in A} u_\alpha(z)$ of a family of functions $u_\alpha$, $\alpha \in A$, that are plurisubharmonic in a domain $D$, is upper semi-continuous in $D$, then it is plurisubharmonic in $D$.

5. For an upper semi-continuous function $u$ to be plurisubharmonic in a domain $D$ it is necessary and sufficient that for each point $z \in D$ and each vector $\omega \in \mathbb{C}^n$ there exist a number $r_0 = r_0(z, \omega)$ such that
$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u\left(z + \omega r e^{it}\right) dt$$

for all $r < r_0$.

6. For any function $u$ that is plurisubharmonic in a neighborhood of a point $z^0 \in \mathbb{C}^n$ the value $u(z^0)$ does not exceed its mean value on the sphere $\{|z - z^0| = r\}$ of sufficiently small radius $r$ :

$$u\left(z^0\right) \leq \int\limits_{\{|z-z^0|=r\}} u(z)\, d\sigma,$$

where $\sigma(r)$ is the area of this sphere and $d\sigma$ is the area element.

7. Any plurisubharmonic function in a domain $D \subset \mathbb{C}^n$ is a subharmonic function of $2n$ real variables, i.e. , for any point $z^0 \in D$ and ball $B = \{|z - z^0| = r\}$ of sufficiently small radius any function $h$ that is harmonic in $B$ and continuous in $\bar{B}$ possesses the property

$$u\mid_{\partial B} \leq h\mid_{\partial B} \Longrightarrow u\mid_{B} \leq h\mid_{B} .$$

8. If the function $u$ is plurisubharmonic in a neighborhood of a point $z^0 \in \mathbb{C}^n$, then its mean value $S(r)$ on the sphere $\{|z - z^0| = r\}$ is an increasing function of $r$.

9. For any function $u$ that is plurisubharmonic in a domain $D \subset \mathbb{C}^n$ we can construct an increasing sequence of open sets $G_\mu$ $(\mu = 1, 2, \cdots), \cup_{\mu=1}^{\infty} G_\mu = D$, and a decreasing sequence of functions $u_\mu \in C^\infty(G_\mu)$, plurisubharmonic in $G_\mu$, converging to $u$ at each point $z \in D$ :

$$u_\mu(z) \to u(z), \; u_{\mu+1} \leq u_\mu.$$

10. If the function $u$ is plurisubharmonic in a domain $D \subset \mathbb{C}^n$, and $v : u(D) \to \mathbb{R}$ is an increasing convex function of class $C^2$, then $v \circ u$ is plurisubharmonic in $D$.

11. The restriction of a plurisubharmonic function $u$ in a domain $D \subset \mathbb{C}^n$ to any $m$-dimensional holomorphic surface $f : G \to \mathbb{C}^n$, $G \subset \mathbb{C}^m$, is also a plurisubharmonic function on an open set $\Omega = \{\zeta \in G : f(\zeta) \in D\}$.

12. (Grauert-Remmert) Any function that is plurisubharmonic in a domain $D \subset \mathbb{C}^n$ everywhere except for an analytic set and is bounded can be extended to a function that is plurisubharmonic in $D$.

A function $u \in P(D)$ is called *maximal* in $D$ or MP-function $(u \in MP(D))$ if for any subdomain $G \Subset D$ and for any function $v \in P(G)$ from $v(z) \leq u(z)$, $z \in \partial G$, it follows that $v(z) \leq u(z)$ in $G$. (1001[24])

#### 3.2.2 Green Pluripotential

Let $E$ be a set in the complex manifold $\Omega$ (see pg. 35). Then the *Green pluripotential* of this set with respect to $\Omega$ is the function

$$\omega(z) = \omega(\Omega, E, z) := \overline{\lim_{\zeta \to z}} \omega^0(\Omega, E, \zeta), \ z \in \Omega, \tag{3.1}$$

where

$$\omega^0(z) = \omega^0(\Omega, E, \zeta) = \sup\{u(z) : u \in P(E, \Omega)\},$$

$$P(E, \Omega) = \{u \in P(\Omega) : u|_E \leq 0; u(z) < 1, \ z \in \Omega\}.$$

The function (3.1) is also called the *pluripotential of a condenser* $(K, D)$.

**Theorem 15** *(Multidimensional Analogue of Hadamard Inequality) Let $\Omega$ be a Stein manifold (see pg. 36), $K$ be a compactum in $\Omega$. If $f$ is a bounded analytic function on $\Omega$ then*

$$|f(z)| \leq (|f|_\Omega)^{\omega(z)} (|f|_K)^{1-\omega(z)}, \ z \in \Omega,$$

*where $\omega(z)$ is the Green pluripotential of $K$ with respect to $\Omega$.*

#### 3.2.3 Pluriregularity

A Stein manifold $\Omega$ is called *pluriregular* (or *strongly pseudoconvex)* if there exists a negative plurisubharmonic function $u \in P(\Omega)$ such that $u(z_\nu) \to 0$ for every sequence $\{z_\nu\} \subset \Omega$ without limit points in $\Omega$. Briefly it will be written by the following

$$\lim_{z \to \partial\Omega} u(z) = 0.$$

A compactum $K$ in a Stein manifold $\Omega$ is said to be

1. *pluriregular* on $\Omega$ if for some open neighborhood $D \Subset \Omega$ of $K$ it follows that $\omega(D, K, z) \equiv 0$ on $K$;

2. *strongly pluriregular* on $\Omega$ if for any open neighborhood $D \Subset \Omega$ of $K$ it follows for the envelope of holomorphy $\tilde{D}$ that

$$\omega\left(\tilde{D}, K, z\right) \equiv 0,\ \ z \in K.$$

**Definition 7** *We say that a pair $(K, \Omega)$ is pluriregular if $K$ is a pluriregular holomorphically convex compactum on the pluriregular Stein manifold $\Omega$ and also every connected component of $\Omega$ has a non-empty intersection with $K$.*

**Theorem 16** *[37] If $(K, \Omega)$ is a pluriregular pair then the function $\omega(z) = \omega(\Omega, K, z)$ is continuous in $\Omega$ and satisfies the conditions*

$$\omega(z) = 0,\ \ z \in K;\ \ 0 < \omega(z) < 1,\ \ z \in \Omega \setminus K;\ \ \lim_{z \to \partial\Omega} \omega(z) = 1.$$

# CHAPTER 4

# SPACES OF ANALYTIC FUNCTIONS

In this chapter, we will define complex manifolds following [12], then introduce spaces of analytic functions. Also, a detailed proof of Grothendieck-Köthe-Silva Duality which realizes the space $A(E)^*$, for any set $E \in \hat{\mathbb{C}}$, as the space of analytic functions $A(E^*)$ where $E^* = \widehat{\mathbb{C}} \setminus E$ will be given for open and compact sets.

## 4.1 Complex Manifolds

A topological space $\Omega$ is called a manifold of dimension $n$ if every point in $\Omega$ has a neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$. The concept of complex analytic manifolds is defined by means of a family of such homeomorphisms:

**Definition 8** *A manifold $\Omega$ (of dimension $2n$) is called a complex analytic manifold of complex dimension $n$ if there is given a family $\mathcal{F}$ of homeomorphisms $\kappa$, called complex analytic coordinate systems, of open sets $\Omega_\kappa \subset \Omega$ on open sets $\tilde{\Omega}_\kappa \subset \mathbb{C}^n$ such that*

1. *If $\kappa$ and $\kappa' \in \mathcal{F}$, then the mapping*

$$\kappa' \kappa^{-1} : \kappa\left(\Omega_\kappa \cap \Omega_{\kappa'}\right) \to \kappa'\left(\Omega_\kappa \cap \Omega_{\kappa'}\right)$$

   *between open sets in $\mathbb{C}^n$ is analytic (interchanging $\kappa$ and $\kappa'$ we find that the inverse mapping is also analytic).*

2. $\cup_{\kappa \in \mathcal{F}} \Omega_\kappa = \Omega$.

3. *If $\kappa_0$ is a homeomorphism of an open set $\Omega_{\kappa_0} \subset \Omega$ onto an open set in $\mathbb{C}^n$ and the mapping*

$$\kappa\kappa_0^{-1} : \kappa_0\left(\Omega_{\kappa_0} \cap \Omega_\kappa\right) \to \kappa\left(\Omega_{\kappa_0} \cap \Omega_\kappa\right)$$

*as well as its inverse are analytic for every $\kappa \in \mathcal{F}$, it follows that $\kappa_0 \in \mathcal{F}$.*

Let $\Omega$ be a Hausdorff topological space. $\Omega$ is said to be countable at infinity if there exists a sequence of compact subsets $K_1, K_2, \cdots$ such that every compact subset of $\Omega$ is contained in some $K_j$, that is $\Omega = \cup_{j=1}^{\infty} K_j$.

**Definition 9** *A complex analytic manifold $\Omega$ of dimension $n$ which is countable at infinity is said to be a Stein manifold if*

1. *$\Omega$ is holomorphically convex, that is,*

$$\hat{K} = \hat{K}_\Omega = \left\{ z : z \in \Omega, |f(z)| \le \sup_K |f| \text{ for every } f \in A(\Omega) \right\}$$

*is a compact subset of $\Omega$ for every compact subset $K$ of $\Omega$.*

2. *$A(\Omega)$ separates the points in $\Omega$, that is for any different pair of points $z_1, z_2$ there exists a function $f \in A(\Omega)$ such that $f(z_1) \neq f(z_2)$.*

3. *Local coordinates can be defined by global analytic functions, that is for every $z \in \Omega$, one can find $n$ functions $f_1, \cdots, f_n \in A(\Omega)$ which form a coordinate system at $z$.*

**Example 1** *Any holomorphically convex domain in $\mathbb{C}^n$ and $\mathbb{C}^n$ itself are Stein manifolds. But, $\widehat{\mathbb{C}}^n$ or $\mathbb{P}^n$ are not Stein manifolds since using Liouville Theorem, only analytic functions in these spaces are constants but these obviously cannot separate points, therefore property 2 fails.*

### 4.2 Spaces of Analytic Functions

Let $\Omega$ be a complex manifold. $A(\Omega)$ is the space of *all analytic functions* on $\Omega$ with the *topology of uniform convergence on compact subsets* of $\Omega$, i.e. with the locally convex topology generated by the system of semi-norms

$$|x|_K := \max\left\{|x(z)| : z \in K\right\}, \; K \in \mathcal{K}(\Omega) \tag{4.1}$$

where $\mathcal{K}(\Omega)$ denotes the set of all compacta on $\Omega$. If $\Omega$ is countable at infinity, then $A(\Omega)$ is a Fréchet space whose topology is produced by the sequence of semi-norms

$$\left\{|x|_{K_s} := \max\{|x(z)| : z \in K_s\}\right\}_{s\in\mathbb{N}}$$

where the $K_s$ are compacta such that $K_s \subset intK_{s+1}$, $s = 1, 2, \cdots$, and $\cup_s K_s = \Omega$.

Let $E$ be an arbitrary subset of $\Omega$. Let $\mathcal{G}(E) = \mathcal{G}_\Omega(E)$ denote the collection of all open neighborhoods of $E$ in $\Omega$. For $D_f, D_g \in \mathcal{G}(E)$, the functions $f \in A(D_f), g \in A(D_g)$ are said to be equivalent $(f \sim g)$ if there exists a $D \in \mathcal{G}(E)$ such that $D \subset D_f \cap D_g$ and $f(z) \equiv g(z)$ for all $z \in D$. A germ of analytic functions, briefly (analytic) germ, is an equivalence class obtained that way. If $x$ is a germ on $E$ and $f \in x$ then we say that $f$ generates the germ $x$. The set of all such germs on $E$ is a vector space. If $E$ is an open set, then $E \in \mathcal{G}(E)$ and every germ on $E$ can be naturally identified with the unique analytic function on $E$, generating the germ.

We denote by $A(E)$ the locally convex space of all analytic germs on $E$ endowed with the inductive limit topology

$$A(E) = \underset{D\in\mathcal{G}(E)}{limind} A(D),$$

that is with the finest topology on $A(E)$ for which all the natural mappings $A(D) \to A(E), D \in \mathcal{G}(E)$, are continuous.

Let $K$ be a compactum in the manifold $\Omega$. Then the space $A(K)$ can be represented as the countable inductive limit

$$A(K) = \underset{s\to\infty}{limind} A(D_s). \tag{4.2}$$

Here $D_s$ is any countable basis of $\mathcal{G}(K)$. It is suitable to choose $D_s$ with the following properties: $D_{s+1} \Subset D_s$ and every $D_s$ does not contain any connected component which is disjoint from $K$. It is sufficient to describe convergent sequences in order to define the topology of this space: $x_k \to x$ in the topology of $A(K)$ if there exists a neighborhood $D \in \mathcal{G}(K)$ (depending on the sequence) such that $x_k \in A(D), x \in A(D)$ and $(x_k)$ is uniformly convergent to $x$ on every compact subset of $D$. Let

$$J : A(K) \to C(K)$$

be the natural homomorphism of the restriction. Let $AC(K)$ be a Banach space obtained as the completion of the set $J(A(K))$ in the space $C(K)$ according to the norm defined in $(4.1)$. In the case when $J$ is a monomorphism, we obtain the injection

$$A(K) \hookrightarrow AC(K),$$

then $K$ is called a set of uniqueness of analytic functions on $K$.

Let $\Omega$ be a complex manifold. Then, if $\Omega$ is countable at the infinity, $A(\Omega)$ can also be defined as

$$A(\Omega) = limproj AC(K_s), \tag{4.3}$$

where the $K_s$ are compacta such that $K_s \subset intK_{s+1}$, $s = 1, 2, \cdots$, and $\cup_s K_s = \Omega$. Also, the space (4.2) can be considered as an inductive limit of Banach spaces

$$A(K) = limind AC(D_s) \tag{4.4}$$

where the $D_s$ are open sets such that $D_s \Supset D_{s+1}$, $s = 1, 2, \cdots$, and $\cap_s D_s = K$.

We will use the following notations for the spaces of analytic functions on disks:

$$A_R = A(\{z : |z| < R\}), \tag{4.5}$$

$$\bar{A}_R = A(\{z : |z| \leq R\}), \tag{4.6}$$

$$\bar{A}_0 = \bar{A}(\{0\}).$$

We can also write the spaces $A_R$ and $\bar{A}_R$ as the following inductive limits up to isomorphism:

$$A_R \simeq limind_{r\downarrow R} l^2(r^n),$$

$$\bar{A}_R \simeq limind_{r\uparrow R} l^2(r^n).$$

### 4.3 Duality

Let $\Omega$ be a Stein manifold. Elements of conjugate space $A'(\Omega) = A(\Omega)^*$, that is linear continuous functionals on $A(\Omega)$, are called analytic functionals (on $\Omega$). In particular for $\Omega = \mathbb{C}$ we obtain the space of analytic functionals $A' = A'(\mathbb{C}^n)$ having well-known importance in the investigation of convolution equation (see [12]). On the other hand analytic functionals have significant part in the investigation of structure of spaces of analytic functions, especially in the basis problem.

If $E$ is an arbitrary subset of Stein manifold $\Omega$ then the natural map

$$J^* = J^*(E, \Omega) : A(E)^* \to A'(\Omega), \tag{4.7}$$

that transforms a functional $x^* \in A(E)^*$ to its restriction on $A(\Omega)$, is a linear continuous map. Since $A(E)$ is reflexive, $J^*$ is dense. In the case when $E$ is a Runge set in $\Omega$ (that is $A(\Omega)$ is dense in $A(E)$) the map in (4.7) is an imbedding.

For a Runge set $E \subset \Omega$ we will identify $A'(E)$ as the image of the space $A(E)^*$ in (4.7). Then for any pair of Runge subsets $E \subset F$ in $\Omega$ such that $A(F)$ is dense in $A(E)$ we have the natural imbeddings $A'(F) \hookrightarrow A'(E) \hookrightarrow A'(\Omega)$.

### 4.4 The Grothendieck-Köthe-Silva Duality (GKS-duality)

The following result, due to Grothendieck, Köthe, and Silva (see [11], [14], [15], [25]) (called shortly GKS-duality) realizes the space $A(E)^*$, for any set $E \in \widehat{\mathbb{C}}$, as the space of analytic functions $A(E^*)$, where $E^* := \widehat{\mathbb{C}} \setminus E$. Usually there is an agreement to suppose that all germs of $A(E)$ are equal to 0 at the point $\infty$ if $\infty \in E$. Here we restrict ourselves to the case when $E$ is open or compact set.

**Theorem 17** *Let $E$ be either an open set or a compactum in $\widehat{\mathbb{C}}$, $E \neq \widehat{\mathbb{C}}$, $E \neq \emptyset$. The space $A(E)^*$, the conjugate of the space $A(E)$, is isomorphic to the space $A(E^*)$. This isomorphism is defined by the formula $x^* \to x'$, where*

$$x^*(x) = \left\langle x', x \right\rangle = \int_\Gamma x'(\zeta)\, x(\zeta)\, d\zeta,\ \ x \in A(E), \tag{4.8}$$

*where* $x^* \in A(E)^*$, $x' \in A(E^*)$, *and* $\Gamma$ *is any contour consisting of a finite number of smooth Jordan curves and separating the singularities of the functions* $x(\zeta)$ *and* $x'(\zeta)$. *Therewith, the formula in* (4.8) *is independent on the choice of the contour* $\Gamma$.

**Proof.** Let $E$ be an open set in $\widehat{\mathbb{C}}$, $E \neq \widehat{\mathbb{C}}$, $E \neq \emptyset$. Using (4.3),

$$A(E) = limproj AC(K_s), \tag{4.9}$$

where $K_s$ are compacta such that $K_s \subset int K_{s+1}$, $s = 1, 2, \cdots$, and $\cup_s K_s = E$.

First we show that for any $x' \in A(E^*)$ the formula (4.8) determines the unique functional $x^* \in A(E)^*$. Let $x' \in A(E^*)$. That means, there exists $s_0 \in \mathbb{N}$ such that $x' \in A\left(K^*_{s_0}\right)$. Then for any $x \in A(E)$, define

$$x^*(x) = \int_\Gamma x'(\zeta)\, x(\zeta)\, d\zeta, \text{ where } \Gamma = \partial K_s \text{ for } s > s_0.$$

Then,

$$|x^*(x)| \leq |x'(\zeta)|_\Gamma \max |x(\zeta)|_\Gamma\, l(\Gamma), \tag{4.10}$$

where $l(\Gamma)$ is the length of the contour $\Gamma$, that is,

$$|x^*(x)| \leq C_s \max |x(\zeta)|_\Gamma = C_s |x|_{K_s}, \text{ where } C_s = |x'(\zeta)|_\Gamma\, l(\Gamma),$$

which implies $x^*$ is bounded in the norm of $\|x\|_s = |x|_{K_s}$.

Let $x_1, x_2 \in A(E)$, $c_1, c_2 \in \mathbb{C}$. Then,

$$\begin{aligned} x^*(c_1x_1 + c_2x_2) &= \int_\Gamma x'(\zeta)(c_1x_1 + c_2x_2)(\zeta)\, d\zeta \\ &= \int_\Gamma x'(\zeta)(c_1x_1(\zeta) + c_2x_2(\zeta))\, d\zeta \\ &= \int_\Gamma x'(\zeta)\, c_1x_1(\zeta)\, d\zeta + \int_\Gamma x'(\zeta)\, c_2x_2(\zeta)\, d\zeta \\ &= x^*(x_1) + x^*(x_2). \end{aligned}$$

So, $x^*$ is linear and thus $x^* \in A(K_s)^* \hookrightarrow A(E)^*$ .Therefore, we observe that for any $x' \in A(E^*)$, the unique functional $x^* \in A(E)^*$ is defined by the formula (4.8).

Now, we will show that for any functional $x^* \in A(E)^*$ the formula $(4.8)$ determines the unique function $x' \in A(E^*)$. Let $x^* \in A(E)^*$. That means, there exists $s_0 \in \mathbb{N}$ such that $x^*$ is bounded in the norm $\|x\|_{s_0} = |x|_{K_{s_0}}$, i.e. $x^* \in AC(K_{s_0})^*$. Also, $x^* \in AC(K_s)^*$, for $s \in \mathbb{N}$, $s \geq s_0$.

According to the Cauchy Integral Formula,

$$x(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{x(\zeta)}{\zeta - z} d\zeta, \text{ where } \Gamma = \partial K_s, \quad z \in int K_s.$$

We denote $\frac{1}{2\pi i}\frac{1}{\zeta - z} =: u_\zeta(z)$. Then, for $\zeta \in K_{s_0}^*$, $u_\zeta$ will be an element of the space $AC(K_{s_0})$, thus $x^*$ can be applied to $u_\zeta$. So, define $x'(\zeta) := x^*(u_\zeta)$. Now, using the linearity and continuity of $x^*$,

$$\begin{aligned}
\lim_{h \to 0} \frac{x'(\zeta + h) - x'(\zeta)}{h} &= \lim_{h \to 0} \frac{x^*(u_{\zeta+h}(z)) - x^*(u_\zeta(z))}{h} \\
&= \lim_{h \to 0} x^*\left(\frac{u_{\zeta+h}(z) - u_\zeta(z)}{h}\right) = x^*\left(\lim_{h \to 0} \frac{u_{\zeta+h}(z) - u_\zeta(z)}{h}\right) \\
&= x^*\left(\frac{d}{d\zeta} u_\zeta(z)\right) = \frac{d}{d\zeta} x^*(u_\zeta(z)) = \frac{d}{d\zeta} x'(\zeta)
\end{aligned}$$

Hence, $x'(\zeta)$ is analytic and is an element of the space $A(K_s^*) \hookrightarrow A(E^*)$. Now, since $x^*$ is linear and continuous,

$$\begin{aligned}
x^*\left(\int\limits_{\Gamma} x(\zeta) u_\zeta(z) d\zeta\right) &= \int\limits_{\Gamma} x^*(x(\zeta) u_\zeta(z)) d\zeta \\
&= \int\limits_{\Gamma} x(\zeta) x^*(u_\zeta(z)) d\zeta = \int\limits_{\Gamma} x(\zeta) x'(\zeta) d\zeta.
\end{aligned}$$

Hence, we have shown that the given mapping $(4.8)$ is a bijection.

Using $(4.9)$, we can represent the space $A(E)^*$ as the countable inductive limit

$$A(E)^* = limind_{s \to \infty} AC(K_s)^*.$$

Also, by definition $A(E^*)$ can be represented as a countable inductive limit as in $(4.4)$. In particular, taking these open sets to be $\{K_s^*\}_{s=1}^{\infty}$, we can represent $A(E^*)$ as

$$A(E^*) = limind_{s \to \infty} A(K_s^*) = limind_{s \to \infty} AC(K_s^*).$$

By $(4.10)$ , for any $s \in \mathbb{N}$, we have

$$|x^*|_{AC(K_s)^*} \leq l\,(\partial K_s) \max |x\,(\zeta)|_{\partial K_s} |x'|_{K_s^*}\,, \tag{4.11}$$

where $l\,(\partial K_s)$ is the length of the contour $\partial K_s$. (4.11) implies that the mapping $x' \to x^*$ is continuous (see, for example [21], page 98).

On the other hand, for any $\zeta \in K_{s+1}$,

$$\begin{aligned} |x'\,(\zeta)| &\leq |x^*\,(u_\zeta)| \leq \|x^*\|_{AC(K_s)^*} |u_\zeta|_{K_s} \\ &\leq \|x^*\|_{AC(K_s)^*} \max_{\substack{\zeta \in K_{s+1} \\ z \in K_s}} \frac{1}{|\zeta - z|} \leq C'_s \|x^*\|_{AC(K_s)^*} \end{aligned}$$

where $C'_s = \frac{1}{\delta_s}$ for $\delta_s = dist\,(K_s, K_{s+1})$ . Thus, we get

$$|x'|_{\partial K_{s+1}} \leq C'_s \|x^*\|_{AC(K_s)^*}\,, \tag{4.12}$$

which implies that the mapping $x^* \to x'$ is continuous. Combining (4.11) and (4.12) we conclude that (4.8) is an isomorphism.

Let $\Gamma$, $\Gamma'$ be two arbitrary contours consisting of a finite number of smooth Jordan curves and separating the singularities of the functions $x\,(\zeta)$ and $x'\,(\zeta)$. Then, since they have the same singularities within, these two contours are homologous. Therefore

$$\int_\Gamma x'\,(\zeta)\,x\,(\zeta)\,d\zeta = \int_{\Gamma'} x'\,(\zeta)\,x\,(\zeta)\,d\zeta$$

for any $x \in A\,(E)\,,\ x' \in A\,(E^*)$ which implies that the formula in (4.8) does not depend on the choice of the contour.

Now, let $K$ be an arbitrary compactum in $\widehat{\mathbb{C}}$, $K \neq \widehat{\mathbb{C}}$, $K \neq \emptyset$. Then, its complement $K^* = E$ is an open set. By the part we have proved, there exists an isomorphism

$$T : A\,(E)^* \to A\,(K) = A\,(E^*)$$

defined by the formula

$$x^*\,(x) = \left\langle x', x \right\rangle = \int_\Gamma x'\,(\zeta)\,x\,(\zeta)\,d\zeta,\ x \in A\,(E)\,,$$

where $x^* \in A(E)^*$, $x^{'} \in A(E^*)$, and $\Gamma$ is any contour consisting of a finite number of smooth Jordan curves and separating the singularities of the functions $x(\zeta)$ and $x^{'}(\zeta)$ and $x' = Tx^*$. Then, the adjoint of the operator $T$ is an isomorphism:

$$T^* : A(K)^* \to A(E)^{**}.$$

Besides, the spaces of analytic functions are Montel, reflexivity of the space $A(E)$ follows therefrom. That is, the natural embedding

$$J : A(E) \to A(E)^{**}$$

is an isomorphism. Then,

$$S : A(K)^* \to A(E) = A(K^*)$$

defines an isomorphism as the superposition of isomorphisms, i.e. $S = J^{-1} \circ T^*$.

Now we have to show the isomorphism $S$ is defined by the formula that is given in the statement. For any $x \in A(E)$

$$F_x(x^*) = \langle x', x \rangle = x^*(x),$$

where $F_x = J^{-1}x$, $x^* \in A(E)^*$, $x' \in A(K)$ and for any $F_x \in A(E)^{**}$, $g = (S)^{-1}x$, the equality

$$g(x') = F_x(x^*)$$

holds. So, the theorem is proved. ■

## CHAPTER 5

## BASES AND ISOMORPHISMS OF SPACES OF ANALYTIC FUNCTIONS IN ONE DIMENSIONAL CASE

### 5.1 Bases

Let $X$ be a complex linear topological space. A *basis* in $X$ is a sequence $\{x_k\}$ such that every vector $x \in X$ has the unique expansion:

$$x = \sum_{k=1}^{\infty} \xi_k x_k, \ \xi_k \in \mathbb{C},$$

which converges in the topology of $X$.

A basis $\{x_k\}$ in $A(E)$, $E \subset \Omega$, is said to be extendible onto a set $F$, $F \neq E$, $F \cap E \neq \varnothing$ (outside if $E \subset F$ and inside if $F \subset E$) if there exists a system of germs $\{\tilde{x}_k\} \subset A(F)$ with properties:

1. $x_k$ and $\tilde{x}_k$ generate the same germs on $F \cap E$,

2. $x_k$ is also a basis in $A(F)$.

For construction of extendible bases in the case of a compactum $E = K \subset \mathbb{C}$ it is convenient to use Newton interpolational polynomials

$$p_0(z) = 1, p_k(z) = (z - \zeta_1) \cdots (z - \zeta_k), \ k \in \mathbb{N},$$

where $\{\zeta_k\}$ is a suitable sequence of points of interpolation on $K$.

**Example 2** $\left\{z^k\right\}_{k=0}^{\infty}$ *is a basis in all the spaces* $A_R$ *and* $\bar{A}_R$*, where* $A_R$ *and* $\bar{A}_R$ *as defined in* (4.5) *and* (4.6)*,* $R \geq 0$*. If* $\varphi : D \to \mathbb{D}_1$ *is a conformal mapping, where* $\mathbb{D}_1$ *is the unit disc and* $D$ *is a domain, the system of functions* $\left\{\varphi\left(z\right)^k\right\}_{k=0}^{\infty}$ *is a basis for the space* $A\left(D\right)$*. The system of functions* $\left\{\varphi\left(z\right)^k\right\}_{k=0}^{\infty}$ *also forms a basis for the spaces* $A\left(D_r\right)$*,* $A\left(\bar{D}_r\right)$*, where* $D_r = \{|\varphi\left(z\right)| < r\}$*,* $0 < r < 1$*. So,* $\left\{\varphi\left(z\right)^k\right\}_{k=0}^{\infty}$ *is an example of extendible bases, since it forms a basis for a family of spaces.*

The existence of extendible bases was established by Walsh [32], Leja [17] for regular and by Zahariuta [34] for polar compacta $K \subset \mathbb{C}$. In both cases we have a basis extendible on $\mathbb{C}$ and moreover onto a family of intermediate domains (or compacta) bounded by level curves of some harmonic function, namely Green's potential.

Let $X$ be a complex linear topological space and $X^*$ be the dual space of $X$. The system $\{x'_k\}_{k=1}^{\infty}$ is said to be biorthogonal to the system $\{x_k\}_{k=1}^{\infty}$ in $X$ if

$$\langle x_k, x'_l \rangle = \delta_{kl}, \text{ for any } k, l \in \mathbb{N},$$

where $\delta_{kl}$ is the Kronecker delta. For the case when the system in $X$ is the Newton polynomials, the biorthogonal system can be constructed in the following way:

**Lemma 7** *Let* $\{\beta_i\}_{i=1}^{\infty}$ *be any bounded sequence in* $\mathbb{C}$ *and*

$$p_k\left(z\right) = \prod_{i=1}^{k}\left(z - \beta_i\right), \quad k = 1, 2, \cdots, \quad p_0\left(z\right) \equiv 1.$$

*Then, the system*

$$p'_k\left(z\right) = \frac{1}{2\pi i}\frac{1}{p_{k+1}\left(z\right)}$$

*is biorthogonal to the system of polynomials* $p_k\left(z\right)$ *for each* $k \in \mathbb{N}$*, namely*

$$\langle p'_l, p_k \rangle = \int\limits_{\Gamma} p'_l\left(\zeta\right) p_k\left(\zeta\right) d\zeta = \delta_{lk},$$

*where* $\Gamma$ *is any arbitrary contour consisting of a finite number of smooth Jordan curves separating the singularities of the functions* $p_k\left(\zeta\right)$ *and* $p'_l\left(\zeta\right)$ *for any* $k, l \in \mathbb{N}$*, i.e.* $\{\beta_i\}_{i=1}^{\infty}$ *are all inside the contour* $\Gamma$*.*

**Proof.** Let $\Gamma$ be any arbitrary contour consisting of a finite number of smooth Jordan curves separating the singularities of the functions $p_k(\zeta)$ and $p'_l(\zeta)$ for any $k, l \in \mathbb{N}$.

Let $k = l$. Then,

$$\begin{aligned} \langle p'_k, p_k \rangle &= \int_\Gamma p'_k(\zeta)\, p_k(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_\Gamma \frac{1}{p_{k+1}(\zeta)} p_k(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\zeta - \beta_{k+1})} d\zeta = \frac{1}{2\pi i}.2\pi i = 1, \end{aligned}$$

by Cauchy Integral Formula.

Let $k > l$. Then,

$$\begin{aligned} \langle p'_k, p_k \rangle &= \int_\Gamma p'_k(\zeta)\, p_k(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_\Gamma \frac{1}{p_{k+1}(\zeta)} p_k(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma (\xi - \beta_{l+2})(\xi - \beta_{l+3}) \ldots (\xi - \beta_k)\, d\xi = 0, \end{aligned}$$

since the integrand is analytic inside the contour $\Gamma$.

Let $k < l$. Then,

$$\begin{aligned} \langle p'_k, p_k \rangle &= \int_\Gamma p'_k(\zeta)\, p_k(\zeta)\, d\zeta = \frac{1}{2\pi i} \int_\Gamma \frac{1}{p_{k+1}(\zeta)} p_k(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{(\xi - \beta_{k+1})(\xi - \beta_{k+2}) \ldots (\xi - \beta_{l+1})} d\xi \\ &= \frac{1}{2\pi i} 2\pi i Res_{\zeta=\infty} \left( \frac{1}{(\xi - \beta_{k+1})(\xi - \beta_{k+2}) \ldots (\xi - \beta_{l+1})} \right). \text{ (see [26], pg 250)} \end{aligned}$$

But, since

$$\frac{1}{(\xi - \beta_{k+1})(\xi - \beta_{k+2}) \ldots (\xi - \beta_{l+1})} = \frac{1}{\zeta^{l-k+1} \left(1 - \frac{\beta_{k+1}}{\zeta}\right)\left(1 - \frac{\beta_{k+2}}{\zeta}\right) \ldots \left(1 - \frac{\beta_{l+1}}{\zeta}\right)}$$

we conclude that

$$Res_{\zeta=\infty} \left( \frac{1}{(\xi - \beta_{k+1})(\xi - \beta_{k+2}) \ldots (\xi - \beta_{l+1})} \right) = 0.$$

Therefore $\langle p'_k, p_k \rangle = 0$ for $k < l$ as well, which proves the lemma. ■

### 5.1.1 Construction of Newton Polynomials For Regular Compacta

**Walsh Knots**

Let $K$ be a compactum in $\mathbb{C}$. For any $n \in \mathbb{N}$, let $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$ be a set of $n+1$ points $z_k^{(n)}$ of $K$ such that the modulus of the *Vandermonde determinant*

$$V_n\left(z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}\right) = \prod_{i<j=1}^{j=n+1}\left(z_i^{(n)} - z_j^{(n)}\right) \tag{5.1}$$

will be maximum, i.e. construct Fekete points for each $n \in \mathbb{N}$. The Vandermonde determinant is continuous in $K$ and since $K$ is compact, maximum is attained in $K$, that is, for any $n \in \mathbb{N}$, there exists $\left\{z_i^{(n)}\right\}_{i=1}^{n+1} \subset K$ such that (5.1) is maximum. These set of points may not be unique for each $n \in \mathbb{N}$, but any set will suffice.

The following set of points

$$\beta_1^{(0)},$$

$$\beta_1^{(1)}, \beta_2^{(1)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

constructed by (5.1) for each $n \in \mathbb{N}$ are called the *Walsh knots*.

We will also enumerate the Walsh knots in the following way:

$$\beta_1 = \beta_1^{(0)}, \beta_2 = \beta_1^{(1)}, \beta_3 = \beta_2^{(1)}, \cdots, \beta_n = \beta_{k(n)}^{j(n)}, \cdots \tag{5.2}$$

**Lemma 8** *Let $K$ be a regular compactum in $\mathbb{C}$ with a connected complement. Assume that $\left\{\beta_i^{(n)}\right\}$, $n \in \mathbb{N}$ is any sequence of finite sets of points in $K$ that satisfy the condition*

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=1}^{n+1}\ln\left|z - \beta_i^{(n)}\right|\right) = g_{K^*}(z,\infty) + \ln C(K), \tag{5.3}$$

*uniformly on any closed subset of $K^*$, where $C(K)$ is the capacity of the compactum $K$ and $g_{K^*}(z,\infty)$ is the Green function. Then,*

$$\lim_{k\to\infty}\frac{1}{k}\left(\sum_{i=1}^{k}\ln\left|z - \beta_i\right|\right) = g_{K^*}(z,\infty) + \ln C(K), \tag{5.4}$$

*where* $\{\beta_j\}_{j=1}^{k}$ *is the sequence obtained by enumeration of* $\beta_i^{(n)}$ *as in* (5.2).

**Proof.** Let $S$ be a compact subset of $K^*$. Consider the sequence of functions $u_i(z) := \frac{\sum_{j=1}^{i+1} \ln z-\beta_j^{(n)}}{i+1}$, $i \in \mathbb{N}$ in $S$. Then, by (5.3), we have

$$\lim_{n\to\infty} u_i(z) = g_{K^*}(z,\infty) + \ln C(K).$$

uniformly on $S$. Denote $k_n = \frac{n(n+1)}{2}$ and construct a new sequence $\{v_i\}_{i=1}^{\infty}$ as follows:

$$\begin{aligned} v_1 &= u_1,\ v_2 = u_2,\ v_3 = u_2, v_4 = u_3, v_5 = u_3, v_6 = u_3, \cdots, \\ v_{k_{n-1}+1} &= u_n, \cdots, v_{k_n} = u_n, v_{k_n+1} = u_{n+1}, \cdots. \end{aligned}$$

That is, the $n^{th}$ term of $\{u_i\}_{i=1}^{\infty}$ will be repeated $n$ times in $\{v_i\}_{i=1}^{\infty}$. It is obvious that also

$$\lim_{n\to\infty} v_i(z) = g_{K^*}(z,\infty) + \ln C(K).$$

uniformly on $S$. Now, take the partial sums $V_m = \sum_{i=1}^{m} v_i$. Then, by Cesaro's Theorem,

$$\lim_{m\to\infty} \frac{V_m}{m} = g_{K^*}(z,\infty) + \ln C(K),$$

uniformly on $S$. Therefore

$$\begin{aligned} \lim_{k_n\to\infty} \frac{\sum_{j=0}^{n}\sum_{i=1}^{j+1} \ln\left|z-\beta_i^{(j)}\right|}{k_n} &= \lim_{k_n\to\infty} \frac{\sum_{j=1}^{k_n} \ln|z-\beta_j|}{k_n} \\ &= g_{K^*}(z,\infty) + \ln C(K). \end{aligned} \tag{5.5}$$

Hence, it is shown that (5.4) holds for the subsequence $k_n$.

Now, let $k \in \mathbb{N}$ such that $k = k_{n-1} + l < k_n$, $l \in \mathbb{N}$. Then, since

$$\frac{1}{k}\sum_{j=1}^{k} \ln|z-\beta_j| = \frac{1}{k}\sum_{j=1}^{k_{n-1}} \ln|z-\beta_j| + \frac{1}{k}\sum_{j=k_{n-1}+1}^{k} \ln|z-\beta_j|,$$

taking the limit as $k \to \infty$, then also $n \to \infty$ and using (5.5), we see

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \ln|z-\beta_j| = g_{K^*}(z,\infty) + \ln C(K) + \lim_{k\to\infty} \frac{1}{k}\sum_{j=k_{n-1}+1}^{k} \ln|z-\beta_j|,$$

Therefore, to prove (5.4) it suffices to show

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=k_{n-1}+1}^{k}\ln|z-\beta_j|=0$$

uniformly on $S$ . Since there exists a constant $0<\Delta<\infty$ such that

$$\frac{1}{\Delta}<|z-\beta_j|<\Delta,\ \text{for } j\in\mathbb{N},\ z\in S.$$

we obtain that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=k_{n-1}1}^{k}\ln|z-\beta_j|\leq\lim_{k\to\infty}\frac{1}{k}\left(k-k_{n-1}\right)\ln\Delta=0.$$

Therefore, (5.4) follows, which completes the proof. ■

**Proposition 9** *Let $K$ be a regular compactum in $\mathbb{C}$ with a connected complement. Let $\{\beta_j : i=1,2,\cdots\}\subset K$ be any sequence such that the relation holds*

$$\lim_{k\to\infty}\frac{1}{k}\left(\sum_{i=1}^{k}\ln|z-\beta_i|\right)=g_{K^*}(z,\infty)+\ln C(K), \tag{5.6}$$

*for each $z\in K^*$; here $g_{K^*}(z,\infty)$ is the Green's function of the region $K^*$ with singularity $\ln|z|$ at infinity. Then the system of polynomials*

$$p_k(z)=\prod_{i=1}^{k}(z-\beta_i),\quad k=1,2,\cdots,\qquad p_0(z)\equiv 1, \tag{5.7}$$

*forms a basis in the space $A(K)$ and in all the spaces $A(D_R)$, $A(K_R)$, and $A(\mathbb{C})$, $1<R<\infty$, where $D_R:=\{z\in K^* : g_{K^*}(z,\infty)<\ln R\}$ and $K_R:=\overline{D_R}$.*

**Proof.** Let (5.6) hold. Then, it follows that

$$C_1(r,\varepsilon)(r-\varepsilon)^n\leq|p_k(z)|_{D_r}\leq C_2(r,\varepsilon)(r+\varepsilon)^n,\quad 1<r<\infty, \tag{5.8}$$

for any $\varepsilon>0$ with some positive constants $C_1(r,\varepsilon), C_2(r,\varepsilon)$. By Lemma 7, $p_k'(z)=\frac{1}{2\pi i}\frac{1}{p_{k+1}(z)}$ is the biorthogonal system of $p_k(z)$. Then, by (5.8), the following is valid:

$$|p_k'(z)|_{D_r}\leq C(r,\varepsilon)\left(\frac{1}{r-\varepsilon}\right)^{k+1}=C_3(r,\varepsilon)\left(\frac{1}{r-\varepsilon}\right)^{k}. \tag{5.9}$$

Let $x \in A(D_R)$ for any $1 < R < \infty$. Then, for $r < \rho < R$, the following estimates are valid:

$$|p_k(z)|_{D_r} \leq C_2(r,\varepsilon)(r+\varepsilon)^k, \tag{5.10}$$

$$|p_k'(z)|_{D\rho} \leq C_3'(\rho,\varepsilon)\left(\frac{1}{\rho-\varepsilon}\right)^k. \tag{5.11}$$

The formal expansion of $x$ is $x = \sum \xi_k p_k(z)$, where $\xi_k = \langle p_k', x\rangle$. Then, by (5.11) and GKS-duality,

$$|\xi_k| \leq |\langle p_k', x\rangle| \leq CC_3'(\rho,\varepsilon)|x|_{D_\rho}\left(\frac{1}{\rho-\varepsilon}\right)^k. \tag{5.12}$$

By (5.10) and (5.12), the general term of the basis expansion of $x$ has the following bound:

$$|\xi_k||p_k(z)|_{D_r} \leq C(r,\rho,\varepsilon)|x|_{D_\rho}\left(\frac{r+\varepsilon}{\rho-\varepsilon}\right)^k.$$

Choose $\varepsilon$ so small that $\frac{r+\varepsilon}{\rho-\varepsilon} < 1$. Then, it is seen that the sum $\sum |\xi_k||p_k(z)|_{D_r}$ is less than the sum of a convergent geometric series, therefore it converges as well. So, the system of polynomials $\{p_k(z)\}$ forms a basis for $A(D_R)$.

Let $x \in A(\mathbb{C})$. Then, for any $R \geq 1$, $x \in A(\mathbb{C}) \hookrightarrow AC(D_R)$. Therefore, the basis expansion of $x$ is $x = \sum \xi_k p_k(z)$, where $\xi_k = \langle p_k', x\rangle$. But it has been shown that it is convergent in the topology of the space $A(D_R)$. Therefore, the system of polynomials $\{p_k(z)\}_{k=0}^{\infty}$ is a basis for the space $A(\mathbb{C})$.

For any $R \geq 1$, consider the space $A(K_R)$, where $K_R := \bar{D}_R$. We assume that $K_1 = K$. Then, each $A(K_R)$ can be represented as the inductive limit

$$A(K_R) = limind_{r\downarrow R} A(D_r).$$

Since the system of polynomials $\{p_k(z)\}_{k=0}^{\infty}$ is a basis for each of the spaces $A(D_r)$, it will also be a basis for their inductive limit. Therefore, the system of polynomials $\{p_k(z)\}_{k=0}^{\infty}$ is a basis for the space $A(K_R)$. ■

**Corollary 1** *Let $K$ be a regular compactum in $\mathbb{C}$ with a connected complement. Let $\{\beta_j : i = 1, 2, \cdots\} \subset K$ be the enumerated Walsh knots as in* (5.2)*. Then, the system of polynomials*

$$p_k(z) = \prod_{i=1}^{k}(z-\beta_i), \quad k = 1, 2, \cdots, \quad p_0(z) \equiv 1, \tag{5.13}$$

*forms a basis in the space* $A(K)$ *and in all the spaces* $A(D_R)$, $A(K_R)$, *and* $A(\mathbb{C})$, $1 < R < \infty$, *where* $D_R := \{z \in K^* : g_{K^*}(z,\infty) < \ln R\}$ *and* $K_R := \overline{D_R}$.

**Proof.** Let $\left\{\beta_i^{(n)}\right\}_{i=1}^{n+1} \subset K$ be Fekete points. Define

$$q_n(z) := \prod_{i=1}^{n+1}\left(z - \beta_i^{(n)}\right).$$

Then, construct

$$u(z) := \frac{1}{n}\ln|q_n(z)| - \frac{1}{n}\ln\|q_n(z)\|_K - g_{K^*}(z,\infty), \text{ where } z \in K^* \setminus \{\infty\}.$$

Since all zeros of $q_n(z)$ lie in $K$, $u$ is harmonic on $\mathbb{C} \setminus K$. Besides,

$$\begin{aligned}\frac{1}{n}\ln|q_n(z)| &= \frac{1}{n}\ln\left(|z^n|\left(1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}\right)\right) \\ &= \ln|z| + \frac{1}{n}\ln\left(1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}\right),\end{aligned}$$

which implies that $u(z)$ is also harmonic at infinity. Therefore, we conclude that $u(z)$ is harmonic in $K^*$. Then, we have

$$\limsup_{z\to\zeta} u(z) \le \frac{1}{n}\ln|q_n(z)| - \frac{1}{n}\ln\|q_n(z)\|_K \le 0, \text{ for } \zeta \in \partial K^*,$$

so by Maximum Principle, $u \le 0$ on $K^*$. Therefore

$$\frac{1}{n}\ln|q_n(z)| \le \frac{1}{n}\ln\|q_n(z)\|_K + g_{K^*}(z,\infty), \text{ for } z \in K^* \setminus \{\infty\}. \tag{5.14}$$

Since $u \le 0$ on $K^*$, we may also apply Harnack's inequality (see [23], pg 13) to $-u$ and obtain

$$u(z) \ge \tau_{K^*}(z,\infty)\, u(\infty) \text{ for } z \in K^*. \tag{5.15}$$

Using Lemma 2,

$$u(\infty) = \ln C(K) - \frac{1}{n}\ln\|q_n(z)\|_K \ge \ln C(K) - \ln\delta_n(K), \tag{5.16}$$

where $\delta_n(K)$ is the n-th diameter of $K$. Then, combining $(5.14)$, $(5.15)$, and $(5.16)$ we get

$$\begin{aligned}\ln\left(\frac{C(K)}{\delta_n(K)}\right)^{\tau_{K^*}(z,\infty)} + g_{K^*}(z,\infty) + \frac{1}{n}\ln\|q_n(z)\|_K &\le \frac{1}{n}\ln|q_n(z)| \qquad (5.17)\\ &\le g_{K^*}(z,\infty) + \frac{1}{n}\ln\|q_n(z)\|_K.\end{aligned}$$

Now, taking the limit as $n \to \infty$ for (5.17) and using Lemma 1 again,

$$g_{K^*}(z,\infty) + \ln C(K) \le \lim_{n\to\infty} \frac{1}{n} \ln |q_n(z)| \le g_{K^*}(z,\infty) + \ln C(K).$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \ln |q_n(z)| = g_{K^*}(z,\infty) + \ln C(K). \tag{5.18}$$

Using Lemma 8, it is seen that

$$\lim_{n\to\infty} \frac{1}{n} \ln |p_n(z)| = g_{K^*}(z,\infty) + \ln C(K),$$

where $p_k(z) = \prod\limits_{i=1}^{k} (z-\beta_i)$, $k = 1,2,\cdots$, $p_0(z) \equiv 1$, $\{\beta_i\}_{i=1}^{\infty}$ are the enumerated Walsh knots as given in $(5.2)$. Then, by Lemma 9, we conclude that the system of polynomials in (5.13) forms a basis for all the spaces $A(K)$, $A(D_R)$, $A(K_R)$, and $A(\mathbb{C})$. That proves the corollary. ■

**Leja Points**

Let $K$ be a compactum in the complex plane $\mathbb{C}$. Take any $\beta_1 \in K$. The sequence $(\beta_i)_{i=1}^{\infty}$ is constructed inductively: If $(n-1)$ points $\{\beta_i\}_{i=1}^{n-1} \subset K$ are chosen, choose $nth$ point $\beta_n \in K$ such that modulus of the Vandermonde determinant satisfies the condition

$$|V_n(\beta_1,\cdots,\beta_{n-1};\beta_n)| = \max\{|V_n(\beta_1,\cdots,\beta_{n-1};\zeta)| : \zeta \in K\}.$$

That is, if $(n-1)$ points are chosen, the $nth$ point $\zeta = \beta_n$ which maximizes the Vandermonde determinant is chosen. Interpolation points constructed in that way are called *Leja points.*

**Corollary 2** *Let $K$ be a regular compactum in $\mathbb{C}$ with a connected complement. Let $\{\beta_j : i = 1,2,\cdots\} \subset K$ be Leja points. Then the system of polynomials*

$$p_k(z) = \prod_{i=1}^{k} (z-\beta_i), \quad k = 1,2,\cdots, \quad p_0(z) \equiv 1,$$

*forms a basis in the space $A(K)$ and in all the spaces $A(D_R)$, $A(K_R)$, and $A(\mathbb{C})$, $1 < R < \infty$, where $D_R := \{z \in K^* : g_{K^*}(z,\infty) < \ln R\}$ and $K_R := \overline{D_R}$.*

**Proof.** Leja proved in [17] that the system of polynomials $\{p_k\}$ satisfy the condition

$$\lim_{k\to\infty}\frac{1}{k}\left(\ln|p_k|\right) = g_{K^*}\left(z,\infty\right) + \ln C\left(K\right).$$

Then, using Proposition 9, we conclude that the system $\{p_k\}$ forms a basis for the spaces $A\left(K\right)$, $A\left(D_R\right)$, $A\left(K_R\right)$, and $A\left(\mathbb{C}\right)$. Therefore, the corollary is proved. ■

### 5.1.2 Construction of Newton Polynomials For Polar Compacta

For a polar compact set $K \subset \mathbb{C}$, as it was shown in [34], a basis in $A\left(K\right)$ can be constructed as sequence of interpolational Newton polynomials:

$$p_n\left(z\right) = \prod_{j=1}^{n}\left(z-\beta_j\right).$$

In this case the sequence of knots is obtained from the sequence of Evans' points $\zeta_\nu$ (see Theorem 14), repeating them in such a way that, roughly speaking, among the first $n$ knots the point $\zeta_\nu$ appears nearly $[\alpha_j n]$ times (proportionally to the weight $\alpha_j > 0$); so $\beta_j$ behaves like a random sequence chosen from the probability space $\{\zeta_\nu\}$ with the discrete probability measure $\mu$ defined by $\mu\left(\zeta_\nu\right) = \alpha_\nu$.

More precisely, it was shown in [34] and [39] that, for a polar compact set $K \subset \mathbb{C}$, the system of polynomials

$$p_k\left(z\right) = \prod_{i=1}^{k}\left(z-\zeta_i\right)^{k_j(n)}, \quad k = 1, 2, \cdots, \quad p_0\left(z\right) \equiv 1, \tag{5.19}$$

will form a basis, where $k_j\left(n\right)$ satisfies the conditions:

1. $k_j\left(1\right) = \delta_{1j}$,
2. $k_j\left(n+1\right) = k_j\left(n\right) + \delta_{j,\mu(n)}$ with some $n \in \mathbb{N}$,
3. $\sum\limits_{j=1}^{\infty}\left|\alpha_j - \frac{k_j(n)}{n}\right| \to 0$ as $n \to \infty$.

**Theorem 18** *(Zahariuta [34], [39]) Let $K$ be a polar compactum in $\mathbb{C}$. Then the system of polynomials*

$$p_k\left(z\right) = \prod_{i=1}^{k}\left(z-\zeta_i\right)^{k_j(n)}, \quad k = 1, 2, \cdots, \quad p_0\left(z\right) \equiv 1,$$

*where $k_j\left(n\right)$ satisfies the conditions above, forms a basis in $A\left(K\right)$.*

**Proof.** Let $K$ be a polar compactum in $\mathbb{C}$. Then, using Theorem 14, there exists a sequence of nonnegative numbers $\{\alpha_j\}$, $\sum\limits_{j=1}^{\infty} \alpha_j = 1$, and a point sequence $\{\zeta_j\} \subset K$ such that the function $\Psi(z) = \sum\limits_{j=1}^{\infty} \alpha_j \ln |z - \zeta_j|$ is subharmonic in the entire plane $\mathbb{C}$, harmonic $\mathbb{C}\backslash K$, and $\Psi(z) \equiv -\infty$ for $z \in K$. We may assume that $\alpha_j \downarrow 0$.

Let $\Delta_\alpha = \{z : \Psi(z) < \alpha\}$. The subharmonic function $\Psi$ is upper semi-continuous in $\mathbb{C}$, so $\Psi(z) \to -\infty$ for $z \to z_0$ and $z_0 \in K$. Thus

$$\cap_\alpha \Delta_\alpha = K, \Delta_\alpha \Subset \Delta_\beta, \alpha < \beta,$$

from which it follows that the system of norms $|x|_{\Delta_\alpha}$, $-\infty < \alpha < \infty$ yields the original topology in $A(K)$.

Consider the system of polynomials

$$p_0 \equiv 1, p_n(z) = \prod_{j=1}^{n} (z - \beta_j)^{k_j(n)}, n = 1, 2, \cdots$$

where $k_j(n)$ satisfies the conditions 1, 2, 3 of Section 5.1.2. Consider

$$\ln |p_n(z)|^{1/n} = \ln \left| \prod_{j=1}^{n} (z - \beta_j)^{k_j(n)} \right|^{1/n} = \frac{1}{n} \sum_{j=1}^{n} k_j(n) \ln |z - \beta_j| .$$

Then,

$$\lim_{n\to\infty} \ln |p_n(z)|^{1/n} = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{k_j(n)}{n} \ln |z - \beta_j| = \sum_{j=1}^{\infty} \alpha_j \ln |z - \beta_j| = \Psi(z) .$$

Hence, we conclude that

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = e^{\Psi(z)}, \ z \in \mathbb{C}. \tag{5.20}$$

We will show that the system of polynomials $\{p_n(z)\}$ forms a basis in $A(K)$. Consider the system of functions

$$q_n(z) = \frac{1}{2\pi i} \frac{1}{p_{n+1}(z)}$$

in the space $A(K^*)$. Using Lemma 7, the system of functions $\{q_n\}$ in the space $A(K^*)$ is biorthogonal to the system $\{p_n\}$ .The formal expansion of an arbitrary $x \in A(K)$ in a series of the $\{p_n(z)\}$ is the Newton interpolation series

$$x(z) = \sum_{n=0}^{\infty} \xi_n p_n(z), \tag{5.21}$$

where

$$\xi_n = \int\limits_{\partial\Delta_\alpha} x(\xi)q_n(\xi)d\xi,\ \alpha = \alpha(x) > -\infty. \tag{5.22}$$

The convergence of the series (5.21) in $A(K)$ for each $x \in A(K)$ remains to be established. From (5.20) and (5.22) we obtain the bounds

$$C_1(\alpha,\varepsilon)e^{(\alpha-\varepsilon)n} \leq |p_n|_{\Delta_\alpha} \leq C_2(\alpha,\varepsilon)e^{(\alpha+\varepsilon)n}, \tag{5.23}$$

$$|\xi_n| \leq \frac{L(\alpha)\,|x|_{\Delta_\alpha}}{2\pi C_1(\alpha,\varepsilon)}e^{-(\alpha-\varepsilon)n} = M(x,\alpha,\varepsilon)e^{-(\alpha-\varepsilon)n}, \tag{5.24}$$

$$0 < C_1(\alpha,\varepsilon),\ C_2(\alpha,\varepsilon) < \infty,\ \ -\infty < \alpha < \infty,\ \varepsilon > 0. \tag{5.25}$$

Taking $\beta = \alpha - 3\varepsilon$, we obtain that

$$\sum |\xi_n|\,|p_n|_{\Delta_\beta} \leq M_1\left(x,\alpha,\varepsilon\right)\sum_{n=0}^{\infty} e^{-\varepsilon n} < \infty, \tag{5.26}$$

$$M_1\left(x,\alpha,\varepsilon\right) = M\left(x,\alpha,\varepsilon\right)C\left(\alpha-3\varepsilon,\varepsilon\right).$$

Inequality (5.26) shows that the series in (5.21) converges in $A\left(K\right)$ for each $x \in A\left(K\right)$. Hence, $\{p_n\}$ is a basis in $A\left(K\right)$. ■

**Remark 2** *If a compactum $L \subset \widehat{\mathbb{C}}$ contains the infinite point then there is no polynomial basis in $A\left(L\right)$. But using results of the sections 5.1.1 and 5.1.2, we can easily construct extendible bases, if $L$ is regular or polar, but $\infty \in L \neq \widehat{\mathbb{C}}$. Indeed, if $\varphi\left(z\right) = \frac{1}{z-a}$ with $a \notin L$, then $K := \varphi\left(L\right) \subset \mathbb{C}$. Due to Corollary 1, or Theorem 18, there exists a polynomial basis $\{p_k\left(z\right)\}$ in $A\left(K\right)$. Then, obviously, the system*

$$q_k\left(z\right) := p_k\left(\frac{1}{z-a}\right).$$

*is a basis in all the spaces $A\left(L\right),\ A\left(L_R\right),\ A\left(G_R\right)$, where $L_R = \varphi^{-1}\left(K_R\right)$ and $G_R = \varphi^{-1}\left(D_R\right)$ and $K_R$, $D_R$ are as defined in Proposition 9.*

### 5.1.3 The Hilbert Methods Proof

**Theorem 19** *Let $K \subset D$ be a regular pair "compact set-domain". Let $H_0,\ H_1$ be such that the dense continuous imbeddings hold:*

$$A\left(K\right) \hookrightarrow H_0 \hookrightarrow AC\left(K\right), \tag{5.27}$$

$$A\left(D^{*}\right)\hookrightarrow H_{1}^{\prime}\hookrightarrow AC\left(D^{*}\right), \tag{5.28}$$

*where $H_1'$ is a GKS-realization of the dual space $H_1^*$. Then the common orthogonal basis $\{e_k(z)\}$ for $H_0, H_1$, normalized in $H_0$ and ordered by non-decreasing of its norms in $H_1$ :*

$$\|e_k\|_{H_0}=1,\ \ \mu_k=\mu_k\left(H_0,H_1\right):=\|e_k\|_{H_1}\nearrow\infty,$$

*is also a common basis in all spaces $A(D)$, $A(D_\alpha)$, $A(K_\delta)$, and $A(K)$ where $D_\alpha, K_\delta$ are the sublevel domains as defined in Lemma 6.*

**Proof.** Let $K,\ D,\ H_0, H_1$ be spaces as given. Using GKS-duality, the left hand side of the imbedding (5.28) can be realized as

$$H_1\hookrightarrow A\left(D\right). \tag{5.29}$$

Then, combining (5.27) and (5.29), the following continuous imbeddings are valid:

$$H_1\hookrightarrow A\left(D\right)\hookrightarrow A\left(K\right)\hookrightarrow H_0\hookrightarrow AC\left(K\right) \tag{5.30}$$

On the other hand, using GKS-duality, the left hand side of the imbedding (5.27) can be realized as

$$H_0^{\prime}\hookrightarrow A\left(K^{*}\right). \tag{5.31}$$

Then, combining (5.28) and (5.31), the following continuous imbeddings are valid:

$$H_0^{\prime}\hookrightarrow A\left(K^{*}\right)\hookrightarrow A\left(D^{*}\right)\hookrightarrow H_1^{\prime}\hookrightarrow AC\left(D^{*}\right). \tag{5.32}$$

The inclusion $H_1\subset H_0$ is a dense linear imbedding and since the spaces $A(D)$, $A(K)$ are nuclear, it is compact. Then, using Theorem 7, there exists a system $\{e_k\}\subset H_1$ which is a common orthogonal basis in $H_1$ and $H_0$ such that

$$\|e_k\|_{H_0}=1,\ \ \mu_k=\mu_k\left(H_0,H_1\right)=\|e_k\|_{H_1}\nearrow\infty.$$

Let $D_q$ be the sublevel open sets defined in Lemma 6. The continuity of the imbedding $H_1\hookrightarrow A(D)$ implies that for any $q$, $0<q<1$, there exists a constant $C_q>0$ such that for any $x\in H_1\hookrightarrow A(D)$,

$$|x|_{D_q}\leq C_q\,\|x\|_{H_1}$$

In particular,

$$|e_k|_{D_q} \leq C_q \|e_k\|_{H_1} = C_q \mu_k \tag{5.33}$$

since $\|e_k\|_{H_1} = \mu_k$. The continuity of the imbedding $H_0 \hookrightarrow AC(K)$ implies that, for any $x \in H_0 \hookrightarrow AC(K)$,

$$|x|_K \leq C \|x\|_{H_0}.$$

In particular,

$$|e_k|_K \leq C \|e_k\|_{H_0} = C \tag{5.34}$$

since $\|e_k\|_{H_0} = 1$. Using Theorem 13 for $e_k$ with the help of the (5.33) and (5.34),

$$|e_k|_{D_{\alpha q}} \leq |e_k|_K^{1-\alpha} |e_k|_{D_q}^{\alpha} = C^{1-\alpha} (C_q \mu_k)^{\alpha} = C^{1-\alpha} C_q^{\alpha} \mu_k^{\alpha}.$$

where $D_{\alpha q} = \{z \in D : 0 < \omega(D, K, z) < \alpha q\}$, $0 < \alpha q < 1$. Since $\alpha q \uparrow \alpha$ as $q \uparrow 1$, we get

$$|e_k|_{D_\alpha} \leq C(\alpha, \varepsilon) \mu_k^{\alpha+\varepsilon}. \tag{5.35}$$

Now, we have to find the estimates for the biorthogonal system. Let $\{e'_k\} \subset H'_0 \subset A(K^*)$. The formal expansion of an arbitrary $x \in H_1$ can be written as

$$x = \sum e_k^*(x) e_k$$

where $\{e_k(z)\} \subset H_0 \subset A(D)$.

Let $K_\delta$ be a sequence of compact sets such that $K_\delta \Downarrow K$ where $K_\delta$ are as defined in Lemma 6. The continuity of the imbedding $H'_0 \hookrightarrow A(K^*)$ implies that for any $\delta$, $0 < \delta < 1$, there exists a constant $C'_\delta > 0$ such that for any $x' \in H'_0 \hookrightarrow A(K^*)$,

$$|x'|_{K_\delta} \leq C'_\delta \|x'\|_{H'_0}.$$

In particular,

$$|e'_k|_{K_\delta} \leq C'_\delta \|e'_k\|_{H'_0} = C'_\delta, \tag{5.36}$$

since $\|e'_k\|_{H'_0} = 1$.The continuity of the imbedding $A(D^*) \hookrightarrow H'_1$ implies that for any $x' \in A(D^*) \hookrightarrow H'_1$,

$$|x'|_{D^*} \leq C' \|x'\|_{H'_1}.$$

In particular,

$$|e'_k|_{D^*} \leq C' \|e'_k\|_{H'_1} = C' \mu_k^{-1}, \tag{5.37}$$

since $\|e'_k\|_{H'_1} = \frac{1}{\mu_k}$.

Again using Theorem 13 for $e'_k$ with the help of the (5.36) and (5.37),

$$|e'_k|_{D^*_{\alpha\delta}} \leq |e_k|^{1-\alpha}_{K^*_\delta} |e_k|^{\alpha}_{D^*} = C'^{(1-\alpha)}_\delta \left(C' \mu_k^{-1}\right)^\alpha .$$

where $D_{\alpha\delta} = \{z \in D : 0 < \omega(D, K, z) < \alpha\delta\}$, $0 < \alpha\delta < 1$. Since $\alpha\delta \uparrow \alpha$ as $\delta \uparrow 1$, we get

$$|e'_k|_{D^*_\alpha} \leq C'(\alpha, \varepsilon) \mu_k^{-\alpha+\varepsilon}. \tag{5.38}$$

Now, we will show that $\{e_k\}$ will be a basis for each of the spaces $A(D)$, $A(D_\alpha)$, $A(K_\gamma)$, and $A(K)$ .We know

$$e'_k \in H'_0 \hookrightarrow A(K^*) \hookrightarrow A(D^*) \hookrightarrow AC(D^*_\alpha) \hookrightarrow AC(D^*) \hookrightarrow H'_1.$$

To show that $\{e_k\}$ is a basis for $A(D)$, first observe that for any $x \in A(D) \hookrightarrow AC(D_\beta)$, and $e'_k \in H'_0 \hookrightarrow A(K^*) \hookrightarrow A(D^*) \hookrightarrow H'_1$

$$|e'_k(x)| \leq |e'_k|^*_\beta |x|_\beta . \tag{5.39}$$

Now, take any $x \in A(D) \hookrightarrow H_0$. Then, since $x$ is also an element of $H_0$, the formal series expansion of $x$ in $H_0$ is

$$x = \sum_{k=1}^{\infty} e'_k(x) e_k \in H_0.$$

We have to show that for any $x \in A(D)$ and any $\alpha < \beta$,

$$\sum_{k=1}^{\infty} |e'_k(x)| |e_k|_\alpha < \infty \tag{5.40}$$

Using (5.39) and estimates (5.35) and (5.38),

$$\sum_{k=1}^{\infty} |e'_k(x)| |e_k|_\alpha \leq \sum |e'_k(x)|^*_\beta |e_k|_\alpha |x|_\beta$$

$$\leq |x|_\beta \sum K(\alpha, \beta, \varepsilon) \frac{1}{\mu_k^{\beta-\varepsilon}} \mu_k^{\alpha+\varepsilon}$$

$$= |x|_\beta K(\alpha, \beta, \varepsilon) \sum \mu_k^{\alpha-\beta+2\varepsilon}$$

where the second inequality follows from the (5.35) and (5.38). Take $\varepsilon$ such that $\alpha - \beta + 2\varepsilon = -\sigma < 0$, then

$$\sum_{k=1}^{\infty} |e_k'(x)| |e_k|_\alpha \leq |x|_\beta K(\alpha, \beta, \varepsilon) \sum_{k=1}^{\infty} \frac{1}{\mu_k^\sigma} < \infty.$$

where $\sum_{k=1}^{\infty} \frac{1}{\mu_k^\sigma} < \infty$ follows from the nuclearity of the operator $H_1 \hookrightarrow H_0$. So, for any $x \in A(D)$, indeed (5.40) is satisfied, therefore the system $\{e_k\}$ is a basis for $A(D)$.

Now, let's show that $\{e_k\}$ is a basis for $A(D_\delta)$, $0 < \delta < 1$. Observe that for any $x \in A(D_\delta) \hookrightarrow AC(D_\beta)$, and $e_k' \in H_0' \hookrightarrow A(K^*) \hookrightarrow A(D^*) \hookrightarrow H_1'$ :

$$|e_k'(x)|_\delta \leq |e_k'|_\beta^* |x|_\beta \tag{5.41}$$

Now, take any $x \in A(D_\delta) \hookrightarrow H_0$. Then, since $x$ is also an element of $H_0$, the formal series expansion of $x$ in $H_0$ is

$$x = \sum_{k=1}^{\infty} e_k'(x) e_k \in H_0$$

We have to show that for any $x \in A(D_\delta)$ and any $\alpha < \beta < \delta$,

$$\sum_{k=1}^{\infty} |e_k'(x)|_\delta |e_k|_\alpha < \infty \tag{5.42}$$

Using (5.41) and estimates (5.35) and (5.38),

$$\sum_{k=1}^{\infty} |e_k'(x)|_\delta |e_k|_\alpha \leq \sum |e_k'(x)|_\beta^* |e_k|_\alpha |x|_\beta$$

$$\leq |x|_\beta \sum K(\alpha, \beta, \varepsilon) \frac{1}{\mu_k^{\beta-\varepsilon}} \mu_k^{\alpha+\varepsilon}$$

$$= |x|_\beta K(\alpha, \beta, \varepsilon) \sum \mu_k^{\alpha-\beta+2\varepsilon}$$

where the second inequality follows from the (5.35) and (5.38). Take $\varepsilon$ such that $\alpha - \beta + 2\varepsilon = -\sigma < 0$, then

$$\sum_{k=1}^{\infty} |e_k'(x)|_\delta |e_k|_\alpha \leq |x|_\beta K(\alpha, \beta, \varepsilon) \sum_{k=1}^{\infty} \frac{1}{\mu_k^\sigma} < \infty.$$

where $\sum_{k=1}^{\infty} \frac{1}{\mu_k^{\sigma}} < \infty$ follows from the nuclearity of the operator $H_1 \hookrightarrow H_0$. So, for any $x \in A(D_\delta)$, indeed (5.42) is satisfied, therefore the system $\{e_k\}$ is a basis for $A(D_\delta)$.

Now, we will show that the system $\{e_k\}$ is a basis for the spaces for the spaces $A(K)$ and $A(K_\rho)$. These spaces can be represented as the inductive limits:

$$A(K) = limind_{\delta \downarrow 0} A(D_\delta),$$

$$A(K_\rho) = limind_{\delta \downarrow \rho} A(D_\delta).$$

Since the system $\{e_k\}$ is a basis for each of the spaces $A(D_\delta)$, it will be a basis for their inductive limits as well. Therefore, the system $\{e_k\}$ is basis for both of the spaces $A(K)$ and $A(K_\rho)$. ■

The restrictions on Hilbert spaces $H_1,\ H_0$ can be considerably weakened if the pair $(K, D)$ satisfies certain additional conditions (see, Theorem 20 below).

**Definition 10** *Let $D$ be a regular domain in $\mathbb{C}$. We say that $D$ is stable from outside if for any $G^{(s)} \Downarrow \bar{D}$ we have*

$$\omega\left(G^{(s)}, K, z\right) \uparrow \omega(D, K, z),\ \forall z \in D.$$

**Definition 11** *Let $K$ be a regular compactum in the domain $D$. We say that $K$ is stable from within if $K^* = \mathbb{C} \setminus K$ is stable from outside.*

Let $K^*$ be stable from outside,

$$1 - \omega(K^*, D^*, \zeta) = \omega(D, K, \zeta),$$

where $K^* = \mathbb{C} \setminus K,\ D^* = \mathbb{C} \setminus D$.Then,

$$1 - \omega\left(K^{(s)*}, D^*, \zeta\right) = \omega\left(D, K^{(s)}, \zeta\right)$$

where $K^{(s)} \Uparrow \mathcal{G},\ K^{(s)*} \Downarrow \mathcal{G}^*,\ K = \bar{\mathcal{G}}$. Since $K^*$ is stable from outside,

$$\omega\left(K^{(s)*}, D^*, \zeta\right) \uparrow \omega(K^*, D, \zeta).$$

So,

$$\omega\left(D, K^{(s)}, \zeta\right) \uparrow \omega(D, K, \zeta)$$

if $K^*$ is stable from outside.

**Theorem 20** *Let $K \subset D$ be a regular pair "compact set-domain", and $K$ be stable from within and $D$ be stable from outside. Let $H_0,\ H_1$ be such that the dense continuous imbeddings hold:*

$$A(K) \hookrightarrow H_0 \hookrightarrow A(intK), \quad A(D^*) \subset H_1' \subset A\left(\bar{D}^*\right),$$

*where $H_1'$ is a GKS-realization of the dual space $H_1^*$. Then the common orthogonal basis $\{e_k(z)\}$ for $H_0,\ H_1$, normalized in $H_0$ and ordered by non-increasing of its norms in $H_1$ :*

$$\|e_k\|_{H_0} = 1,\ \mu_k = \mu_k(H_0, H_1) := \|e_k\|_{H_1} \nearrow \infty$$

*is also a common basis in both spaces $A(D)$ and $A(K)$.*

**Theorem 21** *Let $K \subset D$ be a regular pair "compact set-domain". Under the conditions of Theorem 19, let $\{e_k\}$ be the common basis for the spaces $A(K)$ and $A(D)$ that was constructed therefrom. Then the following asymptotics*

$$\overline{\lim_{\zeta\to z}}\,\overline{\lim_{k\to\infty}}\frac{\ln|e_k(\zeta)|}{\ln\mu_k} = \omega(D, K, z),\ z \in D \setminus K \tag{5.43}$$

*is fulfilled uniformly on every compactum $L \subset D \setminus K$.*

**Proof.** Let the sublevel domain $D_\alpha = \{z \in D : \omega(z) = \omega(D, K, z) < \alpha\}$. Using (5.35), we have

$$|e_k(z)|_{D_\alpha} \leq C(\alpha, \varepsilon)\,\mu_k^{\alpha+\varepsilon}.$$

First, take the logarithm of both sides and divide by $\ln\mu_k$. Then, taking the limit as $k \to \infty$, we get

$$\overline{\lim_{k\to\infty}}\frac{\ln|e_k(z)|}{\ln\mu_k} \leq \alpha,\ \text{for any } z \in D_\alpha.$$

After regularization,

$$u(z) = \overline{\lim_{\zeta\to z}}\,\overline{\lim_{k\to\infty}}\frac{\ln|e_k(z)|}{\ln\mu_k} \leq \alpha = \omega(z),\ \text{for any } z \in D_\alpha.$$

We will prove the other inequality by contradiction. Assume that there exists $z_0 \in D \setminus K$ such that

$$u(z_0) < \omega(z_0) = \alpha. \tag{5.44}$$

Then, there exists an open disk $\mathbb{D}(z_0, \varepsilon) \subset D$ such that

$$u(z) < \alpha - \varepsilon, \text{ for any } z \in \mathbb{D}(z_0, \varepsilon).$$

Using $(5.44)$, we conclude that there exists $k_0 \in \mathbb{N}$ such that

$$\frac{\ln |e_k(z)|}{\ln \mu_k} < \alpha - \delta, \text{ for any } z \in \mathbb{D}(z_0, \varepsilon),$$

where $k \geq k_0$.

Let $x \in A(D_\alpha)$ be a function which is not analytically extendable to any larger domain. The formal expansion of $x$ will be:

$$x(z) = \sum_{k=1}^{\infty} e'_k(x) e_k(z).$$

By $(5.38)$, we have

$$|e'_k| \leq C(\alpha, \varepsilon) \mu_k^{-\alpha+\varepsilon}.$$

Then,

$$\sum_{k=1}^{\infty} |e'_k(x)| |e_k| \leq \sum_{k=1}^{\infty} C(\alpha, \varepsilon) \mu_k^{-\alpha+\varepsilon} C'(\alpha, \delta) \mu_k^{\alpha-\delta}$$
$$\leq C(\alpha, \varepsilon, \delta) \sum_{k=1}^{\infty} \mu_k^{-2\alpha+\varepsilon+\delta}.$$

Choose $\delta$ so that $-2\alpha + \varepsilon + \delta < 0$. Then the convergence of the sum follows from the nuclearity of the operator $H_1 \hookrightarrow H_0$. That shows that $x$ can be analytically extended in the disk $\mathbb{D}(z_0, \varepsilon)$, which contradicts the assumption that $x$ is not extendible. This contradiction shows the validity of $(5.43)$. ■

## 5.2 Isomorphic Classification

### 5.2.1 About Compacta With Infinitely Many Holes

**Theorem 22** *Let $K$ be a compactum, with complement $D = K^*$ consisting of an enumerable collection of mutually exterior regions $D_j$. Then $A(K)$ is isomorphic to the topological sum of the spaces $A(K_j)$, $K_j = (D_j)^*$.*

The isomorphism mentioned is represented by the formula

$$x \leftrightarrow (x_1, x_2, \cdots, x_j, \cdots),$$

where $x_j \in A(K_j)$, and is computed with the aid of the formula

$$x_j(z) = \int\limits_{\Gamma_j} \frac{x(\zeta)\, d\zeta}{\zeta - z},$$

where $z \in \Delta_j$ and $\Delta_j = \Delta_j(x)$ is a neighborhood of the compactum $K_j$, and the contour $\Gamma_j$ consists of a finite number of closed Jordan curves lying in $(\Delta_j)^*$ and containing in its interior all the singular points of the function $x(z)$ situated in $D_j$. For each function $x \in A(K)$, only a finite number of the functions $x_j$ are not identically zero. Thus $x(z) = \sum\limits_{j=1}^{\infty} x_j(z)$ for some neighborhood $\Delta = \Delta(x)$ of the compactum $K$.

**Corollary 3** *The space $A(K)$, where $K$ satisfies the conditions of Theorem 22, is not isomorphic to the space $A(K_0)$ if $K_0$ is a compactum containing in its complement only a finite number of mutually exterior connected components.*

### 5.2.2 Isomorphism to $\overline{A}_1$

**Theorem 23** *Let $K$ be a compactum in $\widehat{\mathbb{C}}$. For the spaces $A(K)$ and $\bar{A}_1$ to be isomorphic, it is necessary and sufficient that (a) the compactum $K$ be regular, and (b) the complement $K^* = \widehat{\mathbb{C}} \setminus K$ consist no more than a finite number of connected components.*

**Proof. Sufficiency**

Let the compactum $K$ satisfy the conditions of Proposition 9. Then, from it follows (5.6) that

$$L_2(\alpha, \varepsilon) C(K)^k e^{(\alpha - \varepsilon)k} \le |p_k(z)|_{\Delta_\alpha} \le L_1(\alpha, \varepsilon) C(K)^k e^{(\alpha + \varepsilon)k}, \tag{5.45}$$

$$\varepsilon > 0, 0 < L_2(\alpha, \varepsilon), L_1(\alpha, \varepsilon) < \infty,$$

where $\Delta_\alpha$ is the interior of the level curve $\Gamma_\alpha = \{z : g_{K^*}(z, \infty) = \alpha\}$, $0 < \alpha < \infty$ and $g_{K^*}(z, \infty)$ is the Green's function. $K$ is regular, hence $g_{K^*}(z, \infty) = 0$ for any

$z \in K$. Define $\Delta_\alpha \Subset \Delta_\beta$ for $\alpha < \beta$, then $K = \cap_{\alpha>0}\Delta_\alpha$. Inductive topology can be defined on $A(K)$ as in (4.4).

Given $x \in A(K)$, consider its basis expansion:

$$x(z) = \sum_{k=0}^{\infty} \xi_k p_k(z).$$

Then, from (5.45) we have

$$|x|_{\Delta_\alpha} \leq L(\alpha, \varepsilon) \sum_{k=0}^{\infty} |\xi_k| \, C(K)^k e^{(\alpha+\varepsilon)k}.$$

Therefore the mapping $T : Y \to A(K)$, where $Y := limind_{\lambda \downarrow 0} l_1(e^{\lambda k})$ defined by the formula $\left(\xi_k C(K)^k\right) \to x$ is continuous. Since both of the spaces $A(K)$ and $Y$ are complete and the operator $T$ is a continuous bijection, by Banach Theorem, we get $T$ is an isomorphism. Hence, $A(K)$ is isomorphic to the space $Y$. Since $\bar{A}_1$ is isomorphic to $Y$, we conclude that $A(K)$ is isomorphic to $\bar{A}_1$. If $\infty \in K$ and $K$ has a simply connected complement, Remark 2 can be used to reduce to the same case.

Now, let $K$ be a regular compactum in $\widehat{\mathbb{C}}$ with complement $D = K^*$, consisting of a finite number of exterior connected components $D_j$, $j = 1, 2, \ldots, r$. Then, $A(D) \simeq \prod_{j=1}^{r} A(D_j)$, consequently

$$A(K) \simeq A(D)^* \simeq \prod_{j=1}^{r} A(D_j)^* \simeq \bigoplus_{j=1}^{r} A(K_j),$$

where $K_j = (D_j)^*$. Using the previous argument, $A(K_j) \simeq \bar{A}_1$, thus $A(K) \simeq \bar{A}_1 \times \bar{A}_1 \times \cdots \times \bar{A}_1$ ($r$ times) and so $A(K) \simeq \bar{A}_1$.

**Necessity of the condition (a)**

Since the case the complement of $K$ has finite number of connected complements can be obtained similarly, we will assume that the complement of $K$ is connected for simplicity. Then the regularity of the compactum $K$ will follow from the existence of an isomorphism between the spaces $A(K)$ and $\bar{A}_1$.

We will prove by contradiction. Assume that $K$ is not regular but there is an isomorphism $T$ of the space $A(K)$ onto $\bar{A}_1$. Then, the adjoint operator $T^*$ is an isomorphism of the space $(\bar{A}_1)^*$ onto $A(K)^*$. Since $A_1$ is Montel space, $(\bar{A}_1)^*$ is isomorphic to $A_1$ and the space $A(K)^*$, according to Theorem 17, is isomorphic to the space $A(D)$ where $D = K^*$. Hence there is an isomorphism $\widetilde{T}$ of the space $A_1$ onto $A(D)$. Now, for $D$, we will define the projective limit topology as:

$$A(D) = limprojAC(D_s),$$

where the $D_s$ are open sets, $D_s \Subset D_{s+1}$, $s = 1, 2, \cdots$ , and $\cup_s D_s$.

Since the system of functions $z^k$ form a basis in $A_1$, their isomorphic images $h_k(z) = \widetilde{T}(z^k)$ is a basis in $A(D)$. Thus, for each $s \geq s_0 = s_0(T)$, choose $r_i = r_i(s) < 1$ and $C_i(s)$, $0 < C_i < \infty$, $i = 1, 2$, such that

$$C_2 r_2^k \leq |h_k|_{D_s} \leq C_1 r_1^k,$$

and $r_i(s) \uparrow 1$ for $s \uparrow \infty$.Define

$$\Psi(z) := \overline{\lim_{\varsigma \longrightarrow z}}\, \overline{\lim_{k \longrightarrow \infty}} \frac{\ln |h_k(\varsigma)|}{k}. \tag{5.46}$$

Since $h_k(\varsigma)$ are analytic in $D$, $\frac{\ln|h_k(\varsigma)|}{k}$ are subharmonic functions in $D$. We have to show that $\frac{\ln|h_k(\varsigma)|}{k}$ are locally uniformly bounded above in the domain $D$. Let $C$ be a compactum in $D$. Then, there exists $s$ such that $C \subset D_s$, so

$$|h_k(\varsigma)|_C \leq |h_k(\varsigma)|_{D_s} \leq C_1 r_1^k, \forall k.$$

$$\Longrightarrow \frac{\ln |h_k(\varsigma)|_C}{k} \leq \frac{\ln C_1}{k} + \ln r_1, \forall k \tag{5.47}$$

Hence, $\frac{\ln|h_k(\varsigma)|}{k}$ are indeed locally uniformly bounded on $D$. Using Theorem 9, we conclude that $\Psi(z)$ is a subharmonic function. Also, using (5.47),

$$\overline{\lim_{\varsigma \longrightarrow z}}\, \overline{\lim_{k \longrightarrow \infty}} \frac{\ln |h_k(\varsigma)|}{k} \leq \overline{\lim_{\varsigma \longrightarrow z}} \left( \overline{\lim_{k \longrightarrow \infty}} \frac{\ln C_1}{k} \right) + \overline{\lim_{\varsigma \longrightarrow z}}\, \overline{\lim_{k \longrightarrow \infty}} \ln r_1$$

$$\Longrightarrow \Psi(z) \leq 0, z \in D, \text{ since } r_1 < 1. \tag{5.48}$$

Also, let $z \in \bar{D}_0$, where $D_0 = D_{s_0}$. Then

$$\Psi(z) \leq \ln r_1(s_0) = -\sigma < 0. \tag{5.49}$$

The set $(\bar{D}_0)^* = G$ is open and consists of a finite number of mutually exterior regular regions $G_j$, $j = 1, 2, ..., m$. Since the compactum $K$ is not regular, for at least one $j$ the compactum $K_j = G_j \cap K$ is not regular.

Consider the generalized solution function $w_j(z) = w(K_j, G_j, z)$ constructed as in Remark 1. By construction, since $w_j(z)$ must coincide with $f$ on the boundaries, $w_j(z) \equiv 1$ on $\partial G_j$. So, for any $z \in \partial G_j$,

$$\Psi(z) \leq -\sigma w_j(z).$$

By the generalized maximum principle, if the maximum value of $\Psi(z)$ exists, it is attained on the boundary. So,

$$\Psi(z) \leq -\sigma w_j(z), \ \forall z \in G_j.$$

Also, since $G_j \backslash K_j \subset G_j$, that also holds for $G_j \backslash K_j$, that is,

$$\Psi(z) \leq -\sigma w_j(z), \ \forall z \in G_j \backslash K_j. \tag{5.50}$$

From (5.46) and (5.50) the existence of the function $C(z)$, defined in $G_j \backslash K_j$, follows, and we have

$$|h_k(z)| \leq C(z)\, e^{-\sigma w_j(z) k}, \ z \in G_j \backslash K_j. \tag{5.51}$$

As $\{h_k(z)\}$ form a basis for $A(D)$, the basis expansion of an arbitrary element $x \in A(D)$ is:

$$x(z) = \sum_{k=0}^{\infty} \xi_k h_k(z).$$

Since $\{h_k(z)\}$ is the isomorphic image of the power basis in $A_1$, the following inequality holds for the coefficients $\xi_k$ :

$$\overline{\lim_{k \longrightarrow \infty}} |\xi_k|^{\frac{1}{k}} \leq 1,$$

i.e., for each $\delta > 0$,

$$|\xi_k| \leq L(x, \delta)\, e^{\delta k}, \ k = 0, 1, \cdots . \tag{5.52}$$

Since $K_j$ is an irregular compactum, it follows from Remark 1 that there exists a point $z_0 \in \partial K_j^*$ and a sequence $\{z_\nu\} \subset G_j \backslash K_j, \ z_\nu \to z_0$, such that $w(z_\nu) \to 2\alpha_0 > 0$. Without loss of generality we can assume $w(z_\nu) \geq \alpha_0, \ \nu = 1, 2, \cdots$. Hence by choosing $\delta$ so small that $\delta - \sigma\alpha_0 < 0$ we obtain, by the use of (5.51) and (5.52), that

$$|x(z_\nu)| \leq L(x, \delta)\, C(z_\nu) \left(1 - e^{\delta - \sigma\alpha_0}\right)^{-1}, \quad x \in A(D).$$

Thus each function $x \in A(D)$ has the bound

$$|x(z_\nu)| = O(C(z_\nu)), \tag{5.53}$$

where $C(z_\nu)$ is a fixed sequence not depending on $x$.

Relation (5.53) cannot be satisfied for all $x \in A(D)$, since it is easy to construct a function analytic except at the point $z_0$ and assuming any previously assigned values at the points $z_k$ [10]. The contradiction thus obtained establishes that the compactum $K$ is regular.

**Necessity of the condition (b)**

Let $T$ be an isomorphism between the spaces $\bar{A}_1$ and $A(K)$. Since the complement of the set $\mathbb{D} = \{z : |z| \leq 1\}$ consists of only one connected component, using Corollary 2, $\bar{A}_1$ cannot be isomorphic to a space $A(K_0)$ for which the complement of the set $K_0$ consists of a countably infinite number of mutually exterior components. Therefore, the complement of $K$ consists no more than a finite number of connected components. ■

**Remark 3** *If $C(K) = 0$, then the space $A_1$ is not isomorphic to any subspace of the space $A(D)$, $D = K^*$, where $K$ satisfy the conditions of Theorem 23 (see [34]).*

### 5.2.3 Isomorphism to $\overline{A}_0$

**Theorem 24** *Let $K$ be a compactum in $\widehat{\mathbb{C}}$. For the spaces $A(K)$ and $\bar{A}_0$ to be isomorphic, it is necessary and sufficient that $C(K) = 0$.*

**Proof. Sufficiency**

Let the capacity of the compactum $K$ be zero. We will show that in that case the spaces $A(K)$ and $\bar{A}_0$ are isomorphic.

For simplicity we will assume that $\infty \notin K$. The general case can be obtained from this one by Remark 2. Since the capacity of the compactum $K$ is zero, it is a polar set. Therefore, using Theorem 18, the system of polynomials

$$p_0 \equiv 1, p_n(z) = \prod_{j=1}^{n} (z - \beta_j)^{k_j(n)}, n = 1, 2, \cdots$$

where $k_j(n)$ satisfies the conditions 1, 2, 3 of Section 5.1.2, $\{\alpha_j\}$ is a sequence of nonnegative numbers, $\sum\limits_{j=1}^{\infty} \alpha_j = 1$, and $\alpha_j \downarrow 0$ forms a basis for the space $A(K)$. We know in that case we have the following bounds:

$$C_1(\alpha,\varepsilon)e^{(\alpha-\varepsilon)n} \leq |p_n|_{\Delta_\alpha} \leq C_2(\alpha,\varepsilon)e^{(\alpha+\varepsilon)n}, \tag{5.54}$$

$$|\xi_n| \leq \frac{L(\alpha)\,|x|_{\Delta_\alpha}}{2\pi C_1(\alpha,\varepsilon)} e^{-(\alpha-\varepsilon)n} = M(x,\alpha,\varepsilon)e^{-(\alpha-\varepsilon)n}, \tag{5.55}$$

$$0 < C_1(\alpha,\varepsilon),\ C_2(\alpha,\varepsilon) < \infty,\ \ -\infty < \alpha < \infty,\ \varepsilon > 0.$$

Using the bounds (5.54)and (5.55), it is seen that $A(K)$ is isomorphic to space $\underset{\alpha\downarrow-\infty}{limind}\left(l_1 e^{\alpha k}\right)$ and hence to the space $\bar{A}_0$, which proves the sufficiency.

**Necessity**

Let $A(K)$ be isomorphic to the space $\bar{A}_0$. We will show that the capacity of the compactum $K$ is zero.

We will prove by contradiction. Assume that $C(K) > 0$. Let $T$ be an isomorphism of $\bar{A}_0$ onto $A(K)$. Then the system of functions $h_k(z) = T\left(z^k\right)$ is a basis in $A(K)$ as it is the isomorphic image of a basis. From this, for each $s \geq s_0$, select $r_i = r_i(s) > 0$ and $C_i = C_i(s),\ 0 < C_i < \infty,\ i = 1, 2,$ such that

$$C_2 r_2^k \leq |h_k|_{G_s} \leq C_1 r_1^k, \tag{5.56}$$

$r_i(s) \downarrow 0$ for $s \uparrow \infty$, and $G_s$ are an open sets as in (4.4).

Define the function

$$\Phi(z) = \overline{\lim_{\zeta\to z}}\,\overline{\lim_{k\to\infty}} \frac{\ln|h_k(\zeta)|}{k} \tag{5.57}$$

in $D = G_{s_0}$. Since $h_k(\zeta)$ are analytic in $D$, $\dfrac{\ln|h_k(\zeta)|}{k}$ are subharmonic functions in $D$. We have to show that $\dfrac{\ln|h_k(\zeta)|}{k}$ are locally uniformly bounded in the domain $D$. Let $C$ be a compactum in $D$.Then, there exists $s$ such that $C \subset G_s$ so

$$|h_k(\varsigma)|_C \leq |h_k(\varsigma)|_{G_s} \leq C_1 r_1^k, \forall k.$$

$$\Longrightarrow \frac{\ln|h_k(\varsigma)|_C}{k} \leq \frac{\ln C_1}{k} + \ln r_1, \forall k. \tag{5.58}$$

Hence, $\dfrac{\ln|h_k(\zeta)|}{k}$ are indeed locally uniformly bounded on $D$. Using Theorem 9, we conclude that $\Phi(z)$ is a subharmonic function. Also, using (5.58),

$$\overline{\lim_{z\longrightarrow z_0}}\,\overline{\lim_{k\longrightarrow\infty}}\frac{\ln|h_k(\varsigma)|}{k}\leq\overline{\lim_{z\longrightarrow z_0}}\left(\overline{\lim_{k\longrightarrow\infty}}\frac{\ln C_1}{k}\right)+\overline{\lim_{z\longrightarrow z_0}}\,\overline{\lim_{k\longrightarrow\infty}}\ln r_1$$

$$\Longrightarrow\Phi(z)\to-\infty,\ z\to z_0,\ z\in D\diagdown K,\ z_0\in\partial K^*,\text{ since } r_1\downarrow 0 \text{ for } z_0\in\partial K^*. \quad (5.59)$$

Again, with the use of (5.58), it is seen that

$$\Phi(z)\leq\ln r_1(s_0)=\sigma<\infty,\ z\in D. \quad (5.60)$$

The open set $D$ consists of a finite number of mutually exterior regular regions $D_j$, $j=1,\ 2,\ \cdots,m$. By the assumption $C(K)>0$, an integer $j$ can be found such that $C(K_j)>0$, where $K_j=D_j\cap K$.

We have $\partial K_j=\partial K\cap D_j$, therefore applying Lemma 2 for the compactum $K_j$ by (5.59) and (5.60), we obtain

$$\Phi(z)\equiv-\infty,\ z\in D_j\setminus K_j. \quad (5.61)$$

Using (5.57) and (5.61), there exists a function $C(z,\varepsilon)<\infty,\ z\in D_j,\ \varepsilon>0$, such that

$$|h_k(z)|\leq C(z,\varepsilon)\,e^{\frac{-k}{\varepsilon}},\ k=0,1,\cdots. \quad (5.62)$$

Substituting the subharmonic minorant $\tilde{C}(z,\varepsilon)$ for $C(z,\varepsilon)$ in (5.62) and then using the local boundedness of the function $\tilde{C}(z,\varepsilon)$, we obtain the existence of an open set $\Delta$ such that $K_j\subset\Delta\Subset D_j$ and

$$|h_k(z)|\leq M(\varepsilon)\,e^{\frac{-k}{\varepsilon}},\ z\in\Delta,\ k=0,1,\cdots, \quad (5.63)$$

where $M(\varepsilon)=\sup\left\{\tilde{C}(z,\varepsilon):z\in\partial\Delta\right\}$.

Now, let $x(z)$ be an arbitrary element of the space $A(K)$, where

$$x(z)=\sum\xi_k h_k(z) \quad (5.64)$$

is its basis expansion. Since $\{h_k\}$ is the isomorphic image of the power basis of $\bar{A}_0$, the coefficients $\xi_k$ have the following bound:

$$|\xi_k|\leq L(x)\,e^{\frac{k}{\delta}},\ \delta=\delta(x)>0. \quad (5.65)$$

Taking advantage of the inequalities (5.63) and (5.65) for $\varepsilon < \delta$, we obtain uniform convergence of the series (5.64) in the region $\Delta$. Thus, an arbitrary element $x \in A(K)$ is an analytic function in the fixed neighborhood $\Delta$ of the compactum $K_j$. A contradiction is reached and this proves that $C(K) = 0$. ■

### 5.2.4 Isomorphism to $\overline{A}_1 \times \overline{A}_0$

**Theorem 25** *Let $K$ be a compactum in $\widehat{\mathbb{C}}$. For the spaces $A(K)$ and $\bar{A}_1 \times \bar{A}_0$ to be isomorphic, it is necessary and sufficient that the compactum $K$ be decomposed into two disjoint non-empty compacta $K^{(1)}$ and $K^{(2)}$, where $K^{(1)}$ is a regular compactum whose complement consists of a finite number of connected components and $C(K^{(2)}) = 0$.*

**Proof. Sufficiency** Let $K$ be decomposed into two disjoint non-empty compacta $K^{(1)}$ and $K^{(2)}$, where $K^{(1)}$ is a regular compactum whose complement consists of a finite number of connected components and $C(K^{(2)}) = 0$, that is, $K = K^{(1)} \cup K^{(2)}$. Then, using the sufficiency parts of the Theorems 23 and 24 and from $A(K) \simeq A\left(K^{(1)}\right) \times A\left(K^{(2)}\right)$, we have that $A(K) \simeq \bar{A}_1 \times \bar{A}_0$.

**Necessity** Let $T$ be an isomorphism between the spaces $\bar{A}_1 \times \bar{A}_0$ and $A(K)$.

Take the natural basis in the space $\bar{A}_1 \times \bar{A}_0$ :

$$e_{2k} = \left(0, z^k\right), \ e_{2k+1} = \left(z^k, 0\right), \ k = 0, 1, \cdots .$$

Then, the system of the functions $h_j(z) = T(e_j), \ j = 0, 1, \cdots$, is a basis in the space $A(K)$ as the isomorphic image of a basis. Therefore, as in the proof of Theorem 23, for each $s \geq s_0$ we can select $\delta_i = \delta_i(s), \ \delta_i > 0$, and $C_i = C_i(s), \ 0 < C_i < \infty$, for $i = 1, 2$, such that

$$C_2 e^{\frac{-k}{\delta_2}} \leq |h_{2k}|_{G_s} \leq C_1 e^{\frac{-k}{\delta_1}}, \ k = 0, 1, \cdots , \tag{5.66}$$

and $\delta_i(s) \downarrow 0$ for $s \uparrow \infty$ and $G_s$ is an open set.

Let $D := G_{s_0}$. For simplicity, assume that $D$ consists of a finite number of mutually exterior regular regions $D_j, \ j = 1, 2, \cdots, r$. Let $K_j := K \cap D_j$. Denote the union of the compacta for which $C(K_j) > 0$ by $K^{(1)}$ and $K^{(2)} := K \setminus K^{(1)}$. Then, denote the corresponding unions of regions $D_j$ by $D^{(1)}$ and $D^{(2)}$.

Without loss of generality, assume that $\left(K^{(1)}\right)^*$ is connected. The case where $\left(K^{(1)}\right)^*$ has a finite number of connected components can be obtained by a similar argument.

Using the construction, it is seen that $K^{(1)} \cap K^{(2)} = \varnothing$ and $C\left(K^{(2)}\right) = 0$. Without loss of generality, denote the $K_j$ for which $C(K_j) > 0$ by $K_j$, $j = 1, \cdots, m$. Then,

$$\begin{aligned}
K^{(2)} &= K \setminus K^{(1)} = K \setminus \cup_{j=1}^{m} (K \cap D_j) \\
&= K \cap \left(\cup_{j=1}^{m} (K \cap D_j)\right)^* \\
&= K \cap \left(\cap_{j=1}^{m} (K \cap D_j)^*\right) \\
&= K \cap \left(\cap_{j=1}^{m} \left(K^* \cup D_j^*\right)\right) \\
&= \cap_{j=1}^{m} \left((K \cap K^*) \cup \left(K \cap D_j^*\right)\right) \\
&= \cap_{j=1}^{m} \left(K \cap D_j^*\right).
\end{aligned}$$

Therefore, $K^{(2)}$ is a compactum as the intersection of compacta. We have to prove that $K^{(1)} \neq \varnothing$, $K^{(2)} \neq \varnothing$, and $K^{(1)}$ is a regular compactum.

Denote $G_s^{(i)} = G_s \cap D^{(i)}$ for $i = 1, 2$ and $s \geq s_0$. Then, each $x \in A(K)$ can uniquely be represented in the form of a sum: $x(z) = \dot{x}(z) + \ddot{x}(z)$ ,where

$$\dot{x}(z) = \begin{cases} x(z), & z \in G_s^{(1)}, \\ 0, & z \in G_s^{(2)}, \end{cases}, s = s(x), \tag{5.67}$$

$$\ddot{x}(z) = \begin{cases} 0, & z \in G_s^{(1)}, \\ x(z), & z \in G_s^{(2)}, \end{cases}, s = s(x). \tag{5.68}$$

We will identify the subspaces of all elements of the forms (5.67) and (5.68) with $A\left(K^{(1)}\right)$ and $A\left(K^{(2)}\right)$, respectively. Then, $A(K) = A\left(K^{(1)}\right) \oplus A\left(K^{(2)}\right)$.

Define the function

$$\Phi(z) := \overline{\lim_{\varsigma \longrightarrow z}}\, \overline{\lim_{k \longrightarrow \infty}} \frac{\ln |h_{2k}(\varsigma)|}{k}, \text{ for } z \in D.$$

Then, in a similar way as in the proof of Theorem 24, $\Phi(z) = -\infty$ for $z \in D^{(1)}$ and

$$|h_{2k}(z)| \leq M e^{\frac{-k}{\varepsilon}},\ z \in G_s^{(i)},\ k = 0, 1, \cdots,\ s \geq s_0 + 1,\ M = M(s, \varepsilon). \tag{5.69}$$

Now, we will construct an operator $S : A(K) \to A(K)$, where

$$S(h_{2k+1}) = h_{2k+1} \text{ and } S(h_{2k}) = \ddot{h}_{2k}.$$

Then, represent $S$ in the form $S = I - B$, where $I$ is the identity operator and by (5.67) and (5.68),

$$B(h_{2k+1}) = 0 \text{ and } B(h_{2k}) = \dot{h}_{2k}.$$

Using the bound in (5.66), we have

$$|Bx|_{G_p} \leq \sum |\xi_{2k}| \left|\dot{h}_{2k}\right|_{G_p^{(1)}} \leq M(p,\varepsilon) \sum |\xi_{2k}| e^{-\frac{k}{\varepsilon}}, \; p \geq s_0 + 1, \tag{5.70}$$

since $|Bh_{2k+1}|_{G_p} = 0$ for any $k \in \mathbb{N}$. For $\varepsilon < \delta_2(s)$ in the left hand side of (5.66),

$$e^{\frac{-k}{\varepsilon}} < e^{\frac{-k}{\delta_2(s)}} \leq \frac{|h_{2k}|_{G_s}}{C_2(s)},$$

therefore using that in (5.70) we get

$$|Bx|_{G_p} \leq \frac{M(s,\delta_2(s))}{C_2(s)} \sum |\xi_{2k}| \, |h_{2k}|_{G_s}, \; p \geq s_0 + 1, \; s = 1, 2, \cdots. \tag{5.71}$$

We get the inductive limit topology in $A(K)$ which is equivalent to the original topology using the system of unbounded norms $\|x\|_p = \sum |\xi_k| \, |h_k|_{G_p}$(see [18]). Therefore, by (5.71) we see that the operator $B$ transforms the space $A(K)$ continuously into the space $AC(G_p)$. The inverse image of the sphere

$$\Sigma = \left\{\gamma \in AC(G_p) : |\gamma|_{G_p} < 1\right\}$$

is a neighborhood $U$ of zero in the space $A(K)$. $\Sigma$ is a compact set in $A(K)$ and thus it has been shown that there exists a neighborhood $U$ of zero in $A(K)$ for which the set $B(U)$ is compact in $A(K)$. Therefore we conclude that $B$ is compact in the space $A(K)$ in the sense of Leray.

From the Riesz theorem, which has been extended to such operators in locally convex spaces (for example, see [21]), it follows, in particular that, $ImS$ is a closed subspace in $A(K)$, $\dim KerS = m_1 < \infty$, and $codimImS = m_2 < \infty$.

Therefore there exists a finite collection $\Gamma = \{i_1, i_2, \cdots, i_{m_1}\}$ of natural numbers such that the system

$$\left\{h_{2k+1}, \ddot{h}_{2i}, k, i = 0, 1, \cdots, i \notin \Gamma\right\} \tag{5.72}$$

is a basis in the closure of its linear span, which has finite codimension $m_2$. There exists a finite collection of elements $x_1, x_2, \cdots, x_{m_2}$ which, when together with the system (5.72), forms a basis $\{g_j, j = 0, 1, \cdots\}$ in $A(K)$ :

$$g_{2i+1} = \begin{cases} x_{i+1}, & i = 0, 1, \cdots, m_2 - 1, \\ h_{2(-m_2+i)+1}, & i = m_2, m_2 + 1, \cdots, \end{cases}$$

where $\{g_{2i}, i = 0, 1, \cdots\}$ is the renumbered subseries of the system $\{h_{2i}, i \notin \Gamma\}$.

Using the bounds (5.66) for $h_{2k}$ and the corresponding bounds for $h_{2k+1}$, the formula $\Phi(e_j) = g_j$ gives an isomorphism $\Phi$ of the space $\bar{A}_1 \times \bar{A}_0$ onto the $A(K)$ such that $\Phi\left(\bar{A}_0\right) \subset A\left(K^{(2)}\right)$. From now on, we will write $\bar{A}_0$ instead of $\{0\} \times \bar{A}_0$.

The subspace $X_0 = \Phi\left(\bar{A}_0\right)$ is complemented in the space $A(K)$ by a subspace spanned by part of the elements of the basis $\{g_j\}$ in $A(K)$. Since $X_0 \subset A\left(K^{(2)}\right)$, $X_0$ is complemented in $A\left(K^{(2)}\right)$ too.

Let $Y_0$ be any subspace which is complementary with $X_0$ in $A\left(K^{(2)}\right)$; i.e. $A\left(K^{(2)}\right) = X_0 \oplus Y_0$. Then

$$A\left(K^{(1)}\right) \oplus A\left(K^{(2)}\right) = \Phi\left(\bar{A}_1\right) \oplus X_0 = \left(A\left(K^{(1)}\right) \oplus Y_0\right) \oplus X_0.$$

Therefore the subspaces $\Phi\left(\bar{A}_1\right)$ and $\left(A\left(K^{(1)}\right) \oplus Y_0\right)$ are isomorphic, being the topological complement of one and the same subspace $X_0$. Hence we obtain that $\bar{A}_1 \simeq A\left(K^{(1)}\right) \oplus Y_0$, or transferring to the conjugate space,

$$A_1 \simeq A\left(D^{(1)}\right) \oplus Z_0, \text{ where } Z_0 = \left\{x' \in A\left(D^{(2)}\right) : \langle x', x\rangle = 0, x \in X_0\right\}.$$

Using an argument similar to the one in the proof of Theorem 23, we observe that if $K^{(1)} \neq \varnothing$, then $K^{(1)}$ is a regular compactum.

Now we have to show that $K^{(1)} \neq \varnothing$. Assume to the contrary: $K^{(1)} = \varnothing$. Then, $A_1 \simeq Z_0$, where $Z_0$ is a subspace of the space $A\left(D^{(2)}\right)$, $D^{(2)} = \left(K^{(2)}\right)^*$, and $C\left(K^{(2)}\right) = 0$. But this contradicts the Remark 3, hence $K^{(1)} \neq \varnothing$.

The fact that the compactum $K^{(2)}$ is non-empty follows from the inclusion $\Phi\left(\bar{A}_0\right) \subset A\left(K^{(2)}\right)$. ■

#### 5.2.5 Dual Result for Open Sets

Using GKS-duality, one can also obtain the following dual results [34]:

**Theorem 26** *Let $D$ be an open set in $\widehat{\mathbb{C}}$. For the spaces $A(D)$ and $A_1$ to be isomorphic, it is necessary and sufficient that the set $D$ be regular and consist of a finite number of connected regions.*

**Theorem 27** *Let $D$ be an open set in $\widehat{\mathbb{C}}$. For the spaces $A(D)$ and $A_\infty$ be isomorphic, it is necessary and sufficient that $C(D^*) = 0$ (or what is the same, that the boundary $\partial D$ consists only of irregular points).*

**Theorem 28** *Let $D$ be an open set in $\widehat{\mathbb{C}}$. For the spaces $A(D)$ and $A_1 \times A_\infty$ to be isomorphic, it is necessary and sufficient that the set $I(D^*)$ of irregular points on $\partial D$ and the set $\partial D \setminus I(D^*)$ be closed and non-empty and that $D$ consists of a finite number of connected components.*

**Theorem 29** *Let $D = \cup_{j=1}^{\infty} D_j$, where the $D_j \neq \varnothing$ and are mutually exterior regions. Then the space $A(D)$ is isomorphic to the topological product $\prod\limits_{j=1}^{\infty} A(D_j)$ ( [21]).*

**Corollary 4** *All of the spaces $A(D)$, where $D$ is a regular set satisfying the conditions of Theorem 29 are isomorphic to each other.*

The following is an example where $A(D)$ is not isomorphic to any of the three canonical spaces $A_1$, $A_\infty$, $A_1 \times A_\infty$.

**Example 3** *Let $K = \{0\} \cup \cup_{j=1}^{\infty} K_j$, where $K_j = \{z : |z - q^j| \leq r_j\}$, $0 < q < 1$, and $r_j \downarrow 0$. If $\sum\limits_{j=1}^{\infty} \frac{j}{\ln(1/r_j)} = \infty$, then $K^* = D$ is regular (see [23], pg 146), therefore $A(D) \simeq A_1$ by Theorem 26. If $r_j \downarrow 0$ rapidly enough, e.g. $\sum\limits_{j=1}^{\infty} \frac{j}{\ln(1/r_j)} < \infty$, then $\{0\}$ will be the irregular point of $\partial D$ (see [23], pg 146). But, $\{0\}$ is not isolated, hence the set of regular points $R(D) = \partial D \setminus I(D)$ is not closed. Therefore, by Theorem 28, $A(D)$ is not isomorphic the space $A_1 \times A_\infty$. Also, since $I(D) \neq \varnothing$, $R(D) \neq \varnothing$ using Theorems 26 and 27, $A(D)$ is not isomorphic to any of the spaces $A_1$ and $A_\infty$.*

# CHAPTER 6

## BASES AND ISOMORPHISMS OF SPACES OF ANALYTIC FUNCTIONS IN MULTI- DIMENSIONAL CASE

In this chapter we will represent, without detailed proofs some results about several complex variables which were proved by Zahariuta [37], [38] (see also [3]).

In multidimensional case, interpolational bases for spaces of analytic functions cannot be found as in the case of one dimensional case. The reason for that is, a multidimensional analogue of GKS-duality does not exist. But, the Hilbert Methods that was suggested in [35] and that we have used in Section 5.1.3 can be applied, as confirmed in [37], [38].

### 6.1 Dragilev Classes of $F$-Spaces

Let $X$ be an $F-$space, $\left\{\|x\|_p\,,\ p \in \mathbb{N}\right\}$ be a system of norms defining its topology. Let us consider the system of non-bounded norms (conorms) in the strong dual space $X^*$ :

$$\|x'\|_p^* := \sup\{|x'(x)| : x \in U_p\}\,,\ x' \in X', p \in \mathbb{N}.$$

where

$$U_p = \left\{x \in X : \|x\|_p \le 1\right\}.$$

We will use the notation:

$$U_p = \left\{x \in X : \|x\|_p \le 1\right\},$$

$$U_p^0 = \left\{ x' \in X^* : \|x'\|_p^* \leq 1 \right\}, \quad p \in \mathbb{N}.$$

We will now discuss two important classes of $F$-spaces denoted by $\mathcal{D}_1$, $\mathcal{D}_2$ which appeared ( [29], [31], [40], [42]) as a development of near concepts introduced by Dragilev [7] under the notations $d_1$, $d_2$ (see also [4], [43]). The system of (co)norms in spaces from one of these classes has the special interpolation estimates of a "middle" (co)norm by extreme "small" and "big" ones.

**Definition 12** *A Fréchet space $X$ belongs:*

1. *to the class $\mathcal{D}_1$ if*

$$\exists p \; \forall q \; \exists r \; \exists C \mid \|x\|_q^2 \leq C \|x\|_p \|x\|_r , \quad x \in X, \tag{6.1}$$

2. *to the class $\mathcal{D}_2$ if*

$$\forall p \; \exists q \; \forall r \; \exists C \mid \left( \|x'\|_q^* \right)^2 \leq \|x'\|_p^* \|x'\|_r^* , \quad x' \in X^*. \tag{6.2}$$

Using the same quantifiers, the conditions (6.1) and (6.2) are equivalent to the following ( [29], [31]), respectively:

$$U_q^0 \subset tU_p^0 + \frac{C}{t} U_r^0, \; t > 0,$$

$$U_q \subset tU_r + \frac{C}{t} U_p, \; t > 0.$$

Using these additive conditions, the following statement was proved:

**Theorem 30** *(Vogt [30]) Let $X$ be a Fréchet-Schwarz space. Then $X \in \mathcal{D}_2$ if and only if there exists a bounded closed absolutely convex set $B \subset X$ such that*

$$\forall p \; \forall \mu : \; 0 < \mu < 1 \; \exists q \; \exists C \mid U_q \subset t^\mu B + \frac{C}{t^{1-\mu}} U_p, \; t > 0. \tag{6.3}$$

That statement was actually proved in [37] under the assumption of the existence of unconditional basis in a countably Hilbert (maybe not Schwartz) space.

Using the same quantifiers, the condition (6.3) can be written in the equivalent form:

$$\|x'\|_q^* \leq C \left( \|x'\|^* \right)^{1-\mu} \left( \|x'\|_p^* \right)^\mu , \; x' \in X^*, \tag{6.4}$$

where $\|x'\|^*$ is the norm in $X^*$ defined as follows:

$$\|x'\|^* := \sup \{|x'(x)| : x \in B\}, \ x' \in X^*. \tag{6.5}$$

Then, we say that a Banach space $E$ that continuously embedded in $X$ is a *Vogt space, or "dead-end" space* $(E \in \mathcal{V}(X))$ if the condition (6.3) holds with the unit ball $B$ in $E$ or that is the same (6.4) holds for the norm $(6.5)$.

The following two theorems describe the connection between the interpolational properties of $F$-spaces $A(\Omega)$, $A(K)^*$ and peculiarity of manifolds $\Omega$ and of compacta $K$ ( [37], [38]).

**Theorem 31** *Let $\Omega$ be a Stein manifold. Then $A(\Omega) \in \mathcal{D}_2$ if and only if $\Omega$ is pluriregular.*

**Theorem 32** *Let $K$ be a compact set on a Stein manifold $\Omega$. Then $A(K)^* \in \mathcal{D}_2$ if and only if $K$ is strongly pluriregular on $\Omega$.*

The proof of these theorems can be done using the two-constant theorems and facts of complex potential theory.

### 6.2 Hilbert Scales of Analytic Function Spaces

The following theorem is the two constant theorem in the case of analytic functionals.

**Theorem 33** *Let $(K, D)$ be a pluriregular pair "compact set-open set" on a Stein manifold $\Omega$ where $D$ is a strongly pluriregular open set on $\Omega$. Then for any $\varepsilon > 0$, $\alpha \in (0,1)$ there exists a constant $C = C(\alpha, \varepsilon)$ such that for any $x^* \in AC(K)^*$ the following estimate holds*

$$|x^*|^*_{D_\alpha} \leq C\left(|x^*|^*_K\right)^{1-\alpha+\varepsilon}\left(|x^*|^*_D\right)^{\alpha-\varepsilon},$$

*where*

$$D_\alpha = \{z \in D : \omega(D, K, z) < \alpha\}.$$

In [42], that result was considered in the implicit form and in [37], it was considered as a result about Hilbert scales for analytic functionals. In [38], it was shown that these two methods are equivalent. Using that, the following theorem about Hilbert scales in multidimensional case can be considered as a corollary of Theorem 33.

**Theorem 34** *( [37])Let $D$ be a strongly pluriregular open set on a Stein manifold $\Omega$, a compactum $K \subset D$ be pluriregular on $D$ and $K = \hat{K}_D$. Let $H_0$, $H_1$ be a pair of Hilbert spaces with the continuous imbeddings*

$$A(K) \hookrightarrow H_0 \hookrightarrow AC(K)$$

$$A\left(\bar{D}\right) \hookrightarrow H_1 \hookrightarrow A(D).$$

*Then the following continuous imbeddings hold:*

$$A(K_\alpha) \subset H^\alpha = (H_0)^{1-\alpha}(H_1)^\alpha \subset A(D_\alpha), \qquad 0 < \alpha < 1, \tag{6.6}$$

*where $H^\alpha = (H_0)^{1-\alpha}(H_1)^\alpha$ is a Hilbert scale generated by the pair $H_1 \subset H_0$ of Hilbert scales with continuous imbedding.*

This theorem was first proved in [37] and Theorem 33 was its corollary. Now, we will give a sketch of a proof where Theorem 34 can be realized as a corollary of Theorem 33.

Let the system $\{e_k(z)\} \subset H_1 \hookrightarrow A(D)$ be the common orthogonal basis for the spaces $H_0$, $H_1$ as in (2.3). Then, using Theorem 15 and Theorem 33, the following estimates can be obtained for the common orthogonal basis $\{e_k(z)\} \subset H_1 \hookrightarrow A(D)$ and the biorthogonal system $\{e'_k\}$ that is realized in $H'_0 \hookrightarrow A(K)^* \hookrightarrow A'(D)$,

$$|e_k|_{D_\alpha} \leq C(\alpha, \varepsilon)\, \mu_k^{\alpha+\varepsilon}, \tag{6.7}$$

$$|e'_k|^*_{D_\alpha} \leq C(\alpha, \varepsilon)\, \mu_k^{-\alpha+\varepsilon}. \tag{6.8}$$

Since the imbedding $H_1 \hookrightarrow H_0$ is nuclear, for any $\delta > 0$,

$$\sum_{k=1}^{\infty} \mu_k^{-\delta} < \infty. \tag{6.9}$$

Then, the estimates of norms providing the continuous imbeddings (6.6) can be obtained.

### 6.3 Bases

**Theorem 35** *( [37], [38]) Let $(K,D)$ be a pluriregular pair "compact set-Stein manifold". Then there exists a common basis $\{x_i(z)\}$ in the spaces $A(D), A(K), A(K_\alpha)$, $A(D_\alpha), 0<\alpha<1$, satisfying the asymptotic estimate*

$$\overline{\lim_{\zeta\to z}}\,\overline{\lim_{i\to\infty}}\frac{\ln\left|x_{i(z)}\right|}{a_i}=\omega(D,K,z),\ \ z\in D\setminus K, \tag{6.10}$$

*where*

$$K_\alpha=\{z\in D:\omega(D,K,z)\le\alpha\},\ \ D_\alpha=\{z\in D:\omega(D,K,z)<\alpha\},0<\alpha<1,$$

*and $\{a_i\}$ is a certain non-decreasing sequence of positive numbers such that with $n=\dim D$,*

$$a_i\asymp i^{\frac{1}{n}},\ \ i\to\infty.$$

Now, a sketch of the proof will be given. Take a common orthogonal basis $\{x_i(z)\}$ for some pair of Hilbert spaces $H_0,\ H_1$ with the continuous imbeddings

$$H_1\hookrightarrow A(D)\hookrightarrow A(K)\hookrightarrow H_0$$

and with the following properties:

$$H_1\in\mathcal{V}(A(D)),\ \ H_0^*\in\mathcal{V}(A(K)^*). \tag{6.11}$$

The dual system $H_0^*$ is naturally embedded in $A(K)^*$. Using Theorems 30, 31, and 32, it can be shown that such spaces in (6.11) exist. Notice that A. Aytuna, using Hörmander $\overline{\partial}$-techniques, suggested in [3] a direct construction of "dead-end" space $H_1$ for the space $A(D)$ if $D$ is a pluriregular Stein manifold: they were realized as weighted L$^2$-spaces of analytic functions in $D$.

Let the system $\{x_i\}$ be normed and ordered in accordance with (2.3) and denote $a_i=\ln\mu_i(H_0,H_1)$. Then the conclusion of the theorem follows from $(6.7)$, $(6.8)$, and $(6.9)$. The asymptotics (6.10) can be obtained in a similar way with the one-dimensional case, but techniques of complex potential theory are needed (see [37], [38]).

### 6.4 Isomorphic Classification

Using the extendible bases that were considered in Theorem 35, one can get the multidimensional analogue of the one dimensional isomorphism result $A(D) \simeq A_1$.

**Theorem 36** *( [37], [38]) Let $\Omega$ be a Stein manifold on dimension $n$. For the isomorphism*

$$A(\Omega) \simeq A(U^n)$$

*it is necessary and sufficient that $\Omega$ is pluriregular and consists of at most finite number of connected components, where $U^n$ is the unit disc in $\mathbb{C}^n$.*

Necessity can be proved using the Theorem 31. On the other hand, let the Stein manifold $\Omega$ satisfy the conditions. Let $K$ be a pluriregular compactum having a non-empty intersection with every connected component of the manifold $\Omega$, $K = \hat{K}_\Omega$. If $\{x_i(z)\}$ is a common basis that exists due to Theorem 35, then the isomorphism $T : A(\Omega) \to A(U^n)$ is established by the correspondence

$$x_i(z) \xrightarrow{T} e^{a_i} e_i(z), \ i \in \mathbb{N},$$

where $\{e_i(z)\}$ is a system of monomials enumerated as in $(2.3)$. Also, Aytuna represented a proof of sufficiency in [3] by a direct construction of a required "dead-end" space.

In the one dimensional case, due to GKS-duality, one can immediately obtain the result about the compacta if the result about open sets is known. But, since a multidimensional analogue of GKS-duality does not exist, the case for compacta should be considered separately.

**Theorem 37** *( [37]) Let $K$ be a compactum on a Stein manifold $\Omega$. Then*

$$A(K) \simeq A\left(\overline{U^n}\right)$$

*if and only if $K$ has a Runge neighborhood in $\Omega$ and moreover $K$ is strongly pluriregular, where $U^n$ is the unit disc in $\mathbb{C}^n$.*

Necessity follows from Theorem 32. For sufficiency, the isomorphism may be obtained by means of the basis from Theorem 35 constructed for the pair $\left(K, \tilde{D}\right)$, where $D$ is some Runge neighborhood of $K$ such that its envelope of holomorphy $\tilde{D}$ is pluriregular.

# Bibliography

[1] Aizenberg L. A., *Spaces of Functions Analytic in* $(p, q)$*-circular Domains*, Soviet Math. Dokl. **2** (1961), 75-82.

[2] Aizenberg L. A., Mityagin B. S. , *The Space of Functions Analytic in Multicircular Domains*, Sibirsk. Mat. Zh. **1** (1960), 153-170 (Russian).

[3] Aytuna A., *Stein Spaces M for Which* $O(M)$ *is Isomorphic to a Power Series Spaces*, in: "Advanced in The Theory Fréchet Spaces" (ed. T. Terzioğlu), Kluwer Academic Publishers, Dordrecht-Boston-London (1988), 115-154.

[4] Bessaga Cz., *Some Remarks on Dragilev's Theorem*, Studia Math, **31** (1968), 307-318.

[5] Bezdudniy G. M., *On Isomorphism and Extendibility of Bases in Spaces of Functions, Holomorphic in p-circular Domains*, Scientific Notes of Moscow Regional Pedagogical Institute, **166**, N10 (1966), 109-138 (Russian).

[6] Bezdudniy G. M., *On Isomorphism of Spaces of Functions, Holomorphic on Some Class of Bicircular Domains*, in: Reports of Third Conference of Rostov Math. Soc. (1969), 66-72 (Russian).

[7] Dragilev M. M., *On Regular Bases in Nuclear Spaces*, Mat. Sbornik 68 (1965), 153-173 (Russian) (Amer. Math. Soc. transl. **93** (1970), 61-82).

[8] Edwards R. E., *Functional Analysis: Theory and Applications,* Dover Publications, Inc., 1994.

[9] Erokhin V. D., *On Conformal Mappings of Annuli and the Principal Basis in the Space of Functions, Analytic in Elementary Neighborhood of a Continuum*, Dokl. AN SSSR, **120** (1958), 689-692 (Russian).

[10] Gel'fond A. O., *Calculus of Finite Differences*, $3^{rd}$ rev. ed., "Nauka", Moscow, 1967 (Russian)

[11] Grothendieck A., *Sur Certain Escapes de Fonctions Holomorphes*, J. Reine und Angrew. Math., **192** (1953), 35-64, 77-95.

[12] Hörmander L., *An Introduction to Complex Analysis in Several Variables*, $3^{rd}$ rev. ed., Elsevier Science, 1990.

[13] Henkin G. M., Mityagin B. S. , *Linear Problems of Complex Analysis*, Russian Math. Surveys **26** (1971), 99-164.

[14] Khavin V. P., *Spaces of Analytic Functions*, Itogi Naugi i Tecniki, VINITI AN SSSR, Mat. Anal. 1964 (1966), 76-164. Engl. transl. Progress in Math., Vol. 1, Plenum Press, New York (1968), 75-167.

[15] Köthe G., *Dualität in der Funktionentheorie*, J. Reine und Angrew. Math. **191** (1953), 30-39.

[16] Krein S. G., Petunin Ju. I. , Semenov E. M. , *Interpolation of Linear Operators*, Translations of Mathematical Monographs Volum.

[17] Leja, F., *Sur Certain Suites Lie'es Aux Ensembles Plans et Leurb Application a la Representation Conforme*, Ann. Soc. Pol. Math. **4** (1957), 8-14.

[18] Mityagin B. S., *Approximate Dimension and Bases in Nuclear Spaces*, Russian Math. Surveys **16** (1961), 59-127.

[19] Meise R. , Vogt D., *Introduction to Functional Analysis,* Oxford Science Publications, Oxford, 1997.

[20] Okun S. D., *On Isomorphism of Spaces of Functions, Analytic in Doubly Circular Domains*, in: Uch. Zap. Mosk. Obl. Pedagog. Inst. **166** (1966), 139-156 (Russian).

[21] Robertson A. P., Robertson W. J., *Topological Vector Spaces*, Cambridge Tracts In Mathematics and Mathematical Physics, London, 1966.

[22] Rolewicz S., *On Spaces of Holomorphic Functions*, Studia Math. **21** (1962), 135-160.

[23] Ransford T., *Potential Theory In The Complex Plane,* Cambridge University Press, Cambridge, 1995.

[24] Sadullaev A., *Plurisubharmonic Functions; Several Complex Variables, II*, Geometric Function Theory, Encycl. Math. Sci. **8** (1989). Translation from Itogi Nauki Tekh. Ser. Sovrem. Probl. Mat., Fundam. Napravlenia 8 (1986).

[25] Silva J. Sebastiao e, *Analytic Functions in Functional Analysis*, Portug. Math. **9** (1950). 1-130.

[26] Silverman Richard A., *Introductory Complex Analysis*, Dover Publications, Inc., New York, 1972.

[27] Shabat B. V., *Introduction to Complex Analysis, Part II, Functions of Several Variables*, Translations of Mathematical Monographs, Volume 110, AMS, 1992.

[28] Vladimirov V. S., *Methods of the Theory of Functions of Many Complex Variables*, English translation, The M.I. T. Press, 1966.

[29] Vogt D., *Charakterisierung der Unterräume von s*, Math Z. **155** (1977), 109-117.

[30] Vogt D., *Charakterisierung der Potenzreihenräume von endlich Typ und ihre Folgrerungen*, Manuscr. Math. **37**, N3 (1982), 269-301.

[31] Vogt D., Wagner M. J., *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. **67** (1980), 225-240.

[32] Walsh, J. L., *Interpolation and Approximation By Rational Functions*, AMS Colloq. Publ. 20, Providence 1960.

[33] Walsh, J. L., Russel P. J., *Hyperbolic Capacity and Interpolating Rational Functions II*, Duke Math. J. **33** (1966).

[34] Zahariuta V. P., *Spaces of Functions of One Variable, Analytic In Open Sets and On Compacta*, Math. USSR Sbornik **7**, 75-88, 1970.

[35] Zahariuta V. P., *On Extendable bases in Spaces of Analytic Functions of One and Several Variables*, Siberian Math. J. **8** (1967), 204-216.

[36] Zahariuta V. P., *Spaces of Analytic Functions and Complex Potential Theory*, in: Linear Topological Spaces and Complex Analysis, METU-TÜBİTAK, Ankara, 74-146, 1991.

[37] Zahariuta V.P., *Extremal Plurisubharmonic Functions, Hilbert Scales, and the Isomorphism of Spaces of Analytic Functions of Several Variables, I, II*, in: Theory of Functions, Functional Analysis and Their Applications, Kharkov **19**, 133-157 (1974); **21**, 65-83 (1974) (Russian).

[38] Zahariuta V. P., *Spaces of Analytic Functions and Maximal Plurisubharmonic Functions*, D. Sc. Dissertation, Rostov-on-Don, 1984.

[39] Zahariuta V. P., Kadampatta S. N. , *On The Existence of Extendible Bases in Spaces of Functions*, Analytic On Compacta, Math. Notes 27 (1980), 334-340.

[40] Zahariuta V. P., *Some Linear Topological Invariants and Isomorphism of Tensor Products of Scale Centers*, Izv. Sev.-Kavkaz Nauch. Centra, 1974, N4, 62-64 (Russian).

[41] Zahariuta V. P., *Compact Operators and Isomorphism of Köthe Spaces*, in: Actual Problems of Math. Analysis, Rostov-onDon, 1978, 62-71 (Russian).

[42] Zahariuta V. P., *Isomorphisms of Spaces of Holomorphic Functions of Several Complex Variables*, Func. Anal. and Appl. 5 (1972), 326-328.

[43] Zahariuta V. P., *On Isomorphism of Cartesian Products of Locally Convex Spaces*, Studia Math. **46** (1973), 201-221.

[44] Zahariuta V. P., *On Bases and the Isomorphism of Spaces of Functions, Analytic in Convex Domains of Several Variables,* in: Theory of Functions, Functional Analysis and Their Applications, Kharkov **5** (1967), 57-76 (Russian).