Factorization of unbounded operators on Köthe spaces

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Abstract

The main result is that the existence of an unbounded continuous linear operator T between Köthe spaces $\lambda(A)$ and $\lambda(C)$ which factors over a third Köthe space $\lambda(B)$, causes the existence of an unbounded continuous quasidiagonal operator from $\lambda(A)$ into $\lambda(C)$ factored over $\lambda(B)$ as a product of two continuous quasidiagonal operators. This fact is a factorized analogue of Dragilev theorem [3, 6, 7, 2] about quasidiagonal characterization of the relation $(\lambda(A), \lambda(B)) \in \mathcal{B}$ (which means that all continuous linear operators from $\lambda(A)$ to $\lambda(B)$ are bounded). The proof is based on the results of [9] where the bounded factorization property $\mathcal{BF}$ is characterized in the spirit of Vogt's result [10] about characterization of $\mathcal{B}$. As an application, it is shown that the existence of an unbounded factorized operator for a triple of Köthe spaces, under some additonal asumptions, causes the existence of a common basic subspaces at least for two of the spaces (this is a factorized analogue of the results for pairs [8, 2]).

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1 Introduction

We denote by $\lambda(A)$ the Köthe space defined by the matrix $A = (a_i^p)$ and by (e_n) the canonical basis of $\lambda(A)$. For a mapping $\sigma : \mathbf{N} \to \mathbf{N}$ and a sequence of scalars (t_n) the operator $D : \lambda(A) \to \lambda(B)$ defined by $D(e_n) =$

$t_n\, e_{\sigma(n)}$, $n \in \mathbf{N}$, is called *quasidiagonal.* Dragilev [3] proved that the existence of an unbounded continuous linear operator from $\lambda(A)$ to $\lambda(B)$, where the both spaces are assumed to be nuclear, implies the existence of a continuous unbounded quasidiagonal operator from $\lambda(A)$ to $\lambda(B)$ (cf. [6, 7]). This result has been recently generalized by Djakov and Ramanujan [2] by omitting the nuclearity assumption.

We recall that the closed linear span of a subbasis (e_{i_n}) is called a *basic subspace* of a Köthe space. If $\lambda(A)$ and $\lambda(B)$ have a common basic subspace, then it is easy to construct a continuous linear operator mapping $\lambda(A)$ into $\lambda(B)$, which is unbounded unless the common basic subspace is a Banach space. Under certain conditions on $\lambda(A)$ and $\lambda(B)$ the converse of this trivial fact is also true. Namely, if the both spaces are nuclear, Nurlu and Terzioğlu [8] proved that the existence of an unbounded continuous linear operator $T : \lambda(A) \to \lambda(B)$ implies, under some additional conditions, the existence of a common basic subspace of $\lambda(A)$ and $\lambda(B)$; this result was generalized by Djakov and Ramanujan in [2] for the non-nuclear case. In these works Dragilev theorem plays a crucial role.

It was discovered in [13, 14] that if the matrices A and B satisfy the conditions d_2, d_1, respectively, then every continuous linear operator from $\lambda(A)$ into $\lambda(B)$ is bounded. This phenomenon was studied extensively by many authors, the most comprehensive result is due to Vogt [10], where all pairs of Fréchet spaces with such property are characterized. Terzioğlu and Zahariuta [9] characterized those triples (X, Y, Z) of Fréchet spaces such that each continuous linear operator $T : X \to Y$, which factors over Y is automatically bounded. The aim of the present work is to prove a factorization analogue of Dragilev theorem [3] and its generalization [2]. Namely, we prove that if there is an unbounded continuous linear operator $T : \lambda(A) \to \lambda(C)$, which factors over $\lambda(B)$, then, in fact, there exists an unbounded continuous quasidiagonal operator $D : \lambda(A) \to \lambda(C)$, factored over $\lambda(B)$ as a product of two continuous quasidiagonal operators. As an application, similarly to [8, 2], we show that the existence of unbounded factorized operator for a triple of Köthe spaces causes that, under some additional conditions, these spaces (or at least two of them) have a common basic subspace.

2 Bounded factorization property and quasidiagonal operators

We shall denote by $L(X,Y)$ and $LB(X,Y)$ the space of all linear continuous operators and the space of all linear bounded operators from the locally convex space X into the locally convex space Y. If for each $S \in L(X,Y)$ and $R \in L(Y,Z)$ we have $T = RS \in LB(X,Z)$, we say (X,Y,Z) satisfies the *bounded factorization property* and write $(X,Y,Z) \in \mathcal{BF}$ ([9]).

Dealing with several Fréchet spaces we always use the same notation $\{|\cdot|_p,\ p \in \mathbf{N}\}$ for a system of seminorms defining their topology and $\{|\cdot|_p^*,\ p \in \mathbf{N}\}$ for the corresponding system of polar norms in the dual spaces. For any operator $T \in L(E,F)$ we consider the following operator seminorms

$$|T|_{p,q} = \sup\{|Tx|_p : |x|_q \le 1\}, \quad p,q \in \mathbf{N},$$

which may take the value $+\infty$. In particular, for any one-dimensional operator $T = x' \otimes y$, $x' \in E'$, $y \in F$, we have $|T|_{p,q} = |x'|_q^* \cdot |y|_p$.

Dealing with a Köthe space $\lambda(A)$ we always assume that the matrix $A = (a_i^p)$ satisfies the condition

$$a_i^p \le a_i^{p+1}, \ i,p \in \mathbf{N}. \tag{1}$$

An operator $T \in L(\lambda(A),\lambda(B))$ is *quasidiagonal* if $T(e_i) = t_i\, e_{\tau(i)},\ i \in \mathbf{N}$ for some map $\tau : \mathbf{N} \to \mathbf{N}$ and scalar sequence (t_i). We shall denote by $Q(A,B)$ the set of all quasidiagonal operators and by $Q_\tau(A,B)$ its subset corresponding to the map τ. We note that $Q_\tau(A,B)$ is a subspace of $L(\lambda(A),\lambda(B))$ whereas $Q(A,B)$ is only a subset.

Our aim is to prove the following characterization of the bounded factorization property for triples of Köthe spaces in terms of quasidiagonal operators, which is a natural generalization of Dragilev's theorem ([3, 2]).

Theorem 1. *We have $(\lambda(A),\lambda(B),\lambda(C)) \in \mathcal{BF}$ if and only if for each $S \in Q(A,B)$ and $R \in Q(B,C)$ the quasidiagonal operator $T = RS$ is bounded.*

The proof will be given in Section 3 after some intermediate results. In what follows we will use the following result from [9].

Proposition 2. *We have $(\lambda(A),\lambda(B),\lambda(C)) \in \mathcal{BF}$ if and only if for each $\pi : \mathbf{N} \to \mathbf{N}$ there is $r \in \mathbf{N}$ such that for every $q \in \mathbf{N}$ there exists $n = n(q) \in$*

$\mathbf{N}$ *so that the inequality*

$$\frac{c_i^q}{a_j^r} \leq n \max_{p=1,\dots,n} \left\{ \frac{b_\nu^p}{a_j^{\pi(p)}} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c_i^p}{b_\nu^{\pi(p)}} \right\} \tag{2}$$

holds for all $(i, j, \nu) \in \mathbf{N}^3$.

Given two Fréchet spaces E and F and a map $\pi : \mathbf{N} \to \mathbf{N}$, we consider the following Fréchet space

$$L_\pi(E, F) := \{T \in L(E, F) : |T|_{p,\pi(p)} < \infty, \ p \in \mathbf{N}\}$$

with the topology generated by the system of seminorms $\{|\cdot|_{p,\pi(p)}, \ p \in \mathbf{N}\}$.

We note that, in the case of Köthe spaces, the intersection

$$Q_\sigma^\pi(A, B) := Q_\sigma(A, B) \cap L_\pi(\lambda(A), \lambda(B))$$

is a closed subspace of $L_\pi(\lambda(A), \lambda(B))$. Let us fix σ, ρ, and π and assume that for each $S \in L_\sigma(A, B)$, $R \in L_\rho(B, C)$ the composition RS is bounded. If we apply Lemma 2.1 from [9] to the bilinear map

$$\theta : Q_\sigma^\pi(A, B) \times Q_\rho^\pi(B, C) \to LB(\lambda(A), \lambda(C)),$$

which simply sends each (S, R) to RS, we obtain the following result.

Proposition 3. *Let* σ *and* ρ *be two maps of* $\mathbf{N}$ *into* $\mathbf{N}$. *If for each* $S \in Q_\sigma(A, B)$ *and* $R \in Q_\rho(B, C)$ *the composition* RS *is bounded, then for each* $\pi : \mathbf{N} \to \mathbf{N}$ *there is* $r \in \mathbf{N}$ *such that for every* $q \in \mathbf{N}$ *there exists* $n = n(q) \in \mathbf{N}$ *such that the inequality*

$$\frac{c_{\rho(\sigma(j))}^q}{a_j^r} \leq n \max_{p=1,\dots,n} \left\{ \frac{b_{\sigma(j)}^p}{a_j^{\pi(p)}} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c_{\rho(\sigma(j))}^p}{b_{\sigma(j)}^{\pi(p)}} \right\} \tag{3}$$

holds for every $j \in \mathbf{N}$.

We note that here the both r and n depend not only on π and q but also on our choice of σ and ρ. This is an obstacle to derive Theorem 1 immediately from Proposition 3. On the other hand, the methods of [9] cannot be applied directly to $Q(A, B)$, since it is not a subspace. So we need some other considerations.

3 Proof of Theorem 1

Suppose $((\lambda(A), \lambda(B), \lambda(C)) \notin \mathcal{BF}$. Then, by Proposition 2, there is a map $\pi : \mathbf{N} \to \mathbf{N}$ such that for each $r \in \mathbf{N}$ there exists $q = q(r) \in \mathbf{N}$ such that for any $n \in \mathbf{N}$ there are $i_n = i_n(r)$, $j_n = j_n(r)$, $\nu_n = \nu_n(r)$ with

$$\frac{c_{i_n}^q}{a_{j_n}^r} > n \max_{p=1,\dots,n} \left\{ \frac{b_{\nu_n}^p}{a_{j_n}^{\pi(p)}} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c_{i_n}^p}{b_{\nu_n}^{\pi(p)}} \right\} \tag{4}$$

For $L = \{n_k\}$ we denote by $J_L(r)$, $I_L(r)$, $N_L(r)$, respectively, the set of all $j = j_{n_k}$, $i = i_{n_k}$, $\nu = \nu_{n_k}$ such that (4) holds for $n = n_k$, $k \in \mathbf{N}$. With this notation we have the following technical result, which is crucial for our proof.

Lemma 4. *For any L the sets $J_L(r)$, $I_L(r)$, and $N_L(r)$ are all infinite if $r \geq r_0 := \pi(1)$.*

Proof. Suppose $J_L(r)$ is finite. Then for the subspace E_L of $\lambda(A)$ spanned by the sub-basis $\{e_j : j \in J_L(r)\}$ we have

$$(E_L, \lambda(B), \lambda(C)) \in \mathcal{BF}.$$

Therefore, by Proposition 2, there is an $r' \geq r$ such that for some $m = m(r)$ the inequality

$$\frac{c_i^{q(r)}}{a_j^{r'}} \leq m \max_{p=1,\dots,m} \left\{ \frac{b_\nu^p}{a_j^{\pi(p)}} \right\} \cdot \max_{p=1,\dots,m} \left\{ \frac{c_i^p}{b_\nu^{\pi(p)}} \right\} \tag{5}$$

holds for each $(j, i, \nu) \in J_L \times \mathbf{N}^2$. Taking $n = n_k$ so that

$$n \geq m \max \left\{ \frac{a_j^{r'}}{a_j^r} : j \in J_L(r) \right\}$$

we easily see from (5) that the inequality

$$\frac{c_i^{q(r)}}{a_j^r} \leq n_k \max_{p=1,\dots,n_k} \left\{ \frac{b_\nu^p}{a_j^{\pi(p)}} \right\} \cdot \max_{p=1,\dots n_k} \left\{ \frac{c_i^p}{b_\nu^{\pi(p)}} \right\}$$

holds for each $(j, i, \nu) \in J_L(r) \times \mathbf{N}^2$. Since this contradicts (4), we conclude that $J_L(r)$ is an infinite set. In a similar fashion we can prove that each $I_L(r)$ is infinite.

Before treating the case $N_L(r)$, notice first that, without loss of generality, we can assume that the Köthe matrix $B = (b_\nu^p)$ satisfies the condition

$$b_\nu^p = b_\nu^{p_\nu}, \text{ for } p \geq p_\nu, \tag{6}$$

for some fixed sequence (p_ν) increasing to infinity.

Now supose that $N_L = N_L(r)$ is finite. Without loss of generality we assume $b_\nu^1 > 0$ for all $\nu \in N_L$ and set

$$\delta := \min \{ \frac{b_\nu^1}{b_\nu^{p_\nu}} : \nu \in N_L(r) \} > 0.$$

By (6) we have for $\nu \in N_L$ and any $s \in \mathbf{N}$

$$b_\nu^1 \geq \delta \, b_\nu^s. \tag{7}$$

For k such that $n_k > q(r)$ and $\delta \, n_k > 1$ from (4) and (7) we get

$$\frac{c_{i_{n_k}}^{q(r)}}{a_{j_{n_k}}^r} \geq n_k \, \frac{b_{\nu_{n_k}}^1}{b_{\nu_{n_k}}^{\pi(n_k)}} \, \frac{c_{i_{n_k}}^{n_k}}{a_{j_{n_k}}^{\pi(1)}} \geq \frac{c_{i_{n_k}}^{n_k}}{a_{j_{n_k}}^r}.$$

Due to $r \geq r_0 = \pi(1)$, this gives $c_{i_{n_k}}^{q(r)} > c_{i_{n_k}}^{n_k}$, though $n_k > q(r)$, which contradicts our assumption (1). $\square$

We are now ready to prove a result which is somewhat stronger than Theorem 1.

Proposition 5. *If $(\lambda(A), \lambda(B), \lambda(C)) \notin \mathcal{BF}$ then there are bijections σ and ρ on $\mathbf{N}$ and operators $S \in Q_\sigma(A, B)$ and $R \in Q_\rho(B, C)$ such that the operator $T = RS$ is unbounded.*

Proof. From our assumption we have (4) with the same notation. Passing to subsequences three times and using Lemma 4, for any fixed $r \geq r_0 := \pi(1)$ we construct a subsequence $L_r = \{n_k(r)\}$ of $\mathbf{N}$ such that each coordinate of $(j_{n_k(r)}, \nu_{n_k(r)}, i_{n_k(r)})$ takes different values for different k. Let us represent each infinite set L_r as a disjoint union of infinite subsets

$$L_r = \bigcup_{\mu=0}^{\infty} L_{r,\mu}.$$

Let us construct now a new sequence of infinite disjoint sets

$$\tilde{L}_r = \{l_\mu(r) : \mu \in \mathbf{N}\} \subset L_r, \ r \geq r_0,$$

in the following inductive way. We form $\tilde{L}_{r_0}$ by taking precisely one element $l_\mu(r_0)$ from each set $L_{r_0,\mu}$, $\mu \in \mathbf{N}$. Let us now assume we have already constructed pairwise disjoint sets $\tilde{L}_s$ for $r_0 \leq s \leq r$, so that each set $\tilde{L}_s$ contains exactly one element from $L_{s,\mu}$ and is disjoint from $L_{s,0}$. We then construct $\tilde{L}_{r+1}$ by taking from each set $L_{r+1,\mu}$, $\mu \in \mathbf{N}$, one element different from every $l_\mu(s)$, $r_0 \leq s \leq r$. By induction this concludes the construction of $\tilde{L}_r$, $r \geq r_0$. The set $I_0 := \mathbf{N} \setminus \bigcup_{r=r_0}^{\infty} I_{\tilde{L}_r}$ is infinite since it contains $I_{L_{r,0}}$ for each $r \geq r_0$. By the same token the sets

$$J_0 := \mathbf{N} \setminus \bigcup_{r=r_0}^{\infty} J_{\tilde{L}_r}, \quad N_0 := \mathbf{N} \setminus \bigcup_{r=r_0}^{\infty} N_{\tilde{L}_r}$$

are also infinite.

Let $\alpha : J_0 \to N_0$ and $\beta : N_0 \to I_0$ be arbitrary bijections. Let us consider the maps $\rho : \mathbf{N} \to \mathbf{N}$ and $\sigma : \mathbf{N} \to \mathbf{N}$ defined by

$$\sigma(j) := \begin{cases} \alpha(j), & \text{if } j \in J_0 \\ \nu_{l_\mu(r)}, & \text{if } j = j_{l_\mu(r)} \in J_{\tilde{L}_r}, \ r \geq r_0 \end{cases}$$

and

$$\rho(\nu) := \begin{cases} \beta(\nu), & \text{if } \nu \in N_0 \\ i_{l_\mu(r)}, & \text{if } \nu = \nu_{l_\mu(r)} \in N_{\tilde{L}_r}, \ r \geq r_0, \end{cases}$$

For an arbitrary r we have

$$\frac{c^{q(r)}_{\rho(\sigma(j))}}{a^r_j} > n \max_{p=1,\dots,n} \left\{ \frac{b^p_{\sigma(j)}}{a^{\pi(p)}_j} \right\} \cdot \max_{p=1,\dots,n} \left\{ \frac{c^p_{\rho(\sigma(j))}}{b^{\pi(p)}_{\sigma(j)}} \right\}$$

for all $j = j_n$ where $n \in \tilde{L}_r$. Hence by Proposition 3, there exist $S \in Q_\sigma(A, B)$ and $R \in Q_\rho(B, C)$ with RS unbounded. □

4 Some consequences

Nurlu and Terzioğlu [8] studied consequences of existence of an unbounded operator between nuclear Köthe spaces. They showed, in particular, that if the spaces satisfy the splitting condition of Apiola type [1], then the existence of an unbounded operator implies the existence of a common basic subspace. Djakov and Ramanujan [2] obtain the same result omitting the assumption of nuclearity and assuming the weaker splitting condition of Krone and Vogt [5].

Before dealing with the main result of this section (see Theorem 10 below) we discuss certain modifications and factorized analogues of some properties, important for studying of the relation $\mathrm{Ext}^1(F, E) = 0$ (see, e.g., [11, 12, 4]). A pair of Fréchet spaces (F, E) satisfies the condition $\mathcal{S}$ if there is a mapping $\tau : \mathbf{N} \to \mathbf{N}$ such that for every $p \in \mathbf{N}$ and $r \in \mathbf{N}$ there exists a constant $C = C(p, r)$ provided that the estimate

$$|T|_{r,\tau(p)} \leq C \max\{|T|_{\tau(p),p}, |T|_{\tau(r),r}\} \tag{8}$$

holds for any one-dimensional operator

$$T = e' \otimes f,\ e' \in E',\ f \in F.$$

It is easy to check that the condition $\mathcal{S}$ is an equivalent slight variation of the Vogt's condition S_2^* ([11]). It is known that the property $\mathrm{Ext}^1(F, E) = 0$ is characterized by $(F, E) \in \mathcal{S}$ in the cases when the both spaces are either Köthe spaces ([5]) or nuclear ([4]). A pair of Köthe spaces $E = \lambda(A)$ and $F = \lambda(B)$ satisfies the conditon $\mathcal{S}$ if and only if the condition (8) holds for the operators $T = e_i' \otimes e_j$, $i, j \in \mathbf{N}$ ([5]).

If the estimate (8) is true for arbitrary operators $T \in L(E, F)$ (with an obvious meaning if some of operator norms equals $+\infty$) then we write $(F, E) \in \bar{\mathcal{S}}$ (in fact, one can see that this condition is sensible only for bounded operators T). It is easy to check that the condition $(F, E) \in \bar{\mathcal{S}}$ coincides with the condition on $LB(E, F)$, considered by Dierolf, Frerick, Mangino, and Wengenroth (see, e.g., [4], the proof of Theorem 2.2); moreover, by Vogt [12], this condition coincides with the condition (wQ) for the natural representation of $LB(E, F)$ as an (LF)-space.

In what follows we shall denote by $\lambda(A)_L$ the basic subspace of a Köthe space $\lambda(A)$ which is a closed linear envelope of $\{e_n : n \in L\}$, $L \subset \mathbf{N}$.

Suppose now $(\lambda(A), \lambda(B), \lambda(C)) \notin \mathcal{BF}$ and $(\lambda(C), \lambda(A)) \in \mathcal{S}$. By Theorem 1 we know that there are $S \in Q_\sigma(A, B)$ and $R \in Q_\rho(B, C)$ with some bijective maps σ and ρ on $\mathbf{N}$ such that $T = RS$ is an unbounded quasidiagonal operator. The theorem of Djakov and Ramanujan [2] implies the existence of infinite subsets J and I of $\mathbf{N}$ such that T maps $\lambda(A)_J$ isomorphically onto $\lambda(C)_I$. Then one can easily check that, for $N := \sigma(J) = \rho^{-1}(I)$, the both operators $S : \lambda(A)_J \to \lambda(B)_N$ and $R : \lambda(B)_N \to \lambda(C)_J$ are also isomorphisms. We have therefore proved the following result.

Proposition 6. *Let $E = \lambda(A)$, $G = \lambda(B)$, and $F = \lambda(C))$. Suppose that $(E, G, F) \notin \mathcal{BF}$ and $(F, E) \in \mathcal{S}$. Then there is a common basic subspace for all three spaces.*

Now we consider a factorized analogue of the condition $\mathcal{S}$. A triple of Fréchet spaces (F, G, E) satisfies the condition $\mathcal{SF}$ (we write $(F, G, E) \in \mathcal{SF}$) if for any one-dimensional operator $T = RS$, with the both operators $S \in L(E, G)$ and $R \in L(G, F)$ being also one-dimensional, the inequality

$$|T|_{r,\tau(p)} \le C \max\{|R|_{\tau(p),p}, |R|_{\tau(r),r}\} \cdot \max\{|S|_{\tau(p),p}, |S|_{\tau(r),r}\} \qquad (9)$$

holds with the same requisite as in (8).

If the condition (9) holds for an arbitrary operator $T = RS$, with $S \in L(E, G)$ and $R \in L(G, F)$ we will write $(F, G, E) \in (\overline{\mathcal{SF}})$ (with the evident meaning when some of the operator norms equals $+\infty$; so, in fact, this condition is reasonable only for bounded operators T).

We note that if $E = G$ or $G = F$ the condition $(F, G, E) \in \mathcal{SF}$ reduces simply to $(F, E) \in \mathcal{S}$ as well as $(F, G, E) \in \overline{\mathcal{SF}}$ does so to $(F, E) \in \overline{\mathcal{S}}$.

Proposition 7. *Let E, G, and F be arbitrary Fréchet spaces. If $(E, G, F) \in \mathcal{BF}$, then $(F, G, E) \in \overline{\mathcal{SF}}$.*

Proof. Suppose that $(E, G, F) \in \mathcal{BF}$. Denote by $\Pi(p)$ the set of all strictly increasing mappings $\pi \in \mathbf{N}^{\mathbf{N}}$ such that $\pi(1) = p$. By Theorem 2.2 from [9], for any $\pi \in \Pi(p)$ there is $q \in \mathbf{N}$ and $\mu \in \mathbf{N}^{\mathbf{N}}$ such that for every $T = RS$ with $S \in L(E, G)$ and $R \in L(G, F)$ the inequality

$$|T|_{r,q} \le \mu(r) \max_{l=1}^{\mu(r)} \{|R|_{l,\pi(l)}\} \cdot \max_{l=1}^{\mu(r)} \{|S|_{l,\pi(l)}\} \qquad (10)$$

holds for each $r \in \mathbf{N}$. By $\Pi_q(p)$ we denote the set of all $\pi \in \Pi(p)$ admitting the condition (10) with a given $q \in \mathbf{N}$. It is obvious that $\Pi(p) = \cup_{q=1}^{\infty} \Pi_q(p)$ and $\Pi_q(p) \subset \Pi_{q+1}(p)$, $q \in \mathbf{N}$. Therefore for each $p \in \mathbf{N}$ there is $q = \rho(p)$ such that $\sup\{\pi(q) : \pi \in \Pi_q(p)\} = \infty$. Now we fix an arbitrary $r \in \mathbf{N}$ and apply

(10) with $q = \rho(p)$ and $\pi \in \Pi_q(p)$ such that $\pi(q) \geq r$. Taking into account that

$$|R|_{l,\pi(l)} \leq \begin{cases} |R|_{q,p} & \text{if } 1 \leq l \leq q \\ |R|_{\mu(r),r} & \text{if } q < l \leq \mu(r), \end{cases}$$

and the same holds for the operator S, we derive from (10) that

$$|T|_{r,\rho(p)} \leq \mu(r) \max\{|R|_{\rho(p),p}, |R|_{\mu(r),r}\} \cdot \max\{|S|_{\rho(p),p}, |S|_{\mu(r),r}\}.$$

From here one can easily conclude that there is $\tau \in \mathbf{N}^{\mathbf{N}}$ and $C = C(p,r)$ such that (9) holds. Thus $(F,G,E) \in \overline{\mathcal{SF}}$. □

In particular, if $F = G$ or $G = E$, we get the following

Corollary 8. *Let E and F be Fréchet spaces. Then $(E,F) \in \mathcal{B}$ implies $(F,E) \in \bar{\mathcal{S}}$.*

This is a generalization of Proposition 3.4 from [5], where the case of Köthe spaces was considered (for Köthe spaces the conditions $\mathcal{S}$ and $\bar{\mathcal{S}}$ coincide): basically, our proof of Proposition 7 is a generalized direct version of the proof ad absurdum from [5]).

Now we compare the conditions $\mathcal{S}$ and $\bar{\mathcal{S}}$ with their factorized versions.

Proposition 9. *Let E, G, F be arbitrary Fréchet spaces. If the couple (F,E) satisfies $\bar{\mathcal{S}}$ (or $\mathcal{S}$), then the triple (F,G,E) satisfies $\overline{\mathcal{SF}}$ (respectively, $\mathcal{SF}$).*

Proof. Because of complete similarity we consider only the case $\bar{\mathcal{S}}$. Suppose that $(F,E) \in \bar{\mathcal{S}}$. Then there is a function $\tau : \mathbf{N} \to \mathbf{N}$ such that for each $T \in L(E,F)$ the estimate

$$|T|_{r,\tau(p)} \leq C \max\{|T|_{\tau(p),p},\ |T|_{\tau(r),r}\} \tag{11}$$

holds for each $p \in \mathbf{N}$, $r \in \mathbf{N}$ with some constant $C = C(p,r)$. Without loss of generality we assume $\tau(p) \geq p$ for every $p \in \mathbf{N}$. Using now the following evident estimate

$$|T|_{\tau(p),p} \leq |S|_{p,p} \cdot |R|_{\tau(p),p} \leq |S|_{\tau(p),p} \cdot |R|_{\tau(p),p},\ p \in \mathbf{N},$$

for any operator $T = RS$, we obtain the estimate (9), which means that $(F,G,E) \in \overline{\mathcal{SF}}$. □

The following example shows that $\mathcal{SF}$ is strictly weaker than $\mathcal{S}$. Here we use the notation $\Lambda_\alpha(a) := K(\exp(\alpha_p a_i))$ with $\alpha_p \uparrow \alpha \le \infty$, $a = (a_i)$.

Example. Let $a = (a_i)$ be a positive sequence increasing to infinity. Since $(\Lambda_1(a), \Lambda_\infty(a)) \in \mathcal{B}$ [14], we have $(\Lambda_1(a), \Lambda_\infty(a)), \Lambda_1(a)) \in \mathcal{BF}$ trivially. Hence $(\Lambda_1(a), \Lambda_\infty(a)), \Lambda_1(a)) \in \mathcal{SF}$ by Proposition 7. However $(\Lambda_1(a), \Lambda_\infty(a)) \notin \mathcal{S}$.

We conclude with a generalizaton of Djakov-Ramanujan's result ([2]; Proposition 3) in the context of the factorization.

Theorem 10. *Suppose $(\lambda(A), \lambda(B), \lambda(C)) \notin \mathcal{BF}$ and $(\lambda(C), \lambda(B), \lambda(A)) \in \mathcal{SF}$. Then one of the pairs $(\lambda(A), \lambda(B)$ or $(\lambda(B), \lambda(C)$ has a common basic subspace.*

Proof. By Theorem 1 there exist quasidiagonal operators $S \in Q_\sigma(A, B)$ and $R \in Q_\rho(B, C)$ with σ and ρ bijective such that $T = RS$ is unbounded. Without loss of generality we assume in what follows that all three operators are identity embeddings, since otherwise we can provide this property considering a new triple of Köthe spaces obtained from the original by some permutations and normalizations of their canonical bases (note that the property $\mathcal{SF}$ is preserved under such reconstruction). As applied to the above embeddings the condition $\mathcal{SF}$ gives the following: there is a map $\tau : \mathbf{N} \to \mathbf{N}$ such that

$$\frac{c_i^r}{a_i^{\tau(p)}} \le C \max\left\{\frac{b_i^{\tau(p)}}{a_i^p}, \frac{b_i^{\tau(r)}}{a_i^r}\right\} \cdot \max\left\{\frac{c_i^{\tau(p)}}{b_i^p}, \frac{c_i^{\tau(r)}}{b_i^r}\right\} \tag{12}$$

for all $(p, r, i) \in \mathbf{N}^3$ with some constant $C = C(p, r)$.

It suffices now to prove that there is an infinite set $I \subset \mathbf{N}$ such that $\lambda(A)_I = \lambda(B)_I$ or $\lambda(B)_I = \lambda(C)_I$. Suppose that this assertion is false. Then for each infinite set $I \subset \mathbf{N}$ and $m \in \mathbf{N}$ there is $r \ge m$ such that

$$\liminf_{i\in I} \frac{b_i^{\tau(r)}}{a_i^r} = \liminf_{i\in I} \frac{c_i^{\tau(r)}}{b_i^r} = 0. \tag{13}$$

We define inductively the sets $N_0 \supset N_1 \supset \cdots$ by

$$N_0 := \mathbf{N};\ N_p := \left\{i \in N_{p-1} : \max\left\{\frac{b_i^{\tau(p)}}{a_i^p}, \frac{c_i^{\tau(p)}}{b_i^p}\right\} \ge 1\right\},\ p \in \mathbf{N}, \tag{14}$$

with τ from (12).

We claim that for each $p \in \mathbf{N}$ the embedding T is unbounded on the basic subspace X_p of $\lambda(A)$ spanned by $\{e_i : i \in N_{p-1} \setminus N_p\}$. If it is not so, then for each $q \in \mathbf{N}$ there is an infinite subset $I_q \subset N_{p-1} \setminus N_p$ and $m(q) \in \mathbf{N}$ with

$$\lim_{i \in I_q} \frac{c_i^{m(q)}}{a_i^q} = \infty. \tag{15}$$

For $I = I_q$ we find $r \geq m(q)$ such that (13) holds. Then there is an infinite set $J_q \subset I_q$ with

$$\max\left\{\frac{c_i^{\tau(r)}}{b_i^r}, \frac{c_i^{\tau(r)}}{b_i^r}\right\} < 1, \ i \in J_q. \tag{16}$$

On the other hand, by (14), we have

$$\max\left\{\frac{c_i^{\tau(p)}}{b_i^p}, \frac{c_i^{\tau(p)}}{b_i^p}\right\} < 1, \ i \in I_q. \tag{17}$$

Applying now (12) with $q = \tau(p)$ and r chosen above and taking into account the estimates (16) and (17), we obtain that

$$\frac{c_i^r}{a_i^q} \leq C$$

for all $i \in J_q$, which contradicts (15). This proves our claim that the embedding T is bounded on each X_p. Hence, for every $p \in \mathbf{N}$, the operator T must be unbounded on the basic subspace Y_p generated by $\{e_i : i \in N_p\}$, which, in particular, implies that N_p is an infinite set.

Now we construct a sequence $I = \{i_p\}$, so that $i_p \in N_p$, $i_{p+1} \neq i_p$, $p \in \mathbf{N}$. Then, due to (14), there is an infinite set $J \subset I$ such that at least one of the inequalities $a_i^p \leq b_i^{\rho(p)}$ or $b_i^p \leq c_i^{\rho(p)}$ holds for all $p \in \mathbf{N}$ and all $i \in J$ such that $i \geq p$, which contradicts the assumption (13). This completes the proof. □

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