Multirectangular invariants for power Köthe spaces

P. Chalov [a], T. Terzioğlu [b], V. Zahariuta [b,*]

[a] *Department of Mathematics, Rostov State University, 344090 Rostov-na-Donu, Russia*
[b] *Sabanci University, 34956 Tuzla, Istanbul, Turkey*

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Abstract

Using some new linear topological invariants, isomorphisms and quasidiagonal isomorphisms are investigated on the class of *first type power Köthe spaces* [Proceedings of 7th Winter School in Drogobych, 1976, pp. 101–126; Turkish J. Math. 20 (1996) 237–289; Linear Topol. Spaces Complex Anal. 2 (1995) 35–44]. This is the smallest class of Köthe spaces containing all Cartesian and projective tensor products of power series spaces and closed with respect to taking of basic subspaces (closed linear hulls of subsets of the canonical basis). As an application, it is shown that isomorphic spaces from this class have, up to quasidiagonal isomorphisms, the same basic subspaces of finite (infinite) type.
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1. Introduction

A matrix $A = (a_{i,p})_{i,p \in \mathbb{N}}$ is called *Köthe matrix* if $0 \leqslant a_{ip} \leqslant a_{ip+1}$, $i, p \in \mathbb{N}$, and $a_{i,p} > 0$ for some $p = p(i)$, $i \in \mathbb{N}$; *Köthe space* $K(A)$ defined by A is a Fréchet space of all sequences $x = (\xi_i)_{i \in \mathbb{N}}$ such that $|x|_p := \sum_{i \in \mathbb{N}} |\xi_i| a_{ip} < \infty$, $p \in \mathbb{N}$, with the topology generated by the seminorms $\{| \cdot |_p \colon p \in \mathbb{N}\}$. The notation $e = (e_i)_{i \in \mathbb{N}}$ will be always

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used for the canonical basis of $K(A)$ regardless of a matrix A. Any closed subspace of $K(A)$ spanned by a subset of a canonical basis is called a *basic subspace*.

An important particular case is represented by so-called *power series spaces* (or *centers of absolute Riesz scales*),

$$E_\alpha(a) := K\big(\exp(\alpha_p a_i)\big), \tag{1}$$

where $a = (a_i)_{i \in \mathbb{N}}$ is a positive sequence, $\alpha_p \uparrow \alpha$, $-\infty < \alpha \leqslant +\infty$.

We say that $K(A)$ is *quasidiagonally isomorphic to* $K(B)$ (*quasidiagonally embedded into* $K(B)$) and write

$$K(A) \overset{\mathrm{qd}}{\simeq} K(B) \quad \big(\text{respectively,} \ K(A) \overset{\mathrm{qd}}{\hookrightarrow} K(B)\big)$$

if there is an isomorphism (respectively, an isomorphic embedding) $T : K(A) \to K(B)$ such that $T e_i := t_i e_{\sigma(i)}$, $i \in \mathbb{N}$, where (t_i) is a sequence of scalars and $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection (respectively, injection). The following problem still remains open on the class of all Köthe spaces (see, e.g., [1,7,13,14,20,21,27,37]).

Problem 1. Let $K(A) \simeq K(B)$. Is it true that $K(A) \overset{\mathrm{qd}}{\simeq} K(B)$?

Partial solutions of this and related problem about quasiequivalence of all absolute bases have been the subject of much research for various classes of Köthe spaces (see, e.g., [7,10–14,17,20,21,23,27–31,37]). Important instruments in those investigations are classical linear topological invariants (approximative and diametrical dimensions, see, e.g., [2,16,20,24]). These invariants are used at their best for *regular Köthe spaces* [10,11,13,14,17,25,27], in particular, for the spaces (1) with $a_i \uparrow \infty$. Problem 1 for non-Montel ($a_i \nrightarrow \infty$) spaces (1), which turns to be beyond powers of the classical invariants, was investigated in [21–23] (with l_2-norms instead of l_1-norms) by means of some new invariants based on spectral behavior of the operator generating the scale; these invariants exerted an influence on further development of linear topological invariants dealing with non-regular spaces, especially on its early stage).

We notice that power series spaces *of finite type* ($-\infty < \alpha < \infty$) and *of infinite type* ($\alpha = \infty$) have very different structure, in particular, a complemented subspace L of a space $E_0(a)$ can be isomorphic to a subspace of $E_\infty(b)$ if and only if L is normed, hence finite-dimensional if $E_0(a)$ is Montel [28,30]. Therefore Köthe spaces of a mixed nature (with the both, finite and infinite type spaces (1) represented as its basic subspaces) are of great interest, since they, as a rule, are irregular and need radically new invariants. In this paper we investigate *power Köthe spaces of first type* [31,37],

$$E(\lambda, a) := K\left(\exp\left(\left(-\frac{1}{p} + \lambda_i\, p\right)a_i\right)\right), \tag{2}$$

where $a = (a_i)_{i \in \mathbb{N}}$, and $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ are sequences of positive numbers. The class $\mathcal{E}$ of such spaces is the smallest class of Köthe spaces, containing all Cartesian and projective tensor products of power series spaces (represented, in a natural way, as Köthe spaces) and closed with respect to taking of basic subspaces. Any *essentially mixed space* (2) (= not reduced to a power series space or Cartesian product of spaces (1)) has quite complicated structure:

it possesses a countable (continuum) base of the "filter" of basic power series subspaces of infinite (finite) type.

Our main goal is the complete solution of the following problem, which gives an important approach to Problem 1 for the class $\mathcal{E}$ (its quite partial solution has been considered in [31–34,37]).

Problem 2. Let $X = E(\lambda, a)$, $Y = E(\mu, b)$ and $X \simeq Y$. Suppose L is a basic subspace of X which is isomorphic to a power series space of finite (respectively, infinite) type. Is it possible to choose a basic subspace M in Y of the same kind so that $L \overset{\text{qd}}{\simeq} M$?

Problem 1, for the spaces (2), was studied in [7,9] by means of m-rectangular characteristics which counts up how many points of the sequence $(\lambda, a) = ((\lambda_i, a_i))_{i \in \mathbb{N}}$, defining the space (2), fall within a union of m rectangles,

$$\bigcup_{k=1}^{m} \left\{ (\xi, \eta) : \delta_k < \xi \leqslant \varepsilon_k; \ \tau_k < \eta \leqslant t_k \right\}. \tag{3}$$

Quasiequivalent isomorphism of spaces from the class $\mathcal{E}$ was characterized there completely in terms of some, uniform by m, equivalence of these characteristics (see Proposition 8 below); but for any isomorphic pair of such spaces it was shown only some weaker equivalence of those characteristics (depending on the number of rectangles).

In the present paper we prove that isomorphism of spaces from the class $\mathcal{E}$ entails much stronger equivalence of multirectangular characteristics (Theorem 11) (with estimates not depending on a number of rectangles but only on a number of different values of δ_k in (3)). The crucial tools are *compound invariants* [4–7,35–37] based on evaluation of classical entropy-like characteristics (inverse to Bernstein diameters) of proper *synthetic absolutely convex sets*, which are quite intricate interpolational constructions made up of sets from given bases of neighborhoods (see the proof of Lemma 10).

As an application of those invariants, we obtain a complete solution of Problem 2 (Theorems 13 and 14). Now we are able also to show that the spaces from the proof of Theorem 5 in [9] are not isomorphic (this was impossible with the invariants considered in [7,9]). On the other hand, we construct a new quite intricate example (in Proposition 15), which shows that there remains a gap (though narrowed down) between characterization of isomorphisms and quasidiagonal isomorphisms.

For not explained here notions we refer to (see, e.g., [15,19]).

2. Preliminaries

2.1. Let $X, \tilde{X}$ be Köthe spaces and $\{f_i\}_{i \in \mathbb{N}}, \{g_i\}_{i \in \mathbb{N}}$ absolute bases in the spaces X and $\tilde{X}$, respectively. We say that these bases are *quasiequivalent* if there exists an isomorphism $T : X \to \tilde{X}$ such that $T f_i = t_i g_{\sigma(i)}$, where (t_i) is a sequence of scalars and $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection. For two sequences of positive numbers $a = (a_i)$ and $\tilde{a} = (\tilde{a}_i)$ we shall write $a \asymp \tilde{a}$ or $a_i \asymp \tilde{a}_i$ if there exists a constant $c > 1$ such that $a_i/c \leqslant \tilde{a}_i \leqslant c a_i$.

In [21–23] B. Mityagin investigated the isomorphic classification of power series spaces (1) and the structure of complemented subspaces of them in terms of the characteristic

$$M_a(\tau, t) := \big|\{i \in \mathbb{N}: \tau < a_i \leqslant t\}\big|, \quad 0 < \tau \leqslant t, \tag{4}$$

where $|A|$ denotes the cardinality of the finite set A and $+\infty$ if A is an infinite set. We shall use the following fact, which is a quite particular case of his results (see also [6, Corollary 3]).

Proposition 3. *Let* $a = (a_i)$ *and* $\tilde{a} = (\tilde{a}_i)$ *be sequences of real numbers such that* $a_i \geqslant 1$, $\tilde{a}_i \geqslant 1$. *Suppose* $X = E_0(a)$ *(or* $X = E_\infty(a)$*) and* $\tilde{X} = E_0(\tilde{a})$ *(or* $\tilde{X} = E_\infty(\tilde{a})$, *respectively*)*. The following conditions are equivalent*:

(i) $X \overset{\text{qd}}{\hookrightarrow} \tilde{X}$;

(ii) $\exists \alpha > 1: \quad M_a(\tau, t) \leqslant M_{\tilde{a}}\left(\dfrac{\tau}{\alpha}, \alpha t\right), \quad t > \tau > 0.$

Dealing with spaces (2) we always assume without loss of generality that

$$a_i > 1, \quad \frac{1}{a_i} \leqslant \lambda_i \leqslant 1, \quad i \in \mathbb{N}. \tag{5}$$

Indeed, the space $E(\lambda, a)$ is identically isomorphic to the space $E(\tilde{\lambda}, \tilde{a})$ satisfying (2), if we define $\tilde{a}, \tilde{\lambda}$ as follows: $\tilde{a}_i$ is equal to $a_i + 1$ if $\lambda_i \leqslant 1$ and to $\lambda_i a_i + 1$ otherwise; $\tilde{\lambda}_i$ is equal to $1/\tilde{a}_i$ if $\lambda_i < 1/\tilde{a}_i$, to 1 if $\lambda_i > 1$ and to λ_i for the rest of i.

2.2. Let $\mathcal{X}$ be a class of locally convex spaces and let Γ be a set with an equivalence relation $\sim$. We say that $\gamma : \mathcal{X} \to \Gamma$ is a *linear topological invariant* if $X \simeq \tilde{X} \Rightarrow \gamma(X) \sim \gamma(\tilde{X})$, $X, \tilde{X} \in \mathcal{X}$.

The invariants considered here are based on the well-known characteristic of a couple of absolutely convex sets U, V in a linear space X,

$$\beta(V, U) := \sup\{\dim L: \ U \cap L \subset V\}, \tag{6}$$

where L runs the set of all finite-dimensional subspaces of $X_V = \overline{\text{span}}\, V$. This characteristic relates to Bernstein diameters $b_n(V, U)$ [26], namely

$$\beta(V, U) = \big|\{n: b_n(V, U) \geqslant 1\}\big|.$$

We shall use the following properties, readily apparent from the definition (6):

$$\text{if } V_1 \subset V \text{ and } U \subset U_1, \text{ then } \beta(V_1, U_1) \leqslant \beta(V, U), \tag{7}$$

$$\beta(\alpha V, U) = \beta\left(V, \frac{1}{\alpha}U\right), \quad \alpha > 0. \tag{8}$$

Let $f = \{f_i\}_{i \in \mathbb{N}}$ be an absolute basis in a Köthe space X. A set

$$B^f(a) := \left\{x = \sum_{i=1}^{\infty} \xi_i f_i \in X: \ \sum_{i=1}^{\infty} |\xi_i| a_i \leqslant 1\right\}$$

is the weighted l_1-ball in X, defined with a given weight sequence of positive numbers $a = (a_i)_{i \in \mathbb{N}}$. For weighted balls the characteristic (6) admits an especially simple computation.

Proposition 4 (see, e.g., [5,21]). *For a couple of weights a, b we have*

$$\beta\big(B^e(b), B^e(a)\big) = \big|\{i\colon b_i \leqslant a_i\}\big|.$$

2.3. In the construction of compound invariants (see Sections 3–5) we shall use the following simple geometrical facts.

Proposition 5. *Let e be an absolute basis of a Köthe space X, $a^{(j)} = (a_i^{(j)})$, $j = 1, \ldots, r$, sequences of positive numbers and $c = (c_i)$, $d = (d_i)$ sequences, defined by the following formulae: $c_i = \max\{a_i^{(j)}\colon j = 1, \ldots, r\}$, $d_i = \min\{a_i^{(j)}\colon j = 1, \ldots, r\}$, $i \in \mathbb{N}$. Then the following relations hold*:

$$B^e(c) \subset \bigcap_{j=1}^{r} B^e(a^{(j)}) \subset r B^e(c), \qquad B^e(d) = \mathrm{conv}\left(\bigcup_{j=1}^{r} B^e(a^{(j)})\right),$$

where $\mathrm{conv}(M)$ *means the convex hull of a set M.*

For a couple $A_\nu = B^e(a^{(\nu)})$, $\nu = 0, 1$, we consider the following one-parameter family of weighted balls:

$$(A_0)^{1-\alpha}(A_1)^{\alpha} := B^e(a^{(\alpha)}),$$

where

$$a^{(\alpha)} := \left(\big(a_i^{(0)}\big)^{1-\alpha}\big(a_i^{(1)}\big)^{\alpha}\right)_{i\in\mathbb{N}}, \quad \alpha \in \mathbb{R}.$$

The following statement is the well-known interpolational fact (see, e.g., [18, IV, Theorem 1.10]) written in a geometrical form.

Proposition 6. *Let f and g be absolute bases of a Köthe space X and $A_\nu = B^f(a^{(\nu)})$, $\tilde{A}_\nu = B^g(\tilde{a}^{(\nu)})$, $\nu = 1, 2$. Then*

$$A_\nu \subset \tilde{A}_\nu, \quad \nu = 1, 2,$$

implies

$$(A_0)^{1-\alpha}(A_1)^{\alpha} \subset (\tilde{A}_0)^{1-\alpha}(\tilde{A}_1)^{\alpha}, \quad \alpha \in (0, 1).$$

3. Multirectangular characteristics and compound invariants

Let $\lambda = (\lambda_i)_{i\in\mathbb{N}}$, $a = (a_i)_{i\in\mathbb{N}}$ be sequences of positive numbers with (5) and $m \in \mathbb{N}$. Following [1,6] (cf., [3,8]), we introduce *m-rectangle characteristic* of a pair (λ, a) as the function

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \left|\bigcup_{k=1}^{m}\{i\colon \delta_k < \lambda_i \leqslant \varepsilon_k,\ \tau_k < a_i \leqslant t_k\}\right|, \tag{9}$$

defined for

$$\delta = (\delta_k), \quad \varepsilon = (\varepsilon_k), \quad \tau = (\tau_k), \quad t = (t_k),$$
$$0 \leqslant \delta_k < \varepsilon_k, \ 0 \leqslant \tau_k < t_k < \infty, \ k = 1, 2, \dots, m. \tag{10}$$

The function (9) calculates how many points (λ_i, a_i) are contained in the union of m rectangles,

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) = \left| \bigcup_{k=1}^{m} \{i \colon (\lambda_i, a_i) \in P_k\} \right| = \left| \left\{ i \colon (\lambda_i, a_i) \in \bigcup_{k=1}^{m} P_k \right\} \right|, \tag{11}$$

where $P_k := (\delta_k, \varepsilon_k] \times (\tau_k, t_k]$, $k = 1, 2, \dots, m$.

Let $\tilde{\lambda} = (\tilde{\lambda}_i)$, $\tilde{a} = (\tilde{a}_i)$ be another couple of positive sequences and m a fixed natural number. Then *functions $\mu_m^{(\lambda,a)}$ and $\mu_m^{(\tilde{\lambda},\tilde{a})}$ are equivalent* (or $\mu_m^{(\lambda,a)} \approx \mu_m^{(\tilde{\lambda},\tilde{a})}$) if there exists a strictly increasing function $\varphi : [0, 2] \to [0, 1]$, $\varphi(0) = 0$, $\varphi(2) = 1$, and a positive constant α such that the following inequalities:

$$\mu_m^{(\lambda,a)}(\delta, \varepsilon; \tau, t) \leqslant \mu_m^{(\tilde{\lambda},\tilde{a})}\left(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t \right), \tag{12}$$

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta, \varepsilon; \tau, t) \leqslant \mu_m^{(\lambda,a)}\left(\varphi(\delta), \varphi^{-1}(\varepsilon); \frac{\tau}{\alpha}, \alpha t \right) \tag{13}$$

hold with $\varphi(\delta) = (\varphi(\delta_k))$, $\varphi^{-1}(\varepsilon) = (\varphi^{-1}(\varepsilon_k))$, $\tau/\alpha = (\tau_k/\alpha)$, $\alpha t = (\alpha t_k)$ for all collections of parameters (10) with $\varepsilon_k \leqslant 1$, $\tau_k \geqslant 1$, $k = 1, \dots, m$ (in line with our agreement (5) we shall suppose always that the parameters (10) satisfy these conditions). If $X = E(\lambda, a)$, we write also μ_m^X in place of $\mu_m^{(\lambda,a)}$.

The following statement shows that each individual m-rectangular characteristic is a linear topological invariant.

Proposition 7 (see [7]). *Let $X = E(\lambda, a)$, $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$, $m \in \mathbb{N}$. If $X \simeq \tilde{X}$, then $\mu_m^X \approx \mu_m^{\tilde{X}}$.*

Systems of characteristics $(\mu_m^{(\lambda,a)})_{m \in \mathbb{N}}$ and $(\mu_m^{(\tilde{\lambda},\tilde{a})})_{m \in \mathbb{N}}$ are equivalent if the function φ and the constant α can be chosen so that the inequalities (12), (13) hold for all $m \in \mathbb{N}$ (we denote this equivalence by $(\mu_m^{(\lambda,a)}) \approx (\mu_m^{(\tilde{\lambda},\tilde{a})})$).

Proposition 8 (see [9, Proposition 1]). *For spaces $X = E(\lambda, a)$ and $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$, the following statements are equivalent*:

(a) $X \overset{\mathrm{qd}}{\simeq} \tilde{X}$;
(b) $\left(\mu_m^X \right) \approx \left(\mu_m^{\tilde{X}} \right)$.

We do not know whether this statement remains true if $\overset{\mathrm{qd}}{\simeq}$ is replaced by $\simeq$, in other words, is the quasidiagonal invariant $\gamma(X) := (\mu_m^X)_{m \in \mathbb{N}}$ also a linear topological invariant on the class $\mathcal{E}$ (with the above notion of equivalence)? Nevertheless we show here that it is possible to get new linear topological invariants, essentially stronger than any invariant (9), simply by taking the same map $\gamma(X)$ but introducing new equivalence relations on the set $\Gamma := \{(\mu_m^X)_{m \in \mathbb{N}} \colon X \in \mathcal{E}\}$.

Definition 9. Let $n \in \mathbb{N}$. We say that systems of characteristics (μ_m^X) and $(\mu_m^{\tilde{X}})$ are n-equivalent (and write $(\mu_m^X) \overset{n}{\approx} (\mu_m^{\tilde{X}})$) if there is a strictly increasing function $\varphi : [0, 2] \to [0, 1]$, $\varphi(0) = 0$, $\varphi(2) = 1$, and a positive constant α such that, for arbitrary $m \in \mathbb{N}$, the inequalities (12) and (13) hold for all collections of parameters (10), satisfying the following additional condition: among the numbers $\delta_1, \delta_2, \ldots, \delta_m$ there are no more than n different.

We consider the maps γ_n from $\mathcal{E}$ onto the set with equivalence $(\Gamma, \overset{n}{\approx})$ which all coincide with the map γ if considered as set maps, $n \in \mathbb{N}$. It will be shown in the next sections that the map γ_n is a linear topological invariant. As in [7] the main tool are *compound invariants*: the characteristic (6) will be applied to some "synthetic" absolutely convex sets V, U, built in a form of some geometrical or interpolational constructions from sets, belonging to a given basis of neighborhoods of zero in the space X. The parameters $\delta, \varepsilon, \tau, t$, satisfying additional condition from Definition 9, will be involved into those constructions in such a manner that, applying properties of the characteristic β, we provide the desired estimates (12), (13), uniformly by m. This plan will be realized in the next two sections within the proofs of Lemma 10 and Theorem 11.

4. Main lemma

Lemma 10. *Let* $X = E(\lambda, a)$, $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$, $n \in \mathbb{N}$. *If* $X \simeq \tilde{X}$, *then there exists an increasing function* $\gamma : [0, 2] \to [0, 1]$, $\gamma(0) = 0$, $\gamma(2) = 1$, *a decreasing function* $M : (0, 1] \to (0, \infty)$ *and a constant* $\alpha > 1$ *such that the inequality*

$$\mu_m^X(\delta, \varepsilon; \tau, t) \leqslant \mu_m^{\tilde{X}}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right) \tag{14}$$

holds for each $m \in \mathbb{N}$ *and all collections of parameters* (10) *satisfying the condition*: *among the numbers* $\delta_1, \delta_2, \ldots, \delta_m$, *there are no more than* n *different.*

Proof. We divide the proof into several parts.

(1) *General scheme.* Let $T : \tilde{X} \to X$ be an isomorphism. We consider two absolute bases of the space X: the canonical basis $e = \{e_i\}_{i \in \mathbb{N}}$ in X and T-image of the canonical basis of $\tilde{X}$: $\tilde{e} = \{\tilde{e}_i\}$, $\tilde{e}_i = Te_i$, $i \in \mathbb{N}$. Then each $x \in X$ has two basis expansions:

$$x = \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \eta_i \tilde{e}_i,$$

and the system of norms $\|x\|_p = \sum_{i=1}^{\infty} |\eta_i| \tilde{a}_{i,p}$, $x \in X$, $p \in \mathbb{N}$, is equivalent to the original system of norms in X: $|x|_p = \sum_{i=1}^{\infty} |\xi_i| a_{i,p}$, $x \in X$, $p \in \mathbb{N}$; here

$$a_{i,p} := \left(\exp\left(\left(-\frac{1}{p} + \lambda_i p\right) a_i\right)\right), \qquad \tilde{a}_{i,p} = \left(\exp\left(\left(-\frac{1}{p} + \tilde{\lambda}_i p\right) \tilde{a}_i\right)\right).$$

To prove the inequality (14) we shall build two pairs of synthetic neighborhoods U, V and $\tilde{U}$, $\tilde{V}$ in the form of certain compound geometrical and interpolational constructions using, as raw materials, the balls $B^e(A_p)$, $B^{\tilde{e}}(\tilde{A}_p)$ with the corresponding weights

$$A_p := (a_{i,p}), \qquad \tilde{A}_p := (\tilde{a}_{i,p}). \tag{15}$$

The sets $U, V, \tilde{U}, \tilde{V}$ will be constructed so that, on the one hand, to provide the inclusions

$$U \supset \tilde{U}, \qquad V \subset \tilde{V}, \tag{16}$$

and, on the other hand, to ensure the estimates

$$\mu_m^X(\delta, \varepsilon; \tau, t) \leqslant \beta\left(V, \frac{1}{n}U\right), \tag{17}$$

$$\beta\left(\tilde{V}, \frac{1}{n}\tilde{U}\right) \leqslant \mu_m^{\tilde{X}}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right). \tag{18}$$

Then the desired estimate (14) will be obtained immediately, since the inclusions (16) imply the inequality

$$\beta\left(V, \frac{1}{n}U\right) \leqslant \beta\left(\tilde{V}, \frac{1}{n}\tilde{U}\right).$$

(2) *Construction of synthetic neighborhoods.* First, we take any $n, m \in \mathbb{N}$. Since the systems of norms are equivalent we can choose an infinite chain of positive integers

$$\begin{aligned} r_0 &< p_0 < s_0 < \tilde{r}_1 < \tilde{p}_1 < \tilde{s}_1 < \cdots < \tilde{r}_l < \tilde{p}_l < \tilde{s}_l < \cdots < \tilde{r}_n \\ &< \tilde{p}_n < \tilde{s}_n < r_{n+1} < p_{n+1} < s_{n+1} < q_1 < \cdots < q_j < \cdots, \end{aligned} \tag{19}$$

so that the following inclusions:

$$\begin{aligned} B^e(A_{\tilde{p}_l}) &\subset C B^{\tilde{e}}(\tilde{A}_{\tilde{r}_l}), \quad B^{\tilde{e}}(\tilde{A}_{\tilde{s}_l}) \subset C B^e(A_{\tilde{p}_l}), \quad l = 1, 2, \ldots, n, \\ B^e(A_{p_l}) &\subset C B^{\tilde{e}}(\tilde{A}_{r_l}), \quad B^{\tilde{e}}(\tilde{A}_{s_l}) \subset C B^e(A_{p_l}), \quad l = 0, n+1, \\ B^e(A_{q_{j+1}}) &\subset C_j B^{\tilde{e}}(\tilde{A}_{q_j}), \quad B^{\tilde{e}}(\tilde{A}_{q_{j+1}}) \subset C_j B^e(A_{q_j}), \quad j \in \mathbb{N}, \end{aligned} \tag{20}$$

are valid with some constants $C = C(n)$, C_j, $j \in \mathbb{N}$. Without loss of generality, we can assume that each consequent number of the chain (19) is four times larger than the preceding one and that the sequence q_j satisfies the condition $4s_0 q_j < q_{j+1}$.

Let $\sigma_1 < \cdots < \sigma_l < \cdots$ represent all different values of the sequence (δ_k), which is always supposed to be non-decreasing. Now we define the numbers $p_k := \tilde{p}_{l_k}$, $r_k := \tilde{r}_{l_k}$, $s_k := \tilde{s}_{l_k}$, where l_k is such that $\delta_k = \sigma_{l_k}$, $k = 1, 2, \ldots, m$. Further we consider the sequence

$$\zeta_0 = 1, \qquad \zeta_j = \frac{1}{q_j}, \quad j \in \mathbb{N}, \tag{21}$$

and choose indices ν_k and j_k so that

$$\zeta_{\nu_k} \leqslant \delta_k < \zeta_{\nu_k - 1}, \quad \zeta_{j_k + 1} < \varepsilon_k \leqslant \zeta_{j_k}, \quad k = 1, 2, \ldots, m. \tag{22}$$

Now we are ready to define the sets serving as elementary blocks in the construction of the sets $U, V, \tilde{U}, \tilde{V}$. Beginning with the first couple of the sets U, V, we consider the blocks $(k = 1, \ldots, m)$

$$W_l^{(k)} = B^e\big(w_l^{(k)}\big), \quad l = 1, 2, \qquad \bar{W}_l^{(k)} = B^e\big(\bar{w}_l^{(k)}\big), \quad l = 1, 2, 3, 4, \tag{23}$$

where each weight-sequence will be responsible for one of the inequalities in (9). First we set

$$w_1^{(k)} = \bar{w}_1^{(k)} = A_{p_k}, \quad k = 1, 2, \ldots, m.$$

The estimates for λ_i from below and from above in (14), (9) are connected with two series of "interpolational" weights $(k = 1, \ldots, m)$

$$w_2^{(k)} = A_{p_0}^{1/2} A_{q_{v_k}}^{1/2}, \qquad \bar{w}_2^{(k)} = \begin{cases} A_{p_0}^{1/2} A_{q_{j_k - 1}}^{1/2} & \text{if } j_k > 3, \\ A_{p_0} & \text{if } j_k \leqslant 3. \end{cases}$$

To meet the estimates of a_i by the parameters τ_k and t_k in (14) we need the following series:

$$\bar{w}_3^{(k)} = \exp\left(\frac{\tau_k}{2 p_0}\right) A_{p_0}, \quad \bar{w}_4^{(k)} = \exp(-2 p_{n+1} t_k) A_{p_{n+1}}, \quad k = 1, 2, \ldots, m.$$

To construct the sets $\tilde{U}, \tilde{V}$ we use the corresponding series of blocks, which are balls with respect to the second basis $\tilde{e}$,

$$\tilde{W}_l^{(k)} = B^{\tilde{e}}\big(\tilde{w}_l^{(k)}\big), \qquad \tilde{\bar{W}}_l^{(k)} = B^{\tilde{e}}\big(\tilde{\bar{w}}_l^{(k)}\big).$$

Their weights we define by the same formulae as for the balls (23) but with the following rules of the substitution: to get the weight $\tilde{w}_l^{(k)}$ (or $\tilde{\bar{w}}_l^{(k)}$) we put $\tilde{A}_{s_k}/C$ (respectively, $C\tilde{A}_{r_k}$) instead of A_{p_k} and $\tilde{A}_{q_{v_k}}/C_{v_k}$ (respectively, $C_{j_k-2}\tilde{A}_{q_{j_k-2}}$) instead of $A_{q_{v_k}}$ (or, respectively, $A_{q_{j_k-1}}$). Putting

$$U^{(k)} = \text{conv}\left(\bigcup_{l=1}^{2} W_l^{(k)}\right), \qquad V^{(k)} = \bigcap_{l=1}^{4} \bar{W}_l^{(k)},$$

$$\tilde{U}^{(k)} = \text{conv}\left(\bigcup_{l=1}^{2} \tilde{W}_l^{(k)}\right), \qquad \tilde{V}^{(k)} = \bigcap_{l=1}^{4} \tilde{\bar{W}}_l^{(k)}$$

with $k = 1, 2, \ldots, m$, we define the sets

$$U = \bigcap_{k=1}^{m} U^{(k)}, \qquad V = \text{conv}\left(\bigcup_{k=1}^{m} V^{(k)}\right),$$

$$\tilde{U} = \bigcap_{k=1}^{m} \tilde{U}^{(k)}, \qquad \tilde{V} = \text{conv}\left(\bigcup_{k=1}^{m} \tilde{V}^{(k)}\right).$$

Taking into account (20), we have the inclusions

$$W_l^{(k)} \supset \tilde{W}_l^{(k)}, \quad l = 1, 2, \qquad \bar{W}_l^{(k)} \subset \tilde{\bar{W}}_l^{(k)}, \quad l = 1, 2, 3, 4,$$
$$k = 1, 2, \ldots, m,$$

which provide the inclusions (16).

(3) *Approximation of sets* $U, V, \tilde{U}, \tilde{V}$ *with the weighted* l_1-*balls.* Unlike elementary blocks, the sets U, V, $\tilde{U}$ and $\tilde{V}$ are not weighted balls. It is why Proposition 4 cannot be used directly for the calculation of $\beta(V, U)$ and $\beta(\tilde{V}, \tilde{U})$. Therefore, using Proposition 5, we approximate these sets with some appropriate weighted balls. For this purpose we consider the sequences $c^{(k)} = (c_i^{(k)})$, $\tilde{c}^{(k)} = (\tilde{c}_i^{(k)})$, $d^{(k)} = (d_i^{(k)})$, $\tilde{d}^{(k)} = (\tilde{d}_i^{(k)})$, $k = 1, 2, \ldots, m$, and the sequences $c = (c_i)$, $\tilde{c} = (\tilde{c}_i)$, $d = (d_i)$, $\tilde{d} = (\tilde{d}_i)$, defined as follows:

$$c_i^{(k)} = \min\{w_{i,l}^{(k)}: l = 1, 2\}, \qquad \tilde{c}_i^{(k)} = \min\{\tilde{w}_{i,l}^{(k)}: l = 1, 2\},$$
$$d_i^{(k)} = \max\{\bar{w}_{i,l}^{(k)}: l = 1, 2, 3, 4\}, \qquad \tilde{d}_i^{(k)} = \max\{\tilde{\bar{w}}_{i,l}^{(k)}: l = 1, 2, 3, 4\},$$
$$c_i = \min\{d_i^{(k)}: k = 1, 2, \ldots, m\}, \qquad \tilde{c}_i = \min\{\tilde{d}_i^{(k)}: k = 1, 2, \ldots, m\},$$
$$d_i = \max\{c_i^{(k)}: k = 1, 2, \ldots, m\}, \qquad \tilde{d}_i = \max\{\tilde{c}_i^{(k)}: k = 1, 2, \ldots, m\}.$$

By Proposition 5 the following relations hold (from now, the superscripts (e) and ($\tilde{e}$) will be omitted, since they are transparent from the context):

$$B(c^{(k)}) = U^{(k)}, \quad B(\tilde{c}^{(k)}) = \tilde{U}^{(k)}, \quad B(d^{(k)}) \subset V^{(k)}, \quad \tilde{V}^{(k)} \subset 4B(\tilde{d}^{(k)}).$$

From the condition for the numbers δ_k, $k = 1, 2, \ldots, m$, it follows that if $\delta_k = \delta_l$, then $j_k = j_l$, $p_k = p_l$, $w_{i,1}^{(k)} = w_{i,1}^{(l)}$, $i \in \mathbb{N}$, $w_{i,2}^{(k)} = w_{i,2}^{(l)}$, $i \in \mathbb{N}$, $c_i^{(k)} = c_i^{(l)}$, $i \in \mathbb{N}$. Since there are no more than n different among the sets $U^{(k)}$, $k = 1, 2, \ldots, m$, we get, using Proposition 5,

$$B(c) \subset V, \quad U \subset nB(d), \quad \tilde{V} \subset 4B(\tilde{c}), \quad B(\tilde{d}) \subset \tilde{U}.$$

Therefore, due to (7), (8), we have

$$\beta\big(B(c), B(d)\big) \leqslant \beta\bigg(V, \frac{1}{n}U\bigg), \tag{24}$$

$$\beta\bigg(\tilde{V}, \frac{1}{n}\tilde{U}\bigg) \leqslant \beta\big(4nB(\tilde{c}), B(\tilde{d})\big). \tag{25}$$

(4) *Estimate* (17). Now we are ready to show how the above construction of synthetic neighborhoods U and V results the estimate (17). Taking into account (24) it is sufficient to prove the inequality

$$\beta\big(B(c), B(d)\big) \geqslant \mu_m^X(\delta, \varepsilon; \tau, t). \tag{26}$$

From Proposition 4 we have

$$\beta\big(B(c), B(d)\big) = \big|\{i: c_i \leqslant d_i\}\big|.$$

By the definition of the sequences c and d, we obtain

$$\beta\big(B(c), B(d)\big) = \left|\bigcup_{k=1}^{m} \bigcup_{l=1}^{m} \{i: d_i^{(k)} \leqslant c_i^{(l)}\}\right|.$$

This implies the estimate

$$\beta\big(B(c), B(d)\big) \geqslant \left| \bigcup_{k=1}^{m} \big\{i\colon d_i^{(k)} \leqslant c_i^{(k)}\big\} \right|. \tag{27}$$

Due to the definition of the sequences $d^{(k)}$ and $c^{(k)}$, $k = 1, 2, \ldots, m$, we get

$$\big\{i\colon d_i^{(k)} \leqslant c_i^{(k)}\big\} = \Big\{i\colon \max_{1 \leqslant l \leqslant 4} \bar{w}_{i,l}^{(k)} \leqslant \min_{l=1,2} w_{i,l}^{(k)}\Big\}. \tag{28}$$

Since $\bar{w}_1^{(k)} = w_1^{(k)}$, the set in the right-hand side of (28) can be written in the following form:

$$\big\{i\colon d_i^{(k)} \leqslant c_i^{(k)}\big\} = \bigcap_{l=2}^{4} \big\{i\colon \bar{w}_{i,l}^{(k)} \leqslant w_{i,1}^{(k)}\big\} \cap \big\{\bar{w}_{i,1}^{(k)} \leqslant w_{i,2}^{(k)}\big\}. \tag{29}$$

To prove the estimate (26) we need to bring out the following inclusions ($k = 1, 2, \ldots, m$):

$$\big\{i\colon \bar{w}_{i,2}^{(k)} \leqslant w_{i,1}^{(k)}\big\} \supset \{i\colon \lambda_i \leqslant \varepsilon_k\}, \tag{30}$$

$$\big\{i\colon \bar{w}_{i,1}^{(k)} \leqslant w_{i,2}^{(k)}\big\} \supset \{i\colon \lambda_i > \delta_k\}, \tag{31}$$

$$\big\{i\colon \bar{w}_{i,3}^{(k)} \leqslant w_{i,1}^{(k)}\big\} \supset \{i\colon a_i > \tau_k\}, \tag{32}$$

$$\big\{i\colon \bar{w}_{i,4}^{(k)} \leqslant w_{i,1}^{(k)}\big\} \supset \{i\colon a_i \leqslant t_k\}. \tag{33}$$

First we consider (30). Due to the definitions of the weights and (15), the inequality in the left member of (30) is equivalent to the following inequality:

$$\frac{1}{2p_0} + \frac{1}{2q_{j_k-1}} - \frac{1}{p_k} \geqslant \lambda_i \left(\frac{1}{2} q_{j_k-1} + \frac{1}{2} p_0 - p_k \right) \quad \text{if } j_k > 3. \tag{34}$$

By the assumption about the chain (19) and by (21), (22) the left side of (34) is larger than $1/(4p_0)$ and the expression in round brackets is less than

$$\frac{q_{j_k}}{4p_0} = \frac{1}{4p_0 \zeta_{j_k}} \leqslant \frac{1}{4p_0 \varepsilon_k}.$$

Together with (34) this implies (30) if $j_k > 3$. In the case $j_k \leqslant 3$ the inclusion (30) is trivial. The inclusion (31) can be obtained analogously.

It remains only to check the inclusion (32), since (33) can be gained similarly. The left side inequality in (32) is equivalent to the inequality

$$\frac{\tau_k}{2p_0} \leqslant \frac{(p_k - p_0)(1 + \lambda_i p_0 p_k)}{p_0 p_k} a_i.$$

Since

$$\frac{(p_k - p_0)(1 + \lambda_i p_0 p_k)}{p_0 p_k} > \frac{1}{2p_0},$$

we get (32). It follows now from (29)–(33) that

$$\big\{i\colon d_i^{(k)} \leqslant c_i^{(k)}\big\} \supset \{i\colon \delta_k < \lambda_i \leqslant \varepsilon_k, \ \tau_k < a_i \leqslant t_k\}.$$

Combining this with (27), we obtain (26), hence (17).

(5) *Estimate* (18). Now we show that the construction of synthetic neighborhoods $\tilde{U}$ and $\tilde{V}$ provides the estimate (18). Due to (25), it is sufficient to check the estimate

$$\beta\left(4nB(\tilde{c}), B(\tilde{d})\right) \leqslant \mu_m^{\tilde{X}}\left(\gamma(\delta) - \frac{M(\delta)}{\tau}, \gamma^{-1}(\varepsilon) + \frac{M(\varepsilon)}{\tau}; \frac{\tau}{\alpha}, \alpha t\right).$$

Applying Proposition 4 and taking into account the definitions of the sequences $\tilde{c}$ and $\tilde{d}$, we get

$$\beta\left(4nB(\tilde{c}), B(\tilde{d})\right) = \left|\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\{i\colon \tilde{d}_i^{(k)} \leqslant 4n\tilde{c}_i^{(l)}\}\right|. \tag{35}$$

For arbitrary $k, l = 1, 2, \ldots, m$, using the definitions of the sequences $\tilde{c}^{(k)}$ and $\tilde{d}^{(l)}$, we obtain

$$\{i\colon \tilde{d}_i^{(k)} \leqslant 4n\tilde{c}_i^{(l)}\} \subset \bigcap_{\rho=1}^{4}\{i\colon \tilde{\tilde{w}}_{i,\rho}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\} \cap \{i\colon \tilde{w}_{i,1}^{(k)} \leqslant 4n\tilde{w}_{i,2}^{(l)}\}. \tag{36}$$

Having regard to the expressions for $\tilde{\tilde{w}}_{i,2}^{(k)}$, $\tilde{w}_{i,1}^{(l)}$ and to (15), it is easy to see that if $j_k > 3$, then the inequality

$$\tilde{\tilde{w}}_{i,2}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)} \tag{37}$$

is equivalent to the inequality

$$\left[\left(\frac{1}{2}r_0 + \frac{1}{2}q_{j_k-2} - s_l\right)\tilde{\lambda}_i - \left(\frac{1}{2r_0} + \frac{1}{2q_{j_k-2}} - \frac{1}{s_l}\right)\right]\tilde{a}_i \leqslant \ln(4nC\sqrt{CC_{j_k-2}}).$$

By the assumptions about the chain (19), the coefficient before $\tilde{\lambda}_i$ can be estimated from below by $q_{j_k-2}/4$, while the next expression in round brackets is less than $1/r_0$. Since

$$\zeta_{j_k-3} = \frac{1}{q_{j_k-3}} > \frac{4}{r_0 q_{j_k-2}},$$

we get the inclusion

$$\{i\colon \tilde{\tilde{w}}_{i,2}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\} \subset \left\{i\colon \tilde{\lambda}_i \leqslant \zeta_{j_k-3} + \frac{4\zeta_{j_k-2}\ln(4nC_{j_k-2}^2)}{\tilde{a}_i}\right\} \tag{38}$$

if $j_k > 3$. In the case $j_l \leqslant 3$, the inequality (37) is equivalent to the inequality

$$\left[\tilde{\lambda}_i(r_0 - s_l) - \left(\frac{1}{r_0} - \frac{1}{s_l}\right)\right]\tilde{a}_i \leqslant \ln(4nC^2).$$

Due to (19), (21), from here we get the inclusion

$$\{i\colon \tilde{\tilde{w}}_{i,2}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\} \subset \{i\colon \tilde{\lambda}_i \leqslant \zeta_0\} \tag{39}$$

if $j_l \leqslant 3$. Using similar arguments we get the inclusions

$$\{i\colon \tilde{\tilde{w}}_{i,1}^{(k)} \leqslant 4n\tilde{w}_{i,2}^{(l)}\} \subset \left\{i\colon \tilde{\lambda}_i \geqslant \zeta_{v_l+2} - \frac{\zeta_{v_l+1}\ln(4nC_{v_l}^2)}{\tilde{a}_i}\right\}. \tag{40}$$

It will be shown below that the estimate (14) will be ensured if we take a constant α, an increasing function $\gamma : [0, 2] \to [0, 1]$ and a decreasing function $M : (0, 1] \to (0, \infty)$, satisfying the following conditions:

$$
\begin{aligned}
&\alpha > \max\{4p_{n+1}\ln(4nC^2), 8p_{n+1}s_{n+1}\}, \\
&\gamma(0) = 0, \qquad \gamma(2) = 1, \qquad \gamma(\zeta_j) = \zeta_{j+4}, \quad j = 0, 1, \ldots, \\
&M(\zeta_j) \geqslant \alpha\zeta_{j+2}\ln\big(4nC_{j+1}^2\big), \quad j = 0, 1, 2, 3, \\
&M(\zeta_j) \geqslant \alpha\max\big\{\zeta_{j+2}\ln\big(4nC_{j+1}^2\big), 4\zeta_{j-2}\ln\big(4nC_{j-2}^2\big)\big\}, \quad j = 4, 5, \ldots. \tag{41}
\end{aligned}
$$

First from (38), (39) and (40) it follows that

$$
\begin{aligned}
&\big\{i : \tilde{\tilde{w}}_{i,2}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\big\} \subset \left\{i : \tilde{\lambda}_i \leqslant \gamma^{-1}(\zeta_{j_k+1}) + \frac{M(\zeta_{j_k})}{\alpha\tilde{a}_i}\right\}, \\
&\big\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leqslant 4n\tilde{w}_{i,2}^{(l)}\big\} \subset \left\{i : \tilde{\lambda}_i > \gamma(\zeta_{v_l-2}) - \frac{M(\zeta_{v_k})}{\alpha\tilde{a}_i}\right\}.
\end{aligned}
$$

Hence, bringing to mind (22), we obtain

$$
\big\{i : \tilde{\tilde{w}}_{i,2}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\big\} \subset \left\{i : \tilde{\lambda}_i \leqslant \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\alpha\tilde{a}_i}\right\}, \tag{42}
$$

$$
\big\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leqslant 4n\tilde{w}_{i,2}^{(l)}\big\} \subset \left\{i : \tilde{\lambda}_i > \gamma(\delta_l) - \frac{M(\delta_l)}{\alpha\tilde{a}_i}\right\}. \tag{43}
$$

Yet we have to examine the following inclusions:

$$
\big\{i : \tilde{\tilde{w}}_{i,3}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\big\} \subset \left\{i : \tilde{a}_i > \frac{\tau_k}{\alpha}\right\}, \tag{44}
$$

$$
\big\{i : \tilde{\tilde{w}}_{i,4}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\big\} \subset \{i : \tilde{a}_i \leqslant \alpha t_k\}. \tag{45}
$$

We prove only the inclusion (44), since the inclusion (45) can be obtained analogously. Having regard to the concrete form of the weights, we see that the inequality in the left-hand side of (44) is equivalent to the inequality

$$
\frac{\tau_k}{2p_0} \leqslant \ln\big(4nC^2\big) + \left[\left(\frac{1}{r_0} - \frac{1}{s_l}\right) + \tilde{\lambda}_i(s_l - r_0)\right]\tilde{a}_i. \tag{46}
$$

Taking into account (20), (41) and the assumption (5) we get that the inequality (46) remains true after replacing its right-hand side by $\alpha\tilde{a}_i$. Therefore, we get (44). After combining (44), (45), (42) and (43), we obtain

$$
\bigcap_{\rho=2}^{4}\big\{i : \tilde{\tilde{w}}_{i,\rho}^{(k)} \leqslant 4n\tilde{w}_{i,1}^{(l)}\big\} \cap \big\{i : \tilde{\tilde{w}}_{i,1}^{(k)} \leqslant 4n\tilde{w}_{i,2}^{(l)}\big\} \subset S_{k,l}, \tag{47}
$$

where

$$
S_{k,l} = \left\{i : \gamma(\delta_l) - \frac{M(\delta_l)}{\tau_k} < \lambda_i \leqslant \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k};\ \frac{\tau_k}{\alpha} < \tilde{a}_i \leqslant \alpha t_k\right\}. \tag{48}
$$

Taking into account the definitions of the sequences $\tilde{\tilde{w}}_1^{(k)}$, $\tilde{w}_1^{(l)}$, and (15), we have

$$\left\{ i\colon \tilde{\tilde{w}}_{i,1}^{(k)} \leqslant 4n \tilde{w}_{i,1}^{(l)} \right\} \subset R_{k,l}, \tag{49}$$

where

$$R_{k,l} = \left\{ i\colon \left[\left(\frac{1}{s_l} - \frac{1}{r_k} \right) + \tilde{\lambda}_i (r_k - s_l) \right] \tilde{a}_i \leqslant \ln(4nC^2) \right\}. \tag{50}$$

Combining (35), (36), (47) and (49) we obtain

$$\beta \left(4n B(\tilde{c}), B(\tilde{d}) \right) \leqslant \left| \bigcup_{k=1}^{m} \bigcup_{l=1}^{m} (S_{k,l} \cap R_{k,l}) \right|. \tag{51}$$

By (19),

$$\left(\frac{1}{s_l} - \frac{1}{r_k} \right) + \tilde{\lambda}_i (r_k - s_l) > \frac{1}{2s_l} > \frac{1}{4p_{n+1}} \quad \text{for } k > l.$$

Hence,

$$R_{k,l} \subset \{ i\colon \tilde{a}_i \leqslant \alpha \} \quad \text{if } k > l. \tag{52}$$

Now we are going to show the inclusions

$$S_{k,l} \cap R_{k,l} \subset S_{k,k}, \quad k, l = 1, \ldots, m. \tag{53}$$

This relation is obvious if $k = l$. Therefore it remains to consider the case when $k \neq l$. Suppose, first, $k < l$; then, by assumption, $\delta_k \leqslant \delta_l$. From the definitions of the functions γ and M it follows that $\gamma(\delta_k) \leqslant \gamma(\delta_l)$, $M(\delta_k) \geqslant M(\delta_l)$. From here and the definition $S_{k,l}$ we get (53) for $k < l$.

Suppose now that $k > l$. Since, by the assumption (5), $\tilde{\lambda}_i \geqslant 1/\tilde{a}_i$ for all $i \in \mathbb{N}$, we derive from (52) that $\tilde{\lambda}_i \geqslant 1/\alpha$. Hence we have

$$S_{k,l} \cap R_{k,l} \subset \left\{ i\colon \frac{1}{\alpha} \leqslant \tilde{\lambda}_i \leqslant \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k}, \ \frac{\tau_k}{\alpha} < \tilde{a}_i \leqslant \alpha t_k \right\}. \tag{54}$$

On the other hand, by the definitions of γ and Δ_k, we have

$$\gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k} < \gamma(\delta_k) < \gamma(\zeta_0) = \zeta_4 = \frac{1}{q_4}. \tag{55}$$

Since the constant α, depends only on n, we can assume the number q_4 chosen so that

$$\frac{1}{q_4} \leqslant \frac{1}{\alpha}. \tag{56}$$

Taking into account (54), (55), (56), we get (53) in the case $k > l$ as well. Thus the relation (53) is proved. Together with (51) it gives the relation

$$\beta \left(4n B(\tilde{c}), B(\tilde{d}) \right) \leqslant \left| \bigcup_{k=1}^{m} S_{k,k} \right|.$$

Remembering (14) we obtain the desired estimate (18). This completes the proof. $\quad\square$

5. Invariance of γ_n

Theorem 11. *Let the spaces $X = E(\lambda, a)$, $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$ be isomorphic. Then $(\mu_m^X) \overset{n}{\approx} (\mu_m^{\tilde{X}})$ for each $n \in \mathbb{N}$.*

Proof. Applying Lemma 10 we are going to establish the estimates (12), (13) for each $m \in \mathbb{N}$ and arbitrary collections of parameters (10) satisfying the condition: among the numbers $\delta_1, \delta_2, \ldots, \delta_m$, there are no more than n different. Therewith the function φ will be chosen in the end of our proof, while the constant α will be the same as in (14).

Because of symmetry we need to prove only the inequality (12). Let us rewrite this estimate, using (11), in the form

$$\left| \left\{ i \colon (\lambda_i, a_i) \in \bigcup_{k=1}^{m} P_k \right\} \right| \leqslant \left| \left\{ i \colon (\tilde{\lambda}_i, \tilde{a}_i) \in \bigcup_{k=1}^{m} Q_k \right\} \right|,$$

where

$$Q_k = \left(\varphi(\delta_k), \varphi^{-1}(\varepsilon_k) \right] \times \left(\frac{\tau_k}{\alpha}, \alpha t_k \right], \quad k = 1, 2, \ldots, m.$$

We cover each rectangle P_k by an appropriate couple of non-intersecting rectangles P_k' and P_k'' (some of them may be empty) and then apply Lemma 10. For construction of above-mentioned rectangles we need to define the decreasing function $\Psi : (0; 1] \to \mathbb{R}_+$, so that

$$\Psi(\xi) > \frac{M(\xi)}{\gamma(\xi)}, \quad 0 < \xi \leqslant 1, \tag{57}$$

where M and γ are as in Lemma 10. We are acting in a different way for each of three cases:

(a) $\tau_k \geqslant \Psi(\delta_k)$;

(b) $\tau_k < \Psi(\delta_k) < t_k$;

(c) $t_k \leqslant \Psi(\delta_k)$.

Setting the notation

$$\tau_k' := \max\{\Psi(\delta_k), \tau_k\}, \qquad t_k' := \min\{\Psi(\delta_k), t_k\},$$

$$\varepsilon_k := \begin{cases} \Psi^{-1}(\tau_k)\} & \text{if } \tau_k \geqslant \Psi(1), \\ 1 & \text{otherwise,} \end{cases}$$

we put

$$P_k' = \begin{cases} (\delta_k, \varepsilon_k] \times (\tau_k', t_k] & \text{in the cases (a) and (b),} \\ \varnothing & \text{otherwise,} \end{cases}$$

and

$$P_k'' = \begin{cases} \varnothing & \text{in the case (a),} \\ (\delta_k, \varepsilon_k'] \times (\tau_k, t_k'] & \text{otherwise.} \end{cases}$$

Applying Lemma 10, we get

$$\left|\left\{i:\ (\lambda_i, a_i) \in \bigcup_{k=1}^{m}(P_k' \cup P_k'')\right\}\right| \leqslant \left|\left\{i:\ (\tilde{\lambda}_i, \tilde{a}_i) \in \bigcup_{k=1}^{m}(\tilde{P}_k' \cup \tilde{P}_k'')\right\}\right|,$$

with

$$\tilde{P}_k' = \begin{cases} (\Delta_k', E_k'] \times \left(\frac{\tau_k'}{\alpha}, \alpha t_k\right] & \text{in the cases (a) and (b),} \\ \varnothing & \text{otherwise,} \end{cases}$$

and

$$\tilde{P}_k'' = \begin{cases} \varnothing & \text{in the case (a),} \\ (\Delta_k, E_k''] \times \left(\frac{\tau_k}{\alpha}, \alpha t_k'\right] & \text{otherwise,} \end{cases}$$

where

$$\Delta_k' = \gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k'}, \qquad E_k' = \gamma^{-1}(\varepsilon_k) + \frac{M(\varepsilon_k)}{\tau_k'},$$

$$\Delta_k = \gamma(\delta_k) - \frac{M(\delta_k)}{\tau_k}, \qquad E_k'' = \gamma^{-1}(\varepsilon_k') + \frac{M(\varepsilon_k')}{\tau_k}.$$

It follows from (57) and the definition of the numbers τ_k' that

$$\Delta_k' \geqslant \frac{1}{2}\gamma(\delta_k).$$

Since $\gamma(\xi) \leqslant \gamma^{-1}(\xi)$ when $\xi \in [0, 1]$, we obtain also the estimate

$$E_k' \leqslant \frac{3}{2}\gamma^{-1}(\varepsilon_k).$$

From $\tilde{\lambda}_i \geqslant 1/\tilde{a}_i$ and (57) it follows that

$$\left\{i:\ (\tilde{\lambda}_i, \tilde{a}_i) \in \tilde{P}_k''\right\} \subset \left\{i:\ (\tilde{\lambda}_i, \tilde{a}_i) \in \left(\frac{1}{2\alpha\Psi(\delta_k)}, E_k''\right] \times \left(\frac{\tau_k}{\alpha}, \alpha t_k\right]\right\}.$$

We can always assume that

$$\tau_k \geqslant \frac{1}{2\varepsilon_k}, \quad k = 1, 2, \ldots, m.$$

Therefore, taking into account (57), the definition of the numbers ε_k and the estimate $\gamma(\xi) \leqslant \gamma^{-1}(\xi)$, $\xi \in [0, 1]$, we obtain that

$$E_k'' \leqslant \frac{3}{2}\gamma^{-1}(\varepsilon_k') \leqslant \frac{3}{2}\gamma^{-1}\left(\Psi^{-1}\left(\frac{1}{2\varepsilon_k}\right)\right).$$

Now we choose an increasing function $\varphi : [0; 2] \to [0; 1]$, $\varphi(2) = 1$, $\varphi(0) = 0$, so that

$$\varphi(\xi) \leqslant \min\left\{\frac{1}{2}\gamma(\xi), \gamma\left(\frac{2}{3}\xi\right), \frac{1}{2\alpha\Psi(\xi)}, \frac{1}{2\Psi\left(\gamma\left(\frac{2}{3}\xi\right)\right)}\right\}, \quad \xi \in [0; 1].$$

Then the estimate (12) holds for each $m \in \mathbb{N}$ and any collection of parameters (10) satisfying the condition: among the numbers $\delta_1, \delta_2, \ldots, \delta_m$, there are no more than n different. This completes the proof. $\quad\square$

6. Basic subspaces of finite and infinite type

In this section we consider only Montel spaces $X = E(\lambda, a)$, that is $a_i \to \infty$ is assumed. Given a subsequence $I = \{i_k\}_{k \in \mathbb{N}}$ of $\mathbb{N}$ a *basic subspace* X_I spanned by $\{e_i : i \in I\}$ is from the same class $\mathcal{E}$. Denote by $\mathcal{E}_0(\lambda, a)$ (respectively, $\mathcal{E}_\infty(\lambda, a)$) the collection of all basic subspaces X_I which are isomorphic to power series spaces of finite (respectively, infinite) type.

Proposition 12 [33,37]. *Let $X = E(\lambda, a)$ be Montel and $I = \{i_k\}$ a subsequence of $\mathbb{N}$. Then a basic subspace X_I belongs $\mathcal{E}_0(\lambda, a)$ (respectively, $\mathcal{E}_\infty(\lambda, a)$) if and only if $\lim_{i \in I} \lambda_i = 0$ (respectively, $\inf_{i \in I} \{\lambda_i\} > 0$). Therewith X_I is quasidiagonally isomorphic to $E_0(c)$ (respectively, $E_\infty(c)$), where $c = (c_k) = (a_{i_k})$.*

Now we are going to give a solution of Problem 2. First we consider the simpler case of basic subspaces of infinite type, which is derivable from one-rectangle invariants (Proposition 7 with $m = 1$).

Theorem 13. *Let $X = E(\lambda, a)$ and $\tilde{X} = E(\tilde{\lambda}, \tilde{c})$ be isomorphic. Then for each $L \in \mathcal{E}_\infty(\lambda, a)$ there exists $M \in \mathcal{E}_\infty(\tilde{\lambda}, \tilde{a})$ such that $L \overset{\text{qd}}{\simeq} M$.*

Proof. Since $X \simeq \tilde{X}$, applying Proposition 7 with $m = 1$, we obtain that there is a function $\varphi : [0; 1] \to \mathbb{R}_+$ and a constant α such that the inequality

$$\mu_1^X(\delta, 1; \tau, t) \leqslant \mu_1^{\tilde{X}}\left(\varphi(\delta), 1; \frac{\tau}{\alpha}, \alpha t\right) \tag{58}$$

holds for any $\delta > 0$ and $1 \leqslant \tau < t$.

Let now $L = X_I \in \mathcal{E}_\infty(\lambda, a)$, $I = \{i_k\}$. Then, by Proposition 12, there is $\delta_0 > 0$ such that $\lambda_i > \delta_0$ if $i \in I$. Consider $c = (c_k) = (a_{i_k})$ and $\tilde{c} = (\tilde{c}_k) := (\tilde{a}_{j_k})$ with $J := \{j_k\} = \{j : \tilde{\lambda}_j > \varphi(\delta_0)\}$. Then, due to (58), we obtain the following estimates for the counting functions (see (4)) of the sequences c and $\tilde{c}$:

$$M_c(\tau, t) \leqslant \mu_1(\delta_0, 1; \tau, t) \leqslant \mu_1^{\tilde{X}}\left(\varphi(\delta), 1; \frac{\tau}{\alpha}, \alpha t\right) = M_{\tilde{c}}\left(\frac{\tau}{\alpha}, \alpha t\right).$$

Therefore, by Proposition 3, we deduce that $L \overset{\text{qd}}{\simeq} E_\infty(c) \overset{\text{qd}}{\hookrightarrow} E_\infty(\tilde{c}) \overset{\text{qd}}{\simeq} \tilde{X}_J$, where $\tilde{X}_J$ is a basic subspace of $\tilde{X}$ spanned by $\{e_j : j \in J\}$. Thus L is quasidiagonally isomorphic to some basic subspace M of $\tilde{X}_J$, which, due to Proposition 12, belongs to $\mathcal{E}_\infty(\tilde{\lambda}, \tilde{a})$. This completes the proof. □

Theorem 14. *Let $X = E(\lambda, a)$ and $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$ be isomorphic spaces. Then for each $L \in \mathcal{E}_0(\lambda, a)$ there exists $M \in \mathcal{E}_0(\tilde{\lambda}, \tilde{a})$ such that $L \overset{\text{qd}}{\simeq} M$.*

Proof. Applying Theorem 11 with $n = 1$ we derive from $X \simeq \tilde{X}$ that there exist an increasing function $\varphi : [0, 2] \to [0, 1]$, $\varphi(2) = 1$, $\varphi(0) = 0$, and a constant $\alpha > 0$ (both independent

of m) such that the estimate (12) holds for all collections of parameters (10) satisfying the condition: $\delta_k = 0$ for all $k = 1, 2, \ldots, m$, $m \in \mathbb{N}$.

Let $L = X_I \in \mathcal{E}_0(\lambda, a)$, $I = \{i_s\}$. Then $L \overset{\text{qd}}{\simeq} E_0(c)$ with $c = (c_s) = (a_{i_s})$. Since $\lambda_{i_s} \to 0$ and $a_i \to \infty$, there is an increasing continuous function $\gamma : \mathbb{R}_+ \to [0, 1]$ such that $\gamma(0) = 0$ and $\lambda_i \leqslant \gamma(a_i)$ if $i \in I$.

Consider two sequences of rectangles:

$$P_k := \big(0; \gamma(\alpha^{k-1})\big] \times (\alpha^{k-1}; \alpha^k],$$
$$Q_k := \big(0; \varphi^{-1}\big(\gamma(\alpha^{k-1})\big)\big] \times (\alpha^{k-2}; \alpha^{k+1}], \quad k \in \mathbb{N},$$

and define $J = \{j_s\} := \bigcup_{k \in \mathbb{N}} \{j : (\tilde{\lambda}_j, \tilde{a}_j) \in Q_k\}$, a sequence $\tilde{c} = (\tilde{c}_s) := (\tilde{a}_{j_s})$ and a subspace $\tilde{X}_J$ spanned in $\tilde{X}$ by $\{e_j : j \in J\}$.

By the above choice of φ and α we have

$$\left| \left\{ i : (\lambda_i, a_i) \in \bigcup_{k \in K} P_k \right\} \right| \leqslant \left| \left\{ j : (\tilde{\lambda}_j, \tilde{c}_j) \in \bigcup_{k \in K} Q_k \right\} \right| \tag{59}$$

for any finite set $K \in \mathbb{N}$.

Take an arbitrary τ and t $(1 \leqslant \tau < t < +\infty)$ and choose $k, l \in \mathbb{N}$ so that

$$\alpha^{k-1} \leqslant \tau < \alpha^k, \qquad \alpha^{l-1} < t \leqslant \alpha^l.$$

Then, by the construction, we have

$$M_c(\tau, t) \leqslant \left| \left\{ i : (\lambda_i, a_i) \in \bigcup_{j=k}^{l} P_j \right\} \right|,$$
$$M_{\tilde{c}}\left(\frac{\tau}{\alpha^2}, \alpha^2 t \right) \geqslant \left| \left\{ i : (\tilde{\lambda}_i, \tilde{c}_i) \in \bigcup_{j=k}^{l} Q_j \right\} \right|.$$

From here, together with (59), we get the estimate

$$M_c(\tau, t) \leqslant M_{\tilde{c}}\left(\frac{\tau}{\alpha^2}, \alpha^2 t \right).$$

Since $\tilde{X}_J \overset{\text{qd}}{\simeq} E_0(\tilde{c})$ we get, by Proposition 3, that L is isomorphic quasidiagonally to some basic subspace $M \subset \tilde{X}_J$, which belongs, by Proposition 12, to $\mathcal{E}_0(\tilde{\lambda}, \tilde{a})$. □

7. Comparison of *n*-equivalence and equivalence of the systems characteristics

In [9] a special pair X and $\tilde{X}$ of spaces (2) was constructed to show that the equivalence of all characteristics μ_m^X and $\mu_m^{\tilde{X}}$ is not sufficient for existence of quasidiagonal isomorphism between these spaces. The question whether these spaces are isomorphic was behind the powers of invariants considered there. Now applying Theorem 11 with $n = 1$ we can easily conclude that they are not isomorphic.

Right now we are going to show that even much stronger invariants based on n-equivalence with arbitrary n still are not sufficient to characterize quasidiagonal isomorphisms of spaces (2). To accomplish this we shall construct an appropriate example.

Proposition 15. *There exist two first type power Köthe spaces $X = E(\lambda, a)$ and $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$, satisfying the following conditions*:

(i) $\left(\mu_m^X\right) \overset{n}{\approx} \left(\mu_m^{\tilde{X}}\right), \quad n \in \mathbb{N}$;

(ii) $\left(\mu_m^X\right) \not\approx \left(\mu_m^{\tilde{X}}\right)$.

Proof. As in [9] the required spaces $X = E(\lambda, a)$ and $\tilde{X} = E(\tilde{\lambda}, \tilde{a})$ will be constructed from finite dimensional blocks

$$X = \bigoplus_{(l,j) \in \mathbb{N}^2} X_{l,j}, \qquad \tilde{X} = \bigoplus_{(l,j) \in \mathbb{N}^2} \tilde{X}_{l,j}.$$

Take an arbitrary number $\alpha > 1$, the sequence $(\xi_i) \downarrow 0$, $\xi_i \in (0, 1)$, the sequence $(\beta_l) \uparrow \infty$ so that $\beta_1 > \alpha$.

Let $l \in \mathbb{N}$, then each block $X_{l,j}(\tilde{X}_{l,j})$ will correspond to some $(2l + 2)$-dimensional vector $k^{(l)} = (k_1^{(l)}, k_2^{(l)}, \ldots, k_{2l+2}^{(l)})$ such that $k_1^{(l)}, k_2^{(l)}, \ldots, k_{2l+2}^{(l)} \in \mathbb{N}$ and $k_1^{(l)} < k_2^{(l)} < \cdots < k_{2l+2}^{(l)}$. The set K_l of all such $(2l+2)$-dimensional vectors we enumerate anyhow: $k^{(l)}(j) = (k_1^{(l)}(j), k_2^{(l)}(j), \ldots, k_{2l+2}^{(l)}(j))$, $j \in \mathbb{N}$. So, the blocks numbered with (l, j) correspond to the vector $k^{(l)}(j)$.

Now we enumerate all elements of the set $\mathbb{N}^2$: (l_p, j_p), $p \in \mathbb{N}$, and select a sequence of positive numbers (η_{l_p, j_p}) so that $\eta_{l_{p+1}, j_{p+1}} \geqslant \beta_{l_p}^{2l_p+4} \eta_{l_p, j_p}$. In the subsequent text we will write l and j instead of l_p and j_p, respectively.

Put $Y_{l,j}^{(\nu)} = \beta_l^{\nu - 1} \eta_{l,j}$, $\nu = 1, 2, \ldots, 2l + 5$. In Fig. 1 it is drawn the set $S_{l,j}$, consisting of horizontal and vertical segments, where all points (λ_i, a_i), $(\tilde{\lambda}_i, \tilde{a}_i)$ corresponding to the blocks $X_{l,j}$ and $\tilde{X}_{l,j}$, will be located.

On the segment $[Y_{l,j}^{(1)}, Y_{l,j}^{(2l+5)}]$ we select a finite α-dense set $M_{l,j}$, including all the points $Y_{l,j}^{(\nu)}$, $\nu = 1, 2, \ldots, 2l + 5$. We remind that a set $A \subset \mathbb{R}$ is said to be α-*dense* in $B \subset \mathbb{R}$ if for each point $b \in B$ there is a point $a \in A$ such that $a/\sqrt{\alpha} \leqslant b \leqslant \sqrt{\alpha} a$.

Let $L_{l,j}$ be the set of all points $(\xi_r, y_{l,j,s}) \in S_{l,j}$ and $(1, y_{l,j,s})$, such that $y_{l,j,s} \in M_{l,j}$ and $1 \leqslant r \leqslant k_{2l+2}^{(l)}(j)$. After enumeration this set we get

$$L_{l,j} = \left\{ (x_{l,j,i}, y_{l,j,i}), \ i = 1, 2, \ldots, n_{l,j} - 1 \right\}$$

with some number $n_{l,j}$. Put two additional points

$$\bar{y}_{l,j} \in \left[Y_{l,j}^{(2)}, Y_{l,j}^{(l+2)} \right] \quad \text{and} \quad \tilde{y}_{l,j} \in \left[Y_{l,j}^{(l+4)}, Y_{l,j}^{(2l+4)} \right].$$

Now define the vectors

$$\lambda^{(l,j)} = \left(\lambda_i^{(l,j)}\right)_{i=1}^{n_{l,j}}, \quad a^{(l,j)} = \left(a_i^{(l,j)}\right)_{i=1}^{n_{l,j}}, \quad \tilde{\lambda}^{(l,j)} = \left(\tilde{\lambda}_i^{(l,j)}\right)_{i=1}^{n_{l,j}} \tilde{a}^{(l,j)} = \left(\tilde{a}_i^{(l,j)}\right)_{i=1}^{n_{l,j}}$$
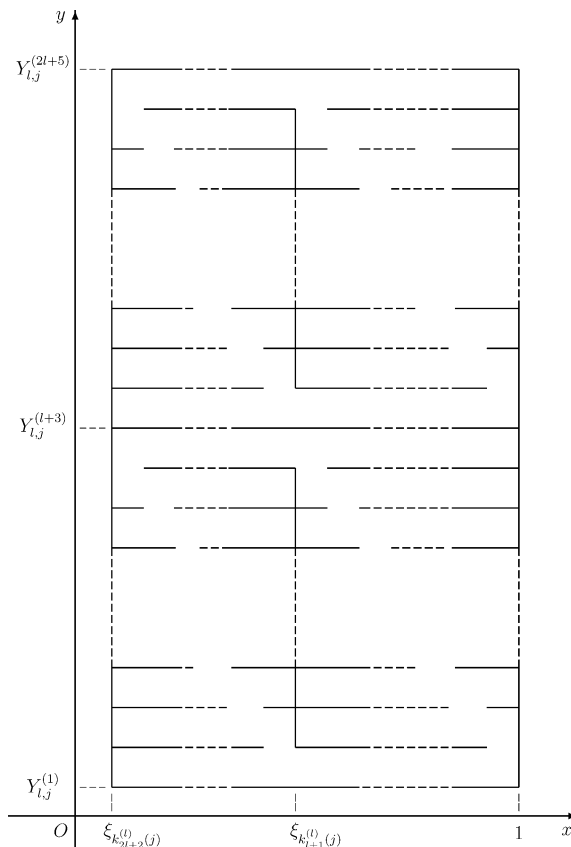
by the following formulae:

Fig. 1. The set $S_{l,j}$.

$$\lambda_i^{(l,j)} = \tilde{\lambda}_i^{(l,j)} = \begin{cases} x_{l,j,i} & \text{if } i = 1, 2, \ldots, n_{l,j} - 1, \\ \xi_{k_{l+1}^{(l)}} & \text{if } i = n_{l,j}, \end{cases}$$

$$a_i^{(j)} = \begin{cases} y_{l,j,i} & \text{if } i = 1, 2, \ldots, n_{l,j} - 1, \\ \bar{y}_{l,j} & \text{if } i = n_{l,j}, \end{cases}$$

$$\tilde{c}_i^{(l,j)} = \begin{cases} y_{l,j,i} & \text{if } i = 1, 2, \ldots, n_{l,j} - 1, \\ \tilde{y}_{l,j} & \text{if } i = n_{l,j}. \end{cases}$$

Finally we construct the sequences $\lambda = (\lambda_i)$, $a = (a_i)$, $\tilde{\lambda} = (\tilde{\lambda}_i)$, $\tilde{a} = (\tilde{a}_i)$ by the rule $\xi_i = \xi_{i-(n_{l_1,j_1}+\cdots+n_{l_{p-1},j_{p-1}})}^{(l,j)}$ for $n_{l_{p-1},j_{p-1}} < i \leqslant n_{l_p,j_p}$ ($n_{l_0,j_0} := 0$), where ξ may be $\lambda, a, \tilde{\lambda}$ or $\tilde{a}$.

To prove the statement (i) we take any increasing function $\varphi : [0, 2] \to [0, 1]$, satisfying the following conditions:

$$\varphi(0) = 0, \quad \varphi(2) = 1, \quad \varphi(1) < \xi_1, \quad \varphi(\xi_i) < \xi_{i+1}.$$

It is easy to check that (i) is true with the function φ and the constant c, which is defined by the formula

$$c = \begin{cases} \alpha & \text{if } n = 1, \\ \beta_{n-1} & \text{if } n = 2, 3, \ldots. \end{cases}$$

On the other hand, show that the statement (ii) also holds, i.e., the system of m-rectangle characteristics, considered with the equivalence $\approx$, distinguishes those spaces. To check this we have to show that for any choice of a positive constant γ, and an arbitrary increasing function $\psi : [0, 2] \to [0, 1]$, $\psi(0) = 0$, $\psi(2) = 1$, there exist $m \in \mathbb{N}$ and a collection of parameters $\delta, \varepsilon, \tau, t$, of kind (10) such that the estimate

$$\mu_m^{(\tilde{\lambda}, \tilde{a})}(\delta, \varepsilon; \tau, t) \geqslant \mu_m^{(\lambda, a)}\left(\psi(\delta), \psi^{-1}(\varepsilon); \frac{\tau}{\gamma}, \gamma t\right) \tag{60}$$

does not hold. To this end we select two natural numbers l_0 and j_0 so that

$$\beta_{l_0} > \gamma, \qquad \xi_{k_1^{(l_0)}(j_0)} < \psi(1),$$

$$\xi_{k_r^{(l_0)}(j_0)} < \psi\left(\xi_{k_{r-1}^{(l_0)}(j_0)}\right), \quad r = 2, 3, \ldots, 2l_0 + 2.$$

Then we put $m = l_0 + 2$ and pick $\delta = (\delta_r)$, $\varepsilon = (\varepsilon_r) \in (0, 1]^m$, $\tau = (\tau_r)$, $t = (t_r) \in \mathbb{R}_+^m$ in the following way:

$$\delta_r = \psi^{-1}\left(\xi_{k_{l_0+r}^{(l_0)}(j_0)}\right), \quad r = 1, 2, \ldots, m - 1, \qquad \delta_m := \psi^{-1}\left(\xi_{k_{l_0+1}^{(l_0)}(j_0)}\right),$$

$$\varepsilon_r := \xi_{k_r^{(l_0)}(j_0)}, \quad r = 1, 2, \ldots, m - 1, \qquad \varepsilon_m := \xi_{k_{l_0+1}^{(l_0)}(j_0)},$$

$$\tau_r := \gamma Y_{l_0, j_0}^{(r)}, \quad r = 1, 2, \ldots, m - 1, \qquad \tau_m := \gamma Y_{l_0, j_0}^{(1)},$$

$$t_r := Y_{l_0, j_0}^{(r+1)}, \quad r = 1, 2, \ldots, m - 1, \qquad t_m := \gamma Y_{l_0, j_0}^{(l_0+2)}.$$

Regarding (11), we obtain

$$\mu_m^{(\tilde{\lambda}, \tilde{a})}(\delta, \varepsilon; \tau, t) = \left\| \left\{ i : (\tilde{\lambda}_i, \tilde{a}_i) \in \bigcup_{r=1}^m P_r \right\} \right\| > \left\| \left\{ i : (\lambda_i, a_i) \in \bigcup_{r=1}^m Q_r \right\} \right\|$$

$$= \mu_m^{(\lambda, a)}\left(\psi(\delta), \psi^{-1}(\varepsilon); \frac{\tau}{\gamma}, \gamma t\right),$$

where

$$Q_r = \left(\psi(\delta_r), \psi^{-1}(\varepsilon_r)\right] \times \left(\frac{\tau_r}{\gamma}, \gamma t_r\right], \quad r = 1, 2, \ldots, m.$$

This means that the inequality (60) is violated, which completes the proof. $\quad\square$

Acknowledgment

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