



Entanglement and second quantization  
in the framework of the fermionic  
projector

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# ENTANGLEMENT AND SECOND QUANTIZATION IN THE FRAMEWORK OF THE FERMIONIC PROJECTOR

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ABSTRACT. We analyze if entangled fermionic states can be described by a projector in the one-particle Hilbert space. A method is developed for realizing entanglement and general second quantized fermionic and bosonic fields in the framework of the fermionic projector. Our constructions are discussed with regard to decoherence phenomena and the measurement problem.

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## 1. INTRODUCTION

In [6] it was proposed to formulate physics based on a new action principle in space-time (see also [8, 9]). One difference of this approach to standard quantum field theory is that a many-particle state no longer corresponds to a vector in the fermionic Fock space, but instead it is described by the so-called fermionic projector, an operator which acts on the one-particle Hilbert space (or more generally on an indefinite inner product space spanned by the one-particle wave functions). Another difference is that the bosonic fields obtained in the so-called continuum limit are only classical. Due to these differences, it is not at all obvious whether the fermionic projector can account for all quantum effects observed in nature. More specifically, is it possible to describe *entanglement*? Can one reproduce the effects of *second quantized fields*?

In this paper, we shall analyze these questions in detail. We will show that it is indeed possible to describe entanglement as well as general second quantized bosonic and fermionic fields in the framework of the fermionic projector. Our considerations lead to the physical concept of a *microscopic mixing of decoherent subsystems*. The physical picture is that space-time is not smooth on the microscopic scale (typically thought of as the Planck scale), but has a non-trivial microstructure. Homogenizing this microstructure, we obtain an effective description of the system by a vector in the fermionic or bosonic Fock space. The key ingredient for making this concept work is that in the framework of the fermionic projector, the usual topological and causal structure of Minkowski space is not a-priori given, but it is induced on the space-time points by the states of the fermionic projector (which form a Dirac sea configuration; see [10]). Thus by bringing the wave functions between certain pairs of space-time points “out of phase”, we obtain decoherence effects which result in a decomposition of the whole system into subsystems between which the usual causal relations are no longer valid. This makes it possible to realize many independent physical systems simultaneously in one spacetime, in such a way that homogenizing on the microscopic scale leads to an effective “superposition” of the subsystems. For technical simplicity, we will describe the microscopic mixing by localizing the subsystems in disjoint space-time regions (see Figure 2 on page 26). But one can also think of the subsystems as being delocalized, similar as if one combines several images in a single hologram (see Section 7.3).

In order to make the paper easily accessible, we begin on an elementary level and develop our ideas step by step, also discussing attempts which turn out not to work. These preliminary attempts are instructive because they motivate and show the necessity of the concept of microscopic mixing. More specifically, in Chapter 2 we begin in the setting of non-relativistic quantum mechanics with the question of whether a vector of the fermionic Fock space can be described by a projector in the one-particle Hilbert space. This chapter also provides the mathematical preliminaries and fixes our notation. We show that every projector in the one-particle Hilbert space corresponds to a Hartree-Fock state (see Proposition 2.16), making it impossible to describe entanglement. In Chapter 3 we show that this shortcoming cannot be overcome by restricting attention to a subsystem and taking the partial trace over the outer system (see Example 3.6). In Chapter 4 we describe entanglement by introducing the concept of microscopic mixing, however on a rather formal level. Since justifying this formalism goes beyond the scope of standard quantum mechanics, in Chapter 5 we generalize to the relativistic setting and explain a few basic concepts behind the framework of the

fermionic projector. In Chapter 6 we justify the formalism of Chapter 4 by introducing a decoherence between space-time regions. As is worked out in Chapter 7, this notion of decoherence also makes it possible to describe second quantized bosonic fields. Finally, in Chapter 8 we give a physical interpretation and discuss our constructions with regard to the measurement problem, decoherence effects, the wave-particle duality and the collapse of the wave function.

## 2. PRELIMINARIES

**2.1. The Fermionic Fock Space and Entanglement.** We denote the one-particle Hilbert space by  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ ; as usual we assume that it is separable. In quantum mechanics and quantum field theory, a many-fermion state is usually described by a vector in the fermionic Fock space, which we now introduce (see also [21, Section II.4] or [23, Section I.1]). We let  $\mathcal{H}^n = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$  be the  $n$ -fold tensor product, endowed with the natural scalar product

$$\langle \psi_1 \otimes \cdots \otimes \psi_n | \phi_1 \otimes \cdots \otimes \phi_n \rangle := \langle \psi_1 | \phi_1 \rangle \cdots \langle \psi_n | \phi_n \rangle. \quad (2.1)$$

Totally anti-symmetrizing the tensor product gives the wedge product

$$\psi_1 \wedge \cdots \wedge \psi_n := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)} \quad (2.2)$$

(here  $S_n$  denotes the set of all permutations and  $\text{sign}(\sigma)$  is the sign of the permutation  $\sigma$ ). The wedge product gives rise to a mapping

$$\Lambda_n : \underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{n \text{ factors}} \rightarrow \mathcal{H}^n : (\psi_1, \dots, \psi_n) \mapsto \psi_1 \wedge \cdots \wedge \psi_n.$$

We denote the image of this mapping by  $\mathcal{F}_n^{\text{HF}}$ . The vectors in  $\mathcal{F}_n^{\text{HF}}$  are called  $n$ -particle *Hartree-Fock states* or *factorizable states*. These states do in general not form a vector space, as the following example shows, which in discussions of spin correlation experiments and Bell's inequalities is often referred to as the EPR singlet state (see for example [1, Section 1.5]).

**Example 2.1.** (*The spatially separated singlet state*) We consider the one-particle Hilbert space  $\mathcal{H} = \mathbb{C}_A^2 \oplus \mathbb{C}_B^2$  of two spins (in quantum information theory called “qubits”), located at the positions of the two observers “Alice” and “Bob”. Choosing in  $\mathbb{C}^2$  the standard basis  $\psi^\uparrow = (1, 0)$  and  $\psi^\downarrow = (0, 1)$  yields the basis  $(\psi_A^\uparrow, \psi_A^\downarrow, \psi_B^\uparrow, \psi_B^\downarrow)$  of  $\mathcal{H}$ . The spatially separated singlet state is the following linear combination of 2-particle Hartree-Fock states

$$\Psi := \frac{1}{\sqrt{2}} \left( \psi_A^\uparrow \wedge \psi_B^\downarrow - \psi_A^\downarrow \wedge \psi_B^\uparrow \right). \quad (2.3)$$

Let us verify in detail that this state is not factorizable. To this end, we assume conversely that  $\Psi$  can be written as a product,

$$\Psi = \psi_1 \wedge \psi_2.$$

Computing this wedge product in the basis representations

$$\psi_1 = \alpha_A^\uparrow \psi_A^\uparrow + \alpha_A^\downarrow \psi_A^\downarrow + \alpha_B^\uparrow \psi_B^\uparrow + \alpha_B^\downarrow \psi_B^\downarrow \quad (2.4)$$

$$\psi_2 = \beta_A^\uparrow \psi_A^\uparrow + \beta_A^\downarrow \psi_A^\downarrow + \beta_B^\uparrow \psi_B^\uparrow + \beta_B^\downarrow \psi_B^\downarrow, \quad (2.5)$$

the vanishing of the term  $\sim \psi_A^\uparrow \wedge \psi_A^\downarrow$  implies that the vectors  $\alpha_A^\uparrow \psi_A^\uparrow + \alpha_A^\downarrow \psi_A^\downarrow$  and  $\beta_A^\uparrow \psi_A^\uparrow + \beta_A^\downarrow \psi_A^\downarrow$  must be linearly dependent. Similarly, the vanishing of the term  $\sim \psi_B^\uparrow \wedge \psi_B^\downarrow$  implies that the vectors  $\alpha_B^\uparrow \psi_B^\uparrow + \alpha_B^\downarrow \psi_B^\downarrow$  and  $\beta_B^\uparrow \psi_B^\uparrow + \beta_B^\downarrow \psi_B^\downarrow$  are linearly dependent. Hence  $\psi_2$  can be written as

$$\psi_2 = \beta_A \left( \alpha_A^\uparrow \psi_A^\uparrow + \alpha_A^\downarrow \psi_A^\downarrow \right) + \beta_B \left( \alpha_B^\uparrow \psi_B^\uparrow + \alpha_B^\downarrow \psi_B^\downarrow \right) \quad (2.6)$$

with suitable complex coefficients  $\beta_A$  and  $\beta_B$ . Taking the wedge product of (2.4) and (2.6) yields

$$\Psi = \psi_1 \wedge \psi_2 = (\beta_B - \beta_A) \left( \alpha_A^\uparrow \psi_A^\uparrow + \alpha_A^\downarrow \psi_A^\downarrow \right) \wedge \left( \alpha_B^\uparrow \psi_B^\uparrow + \alpha_B^\downarrow \psi_B^\downarrow \right).$$

Multiplying out and comparing with (2.3), one sees that the products  $\alpha_A^\uparrow \alpha_B^\downarrow$  and  $\alpha_A^\downarrow \alpha_B^\uparrow$  must be non-zero, and thus none of these four coefficients vanishes. But then the term  $\sim \psi_A^\uparrow \wedge \psi_B^\uparrow$  is non-zero, a contradiction.  $\diamond$

We denote the vector space generated by the  $n$ -particle Hartree-Fock states by

$$\mathcal{F}_n = \overline{\langle \Lambda_n(\mathcal{H}^n) \rangle}.$$

Their direct sum is the *fermionic Fock space*,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n.$$

The non-factorizable vectors  $\Psi \in \mathcal{F}_n \setminus \Lambda_n(\mathcal{H}^n)$  are called *entangled states*. The spatially separated singlet state is the standard example of an entangled state. Entanglement is a basic phenomenon of quantum physics with important potential applications in quantum computing.

**2.2. Mixed States and the Density Operator.** Another important concept, which should not be confused with entanglement, is that of a mixed state. A state  $\Psi \in \mathcal{F}$  (no matter if factorizable or entangled) is referred to as a *pure state*. Keeping in mind that the phase of  $\Psi$  has no physical significance, the physical system is described equivalently by a projector  $P_\Psi$  onto this state,

$$P_\Psi = \frac{1}{\|\Psi\|_{\mathcal{F}}^2} |\Psi\rangle\langle\Psi|. \quad (2.7)$$

The expectation value of an observable  $\mathcal{O}$  can be expressed by the trace over range of the projector,

$$\langle \mathcal{O} \rangle = \text{Tr}(P_\Psi \mathcal{O}). \quad (2.8)$$

Due to a limited knowledge on a physical system, the fermions can often not be described by a pure state, but merely by an ensemble of states, coming with certain probabilities. More precisely, one considers a family of pure states  $\Psi_1, \dots, \Psi_L \in \mathcal{F}$  together with corresponding probabilities  $p_1, \dots, p_L$ , normalized as follows,

$$\|\Psi_l\|_{\mathcal{F}} = 1 \quad \text{and} \quad 0 \leq p_l \leq 1, \quad \sum_{l=1}^L p_l = 1.$$

The expectation value of an observable  $\mathcal{O}$  is then defined by

$$\langle \mathcal{O} \rangle = \sum_{l=1}^L p_l \langle \Psi_l | \mathcal{O} | \Psi_l \rangle. \quad (2.9)$$

We point out that here we do not take the linear combinations of the states  $\Psi_l$ , but linear combinations of the expectation values  $\langle \Psi_l | \mathcal{O} | \Psi_l \rangle$ . It is convenient to introduce the operator

$$\rho = \sum_{l=1}^L p_l |\Psi_l\rangle\langle\Psi_l|.$$

Then the expectation value can be expressed in analogy to (2.8) by

$$\boxed{\langle \mathcal{O} \rangle = \text{Tr}(\rho \mathcal{O}),} \quad (2.10)$$

whereas the conditions (2.9) become

$$\boxed{\rho \geq 0 \quad \text{and} \quad \text{Tr}(\rho) = 1.} \quad (2.11)$$

The operator  $\rho$  is referred to as the *density operator*. If  $\rho$  is a projector, it follows from (2.11) that  $\rho$  has rank one, and thus it can be written in the form (2.7) with a pure state  $\Psi$ . If  $\rho$  is not a projector, it is said to describe a *mixed state*.

To us, mixed states are important because if the whole system is in a pure state, a subsystem is described by a density operator (see Section 3).

**2.3. Projectors and Hartree-Fock States.** We now want to explore in which sense a projector in the one-particle Hilbert space characterizes a many-particle quantum state. Thus let  $P$  be a projector in the Hilbert space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ , for simplicity of finite rank  $f$ , i.e.

$$P^* = P = P^2 \quad \text{and} \quad \dim P(\mathcal{H}) = f. \quad (2.12)$$

In order to get a connection between this projector and the fermionic Fock space formalism, we choose an orthonormal basis  $\psi_1, \dots, \psi_f$  of  $P(\mathcal{H})$  and form the Hartree-Fock state

$$\Psi := \psi_1 \wedge \dots \wedge \psi_f \in \mathcal{F}_f^{\text{HF}}. \quad (2.13)$$

The choice of our orthonormal basis was unique only up to the unitary transformations

$$\psi_i \rightarrow \tilde{\psi}_i = \sum_{j=1}^f U_{ij} \psi_j \quad \text{with} \quad U \in \text{U}(f). \quad (2.14)$$

As the next simple lemma shows, this transformation changes the corresponding Hartree-Fock state only by a phase factor.

**Lemma 2.2.** *The transformed basis vectors (2.14) satisfy the relation*

$$\tilde{\psi}_1 \wedge \dots \wedge \tilde{\psi}_f = \det U \psi_1 \wedge \dots \wedge \psi_f.$$

*Proof.* We compute the wedge product of the transformed basis vectors with (2.2),

$$\tilde{\psi}_1 \wedge \dots \wedge \tilde{\psi}_f = \sum_{j_1, \dots, j_f=1}^f \frac{1}{f!} \sum_{\sigma \in S_f} (-1)^{\text{sign}(\sigma)} U_{\sigma(1)j_1} \dots U_{\sigma(f)j_f} \psi_{j_1} \otimes \dots \otimes \psi_{j_f}.$$

Due to the anti-symmetrization, we only get a contribution if the indices  $j_1, \dots, j_f$  are all different, and in this case the sum over all permutations gives up to a sign the determinant of  $U$ . More precisely, choosing a permutation  $\tau \in S_f$  with  $\tau(i) = j_i$ , we obtain

$$\tilde{\psi}_1 \wedge \dots \wedge \tilde{\psi}_f = \sum_{\tau \in S_f} \frac{1}{f!} (-1)^{\text{sign}(\tau)} \det(U) \psi_{\tau(1)} \otimes \dots \otimes \psi_{\tau(f)},$$

and comparing with (2.2) gives the result.  $\square$

This lemma shows that by (2.13) we can indeed associate to the projector  $P$  a Hartree-Fock state, which is well-defined up to a phase. In order to avoid carrying along irrelevant phases, it is more convenient to consider instead of the  $f$ -particle state (2.13) the projector  $P_f$  onto the corresponding  $f$ -particle state, i.e. in bra-/ket notation

$$P_f = f! |\psi_1 \wedge \cdots \wedge \psi_f\rangle \langle \psi_1 \wedge \cdots \wedge \psi_f| : \mathcal{F}_f \rightarrow \mathcal{F}_f. \quad (2.15)$$

Here the factor  $f!$  comes about because, according to our conventions (2.2) and (2.1),

$$\begin{aligned} \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \wedge \cdots \wedge \psi_f \rangle &= \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots \otimes \psi_f \rangle \\ &= \frac{1}{f!} \sum_{\sigma \in S_f} (-1)^{\text{sign}(\sigma)} \langle \psi_{\sigma(1)} | \psi_1 \rangle \cdots \langle \psi_{\sigma(f)} | \psi_f \rangle = \frac{1}{f!}. \end{aligned}$$

Since the phase freedom drops out when forming the projector (2.15), this operator is well-defined. The next proposition gives an alternative definition of  $P_f$  which does not involve a choice of basis.

**Proposition 2.3.** *For any projector  $P$  in  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  of rank  $f$ , the corresponding operator*

$$P_f : \mathcal{F}_f \rightarrow \mathcal{F}_f : \psi_1 \wedge \cdots \wedge \psi_f \rightarrow (P\psi_1) \wedge \cdots \wedge (P\psi_f) \quad (2.16)$$

*is a projector onto an  $f$ -particle Hartree-Fock state. The mapping  $P \rightarrow P_f$  gives a one-to-one correspondence between projectors in  $\mathcal{H}$  and projectors on Hartree-Fock states in  $\mathcal{F}$ .*

*Proof.* It follows immediately from the definitions that  $P_f$  is symmetric and idempotent, and is thus a projector. To compute the rank of  $P_f$ , we choose an orthonormal basis  $\psi_1, \dots, \psi_f$  of  $P(\mathcal{H})$  and extend it to an orthonormal basis of  $\mathcal{H}$ . As is obvious from (2.16), the operator  $P_f$  applied to any wedge product of basis vectors vanishes unless all basis vectors are elements of the set  $\{\psi_1, \dots, \psi_f\}$ . Hence the vector  $\Psi := \psi_1 \wedge \cdots \wedge \psi_f$  is a basis of the image of  $P_f$ . We conclude that  $P_f$  has indeed rank one and is thus a projector onto the Hartree-Fock state  $\Psi$ .

Now suppose conversely that  $P_f$  is a projector onto a Hartree-Fock state. Representing this operator in the form (2.15), we let  $P$  be the projector in  $\mathcal{H}$  on the subspace  $\langle \psi_1, \dots, \psi_f \rangle$ . Then the operator  $P_f$  has the representation (2.16), concluding the proof.  $\square$

To summarize, a projector  $P$  in  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  of rank  $f$  uniquely describes an  $f$ -particle quantum state, namely the Hartree-Fock state which is in the image of the corresponding operator  $P_f$ .

**2.4. Projectors and Expectation Values.** We now consider how expectation values of observables can be expressed in terms of the projectors  $P$  and  $P_f$ . We begin with a *one-particle observable*  $\mathcal{O}$ , being a self-adjoint operator on the one-particle Hilbert space  $\mathcal{H}$ . By

$$\begin{aligned} \mathcal{O}^{\mathcal{F}}(\psi_1 \wedge \cdots \wedge \psi_n) &:= (\mathcal{O}\psi_1) \wedge \cdots \wedge \psi_n + \psi_1 \wedge (\mathcal{O}\psi_2) \wedge \cdots \wedge \psi_n \\ &\quad + \cdots + \psi_1 \wedge \cdots \wedge \psi_{n-1} \wedge (\mathcal{O}\psi_n) \end{aligned} \quad (2.17)$$

we can define a corresponding operator  $\mathcal{O}^{\mathcal{F}}$  on the Fock space  $\mathcal{F}$ . This operator preserves the number of particles in the sense that it maps the  $n$ -particle subspace  $\mathcal{F}_n$  to itself.

Suppose that an  $f$ -fermion state is described by a projector  $P$  (2.12). The next lemma shows how the expectation values of  $\mathcal{O}^{\mathcal{F}}$  and of products of one-particle operators can be expressed in terms of traces involving the projector  $P$ .

**Lemma 2.4.** *Suppose that  $\mathcal{O}$  and  $\mathcal{O}_{1/2}$  are one-particle observables. Describing a many-fermion state by a projector  $P$  in  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ , we have*

$$\langle \mathcal{O}^{\mathcal{F}} \rangle = \text{Tr}_{\mathcal{H}}(P\mathcal{O}) \quad (2.18)$$

$$\langle \mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}} \rangle = \text{Tr}_{\mathcal{H}}(P\mathcal{O}_1\mathcal{O}_2) + \text{Tr}_{\mathcal{H}}(P\mathcal{O}_1) \text{Tr}_{\mathcal{H}}(P\mathcal{O}_2) - \text{Tr}_{\mathcal{H}}(P\mathcal{O}_1 P \mathcal{O}_2). \quad (2.19)$$

*Proof.* Following Proposition 2.3 and (2.8), the expectation values are obtained by taking the trace of the observables multiplied by the operator  $P_f$ ,

$$\langle \mathcal{O}^{\mathcal{F}} \rangle = \text{Tr}_{\mathcal{F}_f}(P_f \mathcal{O}^{\mathcal{F}}), \quad \langle \mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}} \rangle = \text{Tr}_{\mathcal{F}_f}(P_f \mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}}).$$

Representing  $P_f$  in the form (2.15), it follows that

$$\begin{aligned} \langle \mathcal{O}^{\mathcal{F}} \rangle &= f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \mathcal{O}^{\mathcal{F}} (\psi_1 \wedge \cdots \wedge \psi_f) \rangle \\ &= f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \mathcal{O}^{\mathcal{F}} (\psi_1 \otimes \cdots \otimes \psi_f) \rangle \\ &= \sum_{\sigma \in S_n} \langle \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(f)} | \mathcal{O}^{\mathcal{F}} (\psi_1 \otimes \cdots \otimes \psi_f) \rangle = \sum_{i=1}^f \langle \psi_i | \mathcal{O} \psi_i \rangle, \end{aligned}$$

where in the last step we applied (2.17) together with (2.1) and used the fact that the vectors  $\psi_1, \dots, \psi_f$  are orthonormal. This proves (2.18).

With the same method, we obtain

$$\begin{aligned} \langle \mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}} \rangle &= f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}} (\psi_1 \otimes \cdots \otimes \psi_f) \rangle \\ &= \sum_{k=1}^f f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots \otimes \psi_{k-1} \otimes (\mathcal{O}_1 \mathcal{O}_2 \psi_k) \otimes \psi_{k+1} \cdots \otimes \psi_f \rangle \\ &\quad + \sum_{k \neq l} f! \langle \psi_1 \wedge \cdots \wedge \psi_f | \psi_1 \otimes \cdots \otimes (\mathcal{O}_1 \psi_k) \cdots \otimes (\mathcal{O}_2 \psi_l) \cdots \otimes \psi_f \rangle \\ &= \sum_{k=1}^f \langle \psi_k | \mathcal{O}_1 \mathcal{O}_2 \psi_k \rangle + \sum_{k \neq l} \left( \langle \psi_k | \mathcal{O}_1 \psi_k \rangle \langle \psi_l | \mathcal{O}_2 \psi_l \rangle - \langle \psi_k | \mathcal{O}_1 \psi_l \rangle \langle \psi_l | \mathcal{O}_2 \psi_k \rangle \right). \end{aligned}$$

The last sum can be extended to all  $k, l = 1, \dots, f$ , because the summands for  $k = l$  vanish. We thus obtain (2.19).  $\square$

The method of this lemma immediately extends to higher powers of one-particle observables.

More generally, one can consider *many-particle observables*, described by a self-adjoint operator  $\mathcal{O}$  on the Fock space  $\mathcal{F}$ . In this paper, we shall only consider observables which *preserve the number of particles*, i.e. which are invariant on the  $n$ -particle subspaces  $\mathcal{F}_n$ ,

$$\mathcal{O} : \mathcal{F}_n \rightarrow \mathcal{F}_n. \quad (2.20)$$

This assumption is justified by the physical law of the conservation of the baryon and lepton numbers, stating that the total number of fermions is preserved. Thus by considering a system which is so large that no fermion enters or leaves it, we can arrange that all physical observables satisfy (2.20).



For a many-particle observable satisfying (2.20), the expectation value is again expressed by a trace,

$$\langle \mathcal{O} \rangle = \text{Tr}_{\mathcal{F}_f}(P_f \mathcal{O}) . \quad (2.21)$$

In order to clarify the connection to the fermionic Fock space formalism, we now introduce the creation and annihilation operators, with the aim of expressing the expectation value (2.21) again by traces involving the one-particle projector  $P$ . For a one-particle wave function  $\phi \in \mathcal{H}$ , the *creation operator*  $a^\dagger(\phi)$  is defined on Hartree-Fock states by

$$a^\dagger(\phi)(\psi_1 \wedge \cdots \wedge \psi_n) = \phi \wedge \psi_1 \wedge \cdots \wedge \psi_n .$$

By linearity, it is extended to an operator  $a^\dagger(\phi) : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}$ . Its adjoint  $a(\phi) : \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n$ , the so-called *annihilation operator*, acts on Hartree-Fock states by

$$a(\phi)(\psi_1 \wedge \cdots \wedge \psi_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} \langle \phi | \psi_k \rangle \psi_1 \wedge \cdots \wedge \psi_{k-1} \wedge \psi_{k+1} \wedge \cdots \wedge \psi_{n+1} .$$

A straightforward calculation shows that the creation and annihilation operators satisfy the anti-commutation relations

$$\{a(\phi), a^\dagger(\psi)\} = \langle \phi | \psi \rangle \mathbf{1}_{\mathcal{F}} . \quad (2.22)$$

A one-particle observable (2.17) can be expressed in terms of the creation and annihilation operators by

$$\mathcal{O}^{\mathcal{F}} = \sum_{k,l} a^\dagger(\phi_k) \langle \phi_k | \mathcal{O} \phi_l \rangle a(\phi_l) ,$$

where  $(\phi_l)$  denotes an orthonormal basis of  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ . Products of one-particle observables can be transformed with the anti-commutation rule (2.22); for example,

$$\mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}} = \sum_{k_1, k_2, l_1, l_2} a^\dagger(\phi_{k_1}) a^\dagger(\phi_{k_2}) \langle \phi_{k_1} | \mathcal{O}_1 \phi_{l_1} \rangle \langle \phi_{k_2} | \mathcal{O}_2 \phi_{l_2} \rangle a(\phi_{l_1}) a(\phi_{l_2}) \quad (2.23)$$

$$+ \sum_{k,l} a^\dagger(\phi_k) \langle \phi_k | \mathcal{O}_1 \mathcal{O}_2 \phi_l \rangle a(\phi_l) , \quad (2.24)$$

where in the last line we used the completeness of the basis  $(\phi_l)$ .

A useful rule in quantum field theory is *Wick ordering*, denoted by colons, which states that all creation operators should be written to the left and all annihilation operators to the right, leaving out all terms which would be generated by the anti-commutations. For example, Wick ordering the above product of one-particle observables amounts to omitting the term (2.24),

$$:\mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}}: = \sum_{k_1, k_2, l_1, l_2} a^\dagger(\phi_{k_1}) a^\dagger(\phi_{k_2}) \langle \phi_{k_1} | \mathcal{O}_1 \phi_{l_1} \rangle \langle \phi_{k_2} | \mathcal{O}_2 \phi_{l_2} \rangle a(\phi_{l_1}) a(\phi_{l_2}) . \quad (2.25)$$

A general Wick-ordered two-particle observable can be written as

$$:\mathcal{O}: = \sum_{k_1, k_2, l_1, l_2} a^\dagger(\phi_{k_1}) a^\dagger(\phi_{k_2}) g(k_1, k_2, l_1, l_2) a(\phi_{l_1}) a(\phi_{l_2}) , \quad (2.26)$$

where the function  $g$  is anti-symmetric in its first and last two arguments,

$$g(k_1, k_2, l_1, l_2) = -g(k_2, k_1, l_1, l_2) = -g(k_1, k_2, l_2, l_1) . \quad (2.27)$$

In the next proposition we express the expectation value of this two-particle observable in terms of the one-particle projector  $P$ .

**Proposition 2.5.** *Describing a many-fermion state by a projector  $P$  in  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ , the expectation value of the two-particle operator (2.26) with  $g$  according to (2.27) is given by*

$$\langle : \mathcal{O} : \rangle = \sum_{k \neq l} \left( \langle \phi_k | P \phi_k \rangle \langle \phi_l | P \phi_l \rangle - \langle \phi_k | P \phi_l \rangle \langle \phi_l | P \phi_k \rangle \right) g(k, l, k, l).$$

The expectation value of the Wick ordered product (2.25) is

$$\langle : \mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}} : \rangle = \text{Tr}_{\mathcal{H}}(P \mathcal{O}_1) \text{Tr}_{\mathcal{H}}(P \mathcal{O}_2) - \text{Tr}_{\mathcal{H}}(P \mathcal{O}_1 P \mathcal{O}_2). \quad (2.28)$$

*Proof.* We again represent the projector  $P_f$  in the form (2.15). Using the obvious transformation law of the function  $g$  under basis transformations, we can arrange that  $\phi_1 = \psi_1, \dots, \phi_f = \psi_f$ . Then the results follows from a straightforward calculation.  $\square$

We point out that the formula (2.28) coincides with (2.19), except that the summand  $\text{Tr}_{\mathcal{H}}(P \mathcal{O}_1 \mathcal{O}_2)$  is now missing as a consequence of the Wick ordering.

With the above methods, one can express the expectation values of any many-particle operator satisfying (2.20) in terms of the one-particle projector  $P$ .

### 3. RESTRICTION TO A SUBSYSTEM

According to Proposition 2.3, there is a one-to-one correspondence between projectors in  $\mathcal{H}$  and Hartree-Fock states. Furthermore, we saw in Example 2.1 that entangled states can in general *not* be represented by Hartree-Fock states. Thus the description of a fermionic state using a projector  $P$  in  $\mathcal{H}$  seems to be in conflict with the physical effect of entanglement. In this chapter, we will try to resolve this problem using the fact that in all realistic situations, the relevant measurements are performed only in a small subsystem of the universe. Thus the question is whether restricting attention to a subsystem makes it possible to describe entanglement.

The observation that the restriction to a subsystem gives more freedom to describe the effective fermionic state was already used in [6, Appendix A] to show that if only one-particle measurements are made, describing the system by a one-particle projector  $P$  is equivalent to the general Fock space formalism (this result corresponds to Proposition 3.5 below). Here we generalize the methods to many-particle observables and show that the restriction to a subsystem does *not* make it possible to describe general entangled states (see Example 3.6).

In order to describe the subsystem, we assume that the one-particle observables act only on a proper subspace  $\mathfrak{J} \subset \mathcal{H}$  (the “inner system”). Extending such an observable  $\mathcal{O}$  by zero to all of  $H$ , we obtain a self-adjoint operator which vanishes on the orthogonal complement of  $\mathfrak{J}$ ,

$$\mathcal{O} : \mathcal{H} \rightarrow \mathcal{H} \text{ self-adjoint} \quad \text{and} \quad \mathcal{O}|_{\mathfrak{J}^\perp} \equiv 0. \quad (3.1)$$

Extending this operator by (2.17) to the Fock space, we obtain an operator which preserves the number of particles and vanishes on the orthogonal complement of the space generated by  $\mathfrak{J}$ ,

$$\mathcal{O} : \mathcal{F}_n \rightarrow \mathcal{F}_n \text{ self-adjoint} \quad \text{and} \quad \mathcal{O}|_{\langle \Lambda_n(\mathfrak{J}^n) \rangle^\perp} \equiv 0 \quad (3.2)$$

(for notational convenience the superscript of  $\mathcal{O}^{\mathcal{F}}$  has been omitted). In order to describe many-particle observables, we allow  $\mathcal{O}$  to be a general operator on  $\mathcal{F}$  which satisfies (3.2) and thus leaves the number of particles  $n$  fixed.

We again denote the projector onto the corresponding  $f$ -particle Hartree-Fock state  $\Psi$  by  $P_f$  (see (2.15), (2.13) and Proposition 2.3),

$$P_f = f! |\Psi\rangle\langle\Psi| \quad \text{where} \quad \Psi = \psi_1 \wedge \cdots \wedge \psi_f. \quad (3.3)$$

Setting  $O = \mathfrak{J}^\perp$  (the ‘‘outer system’’), we decompose all one-particle states  $\psi_i$  into their inner and outer parts,

$$\psi_i = \psi_i^{\mathfrak{J}} + \psi_i^O \quad \text{with} \quad \psi_i^{\mathfrak{J}} \in \mathfrak{J}, \psi_i^O \in O. \quad (3.4)$$

Substituting this decomposition into (3.3) and multiplying out, one gets a sum of terms involving wedge products of the  $\Psi_i^{\mathfrak{J}}$  and  $\Psi_i^O$ . In order to keep track of the combinatorics, it is convenient to denote by  $I$  a multi-index

$$I = (i_1, \dots, i_g) \quad \text{with} \quad 1 \leq i_1 < i_2 < \cdots < i_g \leq f,$$

to set  $|I| := g$ , and to define the sign of  $I$  as the sign of a permutation in  $\{1, \dots, f\}$  which maps  $I$  to the set  $\{1, \dots, g\}$ , i.e.

$$\text{sign}(I) = (-1)^{i_1 + \cdots + i_g + \frac{g(g+1)}{2}}.$$

Furthermore, we denote the complement of  $I$  by  $O$ , i.e.

$$O = (j_1, \dots, i_h) \quad \text{with} \quad 1 \leq j_1 < \cdots < j_h \leq f, \quad g + h = f, \quad i_k \neq j_l \quad \forall k, l.$$

Finally, we set

$$\Psi^I = \psi_{i_1}^{\mathfrak{J}} \wedge \cdots \wedge \psi_{i_g}^{\mathfrak{J}} \quad \text{and} \quad \Psi^O = \psi_{j_1}^O \wedge \cdots \wedge \psi_{j_h}^O.$$

Using this notation, we obtain

$$\Psi = \sum_I \text{sign}(I) \Psi^I \wedge \Psi^O,$$

and thus we can write (3.3) as follows,

$$P_f = \sum_{I, I'} f! \text{sign}(I) \text{sign}(I') |\Psi^I \wedge \Psi^O\rangle\langle\Psi^{I'} \wedge \Psi^{O'}|. \quad (3.5)$$

When computing expectation values of operators localized in our subsystem, we can take the partial trace over  $O$  to obtain an equivalent description of our quantum system by a density operator defined in the subsystem. This is made precise in this next lemma.

**Lemma 3.1.** *For any operator  $\mathcal{O}$  on  $\mathcal{F}$  of the form (3.2), the expectation value (2.21) can be expressed by  $\langle\mathcal{O}\rangle = \text{Tr}(\rho \mathcal{O})$ , where  $\rho$  is the density operator*

$$\rho = \sum_{g=0}^f \sum_{I, I' \text{ with } |I|=|I'|=g} g!(f-g)! \text{sign}(I) \text{sign}(I') \langle\Psi^{O'}|\Psi^O\rangle |\Psi^I\rangle\langle\Psi^{I'}|. \quad (3.6)$$

*Proof.* Applying (3.2), we find

$$\begin{aligned} \langle\Psi^{I'} \wedge \Psi^{O'} | \mathcal{O} (\Psi^I \wedge \Psi^O)\rangle &= \langle\Psi^{I'} \wedge \Psi^{O'} | (\mathcal{O}\Psi^I) \wedge \Psi^O\rangle \\ &= \langle\Psi^{I'} \otimes \Psi^{O'} | (\mathcal{O}\Psi^I) \wedge \Psi^O\rangle = \frac{g!(f-g)!}{f!} \langle\Psi^{O'} | \Psi^O\rangle \langle\Psi^{I'} | \mathcal{O}\Psi^I\rangle \\ &= \frac{g!(f-g)!}{f!} \langle\Psi^{O'} | \Psi^O\rangle \text{Tr}(|\Psi^I\rangle\langle\Psi^{I'}| \mathcal{O}). \end{aligned}$$

Using this relation in (3.5) gives the result.  $\square$

We point out that the density operator (3.6) involves states of a variable number of particles  $g = 0, \dots, f$ . The coefficients of the representation (3.6) depend on the inner products  $\langle \Psi^{O'} | \Psi^O \rangle$  of the wave functions in the outer region. Since in the outer region no measurements are possible, we cannot determine the wave functions  $\Psi_i^O$ . Thus we can take the point of view that these wave functions can be chosen arbitrarily. At first sight, this seems to give a lot of freedom to choose the coefficients  $\langle \Psi^{O'} | \Psi^O \rangle$  in (3.6), and thus one might conjecture that with (3.6) it should be possible to describe a general entangled state. We will now show that this conjecture is wrong, basically because many degrees of freedom in choosing the wave functions  $\Psi_i^O$  drop out when carrying out the sums in (3.6). Our method is based on a generalization of Cramer's rule. Since this method will not be needed later on, the hurried reader may skip the remaining proofs in this chapter.

In order to bring the inner products  $\langle \Psi^O | \Psi^{O'} \rangle$  into a more convenient form, we introduce the  $(f \times f)$ -matrix  $A$  by

$$A_{ij} := \langle \psi_i^O | \psi_j^O \rangle; \quad (3.7)$$

it is a positive semi-definite matrix which can be considered as the *Gram matrix* of the one-particle wave functions in the outer system. The matrix  $A$  might be singular. Therefore, it is preferable to add a small multiple of the identity matrix to obtain an invertible matrix. Thus for any  $\varepsilon > 0$  we set

$$A^\varepsilon = A + \varepsilon \mathbf{1} \quad \text{and} \quad B^\varepsilon = (A^\varepsilon)^{-1}. \quad (3.8)$$

By  $B_{I',I}^\varepsilon$  we denote the  $(g \times g)$ -matrix obtained from  $B^\varepsilon$  by deleting all rows and columns except for those corresponding to  $I$  and  $I'$ , respectively.

**Theorem 3.2.** *The expectation value (2.21) can be expressed by*

$$\langle \mathcal{O} \rangle = \text{Tr}(\rho \mathcal{O}),$$

where  $\rho$  is the density operator

$$\rho = \sum_{g=0}^f \sum_{I, I' \text{ with } |I|=|I'|=g} g! \lim_{\varepsilon \searrow 0} \left( \det(A^\varepsilon) \det(B_{I',I}^\varepsilon) \right) |\Psi^I\rangle \langle \Psi^{I'}|, \quad (3.9)$$

and the matrices  $A^\varepsilon$ ,  $B^\varepsilon$  are defined by (3.7) and (3.8).

*Proof.* Setting  $h = f - g$ , the inner product in (3.6) is computed by

$$\langle \Psi^{O'} | \Psi^O \rangle = \frac{1}{h!} \sum_{\sigma \in S_h} \langle \psi_{j'_{\sigma(1)}}^O \otimes \dots \otimes \psi_{j'_{\sigma(h)}}^O | \psi_{j_1}^O \otimes \dots \otimes \psi_{j_h}^O \rangle = \frac{1}{h!} \det(A_{O',O}), \quad (3.10)$$

where in the last step we used (2.1) and the definition of the determinant. Since the determinant is polynomial and thus continuous, it is obvious that

$$\det(A_{O',O}) = \lim_{\varepsilon \searrow 0} \det(A_{O',O}^\varepsilon).$$

Using this relation in (3.10) and (3.6), we conclude that it remains to prove for any invertible matrix  $A$  the identity

$$\det(A_{O',O}) = \text{sign}(I') \text{sign}(I) \det(A) \det(B_{I',I}), \quad (3.11)$$

which relates the minors of  $A$  to the minors of its inverse. For proving (3.11), we first note that in the special case where  $I$  and  $I'$  consist of only one index, this is the well-known Cramer's rule for the inverse of a matrix. In the general case, the

identity (3.11) is stated in [13, Section 0.8.4]. It can be proved as follows. The signs in (3.11) can be understood from the fact that if rows or columns of the matrices  $A$  and  $B$  are permuted without violating the ordering of the multi-indices  $I, I', O$  and  $O'$ , then every such conjugation flips the sign of  $\det(A)$ , and also the sign of one of the functions  $\text{sign}(I)$  or  $\text{sign}(I')$ . With such conjugations we can arrange that

$$I = I' = (1, \dots, g) \quad \text{and} \quad O = O' = (g + 1, \dots, f) \quad (3.12)$$

(but of course, the matrices  $A$  and  $B$  will no longer be Hermitian). It remains to show that in this case,

$$\det(A_{O',O}) = \det(A) \det(B_{I',I}).$$

Using Laplace's formula, in case (3.12) the minor  $\det(A_{O',O})$  can be written as a multiple derivative of  $\det(A)$ ,

$$\det(A_{O',O}) = \frac{\partial}{\partial A_{11}} \cdots \frac{\partial}{\partial A_{gg}} \det A. \quad (3.13)$$

Using the standard formulas

$$\begin{aligned} \frac{\partial}{\partial A_{ii}} \det(A) &= \det(A) B_{ii} \\ \frac{\partial}{\partial A_{ii}} B_{jk} &= \frac{\partial}{\partial A_{ii}} (A^{-1})_{jk} = - \left( B \left( \frac{\partial}{\partial A_{ii}} A \right) B \right)_{jk} = -B_{ji} B_{ik}, \end{aligned}$$

one can iteratively carry out the derivatives in (3.13) to obtain

$$\det(A_{O',O}) = \det(A) \times \left( \text{homogeneous polynomial of degree } g \text{ in } B_{jk} \text{ with } j, k \in I \right).$$

Going through the combinatorial details, one finds that this homogeneous polynomial coincides precisely with  $\det(B_{I',I})$ .  $\square$

We now illustrate this theorem by a few examples and work out simple consequences. We first explain how to recover a Hartree-Fock state.

**Example 3.3.** (*Hartree-Fock state*) Choosing  $\Psi^{\mathcal{J}} = \Psi$  and  $\Psi^O = 0$ , the Gram matrix (3.7) vanishes and thus

$$A^\varepsilon = \varepsilon \quad \text{and} \quad B^\varepsilon = \varepsilon^{-1}.$$

Hence

$$\det(A^\varepsilon) \det(B_{I',I}^\varepsilon) = \varepsilon^{f-|I|} \delta_{I,I'},$$

and this vanishes in the limit  $\varepsilon \searrow 0$  unless  $I = (1, \dots, f)$ . Hence (3.9) reduces to

$$\rho = f! |\Psi\rangle \langle \Psi|,$$

so that the density operator coincides with the projector  $P_f$  (3.3) of the whole system.  $\diamond$

**Example 3.4.** (*A mixed state*) We choose  $f = 3$ ,  $\psi_1^O = 0$ ,  $\|\psi_2^O\| = \|\psi_3^O\| = \kappa \in [0, 1]$  and  $\langle \psi_2^O | \psi_3^O \rangle = 0$ . Thus

$$A^\varepsilon = \text{diag}(\varepsilon, \kappa^2 + \varepsilon, \kappa^2 + \varepsilon),$$

and a short calculation yields that

$$\begin{aligned} \rho &= \kappa^4 |\psi_1^{\mathcal{J}}\rangle \langle \psi_1^{\mathcal{J}}| + 2\kappa^2 \left( |\psi_1^{\mathcal{J}} \wedge \psi_2^{\mathcal{J}}\rangle \langle \psi_1^{\mathcal{J}} \wedge \psi_2^{\mathcal{J}}| + |\psi_1^{\mathcal{J}} \wedge \psi_3^{\mathcal{J}}\rangle \langle \psi_1^{\mathcal{J}} \wedge \psi_3^{\mathcal{J}}| \right) \\ &\quad + 3! |\psi_1^{\mathcal{J}} \wedge \psi_2^{\mathcal{J}} \wedge \psi_3^{\mathcal{J}}\rangle \langle \psi_1^{\mathcal{J}} \wedge \psi_2^{\mathcal{J}} \wedge \psi_3^{\mathcal{J}}|. \end{aligned}$$

Thus the density operator of the subsystem is an ensemble of one, two and three particle states. The one-particle state  $\psi_1^{\mathfrak{J}}$  is always occupied; this is because the corresponding eigenvalue of the matrix  $A$  vanishes. The two-particle component cannot be represented by a pure state; it is a mixed state.  $\diamond$

We next show that when restricting attention to one-particle observables, the expectation value of any vector in the fermionic Fock space can be approximated to arbitrary precision by a one-particle projector  $P$ .

**Proposition 3.5.** *For every normalized  $n$ -particle state  $\Psi$  in our subsystem, i.e.*

$$\Psi \in \overline{\langle \Lambda_n(\mathfrak{J}^n) \rangle} \subset \mathcal{F}_n \quad \text{and} \quad \|\Psi\| = 1,$$

*there is a sequence of projectors  $(P_k)_{k \in \mathbb{N}}$  in  $\mathcal{H}$  of finite rank  $f_k$ , such that for every one-particle observable  $\mathcal{O}$ ,*

$$\langle \Psi | \mathcal{O} \Psi \rangle = \lim_{k \rightarrow \infty} \text{Tr}(P_k \mathcal{O}).$$

*Proof.* Writing  $\Psi$  as a linear combination of Hartree-Fock states and computing the expectation value with (2.17) and (2.1), we obtain

$$\langle \Psi | \mathcal{O} \Psi \rangle = \sum_{k,l} c_{k,l} \langle \psi_k^{\mathfrak{J}} | \mathcal{O} \psi_l^{\mathfrak{J}} \rangle$$

with suitable vectors  $\psi_k^{\mathfrak{J}} \in \mathfrak{J}$  and complex coefficients  $c_{k,l}$ . Using an approximation argument, it clearly suffices to consider finite sums and finite linear combinations. Choosing an orthonormal basis  $\psi_k$  of the subspace  $\langle \psi_k^{\mathfrak{J}} \rangle \subset \mathfrak{J}$ , and expressing the  $\psi_k^{\mathfrak{J}}$  as linear combinations of the  $\psi_k$ , we obtain

$$\langle \Psi | \mathcal{O} \Psi \rangle = \sum_{k,l} \rho_{k,l} \langle \psi_k | \mathcal{O} \psi_l \rangle.$$

Diagonalizing the symmetric matrix  $\rho_{k,l}$  by a unitary transformation, this representation simplifies to

$$\langle \Psi | \mathcal{O} \Psi \rangle = \sum_k \rho_k \langle \psi_k | \mathcal{O} \psi_k \rangle. \quad (3.14)$$

Since the operator  $|\Psi\rangle\langle\Psi|$  is positive and normalized, it follows that

$$\rho_k \geq 0 \quad \text{and} \quad \sum_k \rho_k = 1.$$

We choose an orthonormal family of vectors  $(\phi_k)$  in  $\mathcal{O}$  and set

$$\psi_k^{\text{tot}} = \sqrt{\rho_k} \psi_k + \sqrt{1 - \rho_k} \phi_k.$$

Then the family  $(\psi_k^{\text{tot}})$  is orthonormal, and thus

$$P = \sum_k |\psi_k^{\text{tot}}\rangle\langle\psi_k^{\text{tot}}|$$

defines a projector in  $\mathcal{H}$ . Using (3.1), it is immediately verified that the expectation value  $\text{Tr}(P\mathcal{O})$  coincides with the right side of (3.14).  $\square$

The next example shows that the restriction to a subsystem does *not* make it possible to describe general entangled states.

**Example 3.6.** (*The spatially separated singlet state*) Let us try to realize the spatially separated singlet state of Example 2.1. We assume that the inner system  $\mathfrak{J} \subset \mathcal{H}$  has the orthonormal basis  $(\psi_A^\uparrow, \psi_B^\downarrow, \psi_A^\downarrow, \psi_B^\uparrow)$ . Our goal is to find a projector  $P$  in  $\mathcal{H}$  such that the corresponding density operator of the subsystem (3.9) coincides with the projector onto the singlet state (2.3), i.e.

$$\begin{aligned} \rho = & |\psi_A^\uparrow \wedge \psi_B^\downarrow\rangle \langle \psi_A^\uparrow \wedge \psi_B^\downarrow| + |\psi_A^\downarrow \wedge \psi_B^\uparrow\rangle \langle \psi_A^\downarrow \wedge \psi_B^\uparrow| \\ & - |\psi_A^\uparrow \wedge \psi_B^\downarrow\rangle \langle \psi_A^\downarrow \wedge \psi_B^\uparrow| - |\psi_A^\downarrow \wedge \psi_B^\uparrow\rangle \langle \psi_A^\uparrow \wedge \psi_B^\downarrow|. \end{aligned} \quad (3.15)$$

The simplest way to see that such a projector  $P$  does not exist is to observe that in (3.9) one necessarily gets the contribution involving four particles

$$|\psi_A^\uparrow \wedge \psi_B^\downarrow \wedge \psi_A^\downarrow \wedge \psi_B^\uparrow\rangle \langle \psi_A^\uparrow \wedge \psi_B^\downarrow \wedge \psi_A^\downarrow \wedge \psi_B^\uparrow|,$$

which is not present in (3.15). However, this argument is not fully convincing, because an additional four-particle contribution would not be observable in the standard spin correlation experiments where ‘‘Alice’’ and ‘‘Bob’’ can only detect one particle at a time. Furthermore, by making the matrix elements of  $B^\varepsilon$  small, one could try to arrange that the four-particle contribution is negligible. For these reasons, it is preferable to show that we cannot even realize that the two-particle component of (3.9) coincides with (3.15), as we now explain.

Restricting attention to the two-particle component of (3.9), we need to consider the determinants of the 2-submatrices of the matrix  $B^\varepsilon$ . Noting that in the case  $|I| = |I'| = 2$ , we can write the product of the determinants in (3.9) as

$$\det(A^\varepsilon) \det(B_{I',I}^\varepsilon) = \det\left(\sqrt{\det(A^\varepsilon)} B_{I',I}^\varepsilon\right),$$

it suffices to consider the 2-minors of the matrix

$$B := \lim_{\varepsilon \searrow 0} \sqrt{\det(A^\varepsilon)} B^\varepsilon.$$

Grouping the first two and the last two basis vectors together, we write  $B$  as the block matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

whose entries are  $2 \times 2$  matrices acting on the subspaces  $\langle \{\psi_A^\uparrow, \psi_B^\downarrow\} \rangle$  and  $\langle \psi_A^\downarrow, \psi_B^\uparrow \rangle$ , respectively. In order to realize (3.15), we need to arrange that the determinants of the submatrices  $B_{11}$  and  $B_{22}$  are non-zero, but all other 2-minors must vanish. Diagonalizing  $B_{11}$  and  $B_{22}$  by unitary transformations in the subspaces  $\langle \{\psi_A^\uparrow, \psi_B^\downarrow\} \rangle$  and  $\langle \{\psi_A^\downarrow, \psi_B^\uparrow\} \rangle$ , respectively, we obtain

$$B = \begin{pmatrix} \rho_1 & 0 & \bar{a} & \bar{c} \\ 0 & \rho_2 & \bar{b} & \bar{d} \\ a & b & \rho_3 & 0 \\ c & d & 0 & \rho_4 \end{pmatrix} \quad \text{with} \quad \rho_i > 0.$$

Evaluating the condition  $\det B_{I,I'} = 0$  for  $I = (1,3)$  and  $I' = (1,2)$ , we see that  $b$  must vanish. Similarly, taking  $I = (1,4)$  and  $I' = (1,2)$  yields  $d = 0$ . Repeating this procedure for  $I = (2,3)$  and  $I' = (2,4)$  we find that  $a = c = 0$ . Hence  $B$  is a positive diagonal matrix. But then the submatrix  $B_{I,I}$  with  $I = (1,3)$  has a non-zero determinant, a contradiction.  $\diamond$

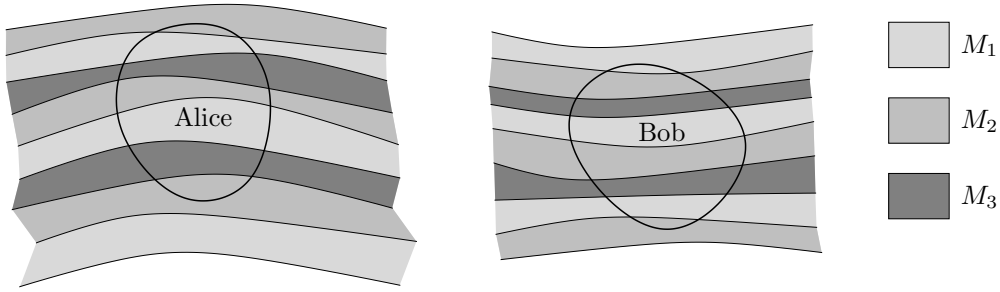


FIGURE 1. Example of microscopic mixing in a spin correlation experiment.

**Remark 3.7.** (*Systems with purely classical bosonic fields*) Let us consider a system where second-quantized fermions are coupled to classical bosonic fields, in the pure sense that also all measurement devices can be described by the classical field equations. As pointed out in [6, Appendix A], the coupling of the fermions to classical fields can be described by one-particle observables (like the expectation value of the Dirac current in Maxwell’s equations or the expectation value of the energy-momentum tensor in Einstein’s equations). In view of Proposition 3.5, the resulting system can be described by a one-particle projector  $P$ . On the other hand, we saw in Example 3.6 that this framework does not allow for the description of general entangled states. We conclude that the physical phenomenon of entanglement makes it necessary to consider second quantized bosonic fields.

This consideration suggests that there is a close connection between entanglement and second quantization of the bosonic fields. This connection will become clearer in the following chapters, because the method of microscopic mixing, which will allow for the description of entanglement (see Chapters 4 and 6), will also make it possible to describe second quantized bosonic fields (see Chapter 7).  $\diamond$

#### 4. MICROSCOPIC MIXING OF SUBSYSTEMS

As we saw in Example 3.6, the restriction to a subsystem does not resolve the basic problem that general entangled states cannot be realized by a projector  $P$  in the one-particle Hilbert space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ . Our next idea for overcoming this problem is to give  $P$  a *non-trivial microscopic structure*, with the hope that “averaging” this microstructure over macroscopic regions of space-time will give rise to an effective kernel  $P(x, y)$  of a more general form which allows for the description of entanglement. More specifically, we consider the situation where space is subdivided into sets  $M_1, \dots, M_L$ , which are fine-grained in the sense that every macroscopic region of space-time intersects several of the sets  $M_a$ . The sets  $M_a$  can be localized, but they can also be extended over a macroscopic region of space-time, for example by forming “layers” or “filaments” connecting the two observers in the spin correlation experiment of Example 2.1 (see Figure 1). The macroscopic physical objects are then introduced by homogenizing over the sets  $M_a$ . We refer to this technique as the *method of microscopic mixing*.

We now work out this method in the non-relativistic setting (Section 4.1). Assuming furthermore that the subsystems have an independent dynamics, we can give a formalism for describing entangled states (Section 4.2). The microscopic justification of this formalism is postponed to Chapters 6 and 8.



**4.1. Microscopic Mixing of the Wave Functions.** We decompose  $M$  into a disjoint union of a finite number  $L$  of subsets  $M_a$ ,

$$M = M_1 \cup \dots \cup M_L \quad \text{and} \quad M_a \cap M_b = \emptyset \quad \text{if } a \neq b. \quad (4.1)$$

Then  $\mathcal{H}$  splits into an orthogonal direct sum of the Hilbert spaces  $\mathcal{H}_a$  of square integrable wave functions on  $M_a$ ,

$$\mathcal{H} = \bigoplus_{a=1}^L \mathcal{H}_a. \quad (4.2)$$

Thus every wave function  $\psi_i$  in the image of  $P$ , (2.13), has the unique decomposition

$$\psi_i = \sum_{a=1}^L \psi_i^{(a)} \quad \text{with} \quad \psi_i^{(a)} \in \mathcal{H}_a. \quad (4.3)$$

Our first attempt is to generalize a macroscopic local one-particle observable  $\mathcal{O}$  (like a position or spin operator) to an operator on  $\mathcal{H}$  being invariant on  $\mathcal{H}_a$ ,

$$\mathcal{O} : \mathcal{H}_a \rightarrow \mathcal{H}_a. \quad (4.4)$$

Then the corresponding one-particle expectation values split into a sum over the subsystems,

$$\langle \psi | \mathcal{O} \psi \rangle = \sum_{a=1}^L \langle \psi^{(a)} | \mathcal{O} \psi^{(a)} \rangle = \sum_{a=1}^L \int_{M_a} \bar{\psi}(x) \mathcal{O} \psi(x) dx,$$

and this can be understood as an ‘‘averaging process’’ over the subregions  $M_a$ .

Following the constructions in Section 2.4, every one-particle operator induces a corresponding operator on the Fock space  $\mathcal{F}$ , and products of such operators yield corresponding many-particle observables. Taking expectation values of such operators in the Fock space again involves an ‘‘averaging process’’ over the subregions  $M_a$ . More specifically, describing the many-particle system by a fermionic projector  $P$ , the expectation value of the Wick-ordered two-particle observables corresponding to ‘‘Alice’’ and ‘‘Bob’’ (see Proposition 2.5) is given by

$$\begin{aligned} \langle : \mathcal{O}_A^{\mathcal{F}} \mathcal{O}_B^{\mathcal{F}} : \rangle &= \sum_{i,j=1}^f \sum_{a,b=1}^L \left( \langle \psi_i^{(a)} | \mathcal{O}_A \psi_i^{(a)} \rangle \langle \psi_j^{(b)} | \mathcal{O}_B \psi_j^{(b)} \rangle \right. \\ &\quad \left. - \langle \psi_i^{(a)} | \mathcal{O}_A \psi_j^{(a)} \rangle \langle \psi_j^{(b)} | \mathcal{O}_B \psi_i^{(b)} \rangle \right). \end{aligned} \quad (4.5)$$

Thus an ‘‘averaging’’ takes place at each of the observers. However, it is important to note that there is no averaging process over *correlations* between the two observers, as would be the case for an expression like

$$\sum_{i,j=1}^f \sum_{a=1}^L \langle \psi_i^{(a)} | \mathcal{O}_A \psi_i^{(a)} \rangle \langle \psi_j^{(a)} | \mathcal{O}_B \psi_j^{(a)} \rangle.$$

This shortcoming is the basic reason why the above method does *not* make it possible to realize general entangled states. Before discussing this point in more detail, we first prove that it is indeed impossible to realize the EPR-singlet state.

**Proposition 4.1.** *In the above setting, there is no projector  $P$  in  $\mathcal{H}$  which reproduces the expectation values of the spatially separated singlet state (2.3) with respect to the spin operators of ‘‘Alice’’ and ‘‘Bob’’ and the corresponding two-particle spin correlation operators.*

*Proof.* We let  $S_\uparrow$  be the spin operator having the expectation values one if the spin is up and zero if the spin is down. Similarly,  $S_\downarrow$  is the operator for spin down. The spin operators corresponding to the observers ‘‘Alice’’ and ‘‘Bob’’ are denoted by  $S_{\uparrow/\downarrow,A}$  and  $S_{\uparrow/\downarrow,B}$ , respectively; they are operators in  $\mathcal{H}$ . Taking for convenience the Wick ordered products of the corresponding operators on the Fock as defined by (2.25), we can compute the following expectation values of the singlet state (2.3),

$$\begin{aligned} \langle S_{\uparrow,A} \rangle &= \langle S_{\downarrow,B} \rangle = \frac{1}{2} \\ \langle :S_{\uparrow,A}^{\mathcal{F}} S_{\downarrow,B}^{\mathcal{F}}: \rangle &= \frac{1}{2}, \quad \langle :S_{\uparrow,A}^{\mathcal{F}} S_{\uparrow,A}^{\mathcal{F}}: \rangle = \langle :S_{\downarrow,B}^{\mathcal{F}} S_{\downarrow,B}^{\mathcal{F}}: \rangle = 0 \end{aligned} \quad (4.6)$$

(all the other expectation values are irrelevant for the proof). Assuming that the statement of the proposition is false, these expectation values can be reproduced by a suitable one-particle projector  $P$ . Applying Lemma 2.4 and Proposition 2.5, we can then express the expectation values by traces over the one-particle Hilbert space  $\mathcal{H}$ ,

$$\langle \mathcal{O}^{\mathcal{F}} \rangle = \text{Tr}(P\mathcal{O}) \quad \text{and} \quad \langle :\mathcal{O}_1^{\mathcal{F}} \mathcal{O}_2^{\mathcal{F}}: \rangle = \text{Tr}(P\mathcal{O}_1) \text{Tr}(P\mathcal{O}_2) - \text{Tr}(P\mathcal{O}_1 P\mathcal{O}_2).$$

Using the idempotence of  $P$ , the arguments of these traces can all be rewritten purely in terms of the operators

$$T_{\uparrow,A} := PS_{\uparrow,A}P \quad \text{and} \quad T_{\downarrow,B} := PS_{\downarrow,B}P, \quad (4.7)$$

which are symmetric and of finite rank. The relations (4.6) give rise to the conditions

$$\text{Tr}(T_{\uparrow,A}) = \text{Tr}(T_{\downarrow,B}) = \frac{1}{2}, \quad \text{Tr}(T_{\uparrow,A}^2) = \text{Tr}(T_{\downarrow,B}^2) = \frac{1}{4}, \quad \text{Tr}(T_{\uparrow,A}T_{\downarrow,B}) = -\frac{1}{4}. \quad (4.8)$$

At this point it is helpful to regard the operators  $T_{\dots}$  as vectors in the real Hilbert space of symmetric Hilbert-Schmidt operators with the scalar product  $\langle A, B \rangle_{HS} = \text{Tr}(AB)$  and corresponding norm  $\|A\|_{HS} = \sqrt{\langle A, A \rangle_{HS}}$ . Then the last two equations in (4.8) imply that

$$\|T_{\uparrow,A} + T_{\downarrow,B}\|_{HS}^2 = \text{Tr}(T_{\uparrow,A}^2) + 2 \text{Tr}(T_{\uparrow,A}T_{\downarrow,B}) + \text{Tr}(T_{\downarrow,B}^2) = 0. \quad (4.9)$$

It follows that  $T_{\uparrow,A} = -T_{\downarrow,B}$ , in contradiction to the first equation in (4.8).  $\square$

We point out that in this proof we did not use that the one-particle operators are invariant on the subsystems (4.4). Thus dropping this assumption would not change the statement of Proposition 4.1.

**4.2. A Formalism for the Description of Entanglement.** The basic shortcoming in the previous consideration was that we did not take into account that the measurement process is a result of an interaction of the system with the measurement device. Assuming that the subsystems have an *independent dynamics* (an assumption which will be justified in Section 6.2 below), also the measurement process should take place independently in the subsystems. Following this idea makes it possible to describe entanglement, as we now explain.

Beginning with the one-particle observables, the assumption of an independent dynamics of the subsystems was already taken into account in (4.4) by the assumption that  $\mathcal{O}$  should be invariant on the subspaces  $\mathcal{H}_a$ . However, for a many-particle observable, it was too simple to take the Wick-ordered product (4.5). Thinking of the situation in Figure 1, ‘‘Alice’’ is built up of fermionic wave functions. Thus considering her as part of the physical system, we should also replace the corresponding measurement operator  $\mathcal{O}_A$  by separate operators  $\mathcal{O}_A^{(a)}$  for each of the subsystems. Proceeding

similarly for “Bob”, in the subsystem  $M_a$  measurements are to be carried out with the operators  $\mathcal{O}_A^{(a)}$  and  $\mathcal{O}_B^{(a)}$ . Thus for correlation measurement in  $M_a$ , we should extend the one-particle observables  $\mathcal{O}^{(a)}$  to operators  $\mathcal{O}^{\mathcal{F}^{(a)}}$  defined on the Fock space  $\mathcal{F}_n^{(a)}$  of the subsystem given by

$$\mathcal{F}_n^{(a)} = \overline{\langle \Lambda_n(\mathcal{H}_n^{(a)}) \rangle} \subset \mathcal{F}_n \quad (4.10)$$

(as explained after (2.20), we again restrict attention to observables which preserve the number of particles) and consider the corresponding two-particle observable

$$:\mathcal{O}_A^{\mathcal{F}^{(a)}} \mathcal{O}_B^{\mathcal{F}^{(a)}}: : \mathcal{F}_n^{(a)} \rightarrow \mathcal{F}_n^{(a)} .$$

More generally, we make the following assumption:

(A) The observables correspond to operators  $\mathcal{O}$  which are invariant on  $\mathcal{F}_n^{(a)}$ ,

$$\mathcal{O} : \mathcal{F}_n^{(a)} \rightarrow \mathcal{F}_n^{(a)} , \quad a = 1, \dots, L .$$

Similar as explained in Section 2.3 for the Fock space  $\mathcal{F}_f$ , we can get a simple connection between the fermionic projector and the Fock spaces  $\mathcal{F}_f^{(a)}$ . Namely, choosing again an orthonormal basis  $\psi_1, \dots, \psi_f$  of  $P(\mathcal{H})$  and decomposing each of the one-particle wave functions according to (4.3), we can construct Hartree-Fock states  $\Psi^{(a)}$  in the  $f$ -particle Fock spaces of the subsystems,

$$\Psi^{(a)} := \psi_1^{(a)} \wedge \dots \wedge \psi_f^{(a)} \in \mathcal{F}_f^{(a)} . \quad (4.11)$$

Exactly as in Lemma 2.2, one sees that these vectors are unique up to a common phase,

$$\Psi^{(a)} \rightarrow e^{i\varphi} \Psi^{(a)} \quad \text{with} \quad \varphi \in \mathbb{R} \text{ independent of } a . \quad (4.12)$$

The setting so far is not sufficient for determining the expectation value of a measurement, because for computing an expectation value we to take an “average” over the subsystems. This process can be described conveniently by the so-called *measurement scalar product*  $(\cdot|\cdot)$ , which we now introduce (for a microscopic derivation and interpretation of the measurement scalar product and the measurement process see Sections 6.3 and 8.2 below). It is defined on the one-particle Hilbert space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  as a positive semi-definite inner product

$$(\cdot|\cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} , \quad (4.13)$$

with respect to which the direct sum decomposition (4.2) need not be orthogonal. The fact that this inner product is only semi-definite models the fact that the measurement process may involve a homogenization process on the microscopic scale, so that fluctuations of the wave functions on the small scale might not enter the measurement process. The measurement scalar product induces on the Fock spaces the bilinear form

$$\begin{aligned} (\cdot|\cdot)^{(a,b)} : \mathcal{F}_n^{(a)} \times \mathcal{F}_n^{(b)} &\rightarrow \mathbb{C} : (\psi_1^{(a)} \wedge \dots \wedge \psi_n^{(a)}, \psi_1^{(b)} \wedge \dots \wedge \psi_n^{(b)}) \\ &\mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} (\psi_{\sigma(1)}^{(a)} | \psi_1^{(b)}) \dots (\psi_{\sigma(n)}^{(a)} | \psi_n^{(b)}) . \end{aligned}$$

We now specify how expectation values are to be computed and state the assumptions which ensure that these expectation values are real.

(B) The expectation value of the measurement of the observable  $\mathcal{O}$  is given by

$$\langle \mathcal{O} \rangle = \frac{\sum_{a,b=1}^L (\Psi^{(a)} | \mathcal{O} \Psi^{(b)})_{(a,b)}}{\sum_{a,b=1}^L (\Psi^{(a)} | \Psi^{(b)})_{(a,b)}} . \quad (4.14)$$

(C) The observables are symmetric possibly up to a microscopic error, meaning that

$$(\Psi^{(a)} | \mathcal{O} \Psi^{(b)})_{\mathcal{F}} = (\mathcal{O} \Psi^{(a)} | \Psi^{(b)})_{\mathcal{F}} + \mathcal{O}(\varepsilon) ,$$

where  $\varepsilon$  is the length scale of the microscopic mixing.

Finally, we need to specify how a state changes in a measurement process. In order to ensure that a repeated measurement of the same observable yields identical results, one usually asserts that after the measurement, the state should be an eigenstate of the observable. In our setting, the situation is a bit more involved because the measurement process may change the number of subsystems, and only the wave function after the homogenization should be an eigenstate of the observable. This is made precise by the following construction. We take the direct sum of the vector spaces  $\mathcal{F}_n^{(a)}$ ,

$$\mathcal{G} := \bigoplus_{a=1}^L \mathcal{F}_n^{(a)} ,$$

and on these spaces we introduce the inner product

$$(\cdot | \cdot)_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C} : \left( (\Psi^{(a)})_{a=1,\dots,L}, (\Psi^{(b)})_{a=1,\dots,L} \right) \mapsto \sum_{a,b=1}^L (\Psi^{(a)} | \Psi^{(b)})_{(a,b)} .$$

This inner product is positive semi-definite, but it need not be definite. Dividing out the null space and taking the completion,

$$\mathcal{F}_n^{\text{eff}} := \overline{\mathcal{G}/\mathcal{G}_0} \quad \text{where} \quad \mathcal{G}_0 = \{u \in \mathcal{G} \text{ with } (u|u)_{\mathcal{G}} = 0\} , \quad (4.15)$$

we obtain a Hilbert space, which we can regard as the effective  $n$ -particle Fock space obtained by homogenization over the subsystems. We denote the natural projection operator by  $\pi_n$ ,

$$\pi_n : \bigoplus_{a=1}^L \mathcal{F}_n^{(a)} \rightarrow \mathcal{F}_n^{\text{eff}} . \quad (4.16)$$

Using linearity together with Assumption (C) above, every observable  $\mathcal{O}$  induces an operator

$$\mathcal{O}^{\text{eff}} : \mathcal{F}_n^{\text{eff}} \rightarrow \mathcal{F}_n^{\text{eff}} , \quad (4.17)$$

uniquely defined possibly up to a microscopic error. To the fermionic projector  $P$  we associate a corresponding state

$$\Psi^{\text{eff}} = \pi_f \left( \Psi^{(1)}, \dots, \Psi^{(L)} \right) \in \mathcal{F}_f^{\text{eff}} \quad (4.18)$$

(with the  $\Psi^{(a)}$  as in (4.11)). According to (4.12), this state is well-defined up to an irrelevant phase.

(D) After a measurement of the observable  $\mathcal{O}$ , the one-particle projector  $P$  takes such a form that the corresponding state  $\Psi^{\text{eff}} \in \mathcal{F}_f^{\text{eff}}$  defined by (4.18) is an eigenstate of the operator  $\mathcal{O}^{\text{eff}}$ , (4.17).

Similar as in the Copenhagen interpretation or the formulation of the measurement process by von Neumann [24, Section V.1], the above Assumptions (A)–(D) are merely working rules to determine the results of measurements. For a conceptually convincing treatment, these assumptions should be derived from the physical equations. This will be discussed in Chapter 8.

We now verify that the above setting indeed makes it possible to realize the EPR singlet state.

**Example 4.2.** (*The spatially separated singlet state*) We choose a microscopic length scale  $\varepsilon > 0$  and subdivide position space  $M = \mathbb{R}^3$  into two subregions  $M_1$  and  $M_2$  which form layers of width  $\varepsilon$ ,

$$M_1 = \{\vec{x} \in \mathbb{R}^3 \text{ with } [x_1/\varepsilon] \in 2\mathbb{Z}\}, \quad M_2 = \{\vec{x} \in \mathbb{R}^3 \text{ with } [x_1/\varepsilon] \in 2\mathbb{Z} + 1\}$$

(where  $[x] = \min\{n \in \mathbb{Z} \mid n \geq x\}$  is the Gauss bracket). We introduce the wave functions

$$\begin{aligned} \psi_1(\vec{x}) &= \psi_A^\uparrow(\vec{x}) \chi_{M_1}(\vec{x}) + \psi_A^\downarrow(\vec{x}) \chi_{M_2}(\vec{x}) \\ \psi_2(\vec{x}) &= \psi_B^\downarrow(\vec{x}) \chi_{M_1}(\vec{x}) - \psi_B^\uparrow(\vec{x}) \chi_{M_2}(\vec{x}), \end{aligned}$$

where  $\psi_{A/B}^{\uparrow/\downarrow}$  are smooth one-particle wave functions supported near ‘‘Alice’’ or ‘‘Bob’’ (and where  $\chi$  is the characteristic function defined by  $\chi_N(x) = 1$  if  $x \in N$  and  $\chi_N(x) = 0$  otherwise). Defining  $P$  as the projector on the subspace spanned by  $\psi_1$  and  $\psi_2$ , the corresponding two-particle wave functions of the subsystems are

$$\Psi^{(1)} = c \left( \psi_A^\uparrow \wedge \psi_B^\downarrow \right) \chi_{M_1 \times M_1} \quad \text{and} \quad \Psi^{(2)} = -c \left( \psi_A^\downarrow \wedge \psi_B^\uparrow \right) \chi_{M_2 \times M_2}$$

with  $c$  a normalization constant.

In order to realize a suitable mixing of the subregions in the measurement process, we introduce the measurement scalar product by

$$(\psi|\phi) = \int_M \overline{\psi(\vec{x})} \phi(\vec{x}) d\vec{x} + \frac{1}{2} \int_M \left( \overline{\psi(\vec{x} + \varepsilon e_1)} \phi(\vec{x}) + \overline{\psi(\vec{x})} \phi(\vec{x} + \varepsilon e_1) \right) d\vec{x}.$$

The spin operators are symmetric with respect to this inner product, whereas the position operators are symmetric up to an error of order  $\varepsilon$ ,

$$(\vec{x}\psi|\phi) - (\psi|\vec{x}\phi) = \frac{\varepsilon}{2} \int_M x_1 \left( \overline{\psi(\vec{x} + \varepsilon e_1)} \phi(\vec{x}) - \overline{\psi(\vec{x})} \phi(\vec{x} + \varepsilon e_1) \right) d\vec{x}.$$

Thus the general observables introduced according to (A) indeed satisfy the condition (C). The expectation values of the spin operators can now be calculated by applying the rule (B). More precisely, the inner products in (4.14) are computed by

$$\begin{aligned} (\Psi^{(1)}|\mathcal{O}\Psi^{(1)})^{(1,1)} &= \langle \Psi^{(1)}|\mathcal{O}\Psi^{(1)} \rangle \\ (\Psi^{(2)}|\mathcal{O}\Psi^{(2)})^{(2,2)} &= \langle \Psi^{(2)}|\mathcal{O}\Psi^{(2)} \rangle \\ (\Psi^{(1)}|\mathcal{O}\Psi^{(2)})^{(1,2)} &= \frac{1}{2} \left( \langle \Psi_+^{(1)}|\mathcal{O}\Psi^{(2)} \rangle + \langle \Psi^{(1)}|\mathcal{O}\Psi_+^{(2)} \rangle \right) \\ (\Psi^{(2)}|\mathcal{O}\Psi^{(1)})^{(2,1)} &= \frac{1}{2} \left( \langle \Psi_+^{(2)}|\mathcal{O}\Psi^{(1)} \rangle + \langle \Psi^{(2)}|\mathcal{O}\Psi_+^{(1)} \rangle \right), \end{aligned}$$

where on the right the scalar product on  $\mathcal{F}$  as defined by (2.1) appears, and the subscript  $+$  denotes that both spatial arguments of the corresponding two-particle wave function have been shifted by  $\varepsilon e_1$ . Note that all these inner products involve

integrals over  $M_1 \times M_1$  or  $M_2 \times M_2$ . Since the wave functions  $\psi_{A/B}^{\uparrow\downarrow}$  are all smooth, we can extend the two-particle wave functions of the subsystems to smooth functions on  $M \times M$ ,

$$\Psi_{\text{eff}}^{(1)} = c \left( \psi_A^\uparrow \wedge \psi_B^\downarrow \right) \quad \text{and} \quad \Psi_{\text{eff}}^{(2)} = c \left( \psi_A^\downarrow \wedge \psi_B^\uparrow \right). \quad (4.19)$$

Shifting the arguments changes these smooth wave functions only by a term of order  $\varepsilon$ . Also, extending the integration range in the above integrals from  $M_1 \times M_1$  or  $M_2 \times M_2$  to  $M \times M$  changes the values of the integrals only by a factor of four, again up to contributions of order  $\varepsilon$ . We thus obtain

$$\sum_{a,b=1}^L (\Psi^{(a)} | \mathcal{O} \Psi^{(b)})^{(a,b)} = 4 \left\langle (\Psi_{\text{eff}}^{(1)} + \Psi_{\text{eff}}^{(2)}) | \mathcal{O} (\Psi_{\text{eff}}^{(1)} + \Psi_{\text{eff}}^{(2)}) \right\rangle + \mathcal{O}(\varepsilon).$$

We conclude that in the limit  $\varepsilon \searrow 0$ , the expectation values as defined by (B) indeed coincide with the expectation values of the spin singlet state.

Moreover, it is straightforward to verify that the space  $\mathcal{F}_2^{\text{eff}}$  as defined by (4.15) can be identified with the ordinary Fock space  $\mathcal{F}_2$ , and that under this identification, the state  $\Psi^{\text{eff}}$  as given by (4.18) goes over to the state  $\Psi_{\text{eff}}^{(1)} + \Psi_{\text{eff}}^{(2)}$  with the wave functions  $\Psi_{\text{eff}}^{(1/2)}$  as in (4.19).  $\diamond$

## 5. A FEW BASICS ON THE FERMIONIC PROJECTOR

The concept of microscopic mixing of subsystems as introduced in the previous chapter goes beyond the scope of ordinary quantum theory. In order to justify this concept, we need to use ideas and methods from the theory of the fermionic projector, which we now outline. We keep the setting as simple as possible and restrict attention to those aspects which will be important for our purpose (for details we refer to [6, 8, 9] or the review papers [7, 10]).

**5.1. The Relativistic Setting, Indefinite Inner Product Spaces.** For the relativistic generalization of the previous constructions we consider Dirac wave functions  $\psi(t, \vec{x})$  in Minkowski space  $(M, \langle \cdot, \cdot \rangle)$ , which satisfy the Dirac equation for example in the presence of an electromagnetic field

$$i\gamma^j (\partial_j - ieA_j)\psi = m\psi$$

(here  $\gamma^j$ ,  $j = 0, \dots, 3$ , are the Dirac matrices,  $A$  is the electromagnetic potential, and  $m$  is the rest mass). At every space-time point, the Dirac wave functions are endowed with the Lorentz invariant inner product  $\langle \psi | \phi \rangle := \bar{\psi} \phi$  (where  $\bar{\psi} = \psi^\dagger \gamma^0$  is the usual adjoint spinor), having the signature  $(2, 2)$ . Integrating this inner product over space-time gives rise to the inner product on the wave functions

$$\langle \psi | \phi \rangle := \int_M \langle \psi(x) | \phi(x) \rangle d^4x. \quad (5.1)$$

Moreover, the non-negative quantity  $\langle \psi | \gamma^0 \psi \rangle$  has the interpretation as the probability density of the Dirac particle. Polarizing and integrating over space yields the scalar product

$$(\psi | \phi) = \int_{t=\text{const}} \langle \psi(t, \vec{x}) | \gamma^0 \phi(t, \vec{x}) \rangle d\vec{x}. \quad (5.2)$$

For solutions of the Dirac equation, current conservation implies that this scalar product is time independent.

In the framework of the fermionic projector (see [6, Chapter 3] or [8]), one considers Dirac wave functions  $\psi_1, \dots, \psi_f$  being vectors in the space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  endowed with the indefinite inner product (5.1) (possibly in finite 4-volume in order to make the space-time integrals finite). Thus the space of one-particle wave functions is no longer a Hilbert space. But we assume that the wave functions  $\psi_1, \dots, \psi_f$  span a *negative definite* subspace of  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  (i.e. the vectors  $\psi_i$  in (3.3) all satisfy the inequality  $\langle \psi_i, \psi_i \rangle \leq 0$ ). Let us verify that our previous results carry over to this framework. We begin by reviewing the restriction to subsystems in Chapter 3.

**Remark 5.1.** (*Restriction to subsystems*) As just explained, we assume that the wave functions  $\psi_1, \dots, \psi_f$  form a negative-definite subspace of  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ . This subspace can be regarded as a Hilbert space. When considering a subsystem, we split up the wave functions according to (3.4) into their inner and outer parts. Now the vectors  $\psi^O$  need no longer be negative definite, and as a consequence, the Gram matrix (3.7) need no longer be semi-definite. This major difference to the setting in Chapter 3 has no consequences on our results, because the semi-positivity of the Gram matrix  $A_{ij}$  was never used in the subsequent arguments. In particular, the consideration in Example 3.6 shows that it is again impossible to realize the density operator of the spatially separated singlet state in (2.3).  $\diamond$

We now verify that the argument in Proposition 4.1 remains valid in indefinite inner product spaces.

**Remark 5.2.** (*Microscopic mixing of the wave functions*) Since the space-time inner product (5.1) is indefinite, the symmetric linear operators no longer form a Hilbert space (in particular, the trace  $\text{Tr}(A^2)$  need no longer be non-negative). As a consequence, the argument after (4.9) in Proposition 4.1 no longer seems to apply. However, at this point we can exploit the fact that the span  $I$  of the wave functions  $\psi_1, \dots, \psi_f$  forms a negative-definite subspace of  $\mathcal{H}$ . Namely, considering the operators in (4.7) as symmetric operators in the Hilbert space  $I$ , all the arguments in the proof of Proposition 4.1 go through.  $\diamond$

We conclude that the “no-go-results” in Chapter 3 and Section 4.1 remain valid in the setting of the fermionic projector. Thus the only possibility to realize entangled states seems to rely on the formalism of Section 4.2.

We finally mention that an indefinite inner product can lead to effects which can be interpreted as a violation of Kolmogorov’s axiom that probabilities take values in the interval  $[0, 1]$ . Such “extended probabilities” make it possible to formulate local and causal quantum theories with hidden variables which do not violate Bell’s inequalities. More precisely, it is possible to formulate a hidden variable theory in which the statistical average of the expectation values coincides precisely with the corresponding expectation values of the spatially separated singlet state (2.3). However, this method does not seem to work for spin  $3/2$ . Moreover, in this formulation the expectation values of the singlet state can be reproduced only in the statistical average, but it is impossible to realize the singlet state itself. As we do not find this situation fully convincing, we shall not follow this approach here. The interested reader is referred to [1, Section 5.2.3], [18] and the detailed review paper [17].

**5.2. Fermion Systems in Discrete Space-Time.** In the discrete setting, we replace Minkowski space by a finite number of space-time points  $M = \{1, \dots, m\}$ . To each space-time point, we associate an indefinite inner product space  $(S_x, \langle \cdot | \cdot \rangle_x)$  of

signature  $(2, 2)$ , referred to as the *spinor space*. A *wave function*  $\psi$  is defined as a function which maps every space-time point  $x$  to a vector  $\psi(x) \in S_x$ . Introducing the inner product

$$\langle \psi | \phi \rangle = \sum_{x \in M} \langle \psi(x) | \phi(x) \rangle, \quad (5.3)$$

the wave functions form an indefinite inner product space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ . To describe the fermions, we consider a negative definite subspace  $I \subset \mathcal{H}$  and introduce the *fermionic projector* as a projector on  $I$ . Choosing a pseudo-orthonormal basis  $\psi_1, \dots, \psi_f$  of  $I$ , the projector on  $I$  can be written in bra/ket-notation as

$$P = - \sum_{j=1}^f |\psi_j\rangle \langle \psi_j|$$

(the minus sign corresponds to the fact that  $I$  is *negative* definite). Using the form of the inner product (5.3), we can also write  $P$  as

$$(P\psi)(x) = \sum_{y \in M} P(x, y) \psi(y) \quad \text{with} \quad P(x, y) = - \sum_{j=1}^f |\psi_j(x)\rangle \langle \psi_j(y)|. \quad (5.4)$$

Due to the similarity to an integral representation, we refer to  $P(x, y)$  as the *discrete kernel* of the fermionic projector. Thus as in the setting of Section 2.3,  $P$  is a projector in the space of one-particle wave functions. However, the inner product on the one-particle vector space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  is no longer positive definite, as can be understood from the analogy to the indefinite inner product on the Dirac wave functions (5.1). We refer to  $P$  on  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  together with the inner product of the form (5.3) as a *fermion system in discrete space-time*.

It is important to keep in mind that according to the above definitions, there are no relations between the space-time points which correspond to a nearest-neighbor relation on a lattice or to the topological or causal structure of Minkowski space. Discrete space-time  $M$  is merely a set of points, without any additional structures. The key for getting such additional structures is to set up an *action principle* where we minimize an action  $\mathcal{S}$  being the space-time sum of a Lagrangian  $\mathcal{L}$ ,

$$\mathcal{S} = \sum_{x, y \in M} \mathcal{L}[P(x, y) P(y, x)] \rightarrow \min. \quad (5.5)$$

The Lagrangian is a non-linear functional of the discrete kernel, and in this way the action depends on the fermionic wave functions  $\psi_1, \dots, \psi_f$ . It turns out that for minimizers of the action, an effect of spontaneous symmetry breaking generates additional structures on space-time, like a notion of causality:

**Definition 5.3. (causal structure)** *Two space-time points  $x, y \in M$  are called **time-like** separated if the spectrum of the product  $P(x, y)P(y, x)$  is real. Likewise, the points are **spacelike** separated if the spectrum of  $P(x, y)P(y, x)$  forms two complex conjugate pairs having the same absolute value.*

In non-technical terms, this effect of *structure formation* can be understood as a self-organization of the wave functions described by our action principle.

The important observation for what follows is that in the framework of the fermionic projector, space-time is *not smooth* on the microscopic scale, but it has an underlying discrete structure. The dynamics is described intrinsically in discrete space-time by the



action principle (5.5). A minimizing fermionic projector has a rich *microscopic structure*, from which one can deduce notions which have a correspondence to macroscopic physics (like the above notion of causality). There are good reasons to believe that for systems involving many particles and many space-time points, these macroscopic notions give rise to the structures of Minkowski space or a Lorentzian manifold [10].

**5.3. The Continuum Limit.** The continuum limit is a mathematical method where one homogenizes the unknown microscopic structure of the fermionic projector with the goal of analyzing the action principle (5.5) for Dirac particles coupled to classical fields. More specifically, one assumes that the fermion configuration of the vacuum can be described after a suitable homogenization process by a Dirac sea configuration in Minkowski space. Thus in the simplest case of one Dirac sea, the kernel of the fermionic projector of the vacuum is given by

$$P^{\text{sea}}(x, y) = \int \frac{d^4 k}{(2\pi)^4} (k_j \gamma^j + m) \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)}, \quad (5.6)$$

where  $\Theta$  is the Heaviside function. In order to introduce particles and anti-particles, one occupies (suitably normalized) positive-energy states and removes states of the sea,

$$P(x, y) = P^{\text{sea}}(x, y) - \frac{1}{2\pi} \sum_{k=1}^{n_f} |\psi_k(x)\rangle \langle \psi_k(y)| + \frac{1}{2\pi} \sum_{l=1}^{n_a} |\phi_l(x)\rangle \langle \phi_l(y)|. \quad (5.7)$$

In the case with interaction, the fermionic projector is described more generally as a solution of a Dirac equation involving a general interaction. For our purposes, it suffices to consider an electromagnetic field,

$$(i\gamma^j (\partial_j - ieA_j) - m) P(x, y) = 0. \quad (5.8)$$

Using the so-called causal perturbation expansion and light-cone expansion, the fermionic projector can be introduced via (5.7) and (5.8) even in the time-dependent setting.

It is important that our setting so far does not involve the field equations; in particular, the electromagnetic potential in the Dirac equation (5.8) does not need to satisfy the Maxwell equations. Our concept is that the field equations should be derived from our action principle (5.5). Indeed, analyzing the corresponding Euler-Lagrange equations, one finds that they are satisfied only if the potentials in the Dirac equation satisfy certain constraints. Some of these constraints are partial differential equations involving the potentials as well as the wave functions of the particles and anti-particles in (5.7). In [9], such field equations are analyzed in detail for a system involving an axial field. In order to keep the setting as simple as possible, we here consider the analogous field equation for the electromagnetic field

$$\partial_{jk} A^k - \square A_j = e \sum_{k=1}^{n_f} \langle \psi_k | \gamma_j \psi_k \rangle - e \sum_{l=1}^{n_a} \langle \phi_l | \gamma_j \phi_l \rangle. \quad (5.9)$$

With (5.8) and (5.9), the interaction as described by the action principle (5.5) reduces in the continuum limit to the *coupled Dirac-Maxwell equations*. The many-fermion state is again described by the fermionic projector. As explained in Section 2.3, this is equivalent to working with a Hartree-Fock state in Fock space. In this sense, we are working with *second-quantized fermions*. The electromagnetic field, however, is a *classical bosonic field*. Treating the coupled Dirac-Maxwell system perturbatively

yields all Feynman diagrams, giving agreement with the high-precision tests of quantum electrodynamics (see [9, Section 8.4]). However, the framework of the continuum limit does *not* make it possible to describe *entanglement* for two different reasons: first because Hartree-Fock states are too restrictive (see Examples 2.1 and 3.6), and second because entanglement does not arise in systems with purely classical fields (see Remark 3.7).

We point out that the fermionic projector in Minkowski space (5.7) fits into the framework in discrete space-time (5.4) if the number of space-time points and the number of particles tends to infinity. Conversely, by considering the system in finite 4-volume and introducing an ultraviolet regularization, the continuum fermionic projector can be associated to a corresponding fermionic projector in a discrete space-time. Note that, since we also count the states forming the Dirac sea, these regularized systems involve an extremely *large number of particles* (typically of the order  $\sim (L/\ell_P)^3$ , where  $L$  is the size of the system and  $\ell_P$  denotes the regularization length).

It is important to keep in mind that in the continuum limit, the causal and metric structure of Minkowski space can be recovered from the fermionic projector using the notion of Definition 5.3. Thus we can say that the *wave functions* of the Dirac particles (also taking into account the states of the Dirac sea) *generate the causal and geometric structure of space-time*. This observation will be helpful in the next section, where by bringing the wave functions in different regions of space-time “out of phase”, we will be able to turn off causal influences between these regions.

## 6. DECOHERENT SPACE-TIME REGIONS

In this chapter we reconsider the concept of microscopic mixing of subsystems (as introduced in Chapter 4) in the framework of the fermionic projector. An effect of decoherence will give rise to an independent dynamics of the subsystems, thus justifying the ad-hoc assumption in Section 4.2. Moreover, we will get a natural explanation for the measurement scalar product (4.13) and see that homogenizing the system on the microscopic scale yields an effective description by superpositions of many-particle wave functions, showing that the Fock space  $\mathcal{F}_n^{\text{eff}}$  defined in (4.15) is the right object to work with. In this way, we will get an explanation for the framework introduced in Section 4.2 for the description of entanglement.

The only point which remains unsatisfactory is that on one hand, the observables are assumed to take part in the independent dynamics of the subsystems (Assumption (A) in Section 4.2). On the other hand, the measurement process involves taking averages over the subsystems and leads to a “collapse” of the wave function into an eigenstate (Assumptions (B) and (D) in Section 4.2), being a process which contradicts an independent dynamics but requires an outside observer. This problem, which appears similarly in ordinary quantum mechanics, will be discussed separately in Chapter 8.

**6.1. Microscopic Mixing in the Relativistic Setting.** In order to extend the method of microscopic mixing of subsystems to the relativistic setting, we decompose Minkowski space  $M$  into two disjoint subregions

$$M = M_1 \cup M_2 \quad \text{with} \quad M_1 \cap M_2 = \emptyset, \quad (6.1)$$

which may be fine-grained as depicted in Figure 2 (for simplicity, we only consider two subregions; the generalization to a finite number of subsystems is straightforward). If one prefers, one can in addition replace the space-time continuum by a discrete set

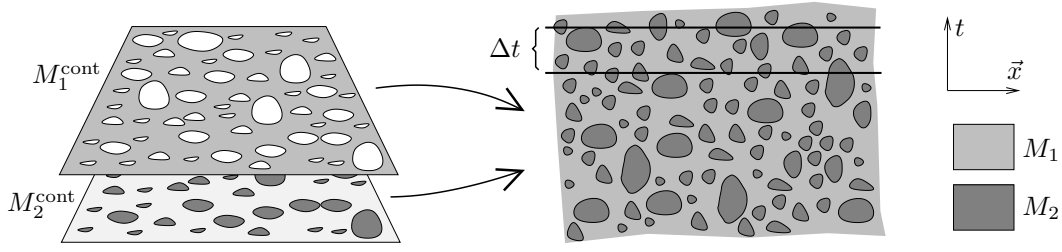


FIGURE 2. Example of a microscopic mixing of two space-time regions.

of points, but the distinction between continuum and discrete space-time will not be of relevance for the following considerations. We again consider a (suitable orthonormalized) family of wave functions  $\psi_1, \dots, \psi_f$ , where in view of the fact that we also count the states of the Dirac sea, the number  $f$  of particles is very large (see the end of Section 5.3). As in (5.4), the fermionic projector takes the form

$$P(x, y) = - \sum_{j=1}^f |\psi_j(x) \succ \prec \psi_j(y)| \quad \text{with} \quad x, y \in M_1 \cup M_2. \quad (6.2)$$

Moreover, splitting up the wave functions similar to (4.3) by

$$\psi_j = \psi_j^{(1)} + \psi_j^{(2)} \quad \text{with} \quad \psi_j^{(a)} = \psi_j \chi_{M_a} \quad (a = 1, 2), \quad (6.3)$$

to each subsystem we can associate similar to (4.11) the many-particle wave function

$$\Psi^{(a)} = \psi_1^{(a)} \wedge \dots \wedge \psi_f^{(a)}. \quad (6.4)$$

**6.2. Justification of an Independent Dynamics of the Subsystems.** Following the concept of discrete space-time of Section 5.2, space-time is not smooth on the microscopic scale, but should have a non-trivial microstructure. In order to explain the possible consequences of such a microstructure in a simple setting, we consider what happens if we choose the wave functions  $\psi_j^{(a)}$  in the two subregions differently. More specifically, we transform the wave functions in the second subsystem by a unitary matrix of determinant one,

$$\psi_j^{(1)} \rightarrow \psi_j^{(1)}, \quad \psi_j^{(2)} \rightarrow \tilde{\psi}_j^{(2)} := \sum_{k=1}^f U_{jk} \psi_k^{(2)} \quad \text{with} \quad U \in \text{SU}(f). \quad (6.5)$$

This transformation has the advantage that it has no effect on the many-particle wave functions (6.4), because (exactly as in Lemma 2.2)

$$\Psi^{(2)} \rightarrow \tilde{\Psi}^{(2)} = \tilde{\psi}_1^{(2)} \wedge \dots \wedge \tilde{\psi}_f^{(2)} = \det U \psi_1^{(2)} \wedge \dots \wedge \psi_f^{(2)} = \Psi^{(2)}. \quad (6.6)$$

Furthermore, the fermionic projector does not change if its two arguments are in the same subsystem, because in the case  $x, y \in M_2$ , the unitarity of  $U$  yields that

$$\begin{aligned}
 P(x, y) &\rightarrow - \sum_{j=1}^f |\tilde{\psi}_j^{(2)}(x)\rangle\langle\tilde{\psi}_j^{(2)}(y)| = - \sum_{j,k=1}^f (UU^\dagger)_{jk} |\psi_j^{(2)}(x)\rangle\langle\psi_k^{(2)}(y)| \\
 &= - \sum_j^f |\psi_j^{(2)}(x)\rangle\langle\psi_j^{(2)}(y)| = P(x, y) .
 \end{aligned}$$

However, if the two arguments of the fermionic projector are in different space-time regions, the operator  $U$  does not drop out. For example, if  $x \in M_2$  and  $y \in M_1$ ,

$$P(x, y) \rightarrow - \sum_{j=1}^f |\tilde{\psi}_j^{(2)}(x)\rangle\langle\psi_j^{(1)}(y)| = - \sum_{j,k=1}^f U_{jk} |\psi_j^{(2)}(x)\rangle\langle\psi_k^{(1)}(y)| . \quad (6.7)$$

In the special case when  $U$  is a diagonal matrix whose entries are phase factors,

$$U = \text{diag}(e^{i\varphi_1}, \dots, e^{i\varphi_f}) \quad \text{with} \quad \sum_{j=1}^f \varphi_j = 0 ,$$

the summations in (6.7) reduce to one sum involving the phase factors,

$$P(x, y) \rightarrow - \sum_{j=1}^f e^{i\varphi_j} |\psi_j^{(2)}(x)\rangle\langle\psi_j^{(1)}(y)| .$$

If the phases  $\varphi_j$  are chosen stochastically, the phases of the summands are random. As a consequence, there will be cancellations in the sum, and keeping in mind that the number of summands is very large, we conclude that  $P(x, y)$  will be very small. More generally, we find that if  $U$  is a random matrix,  $P(x, y)$  becomes small if  $x$  and  $y$  lie in different subsystems (this argument will be quantified in Section 6.4 below by integrating over the space of unitary matrices).

From the physical point of view, the above consideration can be understood using the notion of decoherence. If the one-particle wave functions  $\psi_j^{(1)}$  and  $\psi_j^{(2)}$  are coherent or “in phase”, then the fermionic projector  $P(x, y)$  has the usual form, no matter whether  $x$  and  $y$  are in the same subsystem or not. If however the wave functions in the subregions are decoherent or “out of phase”, then the fermionic projector  $P(x, y)$  will be very small if  $x$  and  $y$  are in different subregion. We refer to this effect as the *decoherence between space-time regions*. It should be carefully distinguished from the decoherence of the many-particle wave function (see for example [14]). Namely, as we saw in (6.6), in our case the many-particle wave functions remain unchanged. Thus they remain coherent, no decoherence in the sense of [14] appears. But the one-particle wave functions become decoherent (6.5), having an influence on the fermionic projector (6.2).

We next consider the influence of the decoherence between space-time regions on the dynamics of our system. We begin by discussing the extreme case where  $P(x, y)$  vanishes identically for  $x$  and  $y$  in different subsystems. Then the action (5.5) splits into the sum of the actions of the two subsystems, so that the interaction takes place independently in the two subsystems. In other words, the *subsystems decouple*. By restricting two different systems in Minkowski space to  $M_1$  respectively  $M_2$ , one can apply the methods of Section 5.3 to both subsystems separately. Then each subsystem is described by an independent continuum limit in terms of a Dirac equation (5.8)

coupled to a classical field (5.9). This explains the assumption of an *independent dynamics of the subsystems* made in Section 4.2. We point out that, following the concept that the wave functions generate the causal and geometric structure of space-time (see the last paragraph in Section 5.3), the decoupling of the subsystems even implies that between the subsystems, the usual causal and topological structure of Minkowski space ceases to exist.

If we merely assume that  $P(x, y)$  is small for  $x$  and  $y$  in different subsystems, our action principle (5.5) does describe a coupling of the two subsystems, which however should be weak. Keeping in mind that the causal structure of Minkowski space is related to the singularities of distributions like (5.6) on the light cone, and that this singular structure will be destroyed by decoherence, we know that the coupling of the two subsystems cannot be described by causal equations formulated in Minkowski space. Although we have a precise mathematical framework (5.5), describing the coupling of the subsystems quantitatively seems very difficult and goes beyond the scope of this paper. But from the mathematical structure of our action it is already clear that we do not get contributions from the boundaries of the two subregions. Therefore, instead of considering the “layers” in Figure 1, it seems more appropriate to draw each subsystem as many disconnected “bubbles” in space-time as shown in Figure 2. In view of the continuum limit, each system has an underlying smooth structure inherited from a corresponding system in Minkowski space, as is indicated in Figure 2 by the two “smooth space-time sheets”  $M_{1/2}^{\text{cont}}$ . But of course, this picture should be considered only as a vain attempt to illustrate an unknown and probably very complicated microscopic structure of space-time (a maybe more realistic picture will be outlined in Section 7.3).

**6.3. Justification of the Superposition of Fock States.** In the framework of fermion systems in discrete space-time, the underlying inner product  $\langle \cdot | \cdot \rangle$  involves the sum over all space-time points (5.3). Likewise, in the continuum limit, the sum is to be replaced by a space-time integral (5.1). This space-time inner product is useful for the mathematical formulation, but it is not suitable for computing the expectation value of a measurement, which usually takes place at a fixed time. Thus in the relativistic setting, it is natural to work in the measurement process with a different scalar product, which in Section 4.2 we referred to as the *measurement scalar product* (4.13). More specifically, in view of the fact that the integrand of (5.2) has the interpretation as the probability density, the scalar product (5.2) seems the correct choice.

Let us consider how to implement the scalar product (5.2) as the measurement scalar product in the presence of a microscopic mixing of two subsystems. First, taking into account that realistic measurements take place in a finite time interval, it seems a good idea to replace the spatial integral in (6.1) by an integral over a strip of width  $\Delta t$  as shown in Figure 2. Moreover, the measurement should involve some kind of *homogenization process* on the microscopic scale. This is clear empirically because the expectation value must be a computable quantity which involves taking averages over the subsystems. The homogenization can also be understood microscopically from the fact that the measurement devices are themselves formed of quantum mechanical wave functions which are spread out in space-time. A typical example for an idealized measurement device is the operator  $|\eta\rangle\langle\eta|$ , where  $\eta$  is a wave function which is supported inside a strip of width  $\Delta t$  as shown in Figure 2. The simplest way to take into account the effect of such a measurement device would be to consider instead

of (5.2) the measurement scalar product

$$(\psi|\phi) = \frac{1}{(\Delta t)^2} \int_{\mathbb{R}^3} d\vec{x} \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 \langle \psi(t_1, \vec{x}) | \gamma^0 \phi(t_2, \vec{x}) \rangle, \quad (6.8)$$

where we take the average over a small time interval before computing the spatial inner product. A more realistic measurement device is of course much more complicated, but fortunately the details are of no relevance. All that matters is that the measurement scalar product involves a homogenization process, with the effect that the direct summands in (4.2) are in general not orthogonal with respect to  $(\cdot|\cdot)$  and that the null space  $\mathcal{G}_0$  in (4.15) becomes non-trivial.

We now want to verify that expectation values computed with respect to the measurement scalar product indeed involve superpositions of Fock states. For simplicity, we again consider the situation for two subsystems  $M_1$  and  $M_2$  and assume that the fermions in each subsystem are described by an  $n$ -particle Hartree-Fock state

$$\psi_1^{(a)} \wedge \cdots \wedge \psi_n^{(a)},$$

whereas the remaining  $f - n$  particles describe the Dirac sea. Thus we choose the one-particle wave functions before microscopic mixing as  $\psi_1^{(a)}, \dots, \psi_n^{(a)}, \psi_{n+1}^{(a)}, \dots, \psi_f^{(a)}$ , where the first  $n$  wave functions describe the particles, whereas the other wave functions form the sea. We again introduce a decoherence between the subsystems by a unitary transformation of all states in the second subsystem (6.5).

Expectation values  $(\psi_i|\psi_j)$  of the one-particle wave functions with respect to the measurement scalar product involve a homogenization process, with the result that wave functions should be identified which differ only by microscopic fluctuations. More specifically, we should not distinguish between the sea states of the two subsystems. Thus introducing on  $\mathcal{H}$  the equivalence relation

$$\psi \cong \tilde{\psi} \quad \iff \quad (\psi - \tilde{\psi} | \psi - \tilde{\psi}) = 0,$$

we assume that

$$\psi_j^{(1)} \cong \psi_j^{(2)} \quad \text{for all } j = n+1, \dots, f.$$

For ease in notation, we make this identification clear simply by omitting the corresponding superscripts <sup>(1)</sup> or <sup>(2)</sup>.

Under this identification, the many-particle wave function of the whole system becomes

$$\begin{aligned} \Psi &= \psi_1 \wedge \cdots \wedge \psi_f \\ &= (\psi_1^{(1)} + \tilde{\psi}_1^{(2)}) \wedge \cdots \wedge (\psi_n^{(1)} + \tilde{\psi}_n^{(2)}) \wedge (\psi_{n+1} + \tilde{\psi}_{n+1}) \wedge \cdots \wedge (\psi_f + \tilde{\psi}_f) \end{aligned} \quad (6.9)$$

Multiplying out, we obtain many contributions. One of them corresponds to the many-particle wave function of the first subsystem

$$\psi_1^{(1)} \wedge \cdots \wedge \psi_n^{(1)} \wedge \psi_{n+1} \wedge \cdots \wedge \psi_f, \quad (6.10)$$

and another is the many-particle wave function of the second subsystem

$$\begin{aligned} \tilde{\psi}_1^{(2)} \wedge \cdots \wedge \tilde{\psi}_n^{(2)} \wedge \tilde{\psi}_{n+1} \wedge \cdots \wedge \tilde{\psi}_f &= \det U \psi_1^{(2)} \wedge \cdots \wedge \psi_n^{(2)} \wedge \psi_{n+1} \wedge \cdots \wedge \psi_f \\ &= \psi_1^{(2)} \wedge \cdots \wedge \psi_n^{(2)} \wedge \psi_{n+1} \wedge \cdots \wedge \psi_f. \end{aligned} \quad (6.11)$$

All the other contributions involve matrix elements of the unitary operator  $U$ . Similar as explained after (6.7), all these contributions become small if  $U$  is a stochastic matrix (for details see Section 6.4).

We conclude that the measurement process involves the sum of the many-particle wave functions (6.10) and (6.11) of the two subsystems. This justifies Assumption (B) in Section 4.2. Moreover, this consideration explains why for measurements one should work in the Fock space  $\mathcal{F}_n^{\text{eff}}$  defined by (4.15).

**6.4. Some Identities Involving Random Matrices.** In order to quantify the above considerations, we now state a few elementary results on random matrices. We first note that the space  $\text{SU}(f)$  of unitary matrices is a compact Lie group, on which we can introduce the normalized Haar measure  $d\mu_{\text{SU}(f)}$  (see for example [3, Section I.5]). For simplicity, we take the probability distribution on  $\text{SU}(f)$  according to the Haar measure (but other choices would also be possible; see for example [16]). Then taking averages over unitary matrices corresponds to integrating with respect to the Haar measure; we write for clarity  $\int_{\text{SU}(f)} \cdots d\mu_{\text{SU}(f)}$ .

We first observe that certain products of matrix elements of  $U$  vanish on average.

**Lemma 6.1.** *Suppose that for any  $p$  in the range  $1 \leq p \leq f - 1$ , we choose indices  $i_1, \dots, i_p$  and  $j_1, \dots, j_p$  with  $i_1 < \dots < i_p$ . Then*

$$\int_{\text{SU}(f)} U_{i_1 j_1} \cdots U_{i_p j_p} d\mu_{\text{SU}(f)} = 0.$$

*Proof.* We let  $k$  be an index which is not contained in  $\{i_1, \dots, i_p\}$  and let  $V$  be the diagonal matrix which has entries one, except that  $V_{i_1 i_1} = V_{kk} = -1$ . Then  $V \in \text{SU}(f)$ , and thus a variable transformation shows that the above integral is invariant under the replacement  $U \rightarrow VU$ . But this transformation flips the sign of the integrand.  $\square$

Applying this lemma to the expression (6.7), we see that the the fermionic projector  $P(x, y)$  indeed vanishes for  $x$  and  $y$  in different subregions, if the mean value over  $\text{SU}(f)$  is taken. The lemma also applies to the contributions obtained by multiplying out (6.9). It shows that all contributions vanish on average, except for the many-particle wave functions (6.10) and (6.11) of the two subsystems.

Since the expectation value of a measurement involves the absolute square of the wave functions, we would also like to integrate the absolute square of (6.7) and (6.9) over  $\text{SU}(f)$ . We begin with a simple integral involving the absolute square of one matrix element of  $U$ .

**Lemma 6.2.** *For any  $j, k \in \{1, \dots, f\}$ ,*

$$\int_{\text{SU}(f)} |U_{jk}|^2 d\mu_{\text{SU}(f)} = \frac{1}{f}. \quad (6.12)$$

*Proof.* By multiplying with suitable unitary operators from the left or the right, we can arbitrarily change the values of the indices  $j$  and  $k$ , without changing the integral. Thus

$$\begin{aligned} \int_{\text{SU}(f)} |U_{jk}|^2 d\mu_{\text{SU}(f)} &= \frac{1}{f^2} \int_{\text{SU}(f)} \sum_{j,k=1}^f |U_{jk}|^2 d\mu_{\text{SU}(f)} \\ &= \frac{1}{f^2} \int_{\text{SU}(f)} \text{Tr}(U^* U) d\mu_{\text{SU}(f)} = \frac{1}{f}, \end{aligned}$$

concluding the proof.  $\square$

Applying this result to (6.7), we see that decoherence typically scales the kernel of the fermionic projector down by a factor  $f^{-\frac{1}{2}}$ . This quantifies that  $P(x, y)$  really becomes small if  $x$  and  $y$  lie in different subregions.

Lemma 6.2 could be extended to integrals involving the absolute squares of  $n$  matrix elements, giving the result

$$\int_{\text{SU}(f)} |U_{j_1 k_1}|^2 \cdots |U_{j_n k_n}|^2 d\mu_{\text{SU}(f)} \sim \frac{1}{f^n} \quad \text{if } n \ll f.$$

This result shows that every fixed summand obtained by multiplying out (6.9) except for (6.10) and (6.11) vanishes in the limit  $f \rightarrow \infty$ . Unfortunately, this is not quite good enough, because the number of summands becomes large if  $f$  increases. Estimating the whole sum of terms for large  $f$  seems a difficult problem which we shall not enter here, also because the result might depend on whether the probability distribution is chosen according to the Haar measure or to another probability measure. Even leaving this point open, we can say that for a generic class of unitary matrices, the many-particle wave function of the total system (6.9) indeed reduces to the sum of the wave functions (6.10) and (6.11) of the two subsystems.

## 7. SECOND QUANTIZATION OF THE BOSONIC FIELD

In Section 6.2 we considered two decoherent subsystems  $M_1$  and  $M_2$  and saw that by analyzing each subsystem in the continuum limit, we could describe the dynamics by the Dirac equation coupled to a classical field. Taking a finite number of such decoherent subsystems, the whole dynamics is described by several classical fields, one for each subsystem. In this section we will show that the resulting framework indeed allows for the description of second quantized bosonic fields.

For simplicity, we only consider an electromagnetic field (the generalization to other bosonic fields is straightforward). We subdivide Minkowski space into  $L$  disjoint regions  $M_1, \dots, M_L$ , which are again assumed to be fine-grained. Similar to (6.2) and (6.3), the fermionic projector can be written as

$$P(x, y) = - \sum_{j=1}^f |\psi_j(x)\rangle \langle \psi_j(y)| \quad \text{with} \quad \psi_j = \sum_{a=1}^L \psi_j^{(a)}, \quad \psi_j^{(a)} := \psi_j \chi_{M_a}, \quad (7.1)$$

where  $f$  is a large number which tends to infinity if the ultraviolet regularization is removed. As in Chapter 6, we arrange by unitary transformations of the form (6.5) that the subsystems are decoherent. Considering each subsystem in the continuum limit, we obtain similar to (5.8) and (5.9) the Dirac-Maxwell equations

$$\begin{aligned} & \left( i\gamma^j (\partial_j - ieA_j^{(a)}) - m \right) P^{(a)}(x, y) = 0 \\ & \partial_j^k A_k^{(a)} - \square A_j^{(a)} = e \sum_{k=1}^{n_f} \langle \psi_k^{(a)} | \gamma_j \psi_k^{(a)} \rangle - e \sum_{l=1}^{n_a} \langle \phi_l^{(a)} | \gamma_j \phi_l^{(a)} \rangle. \end{aligned} \quad (7.2)$$

We note for clarity that according to (6.3), the wave functions  $\psi_k^{(a)}$  and  $\phi_l^{(a)}$  are obtained by restriction to a subregion  $M_a \subset M$  of space-time. But it is reasonable to assume that these wave functions are macroscopic in the sense that they can be extended smoothly to the whole Minkowski space. Similarly, we assume that the potentials  $A^{(a)}$  and the fermionic projectors  $P^{(a)}$  are defined on a whole sheet  $M_a^{\text{cont}}$  of Minkowski space (see Figure 2).



**7.1. Describing a Second Quantized Free Bosonic Field.** In order to get a simple connection to standard textbooks like [28, 15], we begin with a free electromagnetic field (i.e. the situation where no fermionic particles or anti-particles are present). Furthermore, to avoid the technical issues involved in taking an infinite volume limit, we restrict attention to the situation in finite spatial volume by considering a three-dimensional box of length  $L$  with periodic boundary conditions. Working in the Coulomb gauge  $\operatorname{div} \vec{A} = 0$ , the Maxwell equations reduce to the ordinary wave equation for each component of the vector potential,

$$\square \vec{A}(t, \vec{x}) = 0,$$

whereas the electric potential  $A^0$  can be arranged to vanish identically. Decomposing  $\vec{A}$  into Fourier modes of momentum  $\vec{k} \in (2\pi\mathbb{Z}/L)^3$ ,

$$\vec{A}(t, \vec{x}) = \sum_{\vec{k}} \left( \vec{a}(t, \vec{k}) e^{i\vec{k}\vec{x}} + \overline{\vec{a}(t, \vec{k})} e^{-i\vec{k}\vec{x}} \right),$$

the Maxwell equations are solved by

$$\vec{a}(t, \vec{k}) = \vec{a}(\vec{k}) e^{-i\omega t} \quad \text{with} \quad \omega := |\vec{k}|,$$

whereas the Coulomb gauge gives rise to the transversality condition  $\vec{k} \cdot \vec{a}(\vec{k}) = 0$  (see [15, Chapter I, §2]). The two linearly independent solutions of this transversality condition correspond to the two polarizations of the electromagnetic wave; we denote them by an index  $\beta = 1, 2$ . Introducing the canonical field variables

$$q_\beta(\vec{k}) = \frac{1}{4\pi} \left( a_\beta(\vec{k}) + \overline{a_\beta(\vec{k})} \right), \quad p(\vec{k}) = \frac{d}{dt} q_\beta(\vec{k}) = -\frac{i\omega}{4\pi} \left( a_\beta(\vec{k}) - \overline{a_\beta(\vec{k})} \right),$$

the energy  $H$  of the classical electromagnetic field becomes (for details see [15, Chapter I, §2])

$$H = \sum_{\vec{k} \in (2\pi\mathbb{Z}/L)^3} \sum_{\beta=1,2} \frac{1}{2} \left( p_\beta(\vec{k})^2 + \omega^2 q_\beta(\vec{k})^2 \right). \quad (7.3)$$

Here each summand is the Hamiltonian of a harmonic oscillator. Thus we have rewritten the classical electromagnetic field as an infinite collection of classical harmonic oscillators.

The second quantization of the electromagnetic field corresponds to quantizing each harmonic oscillator as in standard quantum mechanics (see for example [28, Part I, Section 1.2]). We now discuss the connection between the classical and the quantum dynamics in detail, for simplicity for a single harmonic oscillator of frequency  $\omega$ . Thus our starting point is the classical Hamiltonian

$$h(p, q) = \frac{1}{2} (p^2 + \omega^2 q^2). \quad (7.4)$$

Here  $q$  and  $p$  are the canonical variables, which together form the classical phase space  $\mathcal{P} = \{(p, q) \text{ with } p, q \in \mathbb{R}\}$ . The classical dynamics is described by Hamilton's equations

$$\frac{d}{dt} q = \frac{\partial h}{\partial p} = p, \quad \frac{d}{dt} p = -\frac{\partial h}{\partial q} = -\omega^2 q.$$

A solution  $(p(t), q(t))$  describes a classical trajectory. Solving Hamilton's equations, the classical dynamics describes a rotation in phase space,

$$\begin{pmatrix} p(t) \\ \omega q(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} p(0) \\ \omega q(0) \end{pmatrix}. \quad (7.5)$$

In order to get a setting similar to that in quantum theory, we next consider on phase space complex-valued functions  $\psi(p, q)$ , referred to as ‘‘classical wave functions.’’ Introducing the scalar product

$$\langle \psi | \phi \rangle_{\text{class}} = \iint_{\mathbb{R} \times \mathbb{R}} \overline{\psi(p, q)} \phi(p, q) dp dq, \quad (7.6)$$

the classical wave functions form a Hilbert space  $(\mathcal{H}_{\text{class}}, \langle \cdot | \cdot \rangle_{\text{class}})$ . The phase flow (7.5) induces a flow on  $\mathcal{H}$ , which is described most conveniently by the *time evolution operator*  $U_{\text{class}}$  defined by

$$\left( U_{\text{class}}(t) \psi \right)(p(t), q(t)) = \left( U_{\text{class}}(0) \psi \right)(p(0), q(0)). \quad (7.7)$$

It is a unitary operator on  $\mathcal{H}_{\text{class}}$ . Before going on, we remark that in classical physics one usually works instead of complex functions with positive functions or densities on phase space. Working with complex-valued functions and the scalar product (7.6), seems unusual but will be very useful for the following considerations. In a somewhat different context, the Hilbert space  $(\mathcal{H}_{\text{class}}, \langle \cdot | \cdot \rangle_{\text{class}})$  is also used in geometric quantization for the so-called prequantization (see [27, Section 5.2]).

For the quantization of the oscillator, one replaces the canonical variables  $p$  and  $q$  by self-adjoint operators  $P$  and  $Q$  which act on a complex Hilbert space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$  and satisfy the canonical commutation relations  $[P, Q] = -i$ . The physical system is now characterized by a state  $\Psi \in \mathcal{H}$ . The dynamics is described by the Schrödinger equation

$$i\partial_t \Psi = H\Psi \quad \text{with} \quad H = \frac{1}{2} (P^2 + \omega^2 Q^2). \quad (7.8)$$

It is most common to represent  $\mathcal{H}$  as the space of square integrable functions with the inner product

$$\langle \Psi | \Phi \rangle = \int_{\mathbb{R}} \overline{\Psi(q)} \Phi(q) dq, \quad (7.9)$$

and to choose the operators  $Q$  and  $P$  as

$$(Q\psi)(q) = q\psi(q) \quad \text{and} \quad P = -i\frac{d}{dq}.$$

Integrating the Schrödinger equation gives rise to the unitary time evolution operator

$$U(t) = e^{-itH} : \mathcal{H} \rightarrow \mathcal{H} : \Psi(0, q) \rightarrow \Psi(t, q). \quad (7.10)$$

With the above formulation we expressed both the classical and the quantum dynamics by a unitary time evolution operator acting on a Hilbert space (see (7.7), (7.6) and (7.10), (7.9)). But the time evolution operators have a completely different form. Furthermore, the Hilbert spaces are different, because the ‘‘classical wave functions’’ depend on both  $q$  and  $p$ . In the quantized theory, however, the Heisenberg uncertainty principle prevents  $P$  and  $Q$  from being simultaneously measurable, as reflected mathematically by the fact that they correspond to non-commuting operators. Since in the classical theory, position and momentum can be chosen independently, there is much more freedom to choose the initial wave function  $\psi(p, q)$  than in quantum theory, where choosing  $\Psi(q)$  automatically determines the corresponding wave function in

momentum space. This raises the question if for a given quantum wave function  $\Psi(q)$  we can choose a corresponding classical wave function  $\psi(p, q)$  such that the classical dynamics of  $\psi$  as described by (7.7) coincides with the time evolution of the quantum wave function (7.10). While the general answer to this question is no, it turns out that for the harmonic oscillator this correspondence can indeed be made, as is specified in the next lemma.

**Lemma 7.1. (Correspondence between classical and quantum dynamics)**

Consider the classical harmonic oscillator (7.4) with dynamics (7.5), (7.7) and the corresponding quantum harmonic oscillator with the dynamics described by the Schrödinger equation (7.8) and (7.10). Then there is an isometric embedding  $\iota : \mathcal{H} \rightarrow \mathcal{H}_{\text{class}}$  which maps the quantum evolution to a corresponding classical evolution, in the sense that

$$U_{\text{class}}(t) \iota = \iota U(t) e^{i\omega t/2} \quad \text{for all } t \in \mathbb{R}. \quad (7.11)$$

Moreover, there are differential operators  $H_{\text{class}}$ ,  $P_{\text{class}}$  and  $Q_{\text{class}}$  in  $\mathcal{H}_{\text{class}}$  such that

$$H_{\text{class}} \iota = \iota H, \quad P_{\text{class}} \iota = \iota P \quad \text{and} \quad Q_{\text{class}} \iota = \iota Q.$$

We point out that the factor  $e^{i\omega t/2}$  in (7.11) corresponds to the zero point energy of the quantum harmonic oscillator. This factor modifies the wave functions only by a joint global phase, without an influence on any observations or any expectation values.

*Proof of Lemma 7.1.* We choose an orthonormal eigenvector basis  $\Psi_n$  of the Hamiltonian in (7.8) (see for example [22, Section 3.1])

$$H\Psi_n = \left(n + \frac{1}{2}\right)\omega\Psi_n, \quad n = 0, 1, \dots$$

Writing the Hamiltonian as  $H = \omega(a^\dagger a + \frac{1}{2})$  with the annihilation and creation operators

$$a = \frac{1}{\sqrt{2\omega}} \left(\omega q + \frac{d}{dq}\right), \quad a^\dagger = \frac{1}{\sqrt{2\omega}} \left(\omega q - \frac{d}{dq}\right), \quad (7.12)$$

the eigenvectors can be obtained by acting with the creation operators on the ground state,

$$\Psi_0 = c_0 \exp\left(-\frac{\omega q^2}{2}\right) \quad \text{and} \quad \Psi_n = c_n (a^\dagger)^n \Psi_0, \quad (7.13)$$

were the  $c_n$  are positive normalization constants. From (7.10) it follows immediately that

$$U(t) e^{i\omega t/2} \Psi_n = e^{-i\omega t} \Psi_n. \quad (7.14)$$

In order to define the mapping  $\iota$ , it suffices to associate to every eigenfunction  $\Psi_n$  a corresponding classical wave functions  $\psi_n \in \mathcal{H}_{\text{class}}$  (then  $\iota$  is determined uniquely by linearity and continuity). First, in order to write the classical dynamics in a simpler form, we rescale the momentum variable by introducing the new phase space variables

$$x = q \quad \text{and} \quad y = \frac{p}{\omega}.$$

Setting  $z = x + iy$ , the time evolution operator (7.7) becomes

$$(U_{\text{class}}(t)\psi)(z) = \psi(e^{i\omega t} z). \quad (7.15)$$

We now define the ‘‘classical annihilation and creation operators’’ on  $\mathcal{H}_{\text{class}}$  by

$$a_{\text{class}} = \frac{1}{2}(a_x + ia_y), \quad a_{\text{class}}^\dagger = \frac{1}{2}(a_x^\dagger + ia_y^\dagger),$$

where  $a_x$  and  $a_x^\dagger$  are given in analogy to (7.12) by

$$a_x = \frac{1}{\sqrt{2\omega}} \left( \omega x + \frac{d}{dx} \right), \quad a_x^\dagger = \frac{1}{\sqrt{2\omega}} \left( \omega x - \frac{d}{dx} \right),$$

whereas the subscript  $y$  refers similarly to the variable  $y$ . We introduce the wave functions  $\psi_n$  by

$$\psi_0 = c_0^2 \exp\left(-\frac{\omega(x^2 + y^2)}{2}\right) \quad \text{and} \quad \psi_n = c_n (a_{\text{class}}^\dagger)^n \psi_0. \quad (7.16)$$

Let us verify that the resulting mapping  $\iota$  has the required properties. First, it is obvious from their definition (7.16) that the functions  $\psi_n$  are orthonormal in  $\mathcal{H}_{\text{class}}$ , and thus  $\iota$  is indeed an isometric embedding. Using a polar decomposition  $z = re^{i\varphi}$ , a short calculation shows that

$$\left[ i\partial_\varphi, a_{\text{class}}^\dagger \right] = a_{\text{class}}^\dagger.$$

Applying this relation in (7.16) and using that  $\psi_0$  is radially symmetric, we obtain

$$i\partial_\varphi \psi_n = n\psi_n \quad \text{and thus} \quad \psi_n(z) = e^{-in\varphi} \phi_n(r)$$

with radially symmetric functions  $\phi_n$ . Thus the classical dynamics (7.15) implies that

$$U_{\text{class}}(t) \psi_n = e^{-i\omega n t} \psi_n.$$

Comparing with (7.14) proves (7.11).

In order to construct the operators  $H_{\text{class}}$ ,  $P_{\text{class}}$  and  $Q_{\text{class}}$ , we first note that both the classical and quantum annihilation and creation operators satisfy the canonical commutation relations

$$\left[ a_{\text{class}}^\dagger, a_{\text{class}} \right] = 1 \quad \text{and} \quad \left[ a^\dagger, a \right] = 1,$$

and in view of (7.13) and (7.16) they correspond to each other in the sense that

$$a_{\text{class}} \iota = \iota a \quad \text{and} \quad a_{\text{class}}^\dagger \iota = \iota a^\dagger.$$

Thus expressing the operators in  $\mathcal{H}$  in terms of  $a$  and  $a^\dagger$ , we obtain the corresponding ‘‘classical’’ operators simply by adding subscripts. Thus we set

$$H_{\text{class}} = \omega \left( a_{\text{class}}^\dagger a_{\text{class}} + \frac{1}{2} \right)$$

and

$$Q_{\text{class}} = \frac{1}{\sqrt{2\omega}} \left( a_{\text{class}} + a_{\text{class}}^\dagger \right), \quad P_{\text{class}} = -i \sqrt{\frac{\omega}{2}} \left( a_{\text{class}} - a_{\text{class}}^\dagger \right),$$

concluding the proof.  $\square$

We remark that the mapping  $\iota$  appears in the mathematical physics literature as the so-called Bargmann transform (see [25, Section 4.3]). But to our knowledge, it has not been used to get a connection between the classical and quantum dynamics.

The above lemma shows that by choosing the ‘‘classical wave function’’  $\phi \in \mathcal{H}_{\text{class}}$  appropriately, we can arrange that the classical dynamics reproduces any quantum dynamics. In simpler terms, the quantum dynamics of the harmonic oscillator can be recovered as a special case of the classical dynamics. However, for making this correspondence, we had to take a somewhat unusual point of view and work on classical phase space with complex-valued functions and the scalar product (7.6). To us, Lemma 7.1 is useful because it makes it possible to *approximate a quantum state by*

a finite number of classical trajectories, if to every classical trajectory we associate a complex number. This can be seen as follows: Suppose that we want to describe a quantum state  $\Psi \in \mathcal{H}$ . According to Lemma 7.1, this state has the same dynamics as the classical wave function  $\psi := \iota\Psi \in \mathcal{H}_{\text{class}}$ . For any  $L \in \mathbb{N}$  and an index  $a = 1, \dots, L$ , we now choose points  $(p^{(a)}, q^{(a)})$  in phase space together with complex coefficients  $\phi(a)$  which approximate  $\psi$  in the sense that

$$\sum_{a=1}^L \phi(a) \delta(p - p^{(a)}) \delta(q - q^{(a)}) \xrightarrow{L \rightarrow \infty} \psi(p, q), \quad (7.17)$$

with convergence in the distributional sense. For these discrete configurations, we can make sense of the scalar product (7.6) by setting

$$\langle (p^{(a)}, q^{(a)}, \phi(a)) | (\tilde{p}^{(b)}, \tilde{q}^{(b)}, \tilde{\phi}(b)) \rangle = \sum_{a,b} \delta_{p^{(a)}, \tilde{p}^{(b)}} \delta_{q^{(a)}, \tilde{q}^{(b)}} \overline{\phi(a)} \tilde{\phi}(b).$$

Then by choosing  $L$  sufficiently large, we can approximate the quantum dynamics of  $\Psi$  by a complex valued function  $\phi$  defined on a finite number of classical trajectories.

Before going on, we point out that the scalar product (7.6) is invariant under the local phase transformations

$$\phi(p, q) \rightarrow e^{i\varphi(p,q)} \phi(p, q), \quad (7.18)$$

where  $\varphi$  is a real-valued function on phase space. Thus the phase of the functions in  $\mathcal{H}_{\text{class}}$  is not of physical relevance. What counts is only the *relative phase* when taking superpositions of two wave functions  $\psi, \phi \in \mathcal{H}_{\text{class}}$ . Similarly, in the discrete approximations in (7.17), the phase of the function  $\phi(a)$  can be changed by

$$\phi(a) \rightarrow e^{i\varphi(a)} \phi(a) \quad \text{under the constraint} \quad (p^{(a)}, q^{(a)}) = (p^{(b)}, q^{(b)}) \implies \varphi(a) = \varphi(b).$$

In the remainder of this section, we carry over the previous results on the harmonic oscillator to the Hamiltonian of the electromagnetic field (7.3). By taking tensor products, the result of Lemma 7.1 immediately extends to a collection of harmonic oscillators as in (7.3). It then states that by considering suitable complex-valued functions on the set of all classical field configurations, one can reproduce the full dynamics of the free second-quantized field. Using an approximation argument similar to (7.17), it suffices to consider a finite number of classical field configurations. Thus our task is to associate to every classical field configuration a complex number  $\phi(a)$ . The point is that in the framework of the fermionic projector, this relative phase information can easily be obtained, as we now explain. Returning to the setting of decoherent subsystems in the continuum limit (7.2), every subsystem involves a classical electromagnetic potential  $A^{(a)}$ . In the considered case without fermions, the field equations reduce to the free Maxwell equations, i.e. again in the Coulomb gauge

$$\square \vec{A}^{(a)}(t, \vec{x}) = 0 \quad (7.19)$$

and  $(A^{(a)})^0(t, \vec{x}) = 0$ . Moreover, we have the Dirac equation for the fermionic projector, which according to (5.7) consists only of the sea states,

$$\left( i\gamma^j (\partial_j - ieA_j^{(a)}) - m \right) P^{\text{sea}}(x, y) = 0 \quad \text{if } x \in M_a. \quad (7.20)$$

Having an ultraviolet regularization in mind, the number  $f$  of sea states is finite, so that  $P^{\text{sea}}$  can be written in the form (7.1). For a given solution  $\vec{A}$  of the free Maxwell equations, we now introduce the following reference system. The causal perturbation

expansion distinguishes a subspace of the solution space of the Dirac equation as being formed by the sea states. Selecting the  $f$ -dimensional subspace of the sea states which is compatible with our regularization and choosing an orthonormal basis  $\check{\psi}_1, \dots, \check{\psi}_f$  of this subspace, we can introduce the many-particle wave function  $\check{\Psi}$  of our reference system by

$$\check{\Psi} = \check{\psi}_1 \wedge \dots \wedge \check{\psi}_f.$$

As in Lemma 2.2, the freedom in choosing the orthonormal basis implies that  $\check{\Psi}$  is determined only up to a phase. Now suppose that  $\check{A}$  coincides with the electromagnetic potential  $A_a$  in one of our subsystems. Then the wave functions  $\psi_1^{(a)}, \dots, \psi_f^{(a)}$  obtained by restricting the wave functions of the fermionic projector of the whole system to the subsystem  $M_a$  span the same subspace of the Dirac solutions as the vectors  $\check{\psi}_1, \dots, \check{\psi}_f$  (probably after suitably modifying the solutions on the microscopic scale or modifying the regularization; a technical issue which for simplicity we will ignore here). Hence the corresponding many-particle wave function (6.4) coincides with  $\check{\Psi}$  up to a complex number,

$$\Psi^{(a)} = \phi(a) \check{\Psi} \quad \text{with} \quad \phi(a) \in \mathbb{C}. \quad (7.21)$$

In this way, we have associated to the field configuration  $A_a$  a complex number  $\phi(a)$ .

Let us consider the phase freedom. As noted above, the phase of the wave function  $\check{\Psi}$  depends on the choice of the basis  $\check{\psi}_1, \dots, \check{\psi}_f$ . Similarly, by transforming the orthonormal basis  $\psi_1, \dots, \psi_f$  of the image of  $P$ , we can also change the phase of  $\Psi^{(a)}$  arbitrarily. Thus (7.21) is well-defined only up to a phase. Now suppose that  $\check{A}$  also coincides with the electromagnetic potential  $A_b$  of another subsystem. Then writing the many-particle wave function of the new subsystem as  $\Psi^{(b)} = \phi(b) \check{\Psi}$ , transforming the bases  $\check{\psi}_1, \dots, \check{\psi}_f$  or  $\psi_1, \dots, \psi_f$  changes the phase of both  $\phi(a)$  and  $\phi(b)$  in the same way. Thus the relative phase of  $\phi(a)$  and  $\phi(b)$  is well-defined. In other words, the complex-valued function  $\phi$  is uniquely defined up to the transformations

$$\phi(a) \rightarrow e^{i\varphi(a)} \phi(a) \quad \text{under the constraint} \quad A_a = A_b \implies \varphi(a) = \varphi(b). \quad (7.22)$$

These transformations can be regarded as local phase transformations on the classical field configurations, just as explained after (7.18) for one harmonic oscillator on phase space.

We conclude that the above construction indeed yields a complex-valued wave function  $\phi(a)$ ,  $a = 1, \dots, L$ , defined on the classical field configurations  $\{A^{(1)}, \dots, A^{(L)}\}$  of the subsystems. It is uniquely determined up to the local phase transformations (7.22). These results make it possible to *approximate a general state of the bosonic Fock space by our decoherent subsystems*, as the following consideration shows: According to Lemma 7.1, the dynamics of a given bosonic Fock state can be described by a complex-valued wave function  $\phi$  on the classical field configurations. By considering similar to (7.17) a sequence of systems where the number of decoherent subsystems tends to infinity, we can approximate  $\phi$  by a finite collection of classical field configurations  $\{A^{(1)}, \dots, A^{(L)}\}$  and a corresponding complex-valued functions  $\phi(a)$ . By suitably adjusting the phases of the sea states  $\psi_1^{(a)}, \dots, \psi_f^{(a)}$  (for given reference systems  $\check{\Psi}$ ), we can arrange that the function  $\phi(a)$  satisfies (7.21). Then the wave functions  $\psi_1, \dots, \psi_f$  of the whole system encode the classical potentials  $\{A^{(1)}, \dots, A^{(L)}\}$  as well as the complex-valued function  $\phi(a)$ , which together approximate the given bosonic Fock state.

**7.2. Describing a Second Quantized Fermion-Boson System.** We will now combine the considerations of the previous section with the constructions of Chapter 6 to obtain a unified framework for describing second quantized fermions and bosons. We again consider  $L$  decoherent subsystems in the continuum limit (7.2). According to (5.7), we can split up the fermionic projector into the particle- and anti-particle as well as the sea states. We begin for clarity in the situation without pair creation where the numbers  $n_a$  and  $n_f$  of particles and anti-particles are constant and coincide in all subsystems (this constraint will be removed below). Then setting  $n = n_a - n_f$ , the corresponding many-particle wave functions of the subsystems (6.4) can be decomposed as

$$\Psi^{(a)} = \left( \psi_1^{(a)} \wedge \cdots \wedge \psi_{n_f}^{(a)} \wedge \phi_1^{(a)} \wedge \cdots \wedge \phi_{n_a}^{(a)} \right) \wedge \left[ \psi_{n+1} \wedge \cdots \wedge \psi_f \right]. \quad (7.23)$$

Here the round brackets can be regarded as the fermionic wave function of the particles and anti-particles. As explained in Section 6.3, measurements involve superpositions of these many-particle wave functions, so that it is reasonable to regard the round brackets in (7.23) as a vector in the fermionic Fock space  $\mathcal{F}^{\text{eff}}$ . Likewise, the square brackets in (7.23) describe the sea. The construction (7.21) yields a corresponding complex wave function  $\phi(a)$  on the classical field configurations, which can be used to describe the dynamics of a second quantized bosonic field. In this way, we have extracted from the fermionic projector both a fermionic and a bosonic quantum field.

We now give a construction which makes it possible to avoid the splitting of the many-particle wave function into the particle/anti-particle component and the sea component. Apart from being simpler and cleaner, this has the advantage of working just as well for fully interacting systems, possibly involving pair creation or annihilation processes. We first recall that in (7.21) we compared the many-particle wave function  $\Psi^{(a)}$  of our subsystem with the wave function  $\check{\Psi}$  of a “reference system” having the same classical field configuration. The proportionality factor  $\phi(a)$  then gave us the desired complex-valued function  $\phi$  on the classical field configurations. Giving up the requirement that the vector space  $\mathcal{H}_{\text{class}}$  should be represented by complex-valued functions, one can work instead of  $\phi(a)$  with the corresponding vector  $\Psi^{(a)} \in \mathcal{F}_f^{\text{eff}}$ . This has no effect on superpositions, because the complex coefficients  $\phi(a)$  and  $\phi(b)$  can be linearly combined only if the corresponding classical field configurations  $A^{(a)}$  and  $A^{(b)}$  coincide. But then the corresponding Fock vectors  $\Psi^{(a)}, \Psi^{(b)} \in \mathcal{F}_f^{\text{eff}}$  are linearly dependent, so that taking their linear combination is the same as taking the linear combinations of the coefficients  $\phi(a)$  and  $\phi(b)$ . This consideration leads us to replace the complex-valued function  $\phi(a)$  constructed in (7.21) by a mapping with values in  $\mathcal{F}_f^{\text{eff}}$ ,

$$\phi : \{1, \dots, L\} \rightarrow \mathcal{F}_f^{\text{eff}} : a \mapsto \psi_1^{(a)} \wedge \cdots \wedge \psi_f^{(a)}. \quad (7.24)$$

In the setting involving particles and anti-particles (7.23), this mapping has the nice property that it involves at the same time the fermionic wave functions of the particles and anti-particles. In free field theory, it can be thought of as the tensor product of a fermionic Fock state and a bosonic state. As desired, two such tensor states are linearly dependent only if both the fermionic and bosonic parts are. Superpositions of these tensor states can be justified exactly as explained in Section 6.3. In a fully interacting system, the mapping (7.24) can no longer be decomposed into a fermionic and a bosonic part, in agreement with the fact that in interacting quantum systems the bosonic and fermionic Fock spaces are coupled together and “mixed” by the Hamiltonian.

Even in this highly complicated situation, the mapping  $\phi$  gives a conceptually simple description of the whole system.

**7.3. Remarks and Outlook.** To avoid confusion, we point out that the constructions in this chapter are not equivalent to the canonical quantization of the bosonic field. In particular, we do not get the physical equations for second quantized fields. Instead, we only show that the dynamics of *free second quantized bosonic fields* can be mimicked by our ensemble of decoherent subsystems, each with a classical dynamics. However, we do not get a justification nor explanation for the physical assumption that electromagnetic wave modes should behave like quantum mechanical oscillators. But we show that this assumption is not in conflict with the framework of the fermionic projector. In particular, it is possible to describe *entangled bosonic states*.

In order to explain why we do not even attempt to reproduce the physical equations for second quantized fields, we now briefly outline how interacting quantum field theory should be formulated in the framework of the fermionic projector. Recently, this formulation of quantum field theory has been worked out in detail for a system involving an axial field [9]. The general strategy is as follows. Instead of quantizing the classical field equations, we describe the interaction and the dynamics of the system by the action principle (5.5). In the continuum limit, the Euler-Lagrange equations corresponding to this action principle give rise to the Dirac equation coupled to classical bosonic field equations (see (5.8) and (5.9)). Treating this coupled system of nonlinear partial differential equations in a perturbation expansion gives rise to all the Feynman diagrams of perturbative quantum field theory (see [9, Section 8.4]). In particular, this gives agreement with the high-precision tests of quantum electrodynamics. We remark that we get additional small corrections to the field equations which are absent in perturbative quantum field theory; the interested reader is referred to [9, Section 8.2 and 8.3].

Since the quantitative aspects are respected, it remains to explain the particular effects of quantized fields. The present paper is concerned with entanglement and shows that entangled fermionic and bosonic states can be described in the framework of the fermionic projector. Other quantum effects related to the measurement problem and the wave-particle duality will be discussed in Chapter 8 below; see also [7, Section 4]. Putting these results together, it seems to us that the framework of the fermionic projector is in agreement with all effects and predictions of quantum field theory (except for the additional corrections discussed in [9, Sections 8.2 and 8.3]). But this agreement cannot be stated in terms of a mathematical equivalence, partly because standard quantum field theory at present has no fully convincing mathematical formulation. Also, many difficulties of quantum field theory clearly remain unsettled in our approach. Thus many conceptual and technical issues need to be debated in the future.

We finally point out that the agreement with free quantized fields in Section 7.1 is obtained only in the limit when the number of subsystems tends to infinity. Thus even for describing the quantum oscillations of the harmonic oscillator corresponding to one mode of the electromagnetic field, one needs to consider a large number of decoherent subsystems. Although this seems possible in principle, it seems hard to imagine that decoherence should really lead to a “fragmentation” of space-time into many disjoint regions with an independent dynamics. This raises the question whether a microscopic mixing of decoherent subsystems might not be a too simple picture for understanding



the mechanisms of space-time on a small scale. Indeed, it might be more appropriate to replace this picture by a more general concept which we now explain in words. Recall that in Section 6.2 the decoherence of subsystems was introduced by inserting a unitary transformation into the fermionic projector,

$$P(x, y) = - \sum_{j,k=1}^f U_{jk} |\psi_j^{(2)}(x)\rangle \langle \psi_k^{(1)}(y)|, \quad (7.25)$$

if  $x$  and  $y$  are in different subsystems (see (6.7)). Since  $P(x, y)$  is a  $4 \times 4$ -matrix, a dimensional argument shows that there is a large class of operators  $U \in \text{SU}(f)$  which do not affect the form of  $P(x, y)$ . In order to make use of this additional freedom, we replace  $U$  by a family of local unitary transformations  $U(x) \in \text{SU}(f)$ , which brings the fermionic projector to the more general form

$$P(x, y) = - \sum_{j,k,l=1}^f U_{jk}(x) U_{kl}^{-1}(y) |\psi_j(x)\rangle \langle \psi_l(y)|. \quad (7.26)$$

By dividing  $M$  into subregions and choosing  $U(x)$  to be constant on each subregion, we get back to the setting of Section 6.2. But if  $U(x)$  is not a piecewise constant function, the situation is more involved. Namely, for any space-time points  $x$  and  $y$ , it is possible that the transformation  $U_{jk}(x) U_{kl}^{-1}(y)$  has no effect on  $P(x, y)$  (similar as discussed in (7.25)); in this case the pair  $(x, y)$  is said to be coherent. Another possibility is that the transformation  $U_{jk}(x) U_{kl}^{-1}(y)$  leads to cancellations in the sum so that  $P(x, y)$  is very small (similar as explained after (6.7)), in which case the pair  $(x, y)$  is said to be decoherent. This notion of decoherence again gives a relation between space-time points. But in contrast to the situation in Section 6.2, this relation is no longer transitive; for example, it is possible that the pairs  $(x, y)$  and  $(y, z)$  are coherent, but the pair  $(x, z)$  is decoherent. As a consequence, decoherence no longer gives rise to a decomposition of space-time into subregions. But for each fixed space-time point  $x$ , one can form the set  $M(x)$  of all space-time points which are coherent to  $x$ . This set can then be divided into subsets  $M_j(x)$  by the condition that any two points  $y, z \in M_j(x)$  should be coherent to each other. On the sets  $M_j(x)$ , one can then again consider the continuum limit to obtain for example the Dirac-Maxwell system (5.8), (5.9). Thus on the coherent space-time points one again gets a description involving classical field equations. Decoherent pairs of space-time points, on the other hand, are not connected by our action principle. Apart from the considered cases of coherence and decoherence, it is also possible that two space-time points are only partially decoherent in the sense that there are cancellations in the sums (7.26), but without  $P(x, y)$  being very small. However, such a partial decoherence should be suppressed by our action principle because the corresponding space-time points should yield a large contribution to the action. The resulting structure resembles the situation in Section 6.2 in that we obtain decoherent subsystems with an independent dynamics. The main difference is that the subsystems are no longer localized in disjoint regions of space-time. Instead, they are all delocalized, and only when picking a pair of space-time points  $(x, y)$ , the phases in the sum (7.26) determine to which subsystem the pair belongs. Due to the obvious analogy to a hologram, we refer to this concept as the *holographic superposition of subsystems* (but it does not seem to be directly related to 't Hooft's holographic principle).

The main advantage of a holographic superposition is that a large number of subsystems no longer leads to a “fragmentation” of space-time into disjoint space-time regions. On the other hand, all the effects considered in this paper can be described just as well by decoherent space-time regions. Therefore, the holographic superposition is not essential for our purposes, and we shall not enter the detailed constructions here.

## 8. PHYSICAL INTERPRETATION

We now explain the previous constructions and results from the physical point of view. For clarity, we try to explain the physical picture in simple examples, without aiming for mathematical rigor nor maximal generality.

**8.1. The Superposition Principle.** In the standard formulation of quantum physics, the one-particle wave functions as well as the whole physical system are described by vectors in a Hilbert space. Taking linear combinations of such vectors, one can form superpositions of wave functions, but also superpositions of physical systems. Thus in this formulation, it is a general physical principle that superpositions of states can be formed (for a good exposition see [14, Section 2.1]). In the framework of the fermionic projector, however, where the whole system is described by a projector in an indefinite inner product space, the validity of the superposition principle is not obvious. We now explain this point by reviewing the previous constructions.

In the framework of the fermionic projector, the superposition principle arises on different levels. On the fundamental level of discrete space-time, the wave functions are vectors in the indefinite inner product space  $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ . Thus superpositions of one-particle wave functions can be formed. In the continuum limit, where the interaction is described by the Dirac equation (5.8) coupled to a classical field (5.9), we thus obtain the superposition principle for the Dirac wave functions. Moreover, since the Maxwell equations are linear, the superposition principle also holds for classical electromagnetic waves.

For many-particle states, however, the superposition principle does not hold on the fundamental level. In particular, it is impossible to form the naive superposition of two physical systems, simply because the linear combination of two projectors in general is no longer a projector. But the superposition principle holds for the effective fermionic many-particle wave function obtained by decomposing the system into decoherent subsystems and homogenizing on the microscopic scale (see Section 6.3). For the free second-quantized electromagnetic field as described in Section 7, the superposition principle corresponds to taking linear combinations of the complex-valued wave function  $\phi$  defined on the classical field configurations. Since the function  $\phi$  is constructed out of the fermionic many-particle wave function (see (7.21) or more generally (7.23) and (7.24)), linear combinations are again justified exactly as for the fermionic many-particle wave function (see Section 6.3).

We conclude that in the framework of the fermionic projector, the superposition principle again holds. It is possible to form linear combinations of macroscopic systems (like a dead and a living cat). Nevertheless, the framework of the fermionic projector differs from standard quantum theory in that for many-particle wave functions, the superposition principle does not hold on the fundamental level, but it is merely a consequence of the microscopic mixing of decoherent subsystems. This means that the superposition principle is overruled in situations when our action principle in discrete

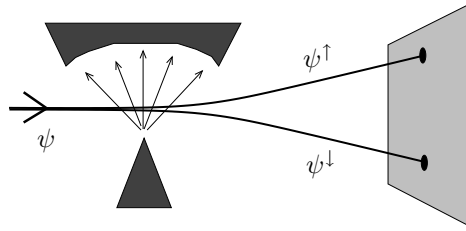


FIGURE 3. The Stern-Gerlach experiment.

space-time (5.5) needs to be analyzed beyond the continuum limit. We will come back to this point in Section 8.4 in the context of collapse phenomena.

**8.2. The Measurement Problem and Decoherence.** One of the most controversial and difficult points in the understanding of quantum physics is the so-called measurement problem. In simple terms, it can be understood from the dilemma that on one side, the dynamics of a quantum system is described by a linear evolution equation in a Hilbert space (for example the Schrödinger equation), and considering the measurement apparatus as part of the system, one would expect that this linear and deterministic quantum evolution alone should give a complete description of physics. But on the other side, the Copenhagen interpretation requires an external observer, who by making a measurement triggers a “collapse” or “reduction” of the wave function to an eigenstate of the observable. It is not obvious how the external observer can be described within the linear quantum evolution. Also, the statistical interpretation of the expectation value in the measurement process does not seem to correspond to the deterministic nature of the quantum evolution. Similarly, in the formalism introduced in Section 4.2, the independent dynamics of the subsystems as well as Assumption (A) seem to contradict the external observer entering Assumptions (B) and (D).

The measurement problem has been studied extensively in the literature, and many different solutions have been proposed (see for example [26, 5, 14, 4, 2, 20]). Here we shall not try to enter an exhaustive discussion or comparison of the different interpretations of quantum mechanics. We only explain how our concepts fit into the picture and give one possible interpretation which corresponds to the personal preference of the author. But it is well possible that the framework of the fermionic projector can be adapted to other interpretations as well.

We begin by considering the *Stern-Gerlach experiment*. Thus a beam of atoms passes through an inhomogeneous magnetic field. Decomposing the wave function  $\psi$  into the components with spin up and down,

$$\psi = \psi^\uparrow + \psi^\downarrow, \quad (8.1)$$

these two components feel opposite magnetic forces. As a consequence, the beam splits up into two beams, leading to two exposed dots on the photographic material (see Figure 3). If the intensity of the beam is so low that only one atom passes through the magnetic field, then either the upper or the lower dot will be exposed, both with probability one half. It is impossible to predict whether the electron will fly up or down; only probabilistic statements can be made.

Let us try to describe the Stern-Gerlach experiment in the framework of the fermionic projector. For simplicity, we replace the atom by an electron (disregarding the Lorentz

force due to the electron's electric charge). Then at the beginning, the system is described by a classical external magnetic field and a Dirac wave function  $\psi(t, \vec{x})$  which at time  $t = 0$  has the form of a wave packet moving towards the magnetic field. This situation is modeled by the fermionic projector in the continuum limit (5.7) for one particle; the dynamics is described by the Dirac equation (5.8) in the given external field. Solving the Dirac equation, the wave function splits up into two components (8.1), which are deflected upwards and downwards, respectively. Writing the contribution of the wave function to the fermionic projector (5.7) as

$$-\frac{1}{2\pi} |\psi(x)\rangle \langle \psi(y)| = -\frac{1}{2\pi} \left( |\psi^\uparrow(x)\rangle \langle \psi^\uparrow(y)| + |\psi^\downarrow(x)\rangle \langle \psi^\downarrow(y)| \right) \quad (8.2)$$

$$+ |\psi^\uparrow(x)\rangle \langle \psi^\downarrow(y)| + |\psi^\downarrow(x)\rangle \langle \psi^\uparrow(y)|, \quad (8.3)$$

one gets contributions of different type. Namely, the two summands in (8.2) are localized at the upper and lower electron beam, respectively. The two summands in (8.3), however, are delocalized and give correlations between the two beams. As observed in [9, Chapter 10], the Euler-Lagrange equations in the continuum limit cannot be satisfied if general delocalized contributions to the fermionic projector are present. This means that there should be a mechanism which tries to avoid nonlocal correlations as in (8.3). A possible method for removing the nonlocal correlations is to divide the system into two decoherent subsystems (as shown in Figure 2, although at this stage they do not necessarily need to be microscopically mixed), in such a way that  $\psi^\uparrow$  belongs to the first and  $\psi^\downarrow$  to the the second subsystem. Then the continuum limit is to be taken separately in the two subsystems. The contribution of the wave function in the two subsystems simply is

$$-\frac{1}{2\pi} |\psi^\uparrow(x)\rangle \langle \psi^\uparrow(y)| \quad \text{and} \quad -\frac{1}{2\pi} |\psi^\downarrow(x)\rangle \langle \psi^\downarrow(y)|,$$

respectively. Thus the delocalized terms (8.3) no longer occur, so that the problem of solving the Euler-Lagrange equations observed in [9, Chapter 10] has disappeared. This consideration gives a possible mechanism for the *generation of subsystems*.

Let us carefully discuss different notions of *decoherence*. First of all, the space-time points of the two subsystems should be decoherent, in the sense that the states of the fermionic projector are unitarily transformed in the second subsystem (6.5). This decoherence has the effect that the fermionic projector  $P(x, y)$  becomes very small if  $x$  and  $y$  are in different subsystems (see (6.7) as well as Lemmas 6.1 and 6.2), implying that the two subsystems have an independent dynamics in the continuum limit. However, the many-particle wave function of the system is not affected by the decoherence between the space-time points (see (6.6)). Rewriting it similar to (7.23) as the wedge product of the one-particle wave function  $\psi$  with the sea states, one sees that the quantum mechanical wave functions of the two subsystems are still coherent. In particular, if the two beams interfered with each other (for example after redirecting them with additional Stern-Gerlach magnets), they could be superposed quantum-mechanically, giving rise to the usual interference effects of the double slit experiment. We also point out that the dynamics of each subsystem is still described by the Dirac equation (5.8) in the external magnetic field. Since the Dirac equation is linear, solving it for  $\psi$  is the same as solving it separately for the two components  $\psi^\uparrow$  and  $\psi^\downarrow$ . Thus at this point, the dynamics is not affected by the decomposition into subsystems; we still have the linear deterministic dynamics as described by the Dirac equation.

As just explained, at this stage the generation of subsystems has no effect on the dynamics of the system. This suggests that it should not be observable whether the subsystems have formed or not. This motivates us to demand that expectation values taken with respect to the measurement scalar product (as introduced in Section 6.3) should not be affected by the generation of subsystems. Keeping in mind that, after a suitable homogenization process, the measurement scalar product coincides with the integral (5.2), we can say alternatively that the process of *generation of subsystems should preserve the probability densities*. This condition also ensures that when we get the connection to the statistical description of the measurement process, the probabilities are indeed given by the spatial integrals of the absolute square of the wave functions (in agreement with Assumption (B) in Section 4.2). Since the probability density is the zero component of the probability current  $\langle \psi | \gamma^j | \psi \rangle$ , we can say equivalently that the generation of subsystems should respect *current conservation*. This assumption seems reasonable, because current conservation holds in each subsystem as a consequence of the Dirac equation, and we merely extend this conservation law to the situation when the number of subsystems changes.

The dynamics becomes more complicated when the wave function approaches the screen, because the interaction with the electrons of the photographic material can no longer be described by an external field. Instead, one must consider the coupled Dirac-Maxwell equations (5.8) and (5.9) for a many-electron system. Since the interaction is no longer linear, it now makes a difference that the two subsystems have an independent dynamics. More specifically, in the first subsystem the wave function in the upper beam interacts with the electrons near the upper impact point, whereas the second subsystem describes the interaction of the lower beam. The whole system is a superposition of these two systems, described mathematically by a vector in the Fock space  $\mathcal{F}^{\text{eff}}$ . Exactly as explained in [14, Chapter 3], the different interaction with the environment leads to a decoherence of the many-particle wave functions of the two subsystems. Thus now it is no longer possible to form quantum mechanical superpositions of the wave functions of the two subsystems. The whole system behaves like a statistical ensemble of the two subsystems. Following the resolution of the measurement problem as proposed in [14], one should regard the human observer as being part of the system. Thus the observer is also decomposed into two observers, one in each subsystem. The two observers measure different outcomes of the experiment. Due to the decoherence of their wave functions, the two observers cannot communicate with each other and do not even experience the existence of their counterparts. From the point of view of the observer, the outcome of the experiment can only be described statistically: the electron moves either up or down, both with probability one half.

Other experiments like the *spin correlation experiment* can be understood similarly. One only needs to keep in mind that if entanglement is present, the subsystems must be microscopically mixed. The quantum state of each subsystem is not entangled. But homogenizing on the microscopic scale leads to an effective description of the system by an entangled state in the Fock space  $\mathcal{F}^{\text{eff}}$  (see Example 4.2).

We finally remark that the mechanism for the generation of subsystems proposed above could be made mathematically precise by analyzing the action principle (5.5) in the discrete setting, going beyond the approximation of the continuum limit. One should keep in mind that on this level, our action principle *violates causality*. Thus it is conceivable that the formation of subsystems depends on later measurements or that subsystems tend to form eigenstates of the subsequent measurement device. However,

such effects cannot be verified or falsified in experiments, and therefore we will not consider them here (for a discussion of a measurable effect of causality violation see [9, Section 8.2]).

**8.3. The Wave-Particle Duality.** Following the above arguments, one can also understand the wave-particle duality in a way where the wave function is the basic physical object, whereas the particle character is a consequence of the interaction as described by the action principle (5.5). To explain the idea, we return to the Stern-Gerlach experiment of Figure 3. In the previous section, we justified that if one electron flies through the magnetic field, the atom will expose either the upper or the lower dot on the screen, much in contrast to the behavior of a classical wave, which would be observable at both dots of the screen at the same time. Repeating the arguments of the previous section on the scale of the atoms of the photographic material, we conclude that more and more subsystems will form, which become decoherent as explained in [14, Chapter 3]. For an observer in one of the subsystems, the electron will not expose the whole dot on the screen uniformly, but it will only excite one atom of the photographic material. As a consequence, the electron appears like a point particle. Again, the outcome of the measurement can only be described statistically.

**8.4. The Collapse of the Wave Function.** The resolution of the measurement problem in Section 8.2 is mathematically convincing and explains the experimental observations. It goes back to Everett’s “relative state interpretation” and has found many different variations (see [26] and [4]). But no matter which interpretation one prefers, there always remains the counter-intuitive effect that all the possible outcomes of experiments are realized as components of the quantum state of the system. Thus when time evolves, the quantum state disintegrates into more and more decoherent components, which should all describe a physical reality. This phenomenon, which is often subsumed under the catchy but oversimplified title “many-worlds interpretation,” is difficult to imagine and hard to accept. Another criticism is that decoherence leads to an effective description by a density operator, which however does not uniquely determine the Fock states of the decoherent components (for details see [11, Section 4] or [2, Section 6]). In order to avoid these problems, it has been proposed to introduce a mechanism which leads to a “collapse” or “reduction” of the wave function. Different mechanisms for a collapse have been discussed, in particular models where the collapse occurs at discrete time steps [12] or is a consequence of a stochastic process [19]. These models have in common that the superposition principle is overruled by a nonlinear component in the dynamics. The nonlinearity is chosen so weakly that it does not contradict the experimental evidence for a linear dynamics. In other words, the nonlinear term is so small that it cannot be detected experimentally. But nevertheless, this term can be arranged to prevent superpositions of macroscopically different wave functions.

In the framework of the fermionic projector, the dynamics as described by the action principle (5.5) is nonlinear. This indeed provides a *new collapse mechanism*, as we now explain. Suppose that our system is described by many decoherent subsystems. Since perfect decoherence seems impossible to arrange, the kernel of the fermionic projector  $P(x, y)$  will in general not vanish identically if  $x$  and  $y$  are in two different subsystems. This gives rise to a contribution to the action (5.5) which, although being small due to decoherence, is strictly positive. These contributions, which “mix” different subsystems, grow quadratically with the number of subsystems, thus penalizing

a very large number of subsystems. This shows that there is a mechanism which tries to reduce the number of subsystems, violating the superposition principle and the independent dynamics of the subsystems. It seems very plausible that this mechanism leads to a collapse of the wave function (although a derivation from the action principle (5.5) or a quantitative analysis has not yet been given). We also point out that, just as in other collapse theories, the collapse itself seems very difficult to observe. Namely, as systems whose quantum states are decoherent no longer interact with each other, it is impossible to decide whether there are other worlds beside the one observed by us, or whether the whole system has collapsed into the state which we observe. For this reason, despite its importance for the interpretation of quantum theory, the issue of the collapse of the wave function is often regarded as being speculative. But at least, the framework of fermion systems in discrete space-time outlined in Section 5.2 provides a well-defined mathematical setting for studying collapse phenomena. Analyzing this framework in more detail might help to overcome the open problems discussed in [20], thus leading to a fully satisfying quantum theory.

To summarize the physical interpretation, the framework of the principle of the fermionic projector seems in agreement with the superposition principle and the decoherence phenomena which explain the appearance of our classical world as well as the wave-particle duality. Our description is more concrete than the usual Fock space formulation because the decoherent components of the quantum state should all be realized in space-time by the states of the fermionic projector. Moreover, we saw qualitatively that our action principle (5.5) provides a mechanism for a collapse of the wave function and a reduction of the number of decoherent components. But this collapse seems very difficult to observe in experiments. Thus the remaining question of how many “different worlds” are realized in our space-time seems of more philosophical nature. The personal view of the author is that the fermionic projector should only realize one macroscopic world, but at present this is mere speculation.

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