Theory of Topological Edges and Domain Walls

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We investigate domain walls between topologically ordered phases in two spatial dimensions. We present a method which allows for the determination of the superselection sectors of excitations of such walls and which leads to a unified description of the kinematics of a wall and the two phases to either side of it. This incorporates a description of scattering processes at domain walls which can be applied to questions of transport through walls. In addition to the general formalism, we give representative examples including domain walls between the Abelian and non-Abelian topological phases of Kitaev’s honeycomb lattice model in a magnetic field, as well as recently proposed domain walls between spin polarized and unpolarized non-Abelian fractional quantum Hall states at different filling fractions.

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Recently, there has been considerable interest in planar systems which exhibit topological phases. It is of great interest to have a clear understanding of the edges of such systems and of domain walls between regions in different phases. In fractional quantum Hall (FQH) systems, where experimental support for the existence of a variety of topological phases is strongest, observations are almost entirely restricted to edge transport, and proposed devices for probing the topological order rely on interference of tunneling currents between edges [1–3]. In such experiments, the electron density is usually not constant throughout the sample, and islands with different filling fractions form, separated by domain walls. In lattice models with several topological phases, one may induce phase boundaries by varying the local couplings.

We present a method to determine the degrees of freedom of boundaries between topological phases and their relation to the bulk degrees of freedom, based on the condensation of bosonic quasiparticles in auxiliary layered systems. Our method is based on the topological symmetry breaking procedure of Refs. [4–7]. In those papers, only transitions between phases at equal topological central charge were considered—this corresponds to a class of transitions caused by perturbations which do not break parity or time reversal. Here we incorporate transitions which do change the central charge by adding an auxiliary layer to part of the system before any Bose condensation. The central charge is then locally the sum of the central charges of the layers. This allows application in a greater variety of physical settings. We work out two such applications, which involve Kitaev’s spin model on the honeycomb lattice [8] and a domain wall between spin polarized and unpolarized non-Abelian FQH liquids [9].

The excitations of topological phases fall into a spectrum of topological sectors a, b, c, etc., distinguished by topological quantum numbers. One such quantum number is the topological spin. In (2 + 1)D, this is a phase factor $\theta_a = e^{i2\pi n_a}$, which acts on the wave function when an excitation of type $a$ is rotated by $2\pi$. The kinematic selection rules for fusion, decay, and scattering of topological excitations are summarized in fusion rules of the form $a \times b \rightarrow \sum c N_{ab}^c$. Here the $N_{ab}^c$ are integer coefficients which give the number of independent channels by which $a$ and $b$ may fuse into $c$. Positive real numbers $d_a$, the quantum dimensions, give a measure of the number of topological degrees of freedom per $a$ particle. A full description of the bulk of a topological phase requires the structure of a (2 + 1)D topological field theory (TQFT), and the boundary of a topological medium can usually be described by a (1 + 1)D conformal field theory (CFT). Here we will make use only of spins and fusion rules, which are usually the same for bulk and boundary sectors.

One way to match two different phases I and II at a domain wall is to treat them as independent systems without interaction. The wall’s sectors are then simply pairs of phase I and phase II sectors. However, this is not always the situation observed in experiments. For example, Camino et al. [10,11] created a setup with FQH states at filling fractions $\nu = 1/3$ and $\nu' = 2/5$. The boundary has excitations of charge $e/15$ and cannot be explained as a simple product of the $\nu = 1/3$ and $\nu = 2/5$ boundaries [12]. To describe more general interfaces, we start with two layers in phases I and III, which we allow to partially overlap as indicated in Fig. 1. If we bring the layers close, we may have some binding between degrees of freedom in phases I and III in the overlap region. In particular, a bosonic composite of excitations from the two layers could occur, and, consequently, a condensate of such bosons may form. This condensation will lead to a different phase for the middle region, which we denote by II. If we are given theories $C_1$ and $C_3$ describing phases I and III, i.e., fusion rules and topological spins for these phases, the topological...
which describes the fusion and braiding of excitations of phase II.

We now make the crucial observation that, after condensation, excitations in all parts of the system can be labeled by sectors of the $T$ theory. More precisely, the bulk excitations of phase II correspond to unconfined $T$ sectors, while those of phases I and III are labeled as before by pairs $(a^I, 1)$ and $(1, b^{III})$ of $C_1 \otimes C_3$ labels (with the vacuum label 1 in the layer that cannot be excited in those phases). These pairs correspond to $T$ sectors by the branching rules. This yields unique $T$ sectors (i.e., no splitting) whenever $C_1$ and $C_3$ do not themselves have bosonic sectors. Boundary excitations correspond to confined $T$ sectors. We can now understand all of the kinematics of processes that may occur when excitations are moved toward or through walls. For example, any $C_1$ particle that is identified with a nonconfined $T$ particle can pass through the phase boundary unnoticed and vice versa, while a $C_1$ particle that corresponds to a confined $T$ particle cannot enter the region in phase II. Reversely, $T$ particles which are confined in phase II but which can be obtained from a $C_1$ sector by branching can pass into the area in phase I after being driven out of phase II. Hence, the true boundary excitations are labeled by confined $T$ sectors which do not correspond to $C_1$ sectors. For processes involving three or more excitations, we need to use the fusion rules of $T$. Any process allowed by these rules could, in principle, occur. For example, a $C_1$ particle corresponding to a confined $T$ sector $c$ could hit the phase boundary and split into a boundary excitation $a$ and a bulk excitation $b$ of phase II, provided that $c \in a \times b$ according to the fusion rules of $T$. The fusion rules of $T$ are valid throughout. For instance, the fusion channel of two particles in phase I should be preserved even if one of the particles is moved into the region in phase II. The full topological state of the multiphase system should be characterized by specifying the amplitudes for the $T$-fusion channels obtained on successive fusions of all of the quasiparticles that are present. To actually perform the fusions involved, it will usually be necessary to bring the quasiparticles from the bulk regions to the boundary.

There are many applications of these general ideas. For example, coset models in CFT can be analyzed in terms of Bose condensates [7]. The construction of these models by condensation, excitations in all parts of the system can be labeled by sectors of the $T$ theory. More precisely, the bulk excitations of phase II correspond to unconfined $T$ sectors, while those of phases I and III are labeled as before by pairs $(a^I, 1)$ and $(1, b^{III})$ of $C_1 \otimes C_3$ labels (with the vacuum label 1 in the layer that cannot be excited in those phases). These pairs correspond to $T$ sectors by the branching rules. This yields unique $T$ sectors (i.e., no splitting) whenever $C_1$ and $C_3$ do not themselves have bosonic sectors. Boundary excitations correspond to confined $T$ sectors. We can now understand all of the kinematics of processes that may occur when excitations are moved toward or through walls. For example, any $C_1$ particle that is identified with a nonconfined $T$ particle can pass through the phase boundary unnoticed and vice versa, while a $C_1$ particle that corresponds to a confined $T$ particle cannot enter the region in phase II. Reversely, $T$ particles which are confined in phase II but which can be obtained from a $C_1$ sector by branching can pass into the area in phase I after being driven out of phase II. Hence, the true boundary excitations are labeled by confined $T$ sectors which do not correspond to $C_1$ sectors. For processes involving three or more excitations, we need to use the fusion rules of $T$. Any process allowed by these rules could, in principle, occur. For example, a $C_1$ particle corresponding to a confined $T$ sector $c$ could hit the phase boundary and split into a boundary excitation $a$ and a bulk excitation $b$ of phase II, provided that $c \in a \times b$ according to the fusion rules of $T$. The fusion rules of $T$ are valid throughout. For instance, the fusion channel of two particles in phase I should be preserved even if one of the particles is moved into the region in phase II. The full topological state of the multiphase system should be characterized by specifying the amplitudes for the $T$-fusion channels obtained on successive fusions of all of the quasiparticles that are present. To actually perform the fusions involved, it will usually be necessary to bring the quasiparticles from the bulk regions to the boundary.
Ising and toric code spins and quantum dimensions.

<table>
<thead>
<tr>
<th>$c = 1/2$</th>
<th>1</th>
<th>$\sigma$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_i$</td>
<td>0</td>
<td>1/16</td>
<td>1/2</td>
</tr>
<tr>
<td>$d_i$</td>
<td>1</td>
<td>$\sqrt{2}$</td>
<td>1</td>
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$\mathbb{Z}_2$ toric code

<table>
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<th>$c = 0$</th>
<th>1</th>
<th>$e$</th>
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<tbody>
<tr>
<td>$h_i$</td>
<td>0</td>
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when a Zeeman term is added to the Hamiltonian and then displays non-Abelian topological order described by the Ising TQFT at $c = 1/2$. The Abelian phase has four sectors with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules, and the Ising model has three sectors labeled $1, \sigma$, and $\psi$, with $1$ denoting the vacuum and with nontrivial fusion rules given by $\sigma \times \sigma = 1 + \psi$, $\sigma \times \psi = \sigma$, and $\psi \times \psi = 1$.

We wish to consider a situation with an island in the Abelian phase surrounded by a medium in the Ising phase. To achieve this, we take a large disk in the Ising phase (phase I) and place a small disk on top of it, which is in a suitable phase III so that a Bose condensate can form, leaving the bulk of the small disk in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ phase (phase II). The addition of the phase III layer should lower the energy cost of creating a Bose condensate in the small disk and, hence, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules. The sectors $\sigma$ and $\psi$ are confined because they cannot be assigned a consistent spin (the corresponding identified Ising $\mathbb{Z}_2$ sectors have spins that differ by a sign). The unconfined sectors $(1, 1), (\sigma, \sigma)_1, (\sigma, \sigma)_2,$ and $(\psi, 1)$ correspond precisely to the toric code sectors $1, e, m$, and $em$, respectively, given in Table I.

Let us now look at the wall in between the phases. Of the nontrivial excitations in the interior bulk, the fermionic $(\psi, 1)$ excitation can freely move out through the wall into the exterior region, and the other two bulk excitations cannot. This corresponds well to the results of Ref. [19], where it was shown that free fermionic excitations occur throughout the phase diagram. The confined excitations are expelled from the interior. One, the $(\sigma, 1)$ excitation, can move into the exterior region, while the other, the $(1, \sigma)$ excitation, is strictly confined to the wall. Now consider a $\sigma$ excitation hitting the boundary. From the $T$ theory's fusion rules, we see that $(\sigma, 1) = (1, \sigma) \times (\sigma, \sigma)_1 = (1, \sigma) \times (\sigma, \sigma)_2$. Hence, the $\sigma$ particle can split into a boundary excitation and either an $e$- or an $m$-type toric code excitation. This corresponds well with the results of Ref. [20], where $\sigma$-like excitations were exhibited in the toric code using superpositions of $e$- and $m$-type excitations. Pushing another $\sigma$ particle through the phase boundary will allow the confined $(1, \sigma)$ excitations to annihilate, yielding either $(1, 1)$ or $(\psi, 1)$. If the two $\sigma$ particles had fusion channel 1, then the two toric code particles that form will have fusion channel 1, respectively, $em \equiv (1, \psi)$, conserving $T$ charge.

Now we turn to the interface between the Moore-Read (MR) Pfaffian FQH state at filling $\nu = 1/2$ or $5/2$ [21] and the non-Abelian spin-singlet (NASS) state of Ardonne and Schoutsen at $\nu = 4/7$ or $18/7$ [22]. This was recently considered in Ref. [9]. We will again realize it as a single-layer–two-layer boundary. We concentrate on the non-Abelian parts of the MR and NASS theories here and leave out the $U(1)$ factors (these can be put back in at any point). Consider a disk with $C_i = \text{Ising}$, corresponding to MR, with on top of that a smaller disk with $C_j = M(4, 5)$. The latter CFT is the minimal model with $c = 7/10$ corresponding to a tricritical Ising model. We give the field content of the Ising and $M(4, 5)$ theories in Tables I and II. For the $M(4, 5)$ fusion rules, we refer to Ref. [23].

The $(\psi, e')$ current is the only bosonic channel in the Ising $\otimes M(4, 5)$ model. Table III shows what happens to the various sectors in the model when it condenses. Of the initial 18 sectors, 16 become pairwise identified [because they are equivalent modulo fusion with $(\psi, e^0)$], and the other two split, giving a total of 12 $T$ sectors, listed at the top of the table. An analysis along the lines of Ref. [7] shows that the fusion rules of $T$ are given by $T = \mathbb{Z}_2 \otimes M(4, 5)$. The $T$ sectors that are not confined correspond to the sectors of the NASS state; see also Table II. The full $T$ theory does not admit a consistent braid group representation, since the confined sectors cannot be assigned unambiguously.
In conclusion, one may describe phase separated topological phase media using auxiliary layers and Bose condensation. An important question is how to fix the appropriate auxiliary theory when the jump in central charge between phases exceeds 1. It would also be of interest to study how our results relate to those of Gils et al. [25].

We thank Professor K. Schoutens for useful discussions.

References

[12] We will deal with this situation and with more general hierarchy states [13-15] in a separate publication [16].
[17] In extensions of the model, a third type of gapped phase has been found [18].