**Multibond Graph Elements in Physical Systems Theory**

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**ABSTRACT:** The multibond graph notation turns out to be a natural and concise way to represent the behaviour of energy, power, entropy and other physical properties of macroscopic multiport systems. A global classification of the multiport elements in such a system is made on the basis of this (physical) behaviour in contrast with the usual classification on the basis of the (mathematical) form of the constitutive equations. Special attention is given to junction multiports.

**I. Introduction**

In this paper the description of specific characteristics of nature will be discussed from a specific point of view and with a specific aim: macroscopic systems will be considered from the point of view of a so-called physical systems theory (1) in order to study energetic relations and behaviour in time (dynamic behaviour). The triple use of the word “specific” indicates that many a priori assumptions or postulates must be formulated in order to reach the level of abstraction necessary to develop a useful language. The grammar of this language has to be mathematics in a general sense together with the basic physical laws. However, idiom and notation will depend on the character of both object and observer, i.e. “macroscopic system” and “engineer” respectively. This intuitive part of the language appears to be very important in the design and analysis of engineering systems, although intuition is often considered to be non-scientific, in spite of the importance epistemologists, as for instance Popper (2), attribute to it. Furthermore, it has been noticed that the modelling approach to physical systems theory is useful, especially in education: the conceptual framework can be used to classify physical devices and to reveal analogies or distinctions between all kinds of theories in a simple and elegant way.

In the following, a multibond graph translation of the fundamental principles of physics will be presented in order to clarify the fact that bond graph modelling is not only an engineering tool, but that it contains some very basic underlying ideas.

**II. Energy and Power in Physical Systems**

The word “system”, which literally means “con-struction”, has been used for almost every kind of material or mental object. Consequently it has no meaning whatsoever without a context. In the area of physical systems theory, which is the context of this paper, a physical system is characterized by a state, an “instantaneous
picture" represented by a finite set of macroscopic state variables ("conserved quantities"). The existence of a set of state variables is the primitive concept of physical systems theory. Furthermore, a system is characterized by an energy function of these state variables which satisfies the "first law of thermodynamics", i.e. energy is a conserved quantity itself (3). The conservation (invariance) of this quantity reflects the necessary assumption that time does not have an absolute zero-point. In other words, an isolated system is time-translation symmetric, which results in the invariant called energy according to Noether's theorem (4).

The class of systems to be studied is further reduced by making the assumptions that neither relativistic nor quantum effects will be of fundamental importance to the dynamic behaviour of the system under study ("classical" space and time scales are used). Two of the consequences of these assumptions are that entropy can be considered as one of the state variables of the energy function and that the "second law of thermodynamics", i.e. entropy is produced during processes, will be valid. Another consequence is that matter cannot be transformed into energy, so that the amount of matter (moles) is a conserved quantity (Lavoisier's principle).

As the word system already implies, a system must have some (relational) structure, i.e. a system is a set of interrelated elements or just one element in the degenerate case. The "building bricks" (elements) of systems in the scope of physical systems theory will consist of the characteristic forms of the constitutive relations of the elements, which are, as usual in macrophysics, supposed to be phenomenological, i.e. obtained by observation (measurement) of the particular phenomenon and not by logical derivation, except for geometric relationships (e.g. coordinate transformations). The set of state variables and accordingly the energy of the whole system may be distributed over a subset of these elements, to be called energic elements, in contrast with the complementary set of non-energic elements, which only transform or transfer energy (Section IV). The "mortar" between the elements will be the restriction of power continuity, which also is a legitimate assumption in macrophysics. It is important to note that this reflects a certain choice, because in electromagnetic fields, for instance, neither energy nor power can be located. Paynter (5, p. 28), cites Heaviside ("Electro-magnetic Theory", pp. 73-77):

The principle of the continuity of energy is a special form of that of its conservation. In ordinary understanding of the conservation principle it is the integral amount of energy that is conserved, and nothing is said about its distribution or its motion. This involves continuity of existence in time, but not necessarily in space also.

But if we can localise energy definitely in space, then we are bound to ask how energy gets from place to place. If it possessed continuity in time only, it might go out of existence at one place and come into existence simultaneously at another. This is sufficient for its conservation. This view, however, does not recommend itself. The alternative is to assert continuity of existence in space also, and to enunciate the principle thus:

When energy goes from place to place it traverses the intermediate space.

This is so intelligible and practical a form of the principle, that we should do our utmost to carry it out.
This citation shows in a clear way the intuitive link between transformation (conservation) and transfer (continuity). The above "Heaviside principle" is based on the analogy between conservation of matter and conservation of energy: matter is not annihilated in one point and produced at the same rate in another point, but is transferred ("flows") from one point to another. However, matter can be located, at least macroscopically, in contrast with energy. Therefore, the choice is to consider power as the "flow of energy", i.e. to assume power continuity. Indeed, by assuming this and by making it an intrinsic property of the notation (bond graphs) "we do our utmost to carry out the principle".

In his work on the inclusion of electromagnetic (EM) phenomena in physical systems theory, Nijen Twilhaar (6) comes to the comparable conclusion that: not only is the electromagnetic energy-density (distribution of the energy) in principle undetermined, but so is the Poynting vector (motion of the energy), which is derived from a certain choice of energy-density, because the addition of terms with a rotation operator to the definition of the Poynting vector does not influence its divergence. Nijen Twilhaar shows that in a systems approach of EM fields \( ED + HD \) as the energy-density and \( E \times H \) as the Poynting vector correspond to the assumption of "power continuity" (where \( E \) and \( H \) are the electric and magnetic field strength respectively, \( D \) the dielectric displacement and \( B \) the magnetic flux density). The multibond graph notation to be discussed next is based on the principle of power continuity.

### III. Multibond Graphs and Conjugate Power Variables

#### III.1. Multibonds and multiport elements

The most important feature (and assumption) of physical systems theory is the concept of what Paynter (5) calls "reticulation" and Kron (7) "tearing" or "diakoptics" (it should be mentioned that Kron's method is often confused with the method of partitioning matrices). In these approaches the assumption is that it is possible to concentrate and separate certain properties of an object and to describe this object as a system of interrelated properties or interconnected elements ("lumps"). This context of the word "system" allows a reduction of the level of abstraction, because it is possible to represent the total power (energy exchange) between elements by the edges of a graph (in this case two parallel lines will be used to represent an edge) and the constitutive relations of the elements by the vertices of this graph, which is called a multibond graph (Fig. 1). The vertices are called multiport
elements and there are distinct types of elementary, constitutive relations, corresponding to distinct types of physical behaviour. Consequently, there are distinct classes and types of multiport elements. They are represented in the graph by a mnemonic code, e.g. "MP E" for multiport element. A port (8) is by definition a powerport, i.e. the connection of a multiport element to an edge which represents a power ("energy flow"). A port enables energy-exchange with or through the multiport element. The edges have been called vectorbonds (9, 10), but are preferably, and herein, called multibonds (11).

III.2. The power postulate

The approach we take is to postulate that the power of a system consists of two constituents, i.e. it is postulated that every powerterm \( P_i \) is the product of two conjugate (power-)variables, an effort \( e_i \) and a flow \( f_i \). It will become clear during this study that this postulate has a distinct, but valid meaning for every type of (multiport) element. The total power \( P \) of a multibond is the sum of products of two distinct types of variables for which always some physical interpretation can be found:

\[
P = \sum_i P_i = \sum_i e_i f_i = e^T f.
\]

The approach to postulate this form of the power has a conceptual advantage: in case one would try to derive that a power is the product of an effort-type and a flow-type variable, one has to prove this \textit{a priori} for all possible applications, which is an impossible task. By contrast, every application can be checked to satisfy the "power postulate" \textit{a posteriori}.

III.3. Notation and orientation of multibonds

Efforts and flows in a (multi-)bond graph are usually written above or below, to the left or to the right of the corresponding multibond respectively, provided the position of the multibond allows this (Fig. 2). A positive value of the total power \( P \) corresponds to a certain "direction" of the energy-exchange represented by the multibond. Hence the multibond needs an orientation in order to represent \( P \) unambiguously. This orientation, which also defines the orientation of the single components (powerterms) of the multibond, is symbolized in the graph by a small
oblique stroke forming a half-arrow with the multibond. This half-arrow gives (multi-)bond graphs their characteristic appearance (Fig. 2).

III.4. The generalized bond graph concept

The physical meaning of effort and flow is "subjective", because it depends on the meaning attributed to the primitive concepts, the macroscopic state variables. The types of macroscopic state variables, corresponding to different physical domains, are discussed extensively in (12) and (13). Effort and flow should not be regarded as other names for across- (two-point-) and through- (one-point-)variables respectively (14–16). Although these concepts are helpful modelling tools for some physical domains (electrical, thermal, hydraulic), a simple generalization to other domains results in paradoxical situations. An example is the controversy between the conventional voltage–force analogy and the voltage–velocity analogy in which both voltage and velocity are across-variables. In other words: Is a (mechanical) force an effort or a flow? This paradox is resolved in (12) by the introduction of a new framework for physical systems theory based on a generalization of the thermodynamic framework, which distinguishes only one type of state variable. A bond graph interpretation of this concept was first called “thermodynamic bond graphs” (TBG) in (17, 12, 18, 19), but appeared to be confusing in the sense that it was associated with thermal processes only. Therefore it was renamed the “generalized bond graph (GBG) concept” (20) and will be referred to as such hereafter.

III.5. Partial dualization and symplectic gyrators

The main difference between the conventional and the GBG approach with respect to the conjugate power variables is the symmetric role of effort and flow in the constitutive relations of the conventional elements versus the asymmetric role of effort and flow in the constitutive relations of the GBG elements. In the conventional approach, the symmetric role of effort and flow makes it possible to interchange the conjugate variables at will. This process is called dualization. If the conjugate effort and flows of all ports of a MP E are interchanged, the result is called the “dual of a MP E” (e.g. the dual of a capacitor is an inductance). If only a subset of the set of ports of a MP E is dualized, this process is called partial dualization (12). In the GBG approach (partial) dualization is not allowed a priori, but is equivalent to the elimination of an interdomain coupler called a sympletic gyrator (SGY). However, this elimination is not always possible. Moreover, the SGY only exists between certain specific physical domains under special assumptions, so that the conventional approach is a special case of the GBG approach, only valid under these assumptions (12).

III.6. Conjugate wave-power variables

Effort and flow can be transformed to other pairs of conjugate variables which are related to the power of the bond in a different way. In (5), but more extensively in an unpublished work (Chapter 2 of his meanwhile “notorious”) “Ergs and Bits: The Flow of Energy and Signals in Engineering Systems”, Paynter starts from a scattering variable approach where the resultant (not the product!) of the incident
and reflected power $P_i$ and $P_r$, expressed in the so-called conjugate wave-power variables $w_i$ and $w_r$, forms the power $P$: i.e.

$$P = P_i + (-P_r) = \left(\frac{w_i^2}{2}\right) + \left(-\frac{w_r^2}{2}\right)$$

$$= (w_i^2 - w_r^2)/2 = \left(\frac{w_i + w_r}{\sqrt{2}}\right)\left(\frac{w_i - w_r}{\sqrt{2}}\right).$$

By a linear transformation the root-power-sum (rps) and the root-power-difference (rpd) are generated and called "intrinsic effort" $\varepsilon$ and "intrinsic flow" $\varphi$ respectively:

$$\begin{bmatrix} \varepsilon \\ \varphi \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_i \\ w_r \end{bmatrix}.$$

It can be observed that the power is the product of these intrinsic power variables, which can be transformed to the "usual" effort and flow by introducing a "scaling resistance" $\alpha$ in order to be able to use the units of the primitive variables:

$$\frac{\varepsilon}{\varphi} = \alpha \left(\frac{\varepsilon}{\varphi}\right)$$

$$P_i = \varepsilon_i \varphi_i = \sqrt{\frac{\varepsilon_i}{\varphi_i}} \alpha = \varepsilon_i f_i.$$

The wave-power variables clearly show that the power of a bond may be the resultant of the powers of two opposite waves, incident and reflected. This approach may have an advantage if there is a real propagation of power in the sense of wave phenomena (hyperbolic partial differential equations). It can also be assumed, and this will be done in Section IV.1, that the time derivative of a state variable is a "flow" itself, but the corresponding bond certainly does not represent propagation of power through space, because in this case the power is a scalar quantity. It can be concluded that there is no all-embracing interpretation of effort and flow. However, all interpretations have in common that there is a conjugate pair of variables from which the power [scalar or (Poynting) vector] can be derived, so that the description of the system corresponds to the principle of energy conservation and the assumption of power continuity. This enables an interdisciplinary approach.

III.7. Multibond graphs: an extension of the bond graph notation

It should be remarked that herein (multi-)bond graphs have been introduced in a more general way than usual. The conventional bond graph with "single bonds" or "1-bonds" (21, 22), originated by Paynter (5), is a simple case of this notation, where a 1-bond, for short "bond", is drawn as one line (edge) and represents one pair of conjugate variables. In other words, multibond graphs are an extension of the notation and of the concept of bond graphs and are consequently more abstract. They allow, however, the representation of complex systems, especially in the case of coupled power terms and ordered structures.

One of the advantages of the (multi-)bond graph notation is that it is not strongly connected to one or more specific physical domains. This is in contrast to the electrical circuit notation which is strongly related to the electrical domain. For
those who are less familiar with bond graphs than with electrical circuits, a simple example is given of the translation of the one into the other, see (23). The simple RLC-circuit of Fig. 3(a) is redrawn in such a way that all power-ports are clearly shown as 2-poles [Fig. 3(b)]. Next, all 2-poles or ports are replaced by bonds (edges) and all elements by vertices, represented by ellipses containing a description of the element [Fig. 3(c)]. This so-called word bond graph can be given an orientation (half arrows) and the description can be replaced by a mnemonic code [Fig. 3(d)]. In this case the (single-)bond graph may be simplified by contracting the so-called 1-junctions representing Kirchhoff's voltage law (KVL) [Fig. 3(e), cf. Section VII.8]. This conventional result can also be “translated" to the GBG representation, which uses the so-called symplectic gyrator (SGY) as a coupler between the electrical and the magnetical domain (12): the storage of magnetic energy in the inductor can be represented by a C of the magnetic domain, while the magnetic domain is coupled to the electrical domain by a SGY [Fig. 3(f)]. Of course, in this case the SGY can be eliminated by partial dualization to yield Fig. 3(e) again. (If one would have started the modelling process by a localization of the different types of energy storage, the GBG approach would have been the most natural one.)

Although in this text (multi-)bond graph terminology is often used, it should be stressed that the ideas presented on physical systems theory do not depend on this notation. The bond graph ideas used in subsequent sections are meant to show that a classification of the vertices of bond graphs, the multiport elements, is possible. From a network point of view they can be considered as generalization of simple concepts like ideal capacitors, transformers, resistors, etc., but this approach is deliberately avoided. Basic physical principles such as conservation laws and entropy production are chosen as a starting point of the classification and result in general properties of multiports with respect to power, energy ("energicness") entropy ("entropicness") and other state variables ("flow continuity"). For further classification, mathematical properties of the constitutive relations are also used, such as the properties that a constitutive relation is either “mixing" or “non-mixing" with respect to the efforts and flows and, locally, either "reciprocal" or "non-reciprocal".

IV. Classification of Multiport Elements

It has been discussed that a (multiport) element, represented by a mnemonic code in a (multi-)bond graph, can have different characteristics with respect to power, effort and flow. A multiport element may store energy (energic multiport element) and consequently be power discontinuous, or it may be power continuous, i.e. it transforms or transfers energy without containing it (non-energic multiport element).

Another way to divide the set of multiport elements into two classes is provided by the "second law": entropy-producing or "entropic" multiport elements and "non-entropic" multiport elements, which do not produce entropy. This distinction means, for instance, that all energic elements are non-entropic, although they may contain entropy, because entropy is a state variable in an energic element (no production or annihilation). Note that entropicness means irreversibility, but that
FIG. 3. (a) Simple RLC-circuit example. (b) Two-pole representation. (c) Word bond graph. (d) Bond graph. (e) Simplified bond graph. (f) Generalized bond graph.
the reverse is not necessarily true, unless all forms of irreversibility are connected to some form of entropy production. This is still one of the basic questions of natural philosophy (24).

IV. 1. **Energetic multiport elements (storage elements)**

Storage of energy is related to the state of the system: the state variables of which the energy is a function will be the “stored properties” of which the time-derivatives are equal to the resultant “flow” of that property into the multiport element in accordance with the (generalized) balance equation of that property for a finite homogeneous subsystem (“lump”):

\[
\frac{d}{dt} q = \sum f_{in} - \sum f_{out} = \sum f_{in} + \sum (-f_{out}).
\]

This time derivative is considered to be a flow itself, so that the balance equation becomes an algebraic relation (summation) between all different types of flows related to the property:

\[
\frac{d}{dt} q = f_q
\]

\[
(-f_q) + \sum f_{in} + \sum (-f_{out}) = \sum f_i = 0.
\]

The latter equation is a generalized form of Kirchhoff’s current law (KCL) and can be represented by a non-energetic multiport element (e.g. the 0-junction in Fig. 3) which is part of the so-called “(generalized) junction structure” to be discussed in Section VII.

Storage of energy is the subject of study in equilibrium thermodynamics (12, 13), in Hamiltonian or Lagrangian mechanics and in quasistationary electromagnetic systems. [The restriction to the quasistationary case (electrical networks) has to be made because the energy density of EM-fields is not unambiguously defined, which means that the energy of an EM-field cannot be located.] It is shown in (12) that these separate theories must be generalized by dropping some intrinsic assumptions, in order to be able to represent all types of energy storage by the same type of multiport element. Analogous to the storage of electrical energy and charge in an ideal electrical capacitor, the energetic multiport element will be called multiport capacitor or C-field [mnemonic code in figures, e.g. Fig. 4(a): “C” in “shaded characters” and in text (for typesetting reasons): “MP C” (10)].

According to the concepts of “state” and “change of state” or “flow” the MP C represents an integration with respect to time of the flows \(f_i\) of its ports resulting in the state variables \(q\). Hence, the formal constitutive relation between effort and flow \([e = e(f)]\) consists of a fixed part between flow and state variable \([q = q(f)]\) and a characteristic part between state variable and effort \([e = e(q)]\), which is usually regarded as the constitutive relation of a MP C. The non-energetic multiport elements are not characterized by states, but allow changes of state (processes) to take place. Therefore the constitutive relations are (in principle) algebraic relations between the effort and the flow vector. However, in the non-linear case the non-energetic
P. C. Breedveld

\[ \begin{align*}
\begin{array}{ll}
a: & e = \frac{d\xi_i}{dt} \quad \text{ capacitance } \\
& f = \frac{\xi_i}{d} \\
& P = \frac{dE}{dt} = \sum \xi_i \frac{d\xi_i}{dt} = \sum e f_i \\
\end{array}
\end{align*} \]

\[ \begin{align*}
b: & e = \frac{d\xi_i}{dt} \quad \text{ resistance } \\
& f = \frac{\xi_i}{d} \\
& P = \frac{dE}{dt} = \sum \xi_i \frac{d\xi_i}{dt} = \sum e f_i \\
& T \rightarrow S \\
& T^{-1} \rightarrow S \\
& e^{\text{diss}} = \sum e f_i \\
& e^{\text{irr}} = \sum e f_i
\end{align*} \]

Fig. 4. (a) Multiport capacitor. (b) Multiport resistor and irreversible transducer.

constitutive relations may be state-dependent, although formally such an element may be regarded as a non-energetic (degenerate) MP C. A rigid mechanism is an example of such a degenerate MP C, although it is represented as a state modulated multiport transformer. This multiport transformer is usually derived from the coordinate transformations invoked by the mechanism, but it can also be regarded as a part of the decomposition of a MP C, from which all 1-port C's are omitted, because the mechanism is non-energetic (non-elastic).

IV.2. Entropic multiport elements (dissipative elements)

Entropy-production is the subject of study in non-equilibrium thermodynamics and it is shown in (12, 25) that the difference between entropic and non-entropic multiport elements can explain some issues of non-equilibrium thermodynamics in a simple way. The entropic, non-energetic, power continuous multiport element is called a multiport irreversible transducer and is intrinsically non-linear, because the produced entropy, represented by a "source-like" port, is the sum of products of the conjugate variables of the other (resistive) ports divided by the absolute temperature \( T \). The resistive ports represent the so-called dissipated power and the source-like port the produced thermal power (heat). In the isothermal case the entropy-production of the system does not have to be modelled explicitly and all (multi-)bonds represent flows of free energy only (13). Hence, an isothermal resistor (dissipator) can be linear. Analogous to an isothermal electrical resistor, the isothermal, entropic, non-energetic, power discontinuous (dissipation of free energy) multiport element will be called multiport resistor or R-field (mnemonic code: MP R). Following Thoma (26) the mnemonic code of an irreversible transducer will be MP RS, a (possibly linear) multiport resistor (R-port) extended with a necessarily nonlinear source-like port (S) which represents the entropy flow discontinuity caused by the entropy production [Fig. 4(b)].

IV.3. Sources

Thus far only those non-energetic multiport elements have been discussed which are power continuous and thus energy conserving (except for the special case of the isothermal resistor). The energy exchange with the environment of the system would violate the power continuity of the bonds, or even the balance equations of the properties, if the environment is not represented explicitly by some multiport element. Such a multiport element, which represents all kinds of boundary conditions, will produce or annihilate power and flows of all properties in such a way that all bonds remain power continuous and all balance equations are satisfied.
Analogous to the modelling of the environment in electrical networks by voltage and current sources, the "environmental", power and flow discontinuous multiport element is called multiport source, which is a collection ("array") of 1-port sources, because a source has by definition no constitutive couplings. Either the effort or the flow is imposed by a source and the resulting two types of sources are called effort-source (mnemonic code: $S_e$) and flow-source (mnemonic code: $S_f$) respectively.

### IV.4. Junction structure

According to the current bond graph terminology, all non-energetic, power continuous, non-entropic multiport elements belong to a multiport element called generalized junction structure (GJS) (27). Before continuing the classification of these junction multiport elements, some basic concepts have to be introduced which are more easily understood in a simple form. Therefore, the general and consequently abstract line of thought followed until now will be left. It is postulated that assumptions can be made in such a way that all multibonds and multiport elements take their simplest form possible. This means that all multibonds become [or can be decomposed (11 into)] 1-bonds or simply "bonds", which are represented by single lines (half-arrows) (Fig. 5).

One of the necessary assumptions is that the system is linear (it will become clear later that classification of the GJS elements is only possible in this linear case) and consequently isothermal, because only in the isothermal case does the thermal domain not have to be modelled explicitly and the nonlinear irreversible transducer can be replaced by a linear MP R. This means that the bonds do not represent the (total) energy flows anymore, but only the flows of free energy, the usual power in isothermal systems like electrical networks, kinematic mechanisms, etc. All multiport capacitors become (or can be decomposed into) linear 1-port capacitors or simply capacitors (mnemonic code C) and all multiport irreversible transducers and resistors become (or can be decomposed into) linear 1-port resistors (mnemonic code R), representing the dissipation of free energy: energy itself cannot be dissipated according to the first law of thermodynamics. Furthermore, the multiport sources already consist of 1-port sources. The multiport elements forming the junction structure turn into elementary junction 2- and 3-ports, because 1-ports are impossible (there is neither storage nor dissipation of free energy in a junction structure) and in order to form a structure other than a chain of bonds and 2-ports ($n$ connected 2-ports can only form a 2-port), the necessary and sufficient number of ports of a junction structure element to form arbitrary $n$-ports is three [$n$ connected 3-ports form an $(n + 2)$-port]. In order to be able to classify these junction 2- and 3-
ports further (Section VII), the concepts of “computational causality” and “modulation” of MP E’s have to be introduced first (Sections V and VI respectively).

V. Computational Causality

V.1. “Assignment statement” form of the constitutive relations

The constitutive relations of MP E’s are (computationally) a-causal: there is only a relation between the conjugate power variables, which means that it is impossible to speak of one variable being the “cause” of the other. However, when the constitutive relations are used in an operational algorithm to compute certain outputs of a model given its inputs, analogous with block diagrams, they take the form of an assignment statement, to use a computer science metaphor. The effort (or flow) to which a certain value is assigned, is a dependent variable of the assignment statement, whereas a flow (or effort) which is used to compute this value, is an independent variable. In general, if a MP E has \( n \) ports and is characterized by a constitutive relation from which \( x \) assignment statements can be derived, then there are \( 2n \) variables, of which \( x \) are dependent and \( (2n - x) \) independent. Due to the conjugacy of the variables with respect to energy and power, the characteristic feature of an \( n \)-port is that \( x = n \), such that there are \( n \) independent and \( n \) dependent variables.

Of all \( 1 \)-ports other than sources (which have no independent variables) either the effort or the flow is independent while the conjugate variable is automatically dependent. Since a system generally contains \( 1 \)-ports, this typical “conjugacy” of \( 1 \)-ports with respect to computational causality can be imposed on every port of the system by convention. In that case the conjugate variables of a bond connecting two ports play the role of independent and dependent variable at one end of the bond while at the other end these roles are interchanged. In other words: at one side of a bond the effort is the independent variable and the flow the dependent variable in the relation represented by the multiport element connected to that side of the bond, while at the other end the flow is the independent variable and the effort dependent.

V.2. The causal stroke

The computational causality of a lumped system (in bond graph texts usually abbreviated to causality) is represented in a bond graph by attaching a small, so-called causal stroke perpendicular to the bond at the end where the effort is the independent variable [Fig. 6(a)]. In his introduction to the bond graph concept Paynter (5) states that power interaction represented by a bond is a bilateral signal flow (of effort and flow). Computational causality thus can be regarded as the orientation of this bilateral signal flow. If all efforts at one end of a multibond are

\[
\begin{align*}
\text{Effort- and flow-causality of a 1P R. (b) Effort- and flow-causality of a MP R.}
\end{align*}
\]
Multibond Graph Elements

FIG. 7. Causal augmentation of Fig. 3(e).

independent (i.e. have the same causality), the same notation can be used for multibonds [Fig. 6(b)]. If not, the multibond has to be decomposed first. This can be done by using a direct sum as discussed in (10).

In Fig. 7 the bond graph from Fig. 3(e) is “causally augmented”. This example shows that there are certain constraints on causality assignment: the sources have a fixed causality, for storage elements an integrative causality is numerically preferable and also reveals dependent storage elements if applied in a consistent way (28). Junctions have restricted causal forms (O-junctions one causal stroke, I-junctions one “open end”), although this restriction can be weakened if computational techniques like iteration are possible [“relaxed causality” (29)]. For further examples and applications the reader is referred to the extensive bond graph literature (30,31).

V.3. Transmission matrices

It was previously mentioned that the opposite computational direction of the conjugate variables is a bond graph convention, only necessary for 1-ports other than sources in principle. When a 2-port system which consists of a chain of 2-ports itself is considered, it is also possible to take the conjugate pair of one port of a 2-port as independent and the conjugate pair of the other bond as dependent variables. This corresponds to the method of determining the transmission matrix of a chain of 2-port systems (32), of which the transmission matrices can be multiplied in order to obtain the transmission matrix of the total chain. Although these matrices are used in assignment statements, they are called “a-causal” by Karnopp and Rosenberg to indicate the impossibility of the use of the causal stroke.

V.4. A physical interpretation of computational causality

Apart from the computational meaning a physical meaning may also be assigned to the causality convention of a 1-port expressed by the causal stroke. It is physically impossible to impose or control both effort and flow of one port. For instance, if the force of some mechanical device, which behaves like a 1-port (load), has to be controlled, it is impossible to control its conjugate velocity (and position) at the same time. From these considerations it can be concluded that it is physically impossible to measure a transmission matrix of a 2-port directly.

VI. Modulation of Multiport Elements

The constitutive relation of a multiport element generally contains parameters. If these parameters are time-dependent, the multiport element is said to be modulated
by a one-directional signal, which carries no power and is represented in the bond graph by a full arrow pointing towards the modulated multiport element (Fig. 8). In the mnemonic code the modulation of a MP E is expressed by the letter M at the first position of the code (Fig. 8). Modulation may be \textit{external}, i.e. not related to the state and/or power variables of the system (time-variant system), or \textit{internal}, i.e. related to the state and/or power variables (non-linear system). In order to be able to classify multiport elements, internal modulation has to be excluded, because every constitutive relation can be constructed by internal modulation. This will be demonstrated by some examples in Section VII.3 and 4. For the sake of simplicity external modulation will be excluded as well, although it does not change the basic character of a multiport element like internal modulation does.

Another way to consider modulation is to treat the modulating parameter as an additional state or power variable so that the signal input becomes a power bond. As long as the modulation is assumed to require negligible power the modulating bond is called an \textit{active bond} (5), and can be represented by a full arrow. This viewpoint has the conceptual advantage that it shows that powerless modulation is an ideal situation: it is physically impossible to measure or control a system without any power exchange, so that an active bond is actually a "low-power bond" compared to the other bonds.

\section*{VII. Junction Structure Elements}

\subsection*{VII.1. Junction 2-ports: transformers and gyrators}

The possible causal forms of a \textit{junction 2-port} (mnemonic code: J2P) are enumerated in Fig. 9(a) (irrespective of the order of its bonds) and correspond to the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{Junction 2-ports: possible causal forms (a), linear constitutive forms (b), 2-port gyrator and transformer (c), unit gyrator (d), symplectic gyrator (e), SGY as partial dualizer (f).}
\end{figure}
following three pairs of constitutive relations:

\begin{align*}
(1) & \quad e_1 = e_1(f_1, f_2), \quad e_2 = e_2(f_1, f_2), \\
(2) & \quad f_1 = f_1(e_1, e_2), \quad f_2 = f_2(e_1, e_2), \\
(3) & \quad e_1 = e_1(e_2, f_1), \quad f_2 = f_2(e_2, f_1).
\end{align*}

Note that if the order of the bonds would be relevant a fourth pair appears (there are \(2^2 = 4\) causal possibilities):

\begin{align*}
(4) & \quad f_1 = f_1(e_1, f_2), \quad e_2 = e_2(e_1, f_2).
\end{align*}

Because linearity is assumed, which is possible in principle because a junction structure (JS) is non-entropic, the demands of power continuity and non-energicness result in:

\begin{align*}
(1) & \quad e_1 = -rf_2, \quad e_2 = +rf_1, \\
(2) & \quad f_1 = +ge_2, \quad f_2 = -ge_1, \\
(3) & \quad e_1 = +ne_2, \quad f_2 = -nf_1,
\end{align*}

where \(r, g\) and \(n\) are constitutive parameters.

If written in vector notation, the constitutive matrices in the equations are antisymmetric, due to the demand of power continuity [Fig. 9(b)]. The first two sets of equations are similar to the constitutive relations of an ideal electrical gyrator in “resistive” and “conductive” causality respectively (33) and the third to the constitutive relations of an ideal electrical transformer. Accordingly, these linear 2-ports are called gyrator and transformer with mnemonic codes GY and TF, respectively [Fig. 9(b)]. Usually the minus signs are eliminated by a change of orientation of one of the ports, such that the power “flows through” a 2-port characterized by one parameter \((r, g \text{ or } n)\) [Fig. 9(c)]. In the following both orientations will be used to illustrate certain analogies.

VII.2. Symplectic gyrators

A unit (degenerate) TF is equal to a bond, whereas a unit gyrator inverts the roles of effort and flow [Fig. 9(d)]. Returning to the original orientation (both bonds towards the element), the characteristic (constitutive) matrix \(J\) is

\[
J = \begin{bmatrix}
0 & -1 \\
+1 & 0
\end{bmatrix}
\]

and is called symplectic in differential geometry (34), where Hamiltonian systems have a so-called “symplectic structure”. A unit gyrator is called a symplectic gyrator [SGY; Fig. 9(e)] (17, 12) if the constitutive parameter \((r = g = 1)\) is dimensionless, i.e. has no physical interpretation itself, but expresses that the SGY represents a certain relation, which is not dependent on material properties (Newton’s second law of motion, Faraday’s induction law, etc.). The representation of these basic physical relations by a SGY has some advantages over the conventional method, which “absorbs” these relations in other elements as discussed in (12).
If connected to a port of an element or system, the SGY "dualizes" this port, i.e. if the SGY is absorbed by the element or system, it interchanges the roles of the conjugate effort and flow of that port in the constitutive relation [Fig. 9(f)]. By dualizing all ports of a multiport element or system the "dual element" or "dual system" (dualogon) is obtained. If only a fraction of the ports is dualized (partial dualization), the result is the "partial dual" of an element or system (cf. III.5). An ineriance or inductance (I) is the dual of a capacitor (C + SGY), the dual of a resistor or gyrator with "resistive causality" is a resistor or gyrator with "conductive causality" respectively and the dual of a transformer is a transformer with inversed causality [Fig. 10(a)]. However, the partial dual of a transformer (in case of a 2-port there is only one possibility) is a gyrator and the partial dual of a gyrator is a transformer [Fig. 10(b)]. The latter equivalence opens the possibility of eliminating gyrators from a network by replacing them by a transformer and a SGY, which might be eliminated itself by (partial) dualization of other elements. The importance of the concepts of "partial dualization" and "SGY" for the modelling of physical systems is elaborated on in (12).

VII.3. The ambiguity of internally modulated transformers and gyrators

If the linear TF and GY would be generalized by allowing a nonlinear constitutive relation, i.e. internal modulation, the characteristic form only depends on the chosen causal form and has no physical meaning whatsoever. This may be demonstrated by the following examples. A linear transformer has the constitutive relations:

\[ e_1 = ne_2; \quad f_2 = nf_1. \]

If these are rewritten in the following illogical causal form (containing an algebraic loop via \( e_1 \) and not containing the constitutive parameter):

\[ e_1 = (e_1/f_2)f_2; \quad e_2 = (e_1/nf_1)f_1 = (e_1/f_2)f_1 \]

they represent an internally modulated gyrator with gyration ratio \((e_1/f_2)\). A second example is a linear capacitor which may be replaced by a peculiar combination of a

![Fig. 10. Dual 1- and 2-ports (a); partial duals of junction 2-ports (b).](image-url)
state-modulated transformer and an effort-source, because
\[ e_c = C^{-1} \int f \, dt = ne_s \]
with
\[ n(q) = \int f \, dt \quad \text{and} \quad e_s = C^{-1}. \]

These examples confirm the proposition of Section VI that internal modulation has to be excluded in order to be able to classify junction multiports.

However, a less peculiar example of internal modulation is Paynter's (35) replacement of a 2-port irreversible transducer with a linear resistive port by an internally modulated gyrator or transformer, depending on its causal form (Fig. 11). In this case there are no (algebraic) loops and the internal modulation can also be regarded as nonlinearity of the 2-port, so that the signals (active bonds) can be omitted. This construct, however, does not satisfy the demand of non-entropenness of a J2P, although it is useful to be kept in mind in order to apply JS equivalence rules to RS's.

The above discussion makes clear that the restriction to the linear (and linear time-variant) case is necessary for the exhaustive classification of JS elements. The linear case is also the most natural one for the structural part of a "lumped" system, because it allows superposition (interconnection) of multiport elements. (In network theory the JS is sometimes called the "connection multiport").

Moreover, only certain types of nonlinear elements which are often used can be given a specific name. An example of such an element is the "triangle structure" of three I-junctions and three gyrators, internally modulated by the opposite junctions (Fig. 12). It represents the gyroscopic forces described by the exterior product in Euler's equations for rotating coordinate frames. Accordingly, it is called a "Eulerian junction structure" (EJS) (36). It is a new type of modulated 3-port gyrator which corresponds to a circulator in network theory (37, 38) and belongs to the class of gyristors (39). In electrical circuit theory other types of nonlinear junction multiports have been introduced by Duinker: conjunctors (40) and traditors (41), to be translated into bond graph terminology in Section VII.4.

VII.4. Bond graph equivalents of conjunctors and traditors

Depending on its computational causality, the conjunctor is a modulated gyrator (conjunctor of type I, II, V and VI) or modulated transformer (conjunctor of type III

![Fig. 11. Irreversible transducer as an internally modulated MGY or MTF.](image_url)
Duinker shows how an ideal amplifier (ideal pentode with infinite plate resistance) can be derived from conjunctors [Fig. 13(a)]. In bond graph form this result turns out to be an externally modulated transformer connected to an internally modulated gyrator, which is in turn connected to a linear (1-port) resistor [Fig. 13(b)]. By connecting gyrators to the ports (partial dualization!) Duinker obtains models of an ideal triode [Fig. 13(c)] and of an ideal transistor [Fig. 13(d)]. In the latter case a "gyrator" is even connected to an active bond (modulating signal). These applications of the conjunctor show that this element is used to generate constitutive relations and has no physical meaning at all. Because internal modulation is used, which should be avoided or at least restricted to special cases, the same results can be obtained in many other ways. The value of this kind of mathematical exercise is therefore questionable.

Because the bond graph representation of a conjunctor revealed its basic character, it is interesting to present also a bond graph interpretation of a second nonlinear junction element introduced by Duinker (41), the traditor, a non-energetic, non-entropic Lagrangian element. (A Lagrangian element is an element derivable from a Lagrangian.) Wyatt and Chua (42) showed that in case of a (non-energetic) junction element, this Lagrangian is first order homogeneous in the generalized velocities (flows). (Duinker made the stronger restriction of linearity in the generalized velocities.) If Duinker's terminology is translated into bond graph (or network) concepts, a traditor of the first degree becomes a zero-valued (because of power continuity!) source, a traditor of the second degree is equivalent with a linear transformer or gyrator and a traditor of the third degree can be decomposed into state-modulated J2P's (transformer or gyrator) and a (0- or 1-)junction (Fig. 14).

FIG. 12. Eulerian junction structure (EJS).
Fig. 13. (a) Conjunctor model of an ideal pentode. (b) Bond graph interpretation and simplification of (a). (c) Bond graph interpretation and simplification of a conjunctor model of an ideal triode. (d) Bond graph interpretation and simplification of a conjunctor model of an ideal transistor.
The property that the efforts (generalized forces) of a system or element can be derived from a Lagrangian is easily mixed up with the property of reciprocity, in which case the equations can be derived from a potential function. However, a Lagrangian element is not necessarily reciprocal. In order to explain this statement and also because reciprocity is an important property for the classification of MP E’s, the concept of reciprocal multiports is discussed separately in Section VII.5.

The above discussion of conjunctor and traditor shows that generalization of the properties of linear elements to non-linear elements is generally not possible. This is also demonstrated by the example of a linear 3-port gyrator, which, like a linear 2-port gyrator, is a traditor, i.e. a linear 3P GY is derivable from a Lagrangian or rather a kinetic co-energy. By contrast, an EJS representing gyroscopic forces is a nonlinear (velocity modulated) 3P GY which is non-Lagrangian. This property of the EJS shows that Karnopp’s conclusion (43) that gyroscopic forces (as represented by the EJS) can be derived from a kinetic co-energy (Lagrangian with zero potential energy) is not true. If one tries to construct such a Lagrangian, terms appear which are second order homogeneous in the velocities. A Lagrangian with such terms cannot describe a power continuous multiport as pointed out by Wyatt and Chua (42). Hence the properties of a linear 3P GY can not simply be extended to the nonlinear case.

VII.5. Reciprocity of mixing and non-mixing multiport elements

Reciprocity of a multiport element is defined by Brayton (44) as:

\[ \rho = \sum_i dy_i \wedge dx_i = 0 \]  

(1)

where \( dy_i \wedge dx_i \) is the exterior product of the exterior derivatives of the conjugate variables in the constitutive relation.† Note that the orientation of the ports has to

† Formally, this definition of reciprocity may be expressed in terms of differential geometry as “the vanishing of the two-form \( \rho \) on the tangent planes of the constitutive manifold”. If a MP E is reciprocal in certain coordinates and in a certain operating point, its constitutive relation can be derived from a potential function (a property called “integrability”): \( w \) is integrable if

\[ w = dF = \sum_i \frac{\partial}{\partial x_i} F(x) \, dx_i = \sum_i y_i \, dx_i \]

where \( F \) is a potential function and \( d \) the exterior derivative. This means that \( w \) is closed according to Poincare’s lemma:

\[ dw = d \cdot dF = \sum_i dy_i \wedge dx_i = 0. \]

From this reciprocity condition it can be derived that

\[ \frac{\partial^2}{\partial x_j \partial x_i} F = \frac{\partial^2}{\partial x_i \partial x_j} F = \frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i} \]

a symmetry relation often used to define reciprocity.
be consistent, preferably towards the MP E, and that:
\[
\begin{align*}
\frac{dy_i}{dx_i} + & \frac{dx_i}{dy_i} = 0 \\
\frac{dy_i}{dx_i} + & \frac{dx_i}{dy_i} = 0.
\end{align*}
\]
From Eq. (1) it can be immediately concluded that every 1-port (C, R, S, S, and I) is reciprocal.

It also follows from Eq. (1) that a (2-port) TF is reciprocal:
\[
\begin{align*}
de_{e_1} & \wedge df_1 + de_{e_1} \wedge df_2 = d(ne_2) \wedge df_1 + de_{e_2} \wedge df(-n_{f_1}) \\
& = n de_{e_2} \wedge df_1 - n de_{e_2} \wedge df_1 = 0.
\end{align*}
\]
and that a (2-port) GY is non-reciprocal:
\[
\begin{align*}
de_{e_1} & \wedge df_1 + de_{e_2} \wedge df_2 = d(-rf_2) \wedge df_1 + d(rf_1) \wedge df_2 = \\
& = \neg r df_2 \wedge df_1 + r df_1 \wedge df_2 = 2r df_1 \wedge df_2 \neq 0.
\end{align*}
\]
In other words: linear J2P’s can also be classified into reciprocal (TF) and non-reciprocal J2P’s (GY). The reciprocal relation which characterizes a TF relates efforts to efforts and flows to flows, whereas the non-reciprocal GY has a constitutive relation which relates efforts to flows or flows to efforts depending on the causal form. For apparent reasons Brayton (44) named these properties “non-mixing” and “mixing” respectively. Brayton also showed that a linear, power continuous MP E (JS) is reciprocal, if and only if it is non-mixing.

Other well-known reciprocal linear MP E’s are the MP C and the MP R, which
are both mixing. Consequently both are power discontinuous: the MP C because of storage of energy and the MP R because of dissipation of free energy: the nonlinear, entropy-producing port of the MP RS causes its non-reciprocity. The reciprocity condition for the MP C:

\[ \rho_C = \sum_i \frac{\partial e_i}{\partial q_i} = 0 \]

is equivalent with the symmetry of the Jacobian of the constitutive relation [the inverse of the (incremental) capacitance matrix], because:

\[ \sum_i e_i \wedge dq_i = \sum_i \sum_{j<i} \frac{\partial e_i}{\partial q_j} dq_j \wedge dq_i = \sum_i \sum_{j<i} \left( \frac{\partial e_i}{\partial q_j} - \frac{\partial e_j}{\partial q_i} \right) dq_j \wedge dq_i = 0 \]

or

\[ \frac{\partial e_i}{\partial q_j} = \frac{\partial e_j}{\partial q_i} \]

In thermodynamics the latter relation is called the Maxwell reciprocity condition of an equilibrium system (3).

Attention should be paid to the fact that the (complete) dual of a reciprocal MP E is still reciprocal although it has “absorbed” non-reciprocal symplectic gyrators, e.g. a MP I (MP C + SGY-array) is reciprocal, because the roles of all conjugate pairs have been interchanged:

\[ \rho_i = -\rho_C = 0. \]

Consequently the important statement can be made that:

**A partial dual of a reciprocal element is always non-reciprocal, unless this MP E consists an array of uncoupled (MP) elements of which some are (completely) dualized and some not.**

This means that a partially dualized 2PC, called IC in conventional terminology, is non-reciprocal. Consequently its linear decomposition (11) contains a gyrator.

The reciprocity condition of the MP R:

\[ \rho_R = \sum_i e_i \wedge df_i = 0 \]

is also equivalent with the symmetry of a Jacobian, the (incremental) resistance matrix, because

\[ \sum_i e_i \wedge df_i = \sum_i \sum_{j<i} \left( \frac{\partial e_i}{\partial q_j} - \frac{\partial e_j}{\partial q_i} \right) df_j \wedge df_i = 0 \]

or

\[ \frac{\partial e_i}{\partial q_j} = \frac{\partial e_j}{\partial q_i} \]

In non-equilibrium thermodynamics these symmetry relations are called the Onsager reciprocal relationships (ORR) (45, 46). In contrast with the MP C, where
the principle of energy conservation leads to Maxwell reciprocity, there seems to be no generally valid physical principle from which Onsager reciprocity might be derived: Onsager provided the reciprocity "only" for a special class of systems, assuming "microscopic reversibility", isotropy, "detailed balance" and absence of external (magnetic) fields and rotating reference frames. As these conditions are not always satisfied, Onsager symmetry can be violated. This leads to a generalization of the gyrator as a non-reciprocal counterpart of a MP R(S) discussed in Section VII.6.

VII.6. Multiport generalizations of GY and TF

In general, a linear, non-energetic, mixing relation can be separated into a symmetric and an anti- or skew-symmetric part: i.e.

\[
e = Bf = B^t f + B^s f.
\]

The symmetric part \((B^s)\) can be represented by an MP R(S), which is entropic, while the antisymmetric part \((B^a)\) is non-entropic because a quadratic form of an antisymmetric matrix is always zero:

\[
f^T B^a f = \sum_i \sum_j f_i B^a_{ij} f_j = \sum_i \sum_{i<j} (B^a_{ij} + B^a_{ji}) f_i f_j = 0.
\]

If the antisymmetric part describes a 2-port, this 2-port is a GY. Consequently MP E's described by antisymmetric matrices will be considered as multiport gyrators (MP GY) (12). Onsager reciprocity can now be expressed as the absence of a MP GY. The MP GY is non-reciprocal, but if reciprocity is defined in this special case as the symmetry of the constitutive matrix, the MP GY may be called "anti-reciprocal", because it is characterized by an antisymmetric matrix. However, in the light of Brayton's general definition Eq. (1), the concept of anti-reciprocity has no meaning.

Like a GY also a TF can be generalized to a MP TF:

\[
e_1 = T^T e_2; \quad f_2 = T f_1
\]

or

\[
\begin{bmatrix}
e_1 \\
f_2
\end{bmatrix} =
\begin{bmatrix}
0 & -T^T \\
T & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
(-e_2)
\end{bmatrix}.
\]

Because in network (and bond graph) theory the gyrator is usually considered to be the counterpart of the transformer, this multiport transformer is normally taken as the starting point to introduce the multiport gyrator. This results, however, in constitutive equations which are a special form of the MP GY:

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} =
\begin{bmatrix}
0 & -G^T \\
G & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
(-f_2)
\end{bmatrix}.
\]

In Refs (12) and (13) this special MP GY is related to Casimir's extension of the Onsager Reciprocal Relationships (45-47), whereas the general MP GY corresponds to symmetry breakdown, for instance due to Lorentz or gyroscopic (Coriolis) forces. In order to appreciate the distinction between a special MP GY and a general MP GY within the context of this paper, the attention should be drawn to the fact that a special MP GY is a MP E with two \((k\text{- and }l\text{-dimensional})\)
multibonds like the MP TF, described by an antisymmetric matrix with a block structure \[2(k \times l)\text{-dimensional off-diagonal blocks}\], while the general MP GY is a MP E with one \((m\text{-dimensional})\) multibond like the MP R, described by any \((m \times m)\text{-dimensional}\) antisymmetric matrix. This distinction is related to the difference between the NES- and the ES-type of J3P’s discussed in Section VII.8.

VII.7. The dependence of reciprocity on coordinates

Oster et al. (48, p. 117) noticed that reciprocity is not an intrinsic property of a MP E, because it is coordinate dependent (and therefore “local” in case of non-linear constitutive relations). However, the example presented by these authors of a “reciprocity-breaking” coordinate transformation from voltage–current or, in bond graph terminology, effort–flow variables \((x_i, y_i)\) to scattering variables \((\xi_i, \eta_i)\)

\[
\xi_i = \frac{(x_i + y_i)}{2}, \quad \eta_i = \frac{(x_i - y_i)}{2}
\]

is not correct, although the authors state: “A short computation shows that

\[
\sum_i d\xi_i \wedge d\eta_i \neq 0.
\]

Reciprocity is only broken, in other words, the symmetry of the constitutive (Jacobian) matrix, in \textit{casu} the scattering matrix \(S\), is only destroyed, if the scattering variables \((a_i, b_i)\) are considered, which are non-normalized (49), i.e.

\[
a_i = \frac{(x_i + r_i y_i)}{2}, \quad b_i = \frac{(x_i - r_i y_i)}{2}
\]

where \(r_i\) is the so-called normalizing constant and with additional constraints: \(r_i \neq r_j\) for at least one \(j \neq i\), and \(\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i} \neq 0\) (49, pp. 243–244). Indeed this may be checked by a short computation:

\[
\rho = \sum_i d\xi_i \wedge d\eta_i = \sum_i (-r_i dx_i \wedge dy_i + r_i dy_i \wedge dx_i)/4
\]

\[
= \sum_i (r_i dy_i \wedge dx_i)/2 = \sum_i \sum_{j < i} (r_i - r_j) \frac{\partial y_j}{\partial x_j} dx_j \wedge dx_i \neq 0.
\]

However, in the normalized case \(r_i = r_j = 1\) and thus

\[
\sum_i d\xi_i \wedge d\eta_i = 0.
\]

Although reciprocity is not an intrinsic property of a multiport element in general, reciprocity in certain coordinates (e.g. effort and flow) is an intrinsic property of a multiport element, which has a physical meaning if these coordinates have a physical meaning. This may be shown by an example from non-equilibrium thermodynamics, illustrated by the multibond graph in Fig. 15 (12, pp. 36–37). Any nonreciprocal, mixing, constitutive flow-effort relation

\[
f = f(e) \quad \text{with} \quad \frac{\partial f_i}{\partial e_j} \neq \frac{\partial f_j}{\partial e_i}
\]

may be separated into a resistive part (reciprocal, with symmetric Jacobian) and a gyrative part (non-reciprocal, with antisymmetric Jacobian). If \(G(e)\) is the constitu-
Fig. 15. Influence of a "coordinate transformation" on the reciprocity of a multiport element.

tive, Jacobian matrix of the gyrator, the transformation

\[ f' = f - G(e)e, \quad e' = e, \]

which is known as Truesdell's "choice of forces and fluxes" (50), eliminates the gyrator and turns the multiport element into a reciprocal resistor. If \( e \) and \( f \) are two sets of conjugate powervariables with some physical meaning, e.g. voltage and current, generally \( f' \) will not have any physical meaning.

VII.8. Junction 3-ports: NES and ES type

Analogous to the junction 2-ports (J2P) also junction 3-ports (J3P) can only be classified exhaustively in the linear case. This means that the constitutive relation becomes a linear matrix equation. As in the case of a J2P, this matrix has to be antisymmetric due to the demand of power continuity: i.e.

\[
\begin{bmatrix}
  e_1 \\
  f_1 \\
  e_2 \\
\end{bmatrix} = \begin{bmatrix}
  0 & -a & b \\
  a & 0 & -c \\
  -b & c & 0 \\
\end{bmatrix}
\begin{bmatrix}
  f_1 \\
  f_2 \\
  e_2 \\
\end{bmatrix}
\]

where \( \dim(e_1) = 3 - \dim(f_2) = \dim(f_1) = 3 - \dim(e_2) \) (51).

The constitutive matrix, like any odd dimensional antisymmetric matrix, is singular, because the rank of an antisymmetric matrix is always even. Consequently the \( 2^3 = 8 \) possible causal forms are not just (partial) inversions of one and the same relation. The eight possible causal forms can be reduced to four by neglecting the order of the bonds. These four possible causal forms are enumerated in Fig. 16. As a consequence of the singularity, inversions are only possible pairwise, i.e. two bonds have to be causally inverted at the same time.

In (51) it is shown that any linear J3P can be decomposed by extraction of transformers and/or gyrators into J2P's and a so-called unit essential junction 3-port (mnemonic code: ES), which is characterized by the matrix:

\[
\begin{bmatrix}
  0 & -1 & +1 \\
  +1 & 0 & -1 \\
  -1 & +1 & 0 \\
\end{bmatrix}
\]
or in J2P's and a unit non-essential junction 3-port (mnemonic code: NES), which is characterized by the matrix:

\[
\begin{bmatrix}
0 & -1 & -1 \\
+1 & 0 & 0 \\
+1 & 0 & 0
\end{bmatrix}
\]

In the first case (ES), the unit 3-port is called essential, because its decomposition in 0- or 1-junctions and J2P's (11) will contain at least one gyrator which cannot be eliminated by partial dualization. Hence, a unit essential junction 3-port is non-reciprocal (27). In the second case (NES), the unit 3-port is non-essential, because its decomposition is a partially dualized 0- or 1-junction and thus contains no gyrators in an essential way. Pairwise causal inversion is restricted to direct coupled ports, i.e. the bonds \(i\) and \(j\) cannot be inverted pairwise, if the matrix elements \(ij\) and \(ji\) are zero.

The unit non-essential J3P is, in Rosenberg's terminology (27), extended reciprocal.

As pointed out in (51), extended non-reciprocity of a linear J3P is related to the non-reciprocity of a "matched", linear, lossless 3-port in scattering theory (e.g. a circulator). This means that lossless, unidirectional power transmission can only take place in a J3P congruent with an ES. This is a kind of irreversibility (asymmetry in time) which is not related with entropy production.

Brayton's definition of reciprocity [Eq. (1)] can be generalized in such a way that the case of extended reciprocity is included by introducing the factor \(\varepsilon_i\), which equals either +1 or −1:

\[
\rho^{\text{ext}} = \sum_i \varepsilon_i \, de_i \wedge df_i = 0.
\]

A factor \(\varepsilon_i = -1\) corresponds to the process of partial dualization of the \(i\)th port. This definition shows that a 2-port gyrator is extended reciprocal \((\varepsilon_1 = -\varepsilon_2)\), as well as a NES \((\varepsilon_1 = -\varepsilon_2 = -\varepsilon_3)\), but an ES is non-reciprocal, because the set of equations \(\varepsilon_1 = -\varepsilon_2 = -\varepsilon_3\) and \(\varepsilon_2 = -\varepsilon_3\) has no solution.

In this study of linear J3P's additional physical properties can be formulated, which lead to special types of J3P's: effort potentiality and flow continuity. Flow continuity (no storage of the corresponding property) can have two forms:

\[
\begin{align*}
(1) \quad \sum_{i=1}^{3} f_i &= 0, \\
(2) \quad f_i &= f_j.
\end{align*}
\]

In combination with the constraint of power continuity, flow continuity of the
second kind results in

\[ \sum_{i=1}^{3} e_i = 0 \]

and this characterizes a 3-port 1-junction.

Effort potentiality means that the J3P represents a (spatial) node characterized by one effort which is a potential, i.e. a scalar property which is independent of the direction (in space) of the conjugate flow:

\[ e_i = e_j \]

From the constraint of power continuity it follows that the effort potential J3P (3-port 0-junction) is also flow continuous of the first kind. Both the 3-port 0- and 1-junction are non-essential J3P's (congruent with a NES) with fixed causal forms (cf. Section V): restricted pairwise causal inversion does not change the basic causal form (Fig. 17): there is always one causal stroke (0) or one open end (1) directed to the junction. It is easily verified that both 0- and 1-junction 3-ports are reciprocal. Hence, effort potentiality means reciprocity, but not every reciprocal MP E is effort potential.

Note that a flow continuous J2P of the first kind is automatically effort potential and is equivalent to an orientation change of a flow for which the 0-junction symbol can be used. A flow continuous J2P of the second kind is equivalent to a change of orientation of the effort, for which the 1-junction symbol can be used.

There exist also physically meaningful J3P's which are flow continuous of the first kind (like a 0-junction), but not effort potential. Such J3P's are congruent with an ES and accordingly non-reciprocal. A hydraulic junction is an example of such a "non-effort potential" J3P, because the effort of the hydraulic domain, the total pressure, contains a dynamic (kinetic) factor which depends on the direction of the conjugate (volume-)flow (51).
VII.9. **Multiport generalizations of 0- and 1-type junction 3-ports**

Flow continuity of the second kind and effort potentiality can easily be generalized to the case of $n$-ports. Flow continuity of the second kind is for an $n$-port 1-junction:

$$f_i = f_j$$

and effort potentiality for an $n$-port 0-junction:

$$e_i = e_j$$

The constraint of power continuity results in the complementary constitutive relations:

$$\sum_{i=1}^{n} e_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} f_i = 0$$

respectively, which correspond to the Kirchhoff voltage law (KVL) for a mesh and the Kirchhoff current law (KCL) for a node in the theory of electrical networks.

The $n$-port 0- and 1-junction may be regarded as a MP TF characterized by a special constitutive matrix $[\text{($n \times 1$)- and ($1 \times n$)-dimensional respectively and containing only } -1 \text{ or } +1 \text{ valued elements}]$. In case of repetitive structures, so-called *junction arrays* can be used which enable the multibond (“vectorbond”) notation (10). Junction arrays are also special types of MP TF’s, characterized by a block-diagonal matrix, where one block characterizes just one junction in the form of a special MP TF as described above. In multibond notation, the junction array necessitates the introduction of a *multibond array*, i.e. an array of multibonds, possibly with different dimensions. Figure 18 illustrates the definition of junction array and multibond array.

A comparable special type of MP TF is the *direct sum* which has been introduced (10) to reorganize (by composition, decomposition and/or reordering) multibonds and multibond arrays. In this case the constitutive matrix of the MP TF is a unit matrix, possibly with reordered columns (*crossing matrix*). In “single bond” notation the direct sum is an array of degenerate 2-port 0- or 1-junctions, which are equal to a bond.

VII.10. **A generalized classification of junction multiports**

Many aspects of junction structures have been discussed in the previous subsections starting from basic properties: non-energicness (power continuity), non-entropieeness, effort potentiality, flow continuity reciprocity and relations being (non-)mixing. This resulted in elementary junction 2- and 3-ports. Usually 0- and 1-junctions and 2-port transformers and gyrators are postulated first and with these “building bricks” the several types of junction multiports are constructed by logical arguments. This provides a classification which can also be considered a generalization of the line of thought of this study in terms of 2- and 3-ports (27, 52):

1. The “Kirchhoff junction structure” or “simple junction structure” (SJS), which consists of bonds, 0- and 1-junctions (junction arrays and direct sums are also
possible in multibond notation) and which is consequently reciprocal (special MP TF).

2) The "Kron junction structure" or "weighted junction structure" (WJS), which is an SJS containing TF's ("weighted bonds") and thus also reciprocal (MP TF).

3) The "Birkhoff junction structure" or "generalized junction structure" (GJS), which is a WJS containing gyrators, i.e. a GJS contains all different types of J2P's and J3P's. A GJS is extended reciprocal if all gyrators are contained in the GJS in a non-essential way and non-reciprocal if one or more gyrators are contained in the GJS in an essential way, in other words, if it contains a J3P congruent with an ES.

VIII. Conclusion

The main purpose of this paper has been the introduction of the fundamental concepts of physical systems theory. The bond graph language has been chosen in order to represent these concepts in a concise way. Because this treatise has been set up from a general point of view it is worthwhile to conclude with a survey of the introduced elements together with their generalizations.

In Table I(aHf) all (multi-)bond graph symbols of the introduced concepts are listed.

In Table II it is indicated how the basic types of elements correspond to certain basic properties. It appears that energicness, entropicness, non-reciprocity, "non-mixingness" and flow discontinuity are the properties from which the MP C, MP R, MP GY, MP TF and S-array are derived respectively. The MP RS is consequently based on both entropicness and flow discontinuity (of the entropy flow).
### TABLE I

**Multibond graph terminology**

<table>
<thead>
<tr>
<th>(a) Edges or BONDS</th>
<th>(b) Vertices or MULTIPORT ELEMENTS (I)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>POWERBONDS</strong> (If P&gt;0, f&gt;0, then P and f in direction of half-arrow):</td>
<td></td>
</tr>
<tr>
<td>-single bond, t-bond, bond (P=ef):</td>
<td>energetic: ( g'.i = \frac{dE}{dt} ) (no modulation possible)</td>
</tr>
<tr>
<td>-multibond, n-bond, (P=p', f):</td>
<td>( C ) capacitor ( q = \int dt ) ( p = \int dt )</td>
</tr>
<tr>
<td>-multibond array (P=( g \cdot i )):</td>
<td>entropic: ( g'.i = T_i ) irreversible transducer ( l_i = l_i(t) )</td>
</tr>
<tr>
<td></td>
<td>boundary: ( g'.i = P ) effort-source (array) ( S_e )</td>
</tr>
<tr>
<td></td>
<td>junction-structure: GJS (ES-type) ( g = 0 ) flow-source (array) ( S_f )</td>
</tr>
<tr>
<td></td>
<td>(non-reciprocal)</td>
</tr>
<tr>
<td></td>
<td>(linear, possibly time-variant)</td>
</tr>
<tr>
<td></td>
<td>(extended recipr.)</td>
</tr>
<tr>
<td></td>
<td>6Y ( g = \frac{dE}{dt} ) generator ( G = -G' )</td>
</tr>
<tr>
<td></td>
<td>GJS (NES-type)</td>
</tr>
<tr>
<td></td>
<td>direct sum</td>
</tr>
<tr>
<td></td>
<td>(non-reciprocal)</td>
</tr>
<tr>
<td>active (single) bond: ( w = \frac{dE}{dt} ) (&quot;powerless signal&quot;)</td>
<td></td>
</tr>
<tr>
<td>active multibond: ( w \cdot i ) decomposition.</td>
<td></td>
</tr>
</tbody>
</table>

**Notation of efforts and flows**

- Single bond decomposition: \( \sum_{i=1}^{n} e_i \)

**Composition, decomposition and reordering by the direct sum**

- Multibond decomposition: \( \sum_{i=1}^{n} e_i \)

**ACTIVE BONDS or Signals (P>0).**

- active (single) bond: \( \dot{e} = \frac{dE}{dt} \)
- active multibond: \( \dot{w} = \frac{dE}{dt} \) single bond decomposition. \( \dot{q} = \frac{dE}{dt} \)

---

**Composifion, decmposition and reordering by the direct sum**

- Multibond decomposition: \( \sum_{i=1}^{n} e_i \)

**Composition, decomposition and reordering by the direct sum**

- Multibond decomposition: \( \sum_{i=1}^{n} e_i \)

**Composition, decomposition and reordering by the direct sum**

- Multibond decomposition: \( \sum_{i=1}^{n} e_i \)
### Table I
(continued)

<table>
<thead>
<tr>
<th>Notation of arrays of vertices by underlining</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_n$</td>
</tr>
<tr>
<td>$E_{m_1}$</td>
</tr>
<tr>
<td>$E_{m_1,m_2}$</td>
</tr>
</tbody>
</table>

*Examples:*

- **MP GY array:**
  - $E_{G_Y}$
  - $E_{G_Y}^{m,n}$
  - $E_{G_Y}^{n}$

- **SGY array:**
  - $E_{S_G_Y}$
  - $E_{S_G_Y}^{m,n}$
  - $E_{S_G_Y}^{n}$

- **O-junction array:**
  - $E_{O}$
  - $E_{O}^{n}$
  - $E_{O}^{m,n}$
### (c) Modulation of a (multiport) element by an active bond or signal

- "M" or "N" as the first character of the (MF) mnemonic code:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>MTF</td>
<td>e(t) → f(t)</td>
</tr>
</tbody>
</table>

### (d) Causal stroke

- Impedance or "effort" causality
- Admittance or "flow" causality

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>E s g(1)</td>
<td>E f i(1)</td>
</tr>
</tbody>
</table>

### Table 1

(continued)

#### (f) Basic forms (1-, 2- and 3-ports) and their electrical analogues

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Electrical Analogue</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
</tr>
<tr>
<td>MTF</td>
<td></td>
</tr>
<tr>
<td>e(t)</td>
<td></td>
</tr>
</tbody>
</table>
Using the concepts and symbols of Table I(a)-(f), it is possible to present a model of a physical system in the sense of this paper. Figure 19(a) shows a generalized junction structure (GJS) to which all kinds of multiports are connected: arrays of MP C's (representing energy-storage), MP R(S)'s (representing entropy-production or dissipation) and a (1-port) source array (representing sources and sinks or boundary conditions). By extraction of the MP GY(-array) the GJS becomes a WJS [Fig. 19(b)] and by extraction of the MP TF(-array) this WJS becomes an SJS [Fig. 19(c)]. If the constitutive relations of the multiports are known, the model of the system is completely determined and can be represented in more detail (single bond notation). Of course, the constitutive relations are different for every specific model, but they obey certain rules or principles according to the character of the MP E (e.g. Maxwell reciprocity of a MP C, non-energicness of a GJS).

The presented structure of basic physical concepts provides the modeller with a powerful tool to find the forms of these constitutive relations in a systematic and meaningful way and to combine the relations properly into a (computable) simulation model with the use of causal augmentation. In the case of linear systems,

![Table II](image)

| multiport element property | MP C | MP R | MP GY | MP TF | S (
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>energetic (storage)</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>entropic (dissipative)</td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>non-reciprocal</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>non-mixing</td>
<td></td>
<td></td>
<td></td>
<td>X (not applc.)</td>
<td></td>
</tr>
<tr>
<td>flow-discontinuous</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 19. Multibond graph of a general system with: a GJS (a), a WJS and extracted MP GY array (b), an SJS and an extracted MP TF array (c).
the dynamic behaviour of a system can already be predicted in a qualitative way even when the constitutive relations (parameters) are not yet known: causal paths in a bond graph correspond to (feedback) loops in a block diagram and thus to relaxational (time constants) or oscillatory behaviour (eigen frequencies). In general, nonlinear systems will have a less predictable dynamic behaviour than linear systems, but using the concepts of physical systems theory and the (multi-)bond graph notation of those concepts, some qualitative impressions of the ("local" or "small signal") character of the system may be obtained, e.g. the prediction of the possibility of (chemical) oscillations and limit cycles.

References

(19) J. J. van Dixhoorn, “Physical modelling on a thermodynamic basis using the bond graph


