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Graph theoretic aspects of music theory

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Abstract

The cycle on twelve points is a well-known representation of the twelve pitch classes of the traditional scale. We treat a more general situation where the number of pitch classes can be different from twelve and where, moreover, other measures of closeness are taken into account. We determine all situations for which the Generalized Hexachord Theorem continues to hold.

1 Introduction

We consider microtonal systems in which the number of tones or pitches in one octave is n. The letter n will be used throughout this paper in that sense. A *pitch class* is a set of pitches where octave equivalent and enharmonically equivalent pitches are identified. The elements of a pitch class are all represented by an integer between 0 and $n-1$ inclusive; $c = 0$, c sharp $= d$ flat $= 1$, etc. So a *pitch class set*, abbreviated *PC set* can be seen as a set of integers reduced modulo n. By the *order* of a PC set we mean the number of pitch classes in the PC set. The *complement* of a PC set S is the set of all pitch classes that are not in the PC set S. The complement of S is denoted by $\overline{S}$.

Two PC sets A and B are said to be *equivalent* if there is a number t such that for each $p \in A$ there is a $q \in B$ such that either $p + t = q(\text{mod } n)$ or $p + q = t(\text{mod } n)$. In the first case B is a *transposition* of A, denoted by $B = T_t A$, in the second case it is an *inversion*, denoted by $B = I_t A$.

The *interval* between two pitch classes p and q with $p < q$ is the minimum of $q-p$ and $n+p-q$. The *interval vector* $V(S)$ of a PC set S is a vector the i-th entry of which is the number of intervals of length i, where i runs from 1 to $\lfloor n/2 \rfloor$. Let S be a PC set of order at most $n/2$. Then the *difference vector* $\triangle V(S)$ is defined as the vector $V(\overline{S}) - V(S)$. If the order of S is greater than $n/2$, $\triangle V(S)$ is defined as $V(S) - V(\overline{S})$.

Example 1 Let $n = 12$ and take S=$\{0, 1, 2, 5, 8\}$. Then $V(S)$=[212221]. The complement $\overline{S}$=$\{3, 4, 6, 7, 9, 10, 11\}$ and $V(\overline{S})$=[434442],
so $V(\overline{S}) - V(S)$=[222221].

The special 'almost constant' form of the difference vector is not accidental. In fact, the Generalized Hexachord Theorem (GHT) says that the difference vector only depends on the <u>number</u> of pitches in S. For details, see Regener (1974).

As is well-known, the twelve pitch classes of the traditional scale can be represented on a *cycle*, consisting of twelve points and a line between two points when the difference between the corresponding pitches is a semitone. More generally, a microtonal system with n pitch classes can be represented by a cycle with n points. In these cases, lines represent 'closeness' of tones in terms of pitch. Other measures of closeness are conceivable, e.g. two C's on different manuals of a church organ. A different example is given by a cycle in which the diagonals that correspond to a fifth have been added.

In the general case, we have a *graph*, that is: a set of points (representing tones or pitch classes) and lines between certain pairs of points (indicating some type of closeness). The GHT refers to the graph C_n: the cycle on n points.

In the next sections we solve the following problem: determine the graphs for which the difference vector $\triangle V(S)$ depends on S only through the order of S. For convenience we call such graphs *robust*.

2 Prerequisites from graph theory

When we replace the well-known cycle by a different graph (e.g. the graph of Figure 1), the concept of 'interval' from music theory is generalized to the concept of 'distance'. The *distance* between points P and Q is the minimum number of lines in a 'walk' between P and Q. The length of the ic-vector is the *diameter* of the graph, i.e. the largest distance occurring in the graph. (The graph of Figure 1 has diameter 3). The *degree* of a point P is the number of lines incident with P. If all points of a graph G have the same degree, G is called *regular*. (The graph of Figure 1 is *3-regular*, i.e. regular of degree 3). Since the points of the graph correspond to pitch classes, the letter n will likewise be used for the number of points of the graph.

Example 2 Now consider a graph G. Take a PC set S of one element P, i.e. $S = \{P\}$. All components of the interval vector $V(S)$ are 0. Since in the graph G, the element P is a point, the first component of $V(\overline{S})$ is equal to the number of lines between points different from P. Hence the first component equals the total number of lines in G minus the degree of P. Therefore $\triangle V(S)$ is independent on P only if the first element of $V(\overline{S})$ is the same for all points P, i.e. if the graph G is regular.

The regularity-condition is not sufficient, as is shown by the following example.

Example 3 The graph G in Figure 1 is regular. However, for the PC set $S = \{A, B\}$ we have $V(S)$=[100], $V(\overline{S})$=[744], $\triangle V(S)$ =[644], whereas $S = \{B, C\}$ gives $V(S)$=[100], $V(\overline{S})$=[753], $\triangle V(S)$ =[653].

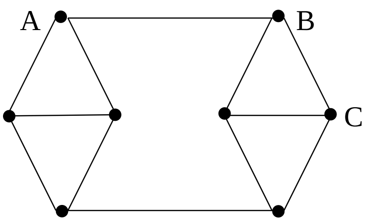

Figure 1: graph G

It turns out that in graph theory a concept exists that is equivalent to robustness.

Definition 1 *A graph G is distance degree regular (DDR) if for each point P of G, the number of points at distance j from P is independent of P.*

Definition 2 *The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (x, y) with $x \in A$ and $y \in B$.*

Example 4 If A is the set of suits from a deck of cards (hence $A = \{$spades, hearts, clubs, diamonds$\}$), and B is the set of values (hence $B = \{2, 3, ..., 10,$ jack, queen, king, ace$\}$), then $A \times B$ is just the deck of cards.

Definition 3 *Consider two graphs G_1 and G_2 with point sets P_1 and P_2 respectively. The product graph $G_1 \times G_2$ is the graph whose point set is the product of sets $P_1 \times P_2$. For any two points (u_1, u_2) and (v_1, v_2) in $P_1 \times P_2$, there is a line between these points in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2v_2 is a line in $G_2]$ or $[u_2 = v_2$ and u_1v_1 is a line in $G_1]$.*

Example 5 The product $K_2 \times C_{12}$ of the graphs K_2 and C_{12} is shown in Figure 2 (The graph K_2 has two points and one line). The two cycles can be seen as the pitch classes of two manuals on a church organ. The lines between the cycles indicate the 'closeness' of the endpoints: they differ in timbre only.

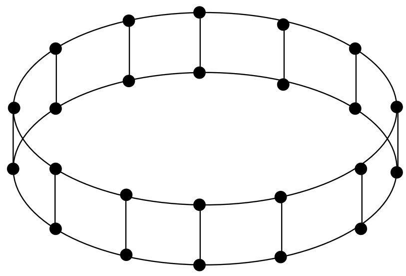

Figure 2: the product graph $K_2 \times C_{12}$

3 The main result

Theorem 1 *If G is DDR, then G is robust, and conversely.*

Proof Let G be a DDR graph. Split its point set into two classes: S and $\overline{S}$. In the following definitions, $d(u, v)$ denotes the distance between the points u and v.
$x_j =$ the number of pairs u, v with $u, v \in S$ and $d(u, v) = j$,
$\overline{x}_j =$ the number of pairs u, v with $u, v \in \overline{S}$ and $d(u, v) = j$,
$y_j =$ the number of pairs u, v with $u \in S$, $v \in \overline{S}$ and $d(u, v) = j$.
Then $x_j + \overline{x}_j + y_j =$ number of pairs in G at distance j. This number depends on G only. Choose a path of length j for each pair of points at distance j. The points at distance j from a fixed point in S lie either in S or in $\overline{S}$. There are x_j S-S-pairs and y_j S-$\overline{S}$-pairs. The total number of end-points in S of such paths is $2x_j + y_j = k_j|S|$, where k_j is a constant (G

is DDR). Since also $x_j + \overline{x}_j + y_j$ is a constant, it follows that $x_j - \overline{x}_j$ is a constant. Hence G is robust.
Conversely, let G be robust. Take $S = P$, hence $|S| = 1$. Obviously, $x_j = 0$ for all j. Since G is robust, $x_j - \overline{x}_j$ depends on $|S|$ only. In this case we conclude that $\overline{x}_j$ is independent of the choice of P. From the fact that $x_j + \overline{x}_j + y_j$ is constant, we know that y_j is independent of P. Hence G is DDR. $\square$

If a graph is point-transitive (roughly speaking: if the points are mutually indistinguishable as in C_n and $K_2 \times C_{12}$), then it is obviously a DDR graph. Hence according to Theorem 1 we have the following result.

Corollary 1 *All point-transitive graphs are robust.*

Remark Point-transitivity refers to equivalence of <u>points</u>, whereas robustness is a property of <u>sets</u> of points!

Theorem 2 *If the graphs G and H are robust, then the graph $G \times H$ is robust.*

Proof Suppose G and H are robust graphs. Let v be a point in G, and w a point in H, hence (v, w) a point in $G \times H$. Let $k_G(d, p)$ denote the number of points in G at distance d from p. Then

$$k_{G\times H}(d, (v, w)) = \sum_{i=1}^{d} k_G(i, v) \times k_H(d - i, w).$$

Since $k_G(i, v)$ and $k_H(d - i, w)$ are independent of v and w, respectively, it follows that $G \times H$ is robust. $\square$

4 Examples

The graph $K_2 \times C_{12}$, see Figure 2, is point transitive, hence robust. So for each set of m points the difference vector has the same form. For example, if $m = 3$, we have $\triangle V(S) = [27, 36, 36, 36, 36, 27, 9]$.
Several types of robust graphs exist that are not point-transitive. For example, <u>each</u> regular graph of diameter 2 is obviously robust. Examples with diameter larger than 2 can be obtained by multiplying such a graph by an arbitrary robust graph (e.g. a point-transitive graph). Still other examples exist. In Figure 3 we present the three 3-regular robust graphs on 12 points that are not point-transitive (this we determined by inspecting the list of all connected 3-regular graphs on 12 vertices in Bussemaker c.s. (1976)).

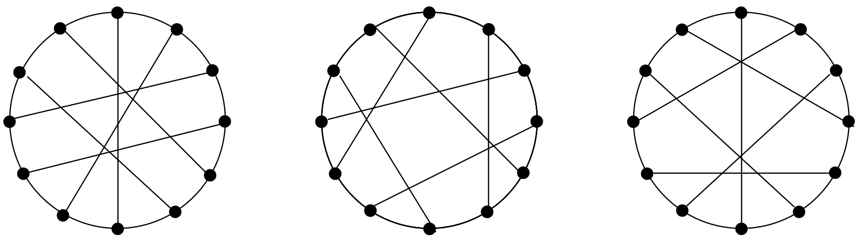

Figure 3: the three 3-regular robust graphs G_1, G_2 and G_3 on 12 points

5 Table

In Table 1 we present difference vectors for some robust graphs. n denotes the number of pitch classes, and m the order of the PC set.

Graph		**Difference vector** $\triangle V(S)$
Cycle C_n	n odd	$[(n-2m)...(n-2m)]$
	n even	$[(n-2m)...(n-2m)(n/2-m)]$
Prism $K_2 \times C_t$	t odd	$[(3n-3m)(4n-4m)...(4n-4m)(2n-2m)]$
$(n=2t)$	t even	$[(3n-3m)(4n-4m)...(4n-4m)(3n-3m)(n-m)]$
G_1		$[(18-3n)(30-5n)(18-3n)]$
G_2		$[(18-3n)(36-6n)(12-2n)]$
G_3		$[(18-3n)(36-6n)(12-2n)]$

Table 1: difference vectors of some robust graphs

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