Boundary controllability and source reconstruction in a viscoelastic string under external traction[1]

Luciano Pandolfi,

Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange",
Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

luciano.pandolfi@polito.it http://calvino.polito.it/∼lucipan

Abstract: Treatises on vibrations devote large space to study the dynamical behavior of an elastic system subject to known external tractions since usually a "system" is part of a chain of mechanisms which disturb the "system" for example due to the periodic rotation of shafts. These "disturbances" affect the horizontal component of traction in a time dependent way.

In this paper we shall study control and source identification problems for a viscoelastic string subject to external traction, using moment theory.

This paper has been published in *Journal of Mathematical Analysis and Applications*, **407** (2013) 464-479.
DOI: 10.1016/j.jmaa.2013.05.051
This version does not contain journal formatting and it may contain minor changes respect to the published version.

1 Introduction

Elastic and viscoelastic systems have been widely studied from the point of view of controllability (see for example [13, 22, 25] and references therein), the simpler "canonical" case being the linearized version of the string equation

$$w_{tt}(\xi,t)=\left(Pc(\xi)w_\xi(\xi,t)\right)_\xi$$

The coefficient $c(\xi)=1/\rho(x)$ depends on the physical properties of the string, i.e. the density, and P is the horizontal traction in the string which, for small oscillations and isolated systems, is constant in time (while c may depend on ξ since the density may not be constant.) It is usually explicitly assumed in control problems that, a part the action of external controls

[1] This paper fits the research plans of INDAM-CNR of the project "Groupement de Recherche en Contrôle des EDP entre la France et l'Italie (CONEDP)".

which does not change the horizontal component of the stress, the string is "isolated" i.e. that P does not depend on time. Control problems for the string equation under external traction have been rarely studied, see [1, 2, 4]. In fact, in general the system is part of a chain of mechanisms which produce "perturbations" which influence the horizontal traction in the string, often in a known way, for example due to the rotation of shafts. For this reason, books on vibrations devote large space to the study of the behaviour of elastic systems under known external tractions, see [11, 20].

In this paper we are going to study a *viscoelastic* string subject to external known tractions. The string is also subject to a known signal $\sigma(t)$ which enters trough an input operator b which is unknown. For example b may be the characteristic function of a certain set and our final goal will be the identification of the input operator b using boundary observations. So, the problem we are going to study boils down to the following equation

$$w_{tt}(\xi,t) = P(t)\left(c(\xi)w_\xi(\xi,t)\right)_\xi + \int_0^t M(t-s)P(s)\left(c(\xi)w_\xi(\xi,s)\right)_\xi \,\mathrm{d}s + b(\xi)\sigma(t) \tag{1}$$

where $P(t)$ represents the effect of the variable external traction and $w = w(\xi,t)$ with $t>0$ and $\xi \in (0,l)$ (unless needed for clarity, dependence of w on its arguments will not be indicated later on).

Remark 1. *The form (1) of the equation implies that the string is at rest for negative times. If not, a known additional term appears in the right hand side. This term has no influence on the arguments below. So, we put it equal to zero.* ∎

Initial and boundary conditions are homogeneous (of Dirichlet type)

$$w(\xi,0)=0\,,\qquad w_t(\xi,0)=0\,,\qquad w(0,t)=w(l,t)=0\,. \tag{2}$$

The observation is the traction at $x=0$, i.e.

$$-c(0)\left(P(t)w_\xi(0,t) + \int_0^t M(t-s)P(s)w_\xi(0,s)\,\mathrm{d}s\right)\,.$$

We shall assume that $P(t)$, $M(t)$ and $c(\xi)$ are smooth and that $P(t)$ and $c(\xi)$ are strictly positive (see below for the assumptions) so that from this "real" observation we can reconstruct the function

$$\eta(t) = w_\xi(0,t) \tag{3}$$

which will be used in the reconstruction algorithm.

In the purely elastic case, this kind of inverse problem has been studied using control ideas in [23] (see also [9, 18]) where it is proved that this inverse problem (in the purely elastic case) depends on the solution of a boundary control problem. So, it is easily guessed that the control problem has to be studied first also in the case that the traction varies with time. This is the second main subject of this paper, which is studied first.

The problem we need to study first is the controllability of the deformation $w(\cdot,T)$*, at a certain time* T*,* when $b=0$, the initial conditions are zero but the boundary conditions are

$$w(0,t)=\frac{f(t)}{c(0)P(t)}\,,\qquad w(l,t)=0\,. \tag{4}$$

Here $f(t)$ is a square integrable control (the denominator is introduced solely for convenience). So, the control problem we shall study first, and which takes the most part of the paper, is as follows: given any prescibed target $W(\xi)\in L^2(0,l)$, we must prove the existence of a square integrable control $f(t)\in L^2(0,T)$ such that $w(\xi,T)=W(\xi)$. Furthermore, we shall see the existence of a *universal* control time T, such that any target in $L^2(0,l)$ can be reached in time T.

The organization of the paper is as follows: the control problem is studied in section 3 while the identification problem is studied in section 4. Assumptions and preliminaries are in Section 2.

1.1 Further references

Control problems for viscoelastic materials have already been studied under several different assumptions. We cite in particular [8, 12] which seems to contain the most general results. The paper [12] studied the case $M=M(t,s)$ but it is explicitly stated that the arguments require constant density. The paper [8] studied the case $M=M(t,s,x)$ when the control is distributed (from which controllability results for boundary control systems with the control acting on the full boundary can be derived). These papers study also the multidimensional case, but the results does not seems to provide a practical construction for the steering control since they depend on compactness arguments (paper [12], which explicitly requires constant density) or Carleman estimates (paper [8]). In this respect we cite also [15], where controllability under boundary control is proved for multidimensional systems. The arguments here are presented in the convolution case and constant density, but are easily extended to non constant density. The proof in this paper being based on compactness arguments is not constructive either. The paper [7], being based on an extension of D'Alambert formula (see last section) is more constructive in spirit, but the elastic operator is assumed to be time invariant. Furthermore, this paper studies only reachability of smooth kernels, which is too weak for the identification problem. The papers [3, 14, 16, 17] instead study the control problem for a viscoelastic string, using moment methods, so that a representation formula for the control can be given (see in particular [16, Section 4] and [14, Section 4]). Our goal here is the extension of these results to a viscoelastic string subject to nonconstant traction, and to show that the results can be used to solve a source identification problem. Source identification problems using control methods have been solved for elastic materials with constant traction first in [23] (see also [9]) and the method has been extended to materials with memory in [18]. When the traction is not constant we need a different idea, presented in section 4.

Finally, we cite the papers [5, 6, 19] in which a new kind of control problem for systems with memory is introduced. This control problem has no counterpart for memoryless systems.

2 Assumptions and preliminaries

The assumptions in this paper are:

1. $P(t)$ is continuous and strictly positive, $P(t) \geq p_0 > 0$ for every $t \geq 0$ and $c(\xi) \in C^1(0, \pi)$ is strictly positive: $c(\xi) \geq c_0 > 0$ for every $\xi \in [0, \pi]$.
2. The kernel $M(t) \in W^{2,2}_{\mathrm{loc}}(0, +\infty)$. Hence
$$N(t) = 1 + \int_0^t M(s)\,\mathrm{d}s$$
has three derivatives ($N'''(t)$ is locally square integrable) and $N(0) = 1$.

We introduce the selfadjoint operator A: $L^2(0,l)\, apsto L^2(0,l)$ defined by
$$\mathrm{dom}\, A = H^2(0,l) \cap H^1_0(0,l)\,, \qquad A\phi = \left(c(\xi)\phi_\xi(\xi)\right)_\xi\,. \tag{5}$$
It is well known that this selfadjoint operator has compact resolvent, and that it has a sequence of eigenvalues $\{-\lambda_n^2\}_{n\geq 1}$, with the following asymptotic estimate [21, p. 173)]
$$\lambda_n = \frac{\pi}{L} n + \frac{H_n}{n}\,, \qquad |H_n| < M\,, \qquad L = \int_0^l \frac{1}{\sqrt{c(s)}}\,\mathrm{d}s\,, \qquad \left|\frac{\phi_n'(0)}{\lambda_n}\right| < M$$

(M does not depend on n).

Note that reference [21] has $n+1$, with $n \geq 0$, while we use n with $n \geq 1$.

Remark 2. *Clearly, the actual length l of the interval will not affect controllability or reconstruction. So, in order to simplify the formula we shall assume that the system has been normalized by taking the length of the interval so to have*

$$L = \int_0^l \frac{1}{\sqrt{c(s)}}\,\mathrm{d}s = \pi\,.$$

With this normalization,

$$\lambda_n = n + \frac{H_n}{n}\,. \tag{6}$$

Alternatively, we can obtain this normalization by changing the unity of measure of the time variable. ∎

2.1 Riesz systems

The arguments of this paper are based on the theory of Riesz systems. We recall the definition: a sequence $\{h_n\}$ in a separable Hilbert space H is a *Riesz basis* when it is the image of an orthonormal basis of H under a linear bounded and boundedly invertible transformation. The key property is that every $h \in H$ ha a unique representation with respect to the Riesz basis $\{h_n\}$,

$$h = \sum \alpha_n h_n$$

and $\{\alpha_n\} \in l^2$. In fact, there are positive numbers m and M such that

$$m \sum |\alpha_n|^2 \leq \|h\|^2 \leq M \sum |\alpha_n|^2\,.$$

If a sequence is a Riesz basis of its closed span (possibly not equal to H) it is called a *Riesz sequence.*

A property we shall need is that if $\{h_n\}$ is a Riesz sequence then the series $\sum \alpha_n h_n$ converges if and only if $\{\alpha_n\} \in l^2$, and the convergence is unconditional. Furthermore we need two perturbation results from [24, Ch. 1]. The first is a Paley-Wiener theorem, adapted to Hilbert spaces and orthonormal bases. It states that if

$$\sum \|h_n - \epsilon_n\|^2 < 1$$

($\{\epsilon_n\}$ orthonormal) then $\{h_n\}$ is a Riesz basis. In fact we shall use the following corollary:

Corollary 3. *Let $\{\tilde{\epsilon}_n\}$ be a Riesz sequence and let the sequence $\{h_n\}$ satisfy*

$$\sum \|h_n - \tilde{\epsilon}_n\|^2 < +\infty \tag{7}$$

then there exists a number N such that $\{h_n\}_{n>N}$ is a Riesz sequence too. This in particular implies that

$$\sum \alpha_n h_n$$

converges in the norm of H if and only if $\{\alpha_n\} \in l^2$.

We stress that the sequence $\{\tilde{\epsilon}_n\}$ in Corollary 3 need not be orthonormal. Condition (7) does not imply that $\{h_n\}$ is a Riesz sequence but

Theorem 4 (Bari Theorem)**.** *Let $\{\tilde{\epsilon}_n\}$ be a Riesz system and let the sequence $\{h_n\}$ satisfy condition (7). If furthermore*

$$\sum \alpha_n h_n = 0 \implies \{\alpha_n\} = 0 \tag{8}$$

then $\{h_n\}$ is a Riesz sequence.

The additional condition (8) is called *ω-independence.* Note that the convergence of the series in (8) is in H so that it must be $\{\alpha_n\} \in l^2$ because $\{h_n\}_{n>N}$ is a Riesz sequence, from Corollary 3.

2.2 Representation of the solution

Now we consider Eq. (1) with zero initial conditions but both the affine term and the boundary condition (4). We need a definition/representation formula for the solutions. We confine ourselves to a formula for $w(\xi, t)$ and we ignore velocity, since only $w(\cdot, T)$ is needed for the identification problem. So, we prefer to write Eq. (1) as a first order equation, integrating both the sides. We get

$$w_t = \int_0^t N(t-s)P(s)\left(c(\xi)w_\xi(s)\right)_\xi \, \mathrm{d}s + b(\xi)g(t)\,, \qquad g(t) = \int_0^t \sigma(s)\, \mathrm{d}S\,. \tag{9}$$

We multiply both the sides of (9) with $\phi_n(\xi)$ and we integrate on $[0, \pi]$. We get

$$w_n'(t) = -\lambda_n^2 \int_0^t N(t-s)P(s)w_n(s)\, \mathrm{d}s + \phi_n'(0) \int_0^t N(t-s)f(s)\, \mathrm{d}s + b_n g(t) \tag{10}$$

where

$$w_n(t) = \int_0^\pi w(\xi,t)\phi_n(\xi)\,\mathrm{d}x\,, \qquad b_n = \int_0^\pi b(\xi)\phi_n(\xi)\,\mathrm{d}\xi\,.$$

Let us fix any time $T>0$. We want a representation formula for $w_n(T)$. Let

$$Q(t) = P(T-t)$$

and let $z_n(t;T)$ solve

$$z_n'(t;T) = -\lambda_n^2 Q(t)\int_0^t N(t-s)z_n(s;T)\,\mathrm{d}s\,, \qquad z_n(0,T)=1\,. \tag{11}$$

Using $w_n(0)=0$, we see that

$$z_n(0;T)w_n(T) = \int_0^T \frac{\mathrm{d}}{\mathrm{d}s}\big(z_n(T-s;T)w_n(s)\big)\,\mathrm{d}s\,.$$

We use Leibniz formula and we replace $w_n'(t)$ and $z_n'(t;T)$ with their expresions. We get the following representation formula for $w_n(T)$:

$$\begin{aligned} w_n(T) = \int_0^T f(T-s)\left[\phi_n'(0)\int_0^s N(s-r)z_n(r;T)\,\mathrm{d}r\right]\,\mathrm{d}s \\ +b_n\int_0^T z_n(T-s;T)g(s)\,\mathrm{d}s\,. \end{aligned} \tag{12}$$

So, we expect the following definition of the solution $w(\xi,T)$:

$$\begin{aligned} w(\xi,T) = \sum_{n=1}^{+\infty}\phi_n(\xi)\int_0^T f(T-s)\left[\phi_n'(0)\int_0^s N(s-r)z_n(r;T)\,\mathrm{d}r\right]\,\mathrm{d}s \\ +\sum_{n=1}^{+\infty}\phi_n(\xi)b_n\int_0^T z_n(T-s;T)g(s)\,\mathrm{d}s\,. \end{aligned} \tag{13}$$

The following Lemma justifies formula (13):

Lemma 5. *The series in (13), as functions of T, converge in $C([0,K];L^2(0,\pi))$ for every $K\ge 0$.*

The proof of this Lemma requires a preliminary study of the functions $z_n(t,T)$ and can be found in the appendix.

Remark 6. *It has an interest to note that $z_n(t;T)$ is related to the resolvent $r(t,s)$ of the kernel $k(t,s) = -\lambda_n^2 N(t-s)P(s)$ as follows:*

$$z_n(t;T) = r(T,T-t)\,.$$

Eq. (11) is the second resolvent equation (see [10, formula (3.3) p. 295]) written in terms of $z_n(t;T)$. ∎

3 The control problem

In this section we study the control problem needed to identify the source term b; i.e. we consider Eq. (1) with $b=0$ and initial and boundary conditions

$$w(\xi,0)=0\,,\quad w_t(\xi,0)=0\,,\qquad w(0,t)=\frac{f(t)}{c(0)P(t)}\,,\qquad w(\pi,t)=0\,. \quad (14)$$

As we already said, we want to identify a time T such that for every $W(\xi)\in L^2(0,\pi)$ there exists a control $f(t)$ with the property that the corresponding solution $w(t)$ satisfies $w(T)=w(\xi,T)=W(\xi)$.

It is convenient to perform several transformations first. In particular, as we are interested in the controllability solely of the deformation $w(\cdot,T)$ and not of the velocity, we preferred to rewrite Eq. (1) in the form of a first order equation, but we note explicitly that the technique we use here can also be used to study the controllability of the pair $(w(\cdot,T),w_t(\cdot,T))$ as done in [14] in the case that the external traction P and the density are constant.

This section on controllability consists of three subsections: first we reduce the control problem to a moment problem. Then we present the transformations which will simplify the problem uder study, in subsection 3.1. Then we prove that a certain sequence of functions is a Riesz sequence in $L^2(0,S)$ wher S is a suitable number to be identified (in Theorem 12) and this will prove controllability, as seen in the subsection 3.1. The final result of controllability is as follows:

Theorem 7. *Let T_0 solve*

$$\int_0^{T_0}\sqrt{P(T-s)}\,\mathrm{d}s=\int_0^l\frac{1}{\sqrt{c(s)}}\,\mathrm{d}s\,. \quad (15)$$

and let $T\geq T_0$. Then, for every $\xi\in L^2(0,\pi)$ there exists a boundary control $f\in L^2(0,T)$ such that the corresponding solution $w(t)$ satisfies $w(T)=\xi$.

The computation of $w(t)$ for every t makes sense thanks to Lemma 5 and the number T_0 exists since

$$\int_0^T\sqrt{P(T-s)}\,\mathrm{d}s\geq Tp_0\,.$$

The control time T_0 depends on the length of the string. If the length has been normalized as in Remark 2, then formula (15) takes the form

$$\int_0^{T_0}\sqrt{Q(s)}\,\mathrm{d}s=\int_0^{T_0}\sqrt{P(T-s)}\,\mathrm{d}s=\pi \quad (16)$$

and this is the formula we use in the proofs, in order to henance readability of the paper.

Remark 8. *In this Section 3 the final time T is fixed once and for all. So, instead of the notation $z_n(t,T)$ we shall use the simpler notation $z_n(t)$, i.e. we put $z_n(t) = z_n(t,T)$ (T fixed). We shall return to the complete notation in Section 4 and in the Appendix.* ▮

3.1 Reduction to a moment problem

We fix a certain time T and we consider the solution $w(\xi,T)$, at this fixed time, when $b = 0$. We have, from (12) and (13):

$$w(\xi,T) = \sum_{n=1}^{+\infty} \phi_n(\xi) \int_0^T f(T-s) \left[\phi_n'(0) \int_0^s N(s-r) z_n(r) \, \mathrm{d}r \right] \, \mathrm{d}s \, . \qquad (17)$$

As we noted, $z_n(t)$ instead of $z_n(t;T)$ since T is fixed.

Equality (17) shows that $w(\xi,T) = W(\xi)$ if we can solve the moment problem

$$\int_0^T f(T-s) \left[\phi_n'(0) \int_0^s N(s-r) z_n(r) \, \mathrm{d}r \right] \, \mathrm{d}s = W_n \qquad (18)$$

where

$$W_n = \int_0^\pi \phi_n(\xi) W(\xi) \, \mathrm{d}\xi \, .$$

The transformation of $L^2(0,\pi) \ni W \mapsto \{W_n\}$ is an isometric isomorphism of $L^2(0,\pi)$ and l^2.

Our main result is that this moment problem is solvable and that the solution f of (18) (which has minimal L^2-norm) depends continuously on ξ, since we have:

Theorem 9. *Let T_0 be as in Theorem 7 and let $T \geq T_0$.*

The sequence

$$\left\{ \phi_n'(0) \int_0^t N(t-r) z_n(r) \, \mathrm{d}r \right\} \qquad (19)$$

is a Riesz sequence in $L^2(0,T)$.

In order to prove the theorem, we perform several transformations.

3.1.1 Transformations

Again we recall that we work with a fixed time T and $0 \le t \le T$ so that $z_n(t;T)$ is simply denoted $z_n(t)$ but it is important for the Appendix that we don't forget the dependence on T.

In this section, we transform the equation of $z_n(t)$.

First we introduce

$$H(t) = -N'(0)t - \log Q(t)\,, \quad \zeta_n(t) = e^{H(t)} z_n(t) = e^{-N'(0)t} \frac{1}{Q(t)} z_n(t)\,. \quad (20)$$

We shall see below the reason for this misterious looking transformation which does depend on T since $Q(t) = P(T-t)$.

The function $\zeta_n(t)$ verifies

$$\zeta_n'(t) - H'(t)\zeta_n(t) = -\lambda_n^2 e^{H(t)} Q(t) \int_0^t N(t-s) e^{-H(s)} \zeta_n(s)\, \mathrm{d}s$$

and so also

$$\zeta_n'' - H'(t)\zeta_n' + \left(\lambda_n^2 Q(t) - H''(t)\right)\zeta_n = G(t) \quad (21)$$

where

$$G(t) = -\lambda_n^2 \left[H'(t)Q(t)e^{H(t)} + e^{H(t)}Q'(t)\right] \int_0^t N(t-s) e^{-H(s)} \zeta_n(s)\, \mathrm{d}s$$
$$-\lambda_n^2 e^{H(t)} Q(t) \int_0^t N'(t-s) e^{-H(s)} \zeta_n(s)\, \mathrm{d}s\,. \quad (22)$$

Now we apply the Liouville transformation to Eq. (21), which requires the introduction of two auxiliary functions, which will be determined later on (see [21, p. 163]): we introduce the function $\tilde{\zeta}_n(t)$ given by

$$a(t)\tilde{\zeta}_n(t) = \zeta_n(t) \quad (23)$$

(here $a(t)$ is the first auxiliary function). Then we have

$$a(t)\tilde{\zeta}_n''(t) + \left[2a'(t) - H'(t)a(t)\right]\tilde{\zeta}_n'(t)$$
$$+ \left[\left(\lambda_n^2 Q(t) - H''(t)\right)a(t) - H'(t)a'(t) + a''(t)\right]\tilde{\zeta}_n(t) = G(t)\,.$$

The second trasformation is a transformation of the time variable: we introduce

$$x = L(t)$$

where $L(t)$ is still unspecified (but we shall see that we can choose $L(t)$ strictly increasing and its inverse transformation will be denoted $M(t)$). We introduce $Y_n(x)$ defined by

$$\tilde{\zeta}_n(t) = Y_n(L(t)) \,.$$

Then we have the following equation for $Y_n(x)$:

$$\begin{aligned} & a(t)\left[L'(t)\right]^2 Y''(L(t)) + \left[a(t)L''(t) + \left(2a'(t) - H'(t)a(t)\right) L'(t)\right] Y'(L(t)) \\ & + \left[\lambda_n^2 Q(t)a(t) + \left(a''(t) - H''(t)a(t) - H'(t)a'(t)\right)\right] Y(L(t)) = G(t) \,. \end{aligned}$$

We must be precise on this point: here and below notations like $Y'(L(t))$ will denote the derivative of $Y(x)$ computed for $x = L(t)$.

Now we relate the functions $L(t)$ and $a(t)$ so to have the coefficient of $Y'(L(t))$ equal to zero, i.e. we impose

$$\left(L'(t)\right)' = -L'(t)\left[2\frac{a'(t)}{a(t)} - H'(t)\right]$$

(this is legitimate since we shall choose $a(t) \neq 0$). In order to satisfy this condition we choose

$$L'(t) = \frac{e^{H(t)}}{a^2(t)} \,.$$

Note that $L(t)$ turns out to be strictly increasing. Furthermore, we choose

$$L(t) = \int_0^t \frac{e^{H(s)}}{a^2(s)} \, \mathrm{d}s$$

so that we have also $L(0) = 0$ and $L(t) > 0$ for $t > 0$.

Once $L(t)$ has been chosen as above, we see that the equation of $Y_n(x)$ is

$$\begin{aligned} & \frac{1}{a^3(t)} e^{2H(t)} Y_n''(L(t)) + \left[\lambda_n^2 Q(t) + \left(a''(t) - H''(t)a(t) - H'(t)a'(t)\right)\right] Y_n(t) \\ & = G(t) \,. \end{aligned}$$

Finally we choose

$$a(t) = e^{H(t)/2} \frac{1}{\sqrt[4]{Q(t)}}$$

(note: it is strictly positive) and we get the final form of the equation for $Y_n(t)$:

$$Y''(L(t)) + \left[\lambda_n^2 + \tilde{V}(t)\right] Y(L(t)) = \frac{e^{-H(t)/2}}{Q(t)^{3/4}} G(t) \qquad (24)$$

where

$$\tilde{V}(t) = \frac{e^{-H(t)/2}}{Q(t)^{3/4}} \left[a''(t) - H''(t)a(t) - H'(t)a'(t) \right] .$$

The definition of $a(t)$ gives

$$L'(t) = \sqrt{Q(t)} \,. \tag{25}$$

Let $M(x)$ be the inverse function of $L(t)$, defined on the interval $[0, L(T)] = [0, S]$ and

$$V(x) = \tilde{V}(M(x)) \,.$$

Using this and the explicit form of $G(t)$ in (22) we see that $Y_n(x)$ satisfies the following integro-differential equation on the interval

$$0 \le x \le S = L(T) \ :$$

$$\begin{aligned}
&Y''(x) + \left(\lambda_n^2 + V(x)\right) Y_n(x) \\
&= -\lambda_n^2 e^{H(M(x))/2} \Big[H'(M(x)) Q^{1/4}(M(x)) \\
&+ Q'(M(x)) Q^{-3/4}(M(x)) \Big] \int_0^{M(x)} N(M(x) - s) e^{-H(s)} \zeta_n(s) \,\mathrm{d}s \\
&- \lambda_n^2 e^{H(M(x))/2} Q^{1/4}(M(x)) \int_0^{M(x)} N'(M(x) - s) e^{-H(s)} \zeta_n(s) \,\mathrm{d}s \,.
\end{aligned}$$

We make the substitution $s = M(r)$ (we recall that $M(r)$ is the inverse function of $L(s)$), we use $0 \le r \le x$ when $0 \le s \le M(x)$ and

$$\frac{\mathrm{d}}{\mathrm{d}r} M(r) = \frac{1}{L'(s)} = \frac{1}{\sqrt{Q(s)}} \qquad \text{where } s = M(r).$$

Then, we have the following equation for $Y_n = Y_n(x)$:

$$Y_n'' + V(x) Y_n + \lambda_n^2 Y_n = -\lambda_n^2 \int_0^x A(x,r) Y_n(r) \,\mathrm{d}r \tag{26}$$

where

$$\begin{aligned}
A(x,r) = e^{H(M(x))/2} Q^{1/4}(M(x)) \big\{ &\left[H'(M(x)) \right. \\
&\left. + Q'(M(x)) Q^{-1}(M(x)) \right] N(M(x) - M(r)) \\
+ N'(M(x) - M(r)) \big\} &e^{-H(M(r))/2} Q^{-3/4}(M(r)) \,.
\end{aligned} \tag{27}$$

The values $Y_n(0) = y_0$ and $Y_n'(0) = y_1$ are easily computed:

$$
\begin{aligned}
y_0 &= Y_n(0) = e^{H(0)/2}\sqrt[4]{Q(0)} = Q(0)^{-1/4} = P(T)^{-1/4}\,, \\
y_1 &= Y_n'(0) = -\frac{1}{4}Q(0)^{-7/4}\left\{2N'(0)Q(0) + Q'(0)\right\} \\
&= \frac{1}{4}P(T)^{-7/4}\left\{P'(T) - 2N'(0)P(T)\right\}\,.
\end{aligned}
$$

In the following, we don't need these explicit expressions, but we shall need the following facts:

- the initial conditions y_0 and y_1 do not depend on n;
- the initial condition y_0 is strictly positive (we shall need that it is different from zero.)
- the initial conditions y_0 and y_1 do depend on T, equivalently on $S = L(T)$.
- In the appendix we shall use the fact that when $T \in [0, K]$ then there exists a number M (which depends on K) such that $|y_0(T)| < M$, $|y_1(T)| < M$ for every $T \in [0, K]$.

We introduce the notation

$$
g_n(x) = y_0 \cos \lambda_n x + \frac{1}{\lambda_n} y_1 \sin \lambda_n x\,. \tag{28}
$$

Using (28) and (26) we can write the following Volterra integral equation for $Y_n(x)$:

$$
\begin{aligned}
Y_n(x) &= g_n(x) - \frac{1}{\lambda_n}\int_0^x \sin \lambda_n(x-s)V(s)Y_n(s)\,\mathrm{d}s \\
&-\lambda_n \int_0^x \sin \lambda_n(x-s)\int_0^s A(s,\nu)Y_n(\nu)\,\mathrm{d}\nu\,\,\mathrm{d}s \\
&= g_n(x) - \left\{\int_0^x \left[A(x,s) + \frac{1}{\lambda_n}\sin \lambda_n(x-s)V(s)\right] Y_n(s)\,\mathrm{d}s\right. \\
&+ \int_0^x \cos \lambda_n(x-s)A(s,s)Y_n(s)\,\mathrm{d}s \\
&+ \left.\int_0^x \cos \lambda_n(x-s)\int_0^s A_{,1}(s,\nu)Y_n(\nu)\,\mathrm{d}\nu\,\,\mathrm{d}s\right\}\,.
\end{aligned}
$$

In this and following formulas, we use the comma notation for the derivative. Hence, $A_{,1}(x,s)$ is the derivative of $A(x,s)$ respect to the first variable.

When $x = r$ the brace in (27) is

$$\left\{H'(M(r)) + Q'(M(r))Q^{-1}(M(r)) + N'(0)\right\} = 0$$

thanks to the choice (20) for $H(t)$. The reason for the choice of the exponential $e^{H(t)}$ is precisely this:

$$A(s, s) = 0\,.$$

So, elaborating further the integral equation for $Y_n(x)$ we have

$$\begin{aligned}
&Y_n(x) = g_n(x) - \int_0^x A_n(x, s)Y_n(s)\,\mathrm{d}s \\
&+\frac{1}{\lambda_n}\int_0^x \sin\lambda_n(x-s)\left[\int_0^s A_{,11}(s,\nu)Y_n(\nu)\,\mathrm{d}\nu\right]\,\mathrm{d}s \\
&= g_n(x) - \int_0^x A_n(x, s)Y_n(s)\,\mathrm{d}s \\
&+\frac{1}{\lambda_n}\int_0^x Y_n(\nu)\left[\int_0^{x-\nu} A_{,11}(x-r,\nu)\sin\lambda_n r\,\mathrm{d}r\right]\,\mathrm{d}\nu\,.
\end{aligned} \tag{29}$$

where

$$A_n(x, s) = \left[A(x, s) + \frac{1}{\lambda_n}\left(V(s) - A_{,1}(s, s)\right)\sin\lambda_n(x-s)\right]\,.$$

Note that

$$A_n(x, x) = 0\,.$$

3.2 Some estimates

The estimates in the following Lemma 10 are used also in the Appendix, where we need to keep track of the dependence on $T \in [0, K]$, hence on S in the corresponding interval $[0, \Xi]$. So, in this lemma dependence on S is explicitly indicated. For this, we recall that the function $Y_n(x) = Y_n(x; S)$ is defined for $0 \leq x \leq S \leq \Xi$. Furthermore we recall that also y_0 and y_1 depend on S: $y_0 = y_0(S)$ $y_1 = y_1(S)$. Using (6) and Gronwall inequality we get:

Lemma 10. *Let $\Xi > 0$. There exists a number $M = M_\Xi$ such that for every $x \in [0, S] \subseteq [0, \Xi]$ the following inequalities hold:*

$$|Y_n(x; S)| < M\,, \qquad |Y_n(x; S) - y_0(S)\cos nx| \leq \frac{M}{n}\,. \tag{30}$$

The proof is in the Appendix.

Using Corollary 3 we get:

Theorem 11. *Let $S \geq \pi$ be fixed. There exists $N > 0$ such that the sequence $\{Y_n(x)\}_{n \geq N} = \{Y_n(x; S)\}_{n \geq N}$ is a Riesz sequence in $L^2(0, S)$.*

This suggests that we can use Bari Theorem, as in [17], in order to prove that $\{Y_n(x)\}_{n \geq 1}$ is a Riesz sequence. This is an intermediate step we shall need below, but we note that the sequence to be studied is (19) which, written in terms of $Y_n(x)$ gives the following sequence of functions of the variable t:

$$\phi_n'(0) \int_0^{L(t)} N(t - M(s)) e^{N'(0)M(s)/2} \frac{1}{\sqrt[4]{Q(M(s))}} Y_n(s) \, \mathrm{d}s \, .$$

The transformation from $\psi(t) \in L^2(0, T)$ to $(\mathcal{L}\psi)(x) = \psi(M(x))$ in $L^2(0, S)$ (where $T = M(S)$) defines a bounded and boundedly invertible transformation of $L^2(0, T)$ onto $L^2(0, S)$ so that we can equivalently study the following sequence in $L^2(0, S)$:

$$\left\{ \phi_n'(0) \int_0^x C(x, s) Y_n(s) \, \mathrm{d}s \right\} \tag{31}$$

where

$$C(x, s) = N(M(x) - M(s)) e^{N'(0)M(s)/2} Q^{-1/4}(M(s)) \tag{32}$$

and we must identify a value of S such that the sequence in (31) is Riesz in $L^2(0, S)$. We shall prove:

Theorem 12. *The sequence (31) is a Riesz sequence in $L^2(0, S)$ for every $S \geq \pi$.*

Using (25) and $L'(0) = 0$ we see that the sequence (19) is a Riesz sequence in $L^2(0, T)$ for every $T \geq M(\pi)$, i.e. for every $T > T_0$ where T_0 solve

$$\int_0^{T_0} \sqrt{Q(s)} \, \mathrm{d}s = \pi \, . \tag{33}$$

This is formula (16).

Thanks to the fact that $C(x, x)$ is strictly positive, we can equivalently study the sequence

$$\left\{ \phi_n'(0) \int_0^x B(x, s) Y_n(s) \, \mathrm{d}s \right\} , \qquad B(x, s) = \frac{C(x, s)}{C(x, x)} \, . \tag{34}$$

The computational advantage of this transformation is that now $B(x, x) = 1$.

3.2.1 The proof of Theorem 12

In this proof we work on a fixed interval $[0, S]$ (hence in a fixed interval $[0, T]$ so that we don't need to keep track of the dependence of $Y_n(x)$ on S). We shall need asymptotic estimates for several different sequences. In order to simplify the notations, M will denote a number which does not depend on the index of the sequences but which in general depends on the interval $[0, S]$ we are working in. Analogously, $\{M_n(x)\}$ will denote a sequence of functions which is bounded on $[0, S]$. The bound can depend on S. These constants and sequences will not be the same at every occurrence, without any possibility of confusion.

We introduce

$$K(s) = V(s) - A_{,1}(s, s) \tag{35}$$

so that $A_n(x, s)$ in formula (29) is

$$A_n(x, s) = A(x, s) + \frac{1}{\lambda_n} K(s) \sin \lambda_n (x - s) .$$

Furthermore, we introduce the notation $Z_n(x)$ (compare (34)):

$$Z_n(x) = \phi_n'(0) \int_0^x B(x, s) Y_n(s) \, \mathrm{d}s . \tag{36}$$

We shall proceed in parallel with the sequences $\{Y_n(x)\}$ and $\{Z_n(x)\}$ in order to prove the following result which, as we noted, implies Theorem 9.

Theorem 13. *Let $S \geq \pi$. Both the sequences $\{Y_n(x)\}$ and $\{Z_n(x)\}$ are Riesz sequences in $L^2(0, S)$.*

The proof require several steps.

Step 1: both the sequences $\{Z_n(x)\}$ and $\{Y_n(x)\}$ are linearly independent. We first prove that if $\{Z_n(x)\}$ is linearly dependent then $\{Y_n(x)\}$ is linearly dependent too; and then we prove linear independence of $\{Y_n(x)\}$.

If $\{Z_n(x)\}$ is linearly dependent then there exist numbers K and α_n such that

$$\sum_{n=1}^{K} \alpha_n Z_n(x) = 0 \quad \text{i.e.} \quad \int_0^x B(x, s) \left[\sum_{n=1}^{K} \phi_n'(0) \alpha_n Y_n(s) \right] \mathrm{d}s = 0 .$$

Hence we have also

$$\sum_{n=1}^{K} \phi_n'(0) \alpha_n Y_n(s) = 0$$

because $B(x,s)$ is smooth and $B(x,x)=1$. We prove that this implies $\alpha_n=0$ by proving that $\{Y_n(x)\}$ is linearly independent. We proceed by contradiction: let K be the first index for which

$$\sum_{n=1}^{K} \alpha_n Y_n(s) = 0\,. \tag{37}$$

Note that $K \geq 2$ since $Y_1(x) \neq 0$.

Using equality (26) we get

$$0 = \sum_{n=1}^{K} \alpha_n Y_n''(x) = -\sum_{n=1}^{K} \lambda_n^2 \alpha_n Y_n - \int_0^x A(x,r) \left(\sum_{n=1}^{K} \lambda_n^2 \alpha_n Y_n(r) \right) \,\mathrm{d}r$$

So we have also

$$\sum_{n=1}^{K} \lambda_n^2 \alpha_n Y_n(x) = 0\,.$$

This equality and (37) contradict the definition of K since they give:

$$\sum_{n=1}^{K-1} (\lambda_K^2 - \lambda_n^2) \alpha_n Y_n(x) = 0\,.$$

Step 2: a new expression for $Z_n(x)$. We integrate by parts in (29) as follows (the definition of $K(s)$ is in (35)):

$$\begin{aligned}
&Y_n(x) = g_n(x) - \int_0^x A(x,s) Y_n(s)\,\mathrm{d}s - \frac{1}{\lambda_n} \int_0^x \sin\lambda_n(x-s) K(s) Y_n(s)\,\mathrm{d}s \\
&-\frac{1}{\lambda_n^2} \int_0^x \left[\int_0^{x-\nu} A_{,11}(x-s,\nu) \frac{\mathrm{d}}{\mathrm{d}s} \cos\lambda_n s\,\mathrm{d}s \right] Y_n(\nu)\,\mathrm{d}\nu \\
&= g_n(x) - \int_0^x A(x,s) Y_n(s)\,\mathrm{d}s - \frac{1}{\lambda_n} \int_0^x \sin\lambda_n(x-s) K(s) Y_n(s)\,\mathrm{d}s \\
&+\frac{1}{n^2} M_n(x)\,.
\end{aligned}$$

Let $R(x,s)$ be the resolvent kernel of $-A(x,s)$. Then we have (recall $\lambda_n \sim n$)

$$\begin{aligned}
Y_n(x) &= \left[g_n(x) - \frac{1}{\lambda_n} \int_0^x \sin\lambda_n(x-s) K(s) Y_n(s)\,\mathrm{d}s \right] \\
&+ \int_0^x R(x,s) \left[g_n(s) - \frac{1}{\lambda_n} \int_0^s K(\nu) \sin\lambda_n(s-\nu) Y_n(\nu)\,\mathrm{d}\nu \right] \,\mathrm{d}s \\
&+\frac{1}{n^2} M_n(x)\,.
\end{aligned} \tag{38}$$

Using smoothness of $R(x,s)$ and $R(x,x)=0$ (a consequence of $A(x,x)=0$), we can integrate by parts twice and we see:

$$\int_0^x R(x,s)g_n(s)\ \mathrm{d}s = \frac{M_n(x)}{n^2}\,.$$

Using Lemma 10, we see that the same holds also for the double integral (exchange the order of integration and integrate by parts once) so that

$$Y_n(x) = \left[g_n(x) - \frac{1}{\lambda_n}\int_0^x \sin\lambda_n(x-s)K(s)Y_n(s)\ \mathrm{d}s\right] + \frac{1}{n^2}M_n(x)\,.$$

We replace this expression in (36) and we see that:

$$\begin{aligned}
&Z_n(x) = \phi_n'(0)y_0\int_0^x B(x,s)\cos\lambda_n s\ \mathrm{d}s + y_1\frac{\phi_n'(0)}{\lambda_n}\int_0^x B(x,s)\sin\lambda_n s\ \mathrm{d}s\\
&-\frac{\phi_n'(0)}{\lambda_n}\int_0^x\left[\int_\nu^x B(x,s)\sin\lambda_n(s-\nu)\ \mathrm{d}s\right]K(\nu)Y_n(\nu)\ \mathrm{d}\nu\\
&+\frac{1}{\lambda_n}\int_0^x B(x,s)M_n(s)\ \mathrm{d}s\,. \qquad (39)
\end{aligned}$$

Integrating by parts, we see that on every interval $[0,S]$ the functions $Z_n(x)$ are sum of a term of the order $1/n$ plus:

$$\phi_n'(0)y_0\int_0^x B(x,s)\cos\lambda_n s\ \mathrm{d}s = y_0\frac{\phi_n'(0)}{\lambda_n}\left[\sin\lambda_n x - \int_0^x B_{,2}(x,s)\sin\lambda_n s\ \mathrm{d}s\right]. \qquad (40)$$

We integrate by parts the last integral and we use the estimates (6). We get the following result:

Lemma 14. *Let $S>0$. There exists M such that for every n and every $x\in[0,S]$ we have*

$$|Z_n(x)| < M\,, \qquad \left|Z_n(x) - y_0\sqrt{\frac{2}{\pi}}\sin nx\right| < \frac{M}{n}\,. \qquad (41)$$

Step 3: we prove that ω-independence of $\{Y_n(x)\}$ on $L^2(0,S)$, $S\geq\pi$, implies that of $\{Z_n(x)\}$. Let $\{\alpha_n\}$ be a sequence such that the following equality holds in $L^2(0,S)$:

$$\sum_{n=1}^{+\infty}\alpha_n Z_n(x) = 0\,. \qquad (42)$$

We prove that if $\{Y_n(x)\}$ is ω-independent, then $\{\alpha_n\} = 0$, i.e. $\{Z_n(x)\}$ is ω-independent too.

The fact that $\{Z_n(x)\}_{n>N}$ is a Riesz sequence in $L^2(0,S)$ implies that $\{\alpha_n\} \in l^2$. In fact, we have more:

Lemma 15. *If (42) holds in $L^2(0,S)$ (any $S>0$) then the series*

$$\sum \phi_n'(0)\alpha_n \sin nx \tag{43}$$

converges in $L^2(0,S)$. So, If (42) holds and if $S \geq \pi$ then there exists $\{\gamma_n\} \in l^2$ such that

$$\alpha_n = \frac{\gamma_n}{\phi_n'(0)}\,.$$

Proof. The second statement follows from the first one, since the sequence $\{\sin \lambda_n x\}_{n>N}$ is a Riesz sequence[2] in $L^2(0,S)$ when $S \geq \pi$ so that convergence of (43) implies $\{\phi_n'(0)\alpha_n\} \in l^2$. So, we prove that equality (42) implies convergence of the series (43).

We prove the first statement. Using (39), equality (42) can be written as

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \frac{\phi_n'(0)}{\lambda_n}\alpha_n \int_0^x \left[\int_\nu^x B(x,s)\sin\lambda_n(s-\nu)\,\mathrm{d}s\right] K(\nu)Y_n(\nu)\,\mathrm{d}\nu \\
&-\sum_{n=1}^{+\infty} \frac{\alpha_n}{\lambda_n}\int_0^x B(x,s)M_n(s)\,\mathrm{d}s - y_1\int_0^x B(x,s)\left(\sum_{n=1}^{+\infty}\frac{\phi_n'(0)}{\lambda_n}\alpha_n \sin\lambda_n s\,\mathrm{d}s\right) \\
&= y_0\left(\sum_{n=1}^{+\infty}\phi_n'(0)\alpha_n\int_0^x B(x,s)\cos\lambda_n s\,\mathrm{d}s\right)
\end{aligned}$$

(these equalities technically have to be intended as finite sums, and the equality holds in the limit if it happens that the series converges. But, every series on the left side side is clearly convergent, so that the series on the right side has to converge too). Using (40) we write the series on the right hand side as

$$y_0\left(\sum_{n=1}^{+\infty}\alpha_n\frac{\phi_n'(0)}{\lambda_n}\sin\lambda_n x\right) - y_0\int_0^x B_{,2}(x,s)\left(\sum_{n=1}^{+\infty}\alpha_n\frac{\phi_n'(0)}{\lambda_n}\sin\lambda_n s\,\mathrm{d}s\right)\,\mathrm{d}s$$

We see that the second series can be differentiated in L^2 and that the differentiated series converge. Hence also the first series can be differentiated

[2] here in fact $N=1$ but we don't need this.

termwise and

$$y_0\left(\sum_{n=1}^{+\infty}\alpha_n\phi_n'(0)\cos\lambda_n x\right)$$

converges to a square integrable function, so that $\{\phi_n'(0)\alpha_n\}\in l^2$ since $\{\cos\lambda_n x\}_{n>N}$ is a Riesz sequence in $L^2(0,S)$ when $S\geq\pi$. ∎

Consequently we have

$$0=\sum_{n=1}^{+\infty}\frac{\gamma_n}{\phi_n'(0)}Z_n(x)=\int_0^x B(x,s)\left[\sum_{n=1}^{+\infty}\gamma_n Y_n(s)\right]\,\mathrm{d}s\,.$$

The fact that $B(x,s)$ is smooth and $B(x,x)=1$ easily implies

$$\sum_{n=1}^{+\infty}\gamma_n Y_n(s)=0\,.$$

If $\{Y_n(x)\}$ is ω-independent then $\{\gamma_n\}=0$ and so $\{\alpha_n\}=0$, i.e. $\{Z_n(x)\}$ is ω-independent too. So, in order to prove that $\{Z_n(x)\}$ is ω-independent, which is our final goal, it is sufficient to prove ω-independence of $\{Y_n(x)\}$. This we do in the last step.

Step 4: the sequence $\{Y_n(x)\}$ is ω-independent. Let the sequence $\{\gamma_n\}$ satisfy

$$\sum_{n=1}^{+\infty}\gamma_n Y_n(x)=0 \tag{44}$$

in $L^2(0,S)$ (so that $\{\gamma_n\}\in l^2$). Using (29) we get

$$\begin{aligned}&\sum\gamma_n g_n(x)=\sum\gamma_n\int_0^x A_n(x,s)Y_n(s)\,\mathrm{d}s\\&-\sum\frac{\gamma_n}{\lambda_n^2}\left\{\int_0^x A_{,11}(x,\nu)Y_n(\nu)\,\mathrm{d}\nu\right.\\&\left.+\int_0^x\cos\lambda_n(x-s)\left[A_{,11}(s,s)Y_n(s)+\int_{x-s}^x A_{,111}(s,\nu)Y_n(\nu)\,\mathrm{d}\nu\right]\,\mathrm{d}s\right\}\,.\end{aligned}$$

We prove that we can compute termwise the derivative of the series on the left hand side. As above, it is sufficient that we prove that the series on the right side can be differentiated termwise. This is clear for the second series.

As to the first one, it can be written as

$$\sum \gamma_n \int_0^x A_n(x,s) Y_n(s)\, \mathrm{d}s = \sum \gamma_n \int_0^x A(x,s) Y_n(s)\, \mathrm{d}s$$
$$- \sum \frac{\gamma_n}{\lambda_n} \int_0^x K(s) \sin \lambda_n (x-s) Y_n(s)\, \mathrm{d}s\,.$$

We have the following result:

Lemma 16. *The series* $\sum (\gamma_n/\lambda_n) \int_0^x K(s) \sin \lambda_n (x-s) Y_n(s)\, \mathrm{d}s$ *converges in* $L^2(0,S)$.

Proof. We note that

$$\sum \frac{\gamma_n}{\lambda_n} \int_0^x K(s) \sin \lambda_n (x-s) Y_n(s)\, \mathrm{d}s$$
$$= \sum \frac{\gamma_n}{\lambda_n} \int_0^x K(s) \left[Y_n(s) - y_0 \cos \lambda_n s \right] \sin \lambda_n (x-s)\, \mathrm{d}s$$
$$+ y_0 \sum \frac{\gamma_n}{\lambda_n} \int_0^x K(s) \sin \lambda_n (x-s) \cos \lambda_n s\, \mathrm{d}s\,.$$

The derivative of the first series converges uniformly, thanks to inequality (30). Using trigonometric formulas we see that the last series is

$$\left(\int_0^x K(s)\, \mathrm{d}s \right) \sum \frac{\gamma_n}{2\lambda_n} \sin \lambda_n x + \sum \frac{\gamma_n}{2\lambda_n} \int_{-x}^x K((x-r)/2) \sin \lambda_n r\, \mathrm{d}r$$

whose derivative is L^2-convergent. ∎

We sum up: $\sum_{n=1}^{+\infty} \gamma_n \cos \lambda_n x \in W^{1,2}(0,S)$ and, proceeding as in previous steps, we see that the following lemma holds:

Lemma 17. *Let equality (44) hold for the sequence* $\{\gamma_n\}$. *Then, there exists* $\{\sigma_n\} \in l^2$ *such that* $\gamma_n = \sigma_n/\lambda_n$.

Now we start a bootstrap argument:

Lemma 18. *The series* $\sum (\sigma_n/\lambda_n) Y_n(x)$ *is uniformly convergent to a* $W^{1,2}(0,S)$ *function, and its derivative can be computed twice termwise.*

Proof. We insert $\gamma_n = \sigma_n/\lambda_n$ in (44) and we use the representation (29) to get

$$\sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n} g_n(x) = \sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n^2} \int_0^x K(s) \sin \lambda_n (x-s) Y_n(s)\, \mathrm{d}s$$
$$- \sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n^2} \int_0^x \sin \lambda_n (x-s) \left[\int_0^s A_{,11}(s,\nu) Y_n(\nu)\, \mathrm{d}\nu \right] \mathrm{d}s$$

We compute a first derivative termwise and we get

$$\sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n} g_n'(x) = \sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n} \int_0^x K(s) \cos \lambda_n (x-s) Y_n(s)\ \mathrm{d}s$$

$$- \sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n} \int_0^x \left[\int_0^{x-\nu} A_{,11}(x-r,\nu) \cos \lambda_n r\ \mathrm{d}r \right] Y_n(\nu)\ \mathrm{d}\nu$$

Clearly we can compute a second derivative of the last series. We prove that also the first series can be again differentiated termwise. In fact, formal differentiation gives

$$K(x) \left(\sum_{n=1}^{+\infty} \frac{\sigma_n}{\lambda_n} Y_n(x) \right) - \sum_{n=1}^{+\infty} \sigma_n \int_0^x K(s) \sin \lambda_n (x-s) Y_n(s)\ \mathrm{d}s\,.$$

The first series is uniformly convergent and the second one is handled as in Lemma 16.

The previous computations prove L^2-convergence of $\sum_{n=1}^{+\infty} (\sigma_n/\lambda_n) g_n''(x)$ so that, as above:

Lemma 19. *If condition (44) holds then there exists a sequence* $\{\delta_n\} \in l^2$ *such that*

$$\gamma_n = \frac{\delta_n}{\lambda_n^2}\,.$$

This lemma combined with (26) shows that the second derivative of the series in (44) can be computed termwise, and of course it is zero. So, from (26) we get

$$\sum_{n=1}^{+\infty} \delta_n Y_n(x) + \int_0^x A(x,r) \left[\sum_{n=1}^{+\infty} \delta_n Y_n(r) \right] \mathrm{d}r = 0 \quad \text{i.e.} \quad \sum_{n=1}^{+\infty} \delta_n Y_n(x) = 0\,.$$

This equality, combined with (44), gives

$$\sum_{n=2}^{+\infty} \left(1 - \frac{1}{\lambda_n^2} \right) \delta_n Y_n(x) = 0\,.$$

This argument can be repeated till we remove N first elements from the series in (44). Using $1 - 1/\lambda_n^2 \neq 0$ for n large we conclude that the series in (44) is in fact a *finite sum*. But, then all its coefficients have to be zero, since we proved linear independence of the sequence $\{Y_n(x)\}$.

This completes the proof of Theorem 13, hence also of Theorem 7.

Using this result, we can now study the identification problem.

4 Source reconstruction

It is now convenient to go back to the full notation $z_n(s;t)$ and we recall that $z_n(s;t)$ is defined for $0 \le s \le t$. So, the solution of Eq. (1) with boundary control $f(t) \equiv 0$ is given by (see (13))

$$w(\xi,t) = \sum_{n=1}^{+\infty} \phi_n(\xi) b_n \int_0^t z_n(t-s;t) g(s)\, \mathrm{d}s\,, \quad b_n = \langle b, \phi_n \rangle = \int_0^\pi b(\xi)\phi_n(\xi)\, \mathrm{d}\xi\,.$$

The output $y(t) = w_x(0,t)$ is

$$\eta(t) = \sum_{n=1}^{+\infty} b_n \phi_n'(0) \int_0^t z_n(t-r;t) g(r)\, \mathrm{d}r \tag{45}$$

We prove that we can compute the output for every t, i.e.:

Lemma 20. *Let $g(t)$ be as in (9). Then, the function $t \to \eta(t)$ is continuous*

The proof is in the Appendix.

Now we choose $g(t)$ of the special form

$$g(t) = \int_0^t N(t-s) f(s)\, \mathrm{d}s\,. \tag{46}$$

Here $f(t)$ is locally square integrable and this is achieved by taking

$$\sigma(t) = f(t) + \int_0^t N'(t-s) f(s)\, \mathrm{d}s$$

(use $N(0) = 1$). Let us fix any $T \ge T_0$ (T_0 is specified in Theorem 7) and let us note that

$$\begin{aligned}
\eta(T) &= \left\langle \sum_{n=1}^{+\infty} \phi_n(x)\phi_n'(0) \int_0^T z_n(T-r;T) \int_0^r N(r-s) f(s)\, \mathrm{d}s\ \mathrm{d}r, b_n(x) \right\rangle \\
&= \left\langle \sum_{n=1}^{+\infty} \phi_n(x) \int_0^T f(T-s) \left[\phi_n'(0) \int_0^s N(s-r) z_n(r;T)\, \mathrm{d}r \right] \mathrm{d}s, b(x) \right\rangle
\end{aligned}$$

(the crochet denotes $L^2(0,\pi)$-inner product). We recall $z_n(r;T) = z_n(r)$ and we compare this formula with (17). We see that the left side of the crochet is $w(T)$ when $f(t)$ is the *boundary control*. Hence, for every k there exists a suitable $f_k(t) \in L^2(0,T)$, i.e. a suitable $\sigma_k(t) \in L^2(0,T)$, such that the corresponding output $\eta_{(k)}(t)$ is such that

$$\eta_{(k)}(T) = \langle \phi_k, b \rangle = b_k \,,$$

the $k-th$ Fourier coefficient of $b(\xi)$ respect to the orthonormal basis $\{\phi_k(\xi)\}$ (or, if we whish, to any prescribed orthonormal basis, thanks to Theorem 7). So, $b(\xi)$ is given by

$$b(\xi) = \sum_{n=1}^{+\infty} \eta_{(k)}(T)\phi_k(\xi) \,.$$

Remark 21. *At first sight it might seem that the previous method is similar to the one developed in [23]. In fact, it is not the same idea and, as expected due to the time varying coefficient $P(t)$, it is less efficient. In fact, the algorithm in [23] does not really use $\sigma(t)$ which has to be known, smooth and with $\sigma(0) = 1$, but fixed once and for all. Using this fixed input the output $y(t)$ is measured for every $t \in [0,T]$ and then a sequence of algorithms applied to this simple observation gives the Fourier coefficients b_k. The algorithm in [23] has been extended to systems with memory, with constant $P(t)$, in [18] but it seems that this algorithm cannot be used to study time varying traction. For this reason we proposed to use the full map $L^2(0,T) \ni \sigma(\cdot) \mapsto y(T)$.* ∎

5 Appendix: The proof of Lemmas 5, 10 and 20

In this appendix it is important to recall that every transformation in Section 3.1.1 is on a fixed interval $[0,T]$ and *does depend on T.* Even the transformation $H(t) = H(t;T)$, since it depends on $Q(t) = P(T-t)$. Now we consider these transformation for every fixed T in an interval $[0,K]$ so that it will be $0 \le t \le T \le K$ and the pair (t,T) will belong to the triangle

$$\tilde{\triangle} = \big\{(t,T) : \quad 0 \le t \le T \le K\big\} \,.$$

We have that

$$L(t) = L(t;T) = \int_0^t \sqrt{P(T-s)}\,\mathrm{d}s \,, \qquad L'(t) = \sqrt{P(T-t)}$$

are bounded on the triangle $\tilde{\triangle}$, uniformly for $0 \le T \le K$.

As in Section 3.1.1, let, for every $T \in [0,K]$,

$$S = L(T) = L(T;T)$$

so that $0 \le x = L(t;T) \le S \le \Xi = L(K,K)$ and the triangle $\tilde{\triangle}$ is transformed to

$$\Delta = \left\{(x,S) : \quad 0 \le x \le S \le \Xi\right\}.$$

So, $M(t;X)$ and $M'(t;X)$ and $A(x,r;X)$ and their derivatives are bounded on $\triangle$. We recall that y_0 and y_1 do depend on S. So we shall write

$$g_n(x) = g_n(x;S) = y_0(S)\cos\lambda_n x + \frac{1}{\phi_n'(0)} y_1(S)\sin\lambda_n x$$

and that $y_0(S)$ and $y_1(S)$ are bounded on $[0,\Xi]$.

Proof of Lemma 10. The first inequality in (30) follows from Gronwall inequality applied to (29). So, we must prove

$$|Y_n(x;S) - g_n(x;S)| \le \frac{M}{n} \tag{47}$$

(the constant M depends on Ξ.)

In order to get inequality (47) we integrate by parts once the last integral in (29) and we add and subtract $g_n(x;S)$ to the first integral. Then we integrate by parts so to get a factor $1/\lambda_n$. We obtain

$$\begin{aligned}
&Y_n(x) - g_n(x;S) = \\
&-\int_0^x A_n(x,r;S)\left[Y_n(r;S) - g_n(r;S)\right]\,\mathrm{d}s \\
&-\int_0^x A_n(x,r;S) g_n(s;S)\,\mathrm{d}r + \frac{1}{\lambda_n^2} M_n(x;S)
\end{aligned}$$

where $M_n(x;S)$ is a function that we don't need to write down explicitly, but such that $|M_n(x,S)| < M$ for every n and $(x,S) \in \triangle$.

The require inequality follows from here, integrating by parts the last integral and using Gronwall inequality. ∎

The proofs of Lemmas 5 and 20. The idea of the proofs of these lemmas is similar. In order to prove Lemma 5 we have to consider the first series in (13) while for Lemma 20 we have to consider the series in (45). The common feature is an integral of the general form

$$\int_0^t V(t,r) z_n(r;T)\, \mathrm{d}r$$
$$= \int_0^{L(t;T)} V(t;M(s;S)) e^{N'(0)M(s;S)} P^{-1/4}(M(s;S)-T)\tilde{\zeta}_n(M(s;S);T)\, \mathrm{d}s\,.$$

Now we use

$$L(M(x;S);T) = x\,, \quad Y_n(s;S) = \tilde{\zeta}_n(M(s;S);T)$$

and finally we get an integral of the form

$$\int_0^x V_1(x,s;X) Y_n(s;X)\, \mathrm{d}s\,.$$

The integral in Lemma 5 is obtained when $V(t,r;T) = N(t-r)$ and then $V_1(x,s;X) = C(x,s;X)$. The integral in (45) is obtained with $V(t,r) = g(t-r)$. Note that $C(x,x;X) \neq 0$ while $g(0) = 0$ and so in the proof of Lemma 20 we shall have $V_1(s,s;X) = 0$.

We present now the proof of Lemma 5 and we leave the similar proof of Lemma 20 (based on inequality (41)) to the reader.

We transform the variable $t \in [0,T] \subseteq [0,K]$ in the series (13) to the variable $x = M(t) \in [0,S] \subseteq [0,\Xi]$ as shown above and we consider the $L^2(0,\pi)$ norm of the resulting series, for each $S \in [0,\Xi]$. Using the fact that $\{\phi_n(\xi)\}$ is orthonormal in $L^2(0,\pi)$ and definition (32), the square of the norm can be written as

$$\sum_{n=1}^{+\infty} \left| \int_0^x \Lambda(x,\nu;S) \left(\phi_n'(0) \int_0^\nu C(\nu,r;S) Y_n(r;S)\, \mathrm{d}r \right) \mathrm{d}\nu \right|^2$$
$$= \sum_{n=1}^{+\infty} \left| \int_0^x \Lambda(x,\nu;S) \left(C(\nu,\nu;S) Z_n(\nu;S) \right) \mathrm{d}\nu \right|^2$$
$$\Lambda(x,\nu;S) = f\big(M(x;S) - M(\nu;S)\big) M'(\nu;S)$$

Using (41) we see that it is sufficient to note

$$\sum_{n=1}^{+\infty} \left| \int_0^x \Lambda(x,\nu;S) C(\nu,\nu;S) \sin ns \right|^2 \leq \int_0^x \left| \Lambda(x,\nu;S) C(\nu,\nu;S) \right|^2 \, \mathrm{d}s\,.$$

The proof is now finished since the right hand side is bounded for $0 \leq s \leq S \leq \Xi$. ∎

References

[1] S. Avdonin, B.P. Belinskiy, Controllability of a string under tension, Discrete and Continuous Dynamical Systems: A Supplement Volume, (2003), 57-67.

[2] S. Avdonin, B.P. Belinskiy, On the basis properties of the functions arising in the boundary control problem of a string with a variable tension, Discrete and Continuous Dynamical Systems: A Supplement Volume, (2005), 40-49.

[3] S. Avdonin, B.P. Belinskiy, On controllability of an homogeneous string with memory, J. Mathematical Analysis Appl., **398** (2013) 254–269.

[4] S. Avdonin, B.P. Belinskiy, L. Pandolfi, Controllability of a Nonhomogeneous String and Ring under Time Dependent Tension, Math. Model. Nat. Phenom. 5 (2010) 4-31.

[5] S. Avdonin and L. Pandolfi, Simultaneous temperature and flux controllability for heat equations with memory, Quarterly Appl. Math., Electronic version DOI 10.1090/S0033-569X-2012-01287-7

[6] S. Avdonin and L. Pandolfi, Temperature and heat flux dependence/independence for heat equations with memory, in"Time Delay Systems - Methods, Applications and New Trend" (Ed. R. Sipahi, T. Vyhlidal, P. Pepe, and S.-I. Niculescu) Lecture Notes in Control and Inform. Sci. 423, Springer-Verlag, (2012), pp. 87–101.

[7] V. Barbu, M. Iannelli, Controllability of the heat equation with memory, Diff. Integral Eq. **13** (2000) 1393-1412.

[8] X. Fu, J. Yong, X. Zhang, Controllability and observability of the heat equation with hyperbolic memory kernel, J. Diff. Eq. 247 (2009) 2395-2439.

[9] M. Grasselli, M. Yamamoto, Identifying a spatial body force in linear elastodynamic via traction measurements, SIAM J. Contr. Optim. 36 (1998) 1190-1206.

[10] G. Gripenberg, S.-O. Londen, O. Staffans, *Volterra integral and functional equations,* Encyclopedia of Mathematics and its Applications, 34. Cambridge University Press, Cambridge, 1990.

[11] J. P. Den Hartog, *Mechanical Vibrations.* McGraw-Hill Book Company, New York, 1956.

[12] J.U. Kim, Control of a second-order integro-differential equation, SIAM J. Control Optim. 31 (1993) 101-110.

[13] I. Lasiecka, R.Triggiani, *Control theory for partial differential equations: continuous and approximation theories. II. Abstract hyperbolic-like systems over a finite time horizon.* Encyclopedia of Mathematics and its Applications, 75. Cambridge University Press, Cambridge, 2000.

[14] P. Loreti, L. Pandolfi, D. Sforza, Boundary controllability and observability of a viscoelastic string, SIAM J. Control Optim. 50 (2012) 820-844.

[15] L. Pandolfi, The controllability of the Gurtin-Pipkin equation: a cosine operator approach, Appl. Math. and Optim. 52 (2005) 143-165 (see a correction in Appl. Math. Optim. 64 (2011) 467-468.)

[16] L. Pandolfi, Riesz systems and controllability of heat equations with memory, Int. Eq. Operator Theory. 64 (2009) 429-453.

[17] L. Pandolfi, Riesz systems and moment method in the study of heat equations with memory in one space dimension, Discr. Cont. Dynamical Systems, Ser. B. 14 (2010) 1487-1510.

[18] L. Pandolfi, Riesz systems and an identification problem for heat equations with memory, Discr. Cont. Dynam. systems. Ser. S. 4 (2011) 745-759.

[19] L. Pandolfi Traction, deformation and velocity of deformation in a viscoelastic string, in print, Evolution Equations and Control Theory.

[20] S. Timoshenko *Thèorie des Vibrations.* Libr. Polytecnique Ch Bèranger, Paris, 1947.

[21] F. Tricomi, *Differential Equations,* Blackie & Sons, Toronto, 1961.

[22] M. Tucsnak, M., Weiss G., *Observation and control for operator semigroups.* Birkhuser Verlag, Basel, 2009.

[23] M. Yamamoto, Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method, Inverse problems. 11 (1995) 481-496.

[24] R.M. Young, *An Introduction to Nonharmonic Fourier Series.* Academic Press, New York, 2001.

[25] E. Zuazua Controllability and observability of partial differential equations: some results and open problems, in *Handbook of differential equations: evolutionary equations. Vol. III,* pp. 527–621, Elsevier/North-Holland, Amsterdam, 2007.