# Toward an efficiently computable formula for the output statistics of MIMO block-fading channels

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***Abstract*—The information that can be conveyed through a wireless channel, with multiple-antenna equipped transmitter and receiver, crucially depends on the channel behavior as well as on the input structure. In this paper, we present very recent analytical results, concerning the probability density function (pdf) of the output of a single-user, multiple-antenna communication. The analysis is carried out under the assumption of an optimized input structure, and assuming Gaussian noise and block-fading. A further simplification of the output pdf expression presented in our last paper is derived, without the need for resorting to involved integration rules over unitary matrices. With respect to the former result, presented at the main track of this conference, the newly derived formula has the appealing feature of being numerically implementable with open access Matlab codes developed at MIT for the evaluation of zonal polynomials.**

## I. Introduction

Several communication and information theoretic performance indices depend on the statistical characterization of the output of a wireless channel. In the MIMO case, this is posing challenging issues due to the need for exploiting sophisticated tools from multivariate statistics and finite-dimensional random matrix theory. This work is intended to describe the ongoing efforts toward an efficiently computable closed form characterization of the joint density of the entries of a matrix variate, which represents the output of a MIMO system affected by AWGN and block-fading. The rationale of the work stems from the fact that, albeit compact, the expression we recently provided for the conditional output probability density function (pdf) of the abovementioned wireless system was not suitable for computer evaluation. Our current aim is, then, to provide a formula whose numerical evaluation could be done via Matlab codes developed for the approximation of zonal polynomials and functions of matrix arguments like those from [1]. The result we introduce later in the paper is the first relevant step toward our main goal.

## II. Notation and System Model

Throughout the work, matrices are denoted by uppercase boldface letters, vectors by lowercase boldface. The pdf of a random matrix $\mathbf{Z}$, $p_{\mathbf{Z}}(\mathbf{Z})$, is simply denoted by $p(\mathbf{Z})$. $(\cdot)^\dagger$ indicates the conjugate transpose operator, $|\cdot|$ and $\mathrm{Tr}(\cdot)$ denote, respectively, the determinant and the trace of a square matrix, and $||\cdot||$ stands for the Euclidean norm[1]. $\Gamma_p(q)$, with $p\leq q$, is the complex multivariate Gamma function [2]

$$\Gamma_p(q) = \pi^{\frac{p(p-1)}{2}} \prod_{\ell=1}^{p} (q-\ell)!$$

and

$${}_pF_q(a_1,\ldots,a_p;b_1\ldots,b_q;\cdot,\ldots,\cdot)\,,$$

with $p$ and $q$ non-negative integers, denotes the generalized hypergeometric function [3]. The arguments of such a function can be either scalars or square matrices; there is in general no limit to the number of arguments, and hypergeometric functions of multiple matrix arguments are defined also for set of square matrices of different size. We denote by $\mathbf{I}_m$ the $m\times m$ identity matrix.

We consider a single-user multiple-antenna communication, with $n_R$ and $n_T$ denoting the number of receive and transmit antennas. For sake of simplicity, let $n_T \leq n_R$, which allows the modeling of a massive-MIMO channel in the large $n_R$ limit[2]. Assuming block-fading of length $n_b$, the channel output can be described by the following linear relationship:

$$\mathbf{Y} = \sqrt{\gamma}\mathbf{H}\,\mathbf{X} + \mathbf{N}. \tag{1}$$

In (1), $\mathbf{Y}$ is the $n_R\times n_b$ output, $\mathbf{X}$ is the complex $n_T\times n_b$ input matrix, and $\mathbf{N}$ is the $n_R\times n_b$ matrix of additive complex circularly symmetric Gaussian noise. $\mathbf{H}$ is the $n_R{\times}n_T$ complex channel matrix, whose entries represent the fading coefficients between each transmit and each receive antenna. Finally, $\gamma = \mathrm{SNR}/n_T$ represents the normalized per-transmit antenna Signal-to-Noise Ratio (SNR).

The input matrix $\mathbf{X}$, unless otherwise stated, is assumed to have a product structure, i.e., $\mathbf{X} = \mathbf{D}^{1/2}\mathbf{\Phi}$, where $\mathbf{D}$ is a random, $n_T$-dimensional, diagonal matrix, which is positive definite w.p. 1. The entries of $\mathbf{D}$ represent the amount of

[1] As applied to a matrix, we mean $||\mathbf{A}||^2 = \mathrm{Tr}(\mathbf{A}^\dagger\mathbf{A})$.

[2] The extension of the results to the case $n_T > n_R$ is straightforward.

transmit power allocated to each of the $n_T$ transmit antennas, while $\mathbf{\Phi}$ is an $n_T \times n_b$ isotropic matrix. As usually done in the literature [4], [5], we will refer to square isotropic matrices as Haar and to rectangular isotropic matrices as Stiefel. We stress that the above structure of the input matrix $\mathbf{X}$ allows to achieve the capacity limit in absence of CSI at both the ends of the communication link [5, Thm2].

## III. STATISTICAL CHARACTERIZATION OF THE CHANNEL OUTPUT

**Theorem 1.** *Given a channel as in (1), the pdf of its matrix-variate output, conditionally on the $n_T$-dimensional (diagonal) input power allocation matrix $\mathbf{D}$ and on the $n_T$-dimensional matrix of the non-zero squared singular values of the channel, $\mathbf{\Sigma}$, can be expressed as*

$$p(\mathbf{Y}|\mathbf{D},\mathbf{\Sigma}) = \frac{e^{-||\mathbf{Y}||^2}}{\widetilde{v}\pi^{n_R[n_T-n_b]^+}\gamma^{n_R n_T}}\frac{|\mathbf{\Sigma}|^{-n_R}}{|\mathbf{D}|^{n_R}} \, {}_0F_0(\mathbf{\Sigma}\mathbf{D}^{-1},\mathbf{D}\mathbf{\Sigma}^{-1},\mathbf{Y}^\dagger\mathbf{Y}) \,, \quad (2)$$

*with $\widetilde{v} = 2^{[n_T-n_b]^+}\pi^{[n_T-n_b]^{+2}}/\Gamma_{[n_T-n_b]^+}([n_T-n_b]^+)$ the volume of the unitary group of dimension $[n_T-n_b]^{+3}$. The hypergeometric function of three matrix arguments in (2) can be expanded in zonal polynomials [2] $\mathcal{C}_\kappa(\cdot)$, following [6, Appendix B] as*

$$ {}_0F_0(\mathbf{\Sigma}\mathbf{D}^{-1},\mathbf{D}\mathbf{\Sigma}^{-1},\mathbf{Y}^\dagger\mathbf{Y}) = \quad (3)$$
$$ (2\pi)^{2q-n_R-n_T}\sum_{k=0}^{+\infty}\sum_{\#\kappa=k}\frac{\mathcal{C}_\kappa(\mathbf{\Sigma}\mathbf{D}^{-1})\mathcal{C}_\kappa(\mathbf{D}\mathbf{\Sigma}^{-1})\mathcal{C}_\kappa(\mathbf{Y}^\dagger\mathbf{Y})}{k!\mathcal{C}^2_\kappa(\mathbf{I}_q)} \,,$$

*with $q = \max\{n_T, n_R, n_b\}$and $\#\kappa$ the cardinality of $\kappa$, which is a partition of the integer $k$.*

**Proof.** Due to the Gaussianity of both the channel and the noise, the conditional output pdf of the MIMO channel (1) can be expressed as follows:

$$p(\mathbf{Y}|\mathbf{X},\mathbf{H}) = \frac{e^{-\mathrm{Tr}\left(\left(\mathbf{Y}-\sqrt{\gamma}\mathbf{H}\mathbf{X}\right)\left(\mathbf{Y}-\sqrt{\gamma}\mathbf{H}\mathbf{X}\right)^\dagger\right)}}{\pi^{n_R n_b}} \,.$$

Notice that, expanding the product in the exponent, decomposing in its singular values/vectors the channel matrix $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}^{1/2}\mathbf{V}^\dagger$, and recalling that $\mathbf{X} = \sqrt{\gamma}\mathbf{D}^{1/2}\mathbf{\Phi}$, one obtains, term by term, $e^{-||\mathbf{Y}||^2}$, which is independent of $\mathbf{H}$ and $\mathbf{X}$;

$$\exp\left\{-\gamma\mathrm{Tr}\left(\mathbf{H}\mathbf{X}\mathbf{X}^\dagger\mathbf{H}^\dagger\right)\right\} = \exp\left\{-\gamma\mathrm{Tr}\left(\mathbf{\Sigma}\mathbf{V}^\dagger\mathbf{D}\mathbf{V}\right)\right\} \,,$$

which depends only on $\mathbf{V}$, and finally

$$\exp\{\sqrt{\gamma}\mathrm{Tr}\left(\mathbf{Y}\mathbf{X}^\dagger\mathbf{H}^\dagger + \mathbf{H}\mathbf{X}\mathbf{Y}^\dagger\right)\} \,,$$

which depends on both $\mathbf{V}$ and $\mathbf{U}$. In turn, the conditional pdf we are interested in can be expressed as

$$p(\mathbf{Y}|\mathbf{D},\mathbf{\Sigma}) = \int p(\mathbf{Y}|\mathbf{\Phi},\mathbf{U},\mathbf{V})p(\mathbf{\Phi})p(\mathbf{U})p(\mathbf{V})d\mathbf{\Phi}d\mathbf{U}d\mathbf{V} \,, \quad (4)$$

with the integral being over the appropriate matrix spaces, which can be performed as described below.

The integration over $\mathbf{V}$ can be carried out by exploiting [7, Integral B.I], that over $\mathbf{U}$ follows by the splitting formula for the hypergeometric functions of matrix argument [2, Formula (92)], while the last one, over $\mathbf{\Phi}$, can be performed by the help of [8, Formula (54)].

## IV. DISCUSSION AND FUTURE WORK

The main expression we obtained for the output law is conditioned to the channel eigenvalues and to the input power allocation, while we were able to average over the input and the channel eigenvectors distributions.

Notice that, tough no determinant representation is known yet for hypergeometric functions of more than two matrix arguments, still the zonal polynomials the output pdf can be expanded in can be represented as ratio of determinants. Moreover, the zonal polynomial of a matrix and this of its inverse are strongly related (see e.g. [2]), and this yields to further simplification of (3). The procedure adopted in [9, Appendix] to perform the average over some of the matrices appearing in the argument of the zonal polynomials may not be fully generalized to our case, however the obtained expression (2) can be already numerically evaluated, as opposite to its counterpart in [10], by exploiting tools in [1].

[3] Here, we assume $[n_T-n_b]^+ = n_T-n_b$ if $n_T \geq n_b$, and $[n_T-n_b]^+ = n_b-n_T$ otherwise.

## REFERENCES


[1] Plamen Koev and Alan Edelman, "The efficient evaluation of the hypergeometric function of a matrix argument," *Math. Comp.* Vol. 75, pp. 833–846, 2006.

[2] A. T. James, "Distribution of Matrix Variates and Latent Roots Derived from Normal Samples," *The Annals of Mathematical Statistics,* Vol. 35, No. 2, pp. 474–501, June 1964.

[3] M. Abramowitz, and I. A. Stegun, "Handbook of Mathematical Functions", *New York: Dover Publications*, 1972.

[4] L. Zheng, D. Tse, "Communicating on the Grassmann Manifold: A Geometric Approach to the Non-coherent Multiple Antenna Channel," *IEEE Trans. on Information Theory*, Vol. 48, No. 2, pp. 359–383, Feb. 2002.

[5] T. L. Marzetta, B. M. Hochwald, "Capacity of a mobile multiple-antenna communication link in Rayleigh flat fading," *IEEE Trans. on Information Theory,* Vol. 45, No. 1, pp. 139–157, Jan. 1999.

[6] L. Hanlen, A. Grant, "Capacity Analysis of Correlated MIMO Channels," *IEEE Trans. Information Theory*, Vol. 58, No. 11, pp. 6773–6787, Nov. 2012.

[7] G. Taricco, "Asymptotic Mutual Information Statistics of Separately Correlated Rician Fading MIMO Channels," *IEEE Trans. Information Theory*, Vol. 54, No. 8, pp. 3490–3504, Aug. 2008.

[8] W. Yang, G. Durisi, E. Riegler, "On the capacity of large-MIMO block-fading channels," *IEEE J. Sel. Areas Commun.*, Feb. 2013, to appear.

[9] M. R. McKay, A. J. Grant, I.B. Collings, "Performance Analysis of MIMO-MRC in Double-Correlated Rayleigh Environments," *IEEE Trans. on Communications*, Vol. 55, No. 3, pp. 497–507, Mar. 2007.

[10] G. Alfano, C.-F.Chiasserini, A. Nordio, S. Zhou, "Output Statistics of MIMO Channels with General Input Distribution," *Proc. of ISIT 2013*, Istanbul, July 7-13, 2013.