LINEAR OPERATOR INEQUALITY AND NULL CONTROLLABILITY WITH VANISHING ENERGY FOR UNBOUNDED CONTROL SYSTEMS

Luciano Pandolfi [1]

Politecnico di Torino, Dipartimento di Matematica
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.
e-mail luciano.pandolfi@polito.it

Enrico Priola [2]

Dipartimento di Matematica, Università di Torino,
via Carlo Alberto 10, 10123, Torino, Italy.
e-mail enrico.priola@unito.it

Jerzy Zabczyk [3]

Instytut Matematyczny, Polskiej Akademii Nauk,
ul. Sniadeckich 8, 00-950, Warszawa, Poland.
e-mail zabczyk@impan.gov.pl

Mathematics Subject Classification (2010): 93C20, 93C25.

Key words: Boundary control systems, Linear operator inequality, Null controllability, Vanishing energy.

This is the author post-print version of an article published on *SIAM J. Control,* Vol 51, n. 1, pp. 629-659, 2013 (ISSN 0363-0129), DOI 10.1137/110846294 .

This version does not contain journal formatting and may contain minor changes with respoect to the published version.

The present verion is accessible on PORTO, the Open Access Repository of the Politecnico di Torino, in compliance with the publisher's copyright policy, as reported in the SHERPA-ROMEO website.

Abstract: We consider linear systems on a separable Hilbert space H, which are null controllable at some time $T_0 > 0$ under the action of a point or boundary control. Parabolic and hyperbolic control systems usually studied in applications are special cases. To every initial state $y_0 \in H$ we associate the minimal "energy" needed to transfer y_0 to 0 in a time $T \geq T_0$ ("energy" of a control being the square of its L^2 norm). We give both necessary and sufficient conditions under which the minimal energy converges to 0 for $T \to +\infty$. This extends to boundary control

[1]Supported in part by Italian MURST and by the project "Groupement de Recherche en Contrôle des EDP entre la France et l'Italie (CONEDP)".

[2]Supported by the M.I.U.R. research project Prin 2008 "Deterministic and stochastic methods in the study of evolution problems".

[3]Supported by the NSF grant DMS-0500270.

systems the concept of *null controllability with vanishing energy* introduced by Priola and Zabczyk (Siam J. Control Optim. 42 (2003)) for distributed systems. The proofs in Priola-Zabczyk paper depend on properties of the associated Riccati equation, which are not available in the present, general setting. Here we base our results on new properties of the quadratic regulator problem with stability and the Linear Operator Inequality.

1. Introduction and preliminaries

The paper [40] introduced and studied the property of "null controllability with vanishing energy", shortly NCVE, for systems with distributed control action, which is as follows: consider a semigroup control system (cf. [4, 6, 23, 24, 39, 46, 47])

$$\dot{y} = Ay + Bu, \qquad y(0) = y_0 \in H,$$

which is null controllable in time $T_0 > 0$ (hence also for every larger time $T > T_0$). This null controllable system is NCVE when for every y_0 and $\epsilon > 0$ there exist a time T and a control u which steers the initial state y_0 to zero in time T and, furthermore, its $L^2(0,T;U)$-norm is less then ϵ. This concept has been already applied in some specific situations (see [17, 18]) and partially extended to the Banach space setting in [32]. Moreover, applications of NCVE property to Ornstein-Uhlenbeck processes are given in [41].

The key result in [40], i.e., Theorem 1.1, shows that, under suitable properties on the operator A stated below, NCVE holds if and only if the system is null controllable and furthermore the spectrum of A is contained in the *closed* half plane $\{\Re e\, \lambda \le 0\}$.

The goal of this paper is to extend this result to a large class of boundary and point control systems which essentially includes all the classes of systems whose null controllability has been studied up to now. Our main results are Theorems 6 and 9. Moreover, Corollary 10 combines these results and gives a necessary and sufficient conditions for NCVE, which applies in the cases most frequently encountered in applications. Finally, Section 3 provides applications of our main results. In particular we establish NCVE for boundary control problems involving systems of parabolic equations recently considered in [10].

The proofs that we give are based on ideas different from those in [40]. Moreover conditions imposed for the sufficiency part are weaker from those used in [40, Theorem 1.1], in the case of distributed control systems.

Now we describe the notations and the class of systems we are studying.

The spaces in this paper are Hilbert, and are identified with their duals unless explicitly stated. The notations are standard. For example, $\mathcal{L}(H,K)$ denotes the Banach space of all bounded linear operators from H into K endowed with the operator norm.

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ and let A be a generator of a C_0-semigroup on H. Due to the fact that the spectrum of A has a role in our arguments, we assume from the outset that H is a *complex* Hilbert space.

Let A^* be the Hilbert space adjoint of A. Its domain with the graph norm

$$|y|^2 = \langle y, y \rangle + \langle A^* y, A^* y \rangle$$

is a Hilbert space *which is not identified with its dual.* It is well known that $(\mathrm{dom} A^*)'$ (the dual of the Hilbert space $\mathrm{dom}\, A^*$) is a Hilbert space and

$$(\mathrm{dom}\, A^*) \subset H = H' \subset (\mathrm{dom} A^*)'$$

(with continuous and dense injections). Moreover, A admits an extension $\mathcal{A}$ to $(\mathrm{dom} A^*)'$, which generates a C_0-semigroup $e^{\mathcal{A}t}$ on $(\mathrm{dom} A^*)'$. The domain of such extension is equal to H (see [23, Section 0.3], [4, Chapter 3] and [46]; see also Appendix B).

The norm in $(\mathrm{dom} A^*)'$ is denoted by $|\cdot|_{-1}$, and it is useful to recall that $|y|_{-1}$ and $|(\omega I - \mathcal{A})^{-1} y|$ are equivalent norms on $(\mathrm{dom} A^*)'$, for every $\omega \in \rho(A) = \rho(\mathcal{A}) = \overline{\rho(A^*)}$ (here ρ indicates the resolvent set and the overbar denotes the complex conjugate). In other words, $(\mathrm{dom} A^*)'$ is the completion of H with respect to the norm $|(\omega I - A)^{-1} \cdot|$, for any $\omega \in \rho(A)$.

Let U be an Hilbert space. A "control" is an element of $L^2_{\mathrm{loc}}(0, +\infty; U)$. Let $B \in \mathcal{L}(U, (\mathrm{dom}\, A^*)')$ and let us consider the control process on $(\mathrm{dom} A^*)'$ described by

$$\dot{y} = \mathcal{A} y + B u, \qquad y(0) = y_0 \in H\,. \tag{1}$$

This equation makes sense in $(\mathrm{dom} A^*)'$, for every $y_0 \in (\mathrm{dom} A^*)'$, but we only consider initial conditions $y_0 \in H$. It is known that the transformation

$$u(\cdot) \longrightarrow (Lu)(t) \qquad \text{where} \quad (Lu)(t) := \int_0^t e^{\mathcal{A}(t-s)} B u(s)\, \mathrm{d}s \tag{2}$$

is continuous from $L^2(0, T; U)$ into $C([0, T]; (\mathrm{dom} A^*)')$, for every $T > 0$. The class of systems we study is identified by pairs (A, B) with the following property:

Assumption 1. *We have $B \in \mathcal{L}(U, (\mathrm{dom}\, A^*)')$ and, for every $T > 0$, the transformation (2) is linear and continuous from $L^2(0, T; U)$ into $L^2(0, T; H)$.*

Clearly, the case of distributed controls, i.e., $B \in \mathcal{L}(U, H)$, fits Assumption 1 (in such case, the transformation (2) is linear and continuous from $L^2(0, T; U)$ into $C([0, T]; H)$). Examples of boundary control systems which satisfy our condition are in Section 1.1.

From now on we consider $\omega \in \rho(A)$, *which is fixed once and for all,* and introduce the operator

$$D = (\omega I - \mathcal{A})^{-1} B \in \mathcal{L}(U, H)\,. \tag{3}$$

By definition, the solution of system (1) is

$$y^{y_0, u}(t) = e^{At} y_0 + \int_0^t e^{\mathcal{A}(t-s)} B u(s)\, \mathrm{d}s = e^{At} y_0 + \int_0^t e^{\mathcal{A}(t-s)} (\omega I - \mathcal{A}) D u(s)\, \mathrm{d}s\,. \tag{4}$$

This is a continuous $(\mathrm{dom} A^*)'$-valued function and belongs to $L^2_{\mathrm{loc}}(0, +\infty; H)$ thanks to Assumption 1. Integration by parts shows that:

Lemma 2. *If $u \in C^1([0, +\infty); U)$ then $y^{y_0, u}$ belongs to $C([0, +\infty); H)$.*

Now we give the definitions of null controllability and NCVE, adapted to our system, by taking into account the fact that if $u \in L^2_{\mathrm{loc}}(0, +\infty; U)$ then the integrals in (4) belong to $L^2_{\mathrm{loc}}(0, +\infty; H)$, and point-wise evaluation in H in general is meaningless.

Definition 3. *We say that $y_0 \in H$ can be steered to the rest in time (at most) T if there exists a control $u \in L^2_{\rm loc}(0,+\infty;U)$ whose support is contained in $[0,T]$ and such that the support of the corresponding solution (4) is contained in $[0,T]$ too.*

System (1) is null controllable if every $y_0 \in H$ can be steered to the rest in a suitable time T_{y_0} at most.

System (1) is null controllable in time (at most) T if every $y_0 \in H$ can be steered to the rest in time at most T.

In connection with this definition see also Lemma 11.

Controllability in time T implies controllability at every larger time. Note that if u steers y_0 to the rest in time at most T, then we have

$$\int_0^T e^{\mathcal{A}(t-s)}Bu(s)\,{\rm d}s = -e^{At}y_0, \qquad \text{a.e.} \quad t>T,$$

and so the integral is represented by a continuous function for $t>T$.

The control u which steers y_0 to zero in time T needs not be unique. Then, we define:

Definition 4. *Let $y_0 \in H$ be an element which can be steered to the rest. We say that this element is NCVE if for every $\epsilon > 0$ there exists a control u_ϵ such that*

- *it steers y_0 to the rest in time at most T_ϵ (i.e., the control time T depends on ϵ, $T = T_\epsilon$, and the support of u is in $[0,T_\epsilon]$);*
- *the $L^2(0,+\infty;U)$ norm of u is less then ϵ:*

$$\int_0^{+\infty} |u(s)|^2\,{\rm d}s = \int_0^{T_\epsilon} |u(s)|^2\,{\rm d}s \le \epsilon^2\,.$$

If every element of H is NCVE, then we say that system (1) is NCVE.

As a variant to Definitions 3 and 4, we introduce also:

Definition 5. *Let $\mathcal{D}$ be a subspace of H. If every initial condition $y_0 \in \mathcal{D}$ can be steered to the rest in time T then we say that the system is null controllable on $\mathcal{D}$ in time T (note that we don't require that the trajectory which joins y_0 to zero remains in the set $\mathcal{D}$).*

We say that the system is NCVE on $\mathcal{D}$ if for every $y_0 \in \mathcal{D}$ and every $\epsilon > 0$ there exists a control u_ϵ such that

- *it steers y_0 to the rest in time T_ϵ (i.e., the control time T depends on ϵ, i.e., $T = T_\epsilon$);*
- *the $L^2(0,+\infty;U)$ norm of u is less then ϵ:*

$$\int_0^{+\infty} |u(s)|^2\,{\rm d}s = \int_0^{T_\epsilon} |u(s)|^2\,{\rm d}s \le \epsilon^2\,.$$

1.1. **Classes of systems which fit our framework.** Essentially, controllability has been studied for "parabolic" and "hyperbolic" type systems.

(i) *Parabolic systems* can be described, in a unified way, as follows.

The operator A generates a holomorphic semigroup and, following [23, Section 0.4 and Chapter 1], there exists $\omega \in \rho(A) = \rho(\mathcal{A})$ and $\gamma \in [0,1)$ such that

$$B \in \mathcal{L}\left(U, \left({\rm dom}\,(\omega - A^*)^\gamma\right)'\right). \tag{5}$$

Note that (5) implies the estimate

$$\|e^{\mathcal{A}t}B\|_{\mathcal{L}(U,H)} \leq \frac{Me^{\omega_1 t}}{t^\gamma}, \quad t > 0. \tag{6}$$

for some $M > 0$, $\omega_1 \in \mathbb{R}$ (see [23, Section 0.3], [4, Chapter 3] and [46]; see also Appendix B); recall that $\left(\mathrm{dom}\,(\omega - A^*)^\gamma\right)' \subset (\mathrm{dom}A^*)'$ with continuous and dense injection).

Using (6), one can show that *Assumption 1 holds in this case.* Indeed, the integral in (4) does not converge in the space H for every t but, using the Young inequality for convolutions, it defines an H-valued locally square integrable function for every locally square integrable input u. Formula (4) defines the unique solution of eq. (1) with values in H, which however does not have a pointwise sense in general.

A recent example of parabolic system will be considered in Section 3.

The singular inequality (6) holds for certain important classes of interconnected systems, as studied for example in [5, 25], even if they do not generate holomorphic semigroups.

(ii) *Hyperbolic systems* are further important examples of systems which fit our framework, see [24, 29] and [46, p. 122]. In spite of the fact that this class lacks of a plain unification, it turns out that in this case the following important property, first proved for the wave equation with Dirichlet boundary control in [20, 21], holds: the function $y(t)$ is even continuous in time.

We listed earlier systems which fit our Assumption 1. However, null controllability cannot be studied "in abstract": it has to be studied separately in concrete cases and these are too many to be cited here. So, we confine ourselves to note that controllability for several hyperbolic type problems is studied in [2, 22, 27, 29, 30, 44]; controllability for parabolic type equations is studied in [10, 31, 45, 48] and references therein. Note that controllability for heat-type equations is often achieved using smooth controls, so that the resulting trajectory $y(t)$ is even continuous.

An overview on controllability both of hyperbolic and parabolic type equations is [26, 49].

1.2. **Key results and discussion.** As we have already said, our point of departure is paper [40] which proves the following result, in the case of distributed controls, i.e., the case that $\mathrm{im}\, D \subseteq \mathrm{dom}\, A$ (recall that D is defined in (3)) so that $B \in \mathcal{L}(U, H)$: *under suitable assumptions on the spectral properties of the operator A, NCVE is equivalent to null controllability at some time $T > 0$.* An interesting interpretation of this result is that for this class of systems *NCVE does not depend on the control operator provided that this operator is so chosen to guarantee null controllability at a certain time T.*

Now we state our main results, which we split in Theorems 6 and 9. We don't try to unify them, since they are proved using different ideas but, in the most important cases for the applications, they can be combined to get a necessary and sufficient condition for NCVE, see Corollary 10.

We recall that a *reducing subspace* E for a C_0-semigroup e^{At} on H is a closed subspace of H such that both E and one of its complementary subspaces are invariant for the semigroup:

$$e^{At}x \in E, \qquad \forall x \in E\,,\ \forall t \geq 0$$

and the same for one complement of E.

It is possible to prove that the restriction of e^{At} is a C_0-semigroup on E and that $A\,(E \cap (\mathrm{dom}\,A)) \subseteq E$ (the restriction of A to E is the infinitesimal generator of e^{At} on E).

The necessary condition for NCVE is given by the next theorem:

Theorem 6. *Let Assumption 1 hold and suppose the existence of a reducing subspace E for e^{At}, such that e^{-At} generates a C_0-*group *on E which is* exponentially stable *(for $t \to +\infty$). Then, the system* (1) *is not NCVE.*

A consequence is:

Corollary 7. *Let Assumption 1 hold. If $\sigma(A)$ has an isolated point with* positive *real part, then the system* (1) *is* not *NCVE.*

In fact, [40] proves the existence of the subspace E in Theorem 6, under the assumption of the corollary.

Now we come to the second theorem. We recall that $x \in H$ is a generalized eigenvector of A associated to the eigenvalue $\lambda \in \mathbb{C}$ if $x \in \bigcup_{k\geq 1}\mathrm{Ker}[(\lambda I - A)^k]$ and we recall the standard notation for the spectral bound

$$s(A) = \sup\{\Re e\,\lambda\,, \quad \lambda \in \sigma(A)\}.$$

where $s(A) = -\infty$ if $\sigma(A)$ is empty.

Now we introduce the following assumption, which slightly generalizes the one in [40, (ii) Hypothesis 1.1].

Assumption 8. *There exist closed linear subspaces H_s, H_1 of H such that:*

- *$H = H_s \oplus H_1$;*
- *for every $x \in H_s$ we have*

$$\lim_{t\to+\infty} e^{At}x = 0\,;$$

- *the subspace H_1 is invariant for the semigroup and the set of all the generalized eigenvectors of A contained in H_1 is linearly dense in H_1.*

In the definition the subspace H_1 can be $\{0\}$. If $\sigma(A) = \emptyset$ we set $H_1 = \{0\}$.

We note that the assumption in [40] is slightly stronger in that [40] assumes that H_s is an invariant subspace for the semigroup, and that the semigroup restricted to H_s is exponentially stable.

We have:

Theorem 9. *Assume Hypotheses 1 and 8 and furthermore suppose that $s(A) \leq 0$. If system (1) is null controllable at some time $T > 0$, then it is NCVE.*

The ideas used in the proof of both Theorems 6 and 9 are different from those used in the proofs of the corresponding results in [40]. In particular, the proof of Theorem 9 relies on the *Yakubovich theory of the regulator problem with stability*, and the corresponding *Linear Operator Inequality,* that can be found in [28, 35, 36, 37].

Clearly, the spectral condition in Assumption 8 is satisfied by most of the systems encountered in practice, when the "dominant part" of the spectrum is a sequence of eigenvalues (in particular, if A has compact resolvent). Hence, for all these systems, Theorems 6 and 9 can be combined to get a *necessary and sufficient* condition for NCVE which depends only on the spectrum of A, provided that null controllability holds. For example we can state the following Corollary. Recall that a C_0-semigroup e^{At} is called *eventually compact* if there exists $t_0 > 0$ such that e^{At} is a compact operator for any $t \geq t_0$; moreover any differentiable semigroup such that its generator has compact resolvent is in particular an eventually compact semigroup, see [38, Theorem 3.3, page 48].

Corollary 10. *Let Assumption 1 hold and suppose that the semigroup is eventually compact. If $s(A) \leq 0$ then null controllability and NCVE are equivalent properties. When $s(A) > 0$ the system is not NCVE.*

Proof.
The spectrum of A is a sequence of eigenvalues (this is well known when $(\omega I - A)^{-1}$ is compact, and it is true also if the semigroup is eventually compact, see [7, p. 330]; note that the spectrum might be empty in this case). Furthermore, under the stated assumptions we have (see [7, p. 330]), for any $r \in \mathbb{R}$, the set

$$\{\mu \in \sigma(A) \, : \, \mathrm{Re}(\mu) \geq r\} \quad \text{is finite or empty.}$$

As we noted, when the semigroup is eventually compact, the spectrum of A might be empty. In this case we can choose $H_1 = 0$ and $H_s = H$ since the semigroup is exponentially stable on H, see [7, p. 250-252]. So, if null controllability holds we have also NCVE.

Let the spectrum be not empty. If $s(A) > 0$ then there exists an eigenvalue λ with positive real part: one can easily show (see for example [40, Sect. 2.1]) that the subspace E of all generalized eigenvectors associated to λ is reducing for e^{At} and that e^{-At} generates a group on E which is exponentially stable. We are in the case of Theorem 6 and NCVE does not hold.

Let now $s(A) \leq 0$ and take $c < s(A)$. Let H_1 be the invariant subspace of H spanned by all the generalized eigenvectors associated to the (finitely many) eigenvalues with real part larger then c. Let P_{H_1} be the corresponding spectral projection and set $H_s = (I - P_{H_1})\, H$. Then, from [6, p. 267], the semigroup is even exponentially stable on H_s and the conditions of Theorem 9 are satisfied, hence NCVE holds. ∎

We conclude this introduction with the following observation which extends a property of null controllable systems proved by many people for distributed controls (see [11, 42, 32]) and likely known at least for some boundary control systems, in spite of the fact that we cannot give a precise reference:

Lemma 11. *Let Assumption 1 hold and suppose that every $y \in H$ can be steered to rest in a time T_y. Then:*

- *there exists a time T_0 such that system (1) can be steered to the rest in time T_0;*
- *there is a ball $B(0, r)$ (centered at 0, radius $r > 0$) and a number N such that every element of $B(0, r)$ can be steered to the rest using a control whose L^2-norm is less then N.*

Proof. The proof is the same as for distributed systems: we introduce the sets $E_{T,N}$ of those elements $y \in H$ which can be steered to the rest in time (at most) T and using controls of norm at most N. These sets are closed, convex and balanced. Furthermore, they grow both with T and with N.

Every y belongs to a suitable $E_{T,N}$ so that

$$H = \cup E_{N,N} \, .$$

Baire Theorem implies the existence of N_0 such that E_{N_0,N_0} has interior points.

The set E_{N_0,N_0} being convex and balanced, 0 is an interior point, i.e., any point of a ball centered at zero can be steered to the rest in time $T = N_0$ and the L^2-norm of the corresponding control is less then N_0. This is the second statement and it implies that every $y \in H$ can be steered to the rest in time $T = N_0$. ∎

In conclusion, we see that null controllability and null controllability at a fixed time $T > 0$ are equivalent concepts.

2. Proof of the main results

First we state two lemmas which have an independent interest.

Let $y(t)$ solve equation (1). Then, $x(t) = (\omega I - \mathcal{A})^{-1} y(t)$ solves the equation

$$\dot{x} = Ax + Du\,, \qquad x(0) = x_0 = (\omega I - A)^{-1} y_0 \in \operatorname{dom} A\,. \tag{7}$$

Consequently, every control which steers y_0 to zero, steers also x_0 to zero, and conversely. Therefore, we have:

Lemma 12. *Let Assumption 1 hold. There exists $T > 0$ such that system (1) is null controllable in time T if and only if system (7) is null controllable on $\mathcal{D} = \operatorname{dom} A$ in the same time T; system* (1) *is NCVE if and only if system (7) is NCVE on $\mathcal{D} = \operatorname{dom} A$.*

¿From now on, $\mathcal{D}$ will always denote $\operatorname{dom} A$, i.e.,

$$\mathcal{D} = \operatorname{dom} A\,.$$

The second preliminary result is the following lemma:

Lemma 13. *Let Assumption 1 hold and suppose that system (1) is null controllable in time T. Then there exists a number $M > 0$ such that for every $y_0 \in H$ there exists a control $u^{y_0,T}(t)$ which steers y_0 to 0 in time T and such that:*

$$\int_0^T |u^{y_0,T}(t)|^2 \, \mathrm{d}t \le M |y_0|^2\,.$$

<u>Proof</u>. We already noted that we can equivalently control system (7) to zero on $\mathcal{D}$, i.e., we can solve

$$-e^{AT}(\omega I - A)^{-1} y_0 = \int_0^T e^{A(T-s)} Du(s) \, \mathrm{d}s \tag{8}$$

and by assumption this equation is solvable for every $y_0 \in H$. We introduce the operator $\Lambda_T : L^2(0,T;U) \to H$ as

$$\Lambda_T u = \int_0^T e^{\mathcal{A}(T-s)} Du(s) \, \mathrm{d}s\,. \tag{9}$$

So, null controllability at time T is equivalent to

$$\operatorname{im} e^{AT}(\omega I - A)^{-1} \subseteq \operatorname{im}\Lambda_T.$$

The operator Λ_T is continuous,

$$\Lambda_T \in \mathcal{L}\left(L^2(0,T;U), H\right).$$

Let us introduce the continuous operator $Q_T = \Lambda_T \Lambda_T^*$. Its kernel is closed and its restriction to the orthogonal of the kernel is invertible with closed inverse. Let us denote $Q_T^\dagger$ this inverse, so that the control which steers $(\omega I - A)^{-1}y_0$ to zero in time T and which has minimal $L^2(0,T)$ norm is

$$u^{y_0,T}(t) = -\Lambda_T^* Q_T^\dagger e^{AT}(\omega I - A)^{-1}y_0 = -D^* e^{A^*(T-t)} Q_T^\dagger e^{AT}(\omega I - A)^{-1}y_0 \,. \quad (10)$$

The closed operator $Q_T^\dagger e^{AT}(\omega I - A)^{-1}$ being everywhere defined, it is continuous, so that

$$\|u^{y_0,T}\|_{L^2(0,T;U)} \le M|y_0|\,, \qquad M = M_T\,,$$

as wanted. ∎

Remark 14. *We note:*

- *The function $u^{y_0,T}(t)$, extended with 0 for $t > T$, produces a solution $y(t)$ to Eq. (1), which has support in $[0,T]$.*
- *We can work with any initial time τ instead of the initial time 0. If the system is null controllable in time at most T, then any "initial condition" assigned at time τ can be steered to rest on a time interval still of duration T, i.e., at the time $T+\tau$ and the previous Lemma 13 still holds, with the constant M depending solely on the length of the controllability time, i.e., the same constant M_T can be used for every initial time τ.*

2.1. **Proof of Theorem 6,** *i.e., if NCVE holds then the subspace E does not exist.* The proof in [40] relays on a precise study of the quadratic regulator problem and the associated Riccati equation. Here we follow a different route: we prove that the existence of the subspace E implies that system (1) is not NCVE.

Let E^C be the complementary subspace of E which is invariant for the semigroup. Let $y_0 \neq 0$ be any point of E. If it cannot be steered to 0 then system (1) is not null controllable, hence even not NCVE. So, suppose that there exists a control u which steers y_0 to zero in time T. Then we have, for every $t > T$,

$$e^{At}y_0 = -\int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s,$$

i.e.,

$$e^{AT}(\omega I - A)^{-1}y_0 = -\int_0^T e^{A(T-s)}Du(s)\,\mathrm{d}s\,.$$

Let now P_E be the projection of H onto E along E^C. We have

$$e^{AT}(\omega I - A)^{-1}y_0 = -\int_0^T e^{A(T-s)}P_E Du(s)\,\mathrm{d}s - \int_0^T e^{A(T-s)}(I-P_E)Du(s)\,\mathrm{d}s\,. \quad (11)$$

The left hand side belongs to E so that the last integral is zero since $(I - P_E)$ commutes with the semigroup due to the fact that E is a reducing subspace. Then,

$$e^{AT}(\omega I - A)^{-1}y_0 = -\int_0^T e^{A(T-s)}P_E Du(s)\,\mathrm{d}s\,.$$

Hence we have also

$$(\omega I - A)^{-1}y_0 = -\int_0^T e^{-As}P_E Du(s)\,\mathrm{d}s$$

since A generates a group on E. Note that this equality in particular implies that $P_E D \neq 0$ since the left hand side is not zero.

We assumed that e^{-At} is exponentially stable on E, i.e., we assumed the existence of $M>1$ and $\gamma>0$ such that

$$\left|e^{-As}y\right|\le Me^{-\gamma s}|y| \qquad \text{for all } s>0 \text{ and for all } y\in E\,.$$

So, using Schwarz inequality we see that:

$$\left|(\omega I-A)^{-1}y_0\right|\le \frac{M\|P_E D\|_{\mathcal{L}(U,H)}}{\sqrt{2\gamma}}\|u\|_{L^2(0,T;U)}\,.$$

This is an estimate *from below* for the $L^2(0,T)$-norm of any control which steers y_0 to the rest, and this estimate does not depend on T. Hence, the system is not NCVE, as we wished to prove. □

2.2. **Proof of Theorem 9,** *i.e., null controllability and $s(A)\le 0$ implies NCVE***.** We introduce a new notation. Since we need to consider solutions of equation (1) with initial time τ, possibly different from 0, we introduce

$$y(t;\tau,y_0,u)$$

to denote the solution of the problem

$$y'=\mathcal{A}y+Bu \qquad t>\tau\,, \qquad y(\tau)=y_0\,.$$

Furthermore, when $\tau=0$, we shall write $y(t;y_0,u)$ instead of $y(t;0,y_0,u)$. Comparing with (4), we have

$$y(t;y_0,u)=y^{y_0,u}(t)\,.$$

We first give a different formulation of the problem under study. To this purpose we introduce the following functionals $I(y_0)$ and $Z(y_0)$:

$$I(y_0)=\inf_{u\in\mathcal{U}(y_0)}J(y_0;u)\,, \qquad J(y_0;u)=\int_0^{+\infty}|u(s)|^2\,\mathrm{d}s \tag{12}$$
$$\mathcal{U}(y_0)=\left\{u\in L^2(0,+\infty;U)\ :\ y(\cdot;y_0,u)\in L^2(0,+\infty;H)\right\}$$

and

$$Z(y_0)=\inf_{t>T}\left(\inf\left\{\int_0^t|u(s)|^2ds\right\}\right)\,, \tag{13}$$

where, for each $t>T$, the infimum in braces is computed on those controls u which steers y_0 to the rest in time at most t (i.e., the supports of u and $y^{y_0,u}$ have to be contained in $[0,t]$) (recall Lemma 11).

Using Lemma 13, we can prove:

Theorem 15. *Let Assumption 1 hold. Let system (1) be null controllable in time T. Then we have*

$$I(y_0)=Z(y_0)\,.$$

Proof. It is clear that $I(y_0)\le Z(y_0)$ for every $y_0\in H$. We prove the converse inequality.

Let us fix any $y_0\in H$. Null controllability implies that $I(y_0)<+\infty$ for every y_0 so that for every $\epsilon>0$ there exist a control $u_\epsilon\in\mathcal{U}(y_0)$ such that for every $S>0$ we have

$$\int_0^S|u_\epsilon(t)|^2\,\mathrm{d}t<I(y_0)+\epsilon\,. \tag{14}$$

The condition $u_\epsilon\in\mathcal{U}(y_0)$ implies $y^{y_0,u_\epsilon}\in L^2(0,+\infty;H)$ and so for every $\sigma>0$ we have $|y^{y_0,u_\epsilon}|^2_{L^2(R,R+1;H)}<\sigma$ for R sufficiently large.

There exists a sequence $\{u_n\}$ in $C^1([0,R+1];U)$, which converges to u_ϵ in $L^2(0,R+1;U)$. So, there exists an index N such that inequality (14) holds for u_N and furthermore $|y^{y_0,u_N}|^2_{L^2(R,R+1;H)} < \sigma$. Hence we can find $S_\sigma \in [R,R+1]$ such that the continuous function $y^{y_0,u_N}(t)$ satisfies

$$|y^{y_0,u_N}(S_\sigma)|^2 = |y(S_\sigma,y_0,u_N)|^2 < \sigma\,.$$

Null controllability holds also on $[S_\sigma, S_\sigma+T]$ and Lemma 13 can be applied on this interval (see also Remark 14). Hence, there exists a control $\tilde{u}$ with support in $[S_\sigma, S_\sigma+T]$ which steers to the rest in time T the "initial condition" $y^{y_0,u_N}(S_\sigma) = y(S_\sigma,y_0,u_N)$, assigned at the "initial time" S_σ. Lemma 13 shows that the square norm of this control is less then $M\sigma$.

Now we apply first the control u_N, on $[0,S_\sigma]$, and after that the control $\tilde{u}$. In this way we steer y_0 to the rest in time at most $S_\sigma+T$ and the square of the L^2 norm of the control is less then $I(y_0)+\epsilon+M\sigma$. This shows that

$$Z(y_0) \le I(y_0)+\epsilon+M\sigma\,.$$

The required inequality follows since $\epsilon>0$ and $\sigma>0$ are arbitrary. ∎

This theorem shows:

Corollary 16. *Let Assumption 1 hold and suppose that system (1) is null controllable in time T. Then System (1) is NCVE if and only if $I(y_0)=0$, for every $y_0 \in H$.*

So, our goal now is the proof that, under the assumptions of Theorem 9, we have $I(y_0)=0$.

The study of the value function $I(y_0)$ defined above is the object of the so-called theory of the *quadratic regulator problem with stability* or *Kalman-Yakubovich-Popov Theory.* It has been studied, for special classes of distributed control systems, in [28, 33, 35, 36, 37]. But, we need an improved version of the results of this theory, i.e., we need the following theorem, which is proved in Appendix A.

Theorem 17. *Let Assumption 1 hold and suppose that system (1) is null controllable. Then there exists $P=P^* \in \mathcal{L}(H)$ such that for every $y_0 \in H$ we have*

$$I(y_0) = \langle y_0, Py_0\rangle\,. \tag{15}$$

Furthermore, the operator P satisfies the following inequality, for any $y_0 \in H$, $u \in L^2_{loc}(0,+\infty;U)$,

$$\langle Py(t;y_0,u), y(t;y_0,u)\rangle - \langle Py_0,y_0\rangle + \int_0^t |u(s)|^2\,\mathrm{d}s \ge 0 \quad a.e.\;\; t\ge 0\,. \tag{16}$$

Inequality (16) is called *Linear Integral Inequality*—shortly (LOI)—or *Dissipation inequality*—shortly (DI)—in integral form.

Combining Corollary 16 and Theorem 17, we get:

Corollary 18. *Let Assumption 1 hold and suppose that system (1) is null controllable. Then System (1) is NCVE if and only if $P=0$.*

These are the preliminaries we need in order to prove Theorem 9 an equivalent formulation of which is as follows:

Theorem 19. *Assume Hypotheses 1 and 8 and furthermore suppose that $s(A)\le 0$. If system (1) is null controllable at some time $T>0$, then $P=0$.*

Proof. We decompose $H = H_s \oplus H_1$ according to Assumption 8 and we show that the restrictions of P respectively to H_s and H_1 are zero.

If $y_0 \in H_s$, then by (LOI) with $u = 0$ we get

$$\left\langle Pe^{At}y_0, e^{At}y_0 \right\rangle \geq \langle Py_0, y_0 \rangle \geq 0.$$

By assumption, when $y_0 \in H_s$ we have

$$\lim_{t\to+\infty} e^{At}y_0 = 0\,.$$

Hence, letting $t \to +\infty$, we find $\langle Py_0, y_0\rangle = 0$ and so $Py_0 = 0$.

If $H_1 = \{0\}$, in particular if $\sigma(A) = \emptyset$, then $H = H_1$ and we are done. Otherwise, we prove that P is zero on H_1.

In order to prove that P is zero on H_1, *it is enough to verify that $Pz = 0$ on every generalized eigenvector z of A which belongs to H_1.*

Indeed the subspace generated by all the generalized eigenvectors of A which belong to H_1 is dense in this space and, moreover, P is continuous on H_1.

We recast the definitions of the generalized eigenvectors in the form that we need. Let y_0 be an eigenvector of A, $Ay_0 = \lambda y_0$. We associate to y_0 the "Jordan chain" whose elements are the vectors y_k which, for $k \geq 1$, are defined by

$$Ay_k = \lambda y_k - y_{k-1}\,. \tag{17}$$

This process ends at the index n if $y_{n+1} = 0$. So, a Jordan chain may be infinite or finite (possible of length 1, reduced to y_0). The generalized eigenvectors of A are the elements of a Jordan chain.

Note that the chain is identified by both λ and its eigenvector y_0: an eigenvalue λ has as many (independent) Jordan chains as independent eigenvectors.

A fact we shall use is that when y_k is an element of a Jordan chain (corresponding to an eigenvalue λ) then

$$e^{At}y_k = e^{\lambda t}y_k + q_k(t)\,, \qquad q_k(t) = e^{\lambda t}\sum_{n=1}^{k} \alpha_{k,n} y_{k-n} t^n \tag{18}$$

for suitable coefficients $\alpha_{k,n}$. The important point to be noted is that *only the elements y_r of the chain, with $r < k$, appear in the expression of $q_k(t)$.*

Now we prove that $Py = 0$ if y is any generalized eigenvector of A in H_1. We distinguish two cases, that the eigenvalue has negative real part, or null real part (positive real part is impossible, due to our assumption $s(A) \leq 0$).

2.2.1. *the case $\Re e\,\lambda < 0$.* Let y_N be a generalized eigenvector, defined by the sequence of the equalities (17) and let

$$Y_N = \operatorname{span}\{y_k\}_{0\leq k\leq N}\,.$$

The subspace Y_N is invariant for A and there exist *positive* number M and σ such that

$$y \in Y_N \implies \left|e^{At}y\right| \leq Me^{-\sigma t}|y| \qquad \forall y \in Y_N\,.$$

So, the same argument as used above for the case of the subspace H_s can be used here: (LOI) with $u = 0$ gives

$$0 \leq \langle Py, y\rangle \leq \langle Pe^{At}y, e^{At}y\rangle \leq M^2 e^{-2\sigma t}\|P\|\,|y|$$

and the right hand side tends to 0 for $t \to +\infty$.

This proves that $P = 0$ on Y_N, as wanted, when the corresponding eigenvalue has negative real part.

2.2.2. *the case* $\Re e\,\lambda = 0$. We recall the notation $(Lu)(t)$ from (2) so that

$$y(t;y_0,u) = e^{At}y_0 + (Lu)(t)$$

and (LOI) can be written as

$$\left\{\left\langle e^{At}y_0, Pe^{At}y_0\right\rangle - \langle y_0, Py_0\rangle\right\} + 2\Re e\,\left\langle (Lu)(t), Pe^{At}y_0\right\rangle \\ + \left\{\int_0^t \|u(s)\|^2\,\mathrm{d}s + \langle (Lu)(t), P(Lu)(t)\rangle\right\} \geq 0\,. \qquad (19)$$

Now let y_0 be any eigenvector of the eigenvalue $i\omega$, $\omega \in \mathbb{R}$. Then

$$\left\langle e^{At}y_0, Pe^{At}y_0\right\rangle = \left\langle e^{i\omega t}y_0, Pe^{i\omega t}y_0\right\rangle = \langle y_0, Py_0\rangle$$

and the first brace in (19) is equal 0. Replacing y_0 by μy_0, $\mu \in \mathbb{R}$, we see that

$$\langle P(Lu)(t), y_0\rangle = 0.$$

So, for every $u \in L^2_{\mathrm{loc}}(0,+\infty)$ we have, for a.e. $t \geq 0$,

$$\left\langle y_0, P\left[e^{At}y_0 + (Lu)(t)\right]\right\rangle = \left\langle y_0, P\left[e^{i\omega t}y_0 + (Lu)(t)\right]\right\rangle = e^{-i\omega t}\langle y_0, Py_0\rangle. \qquad (20)$$

The system being controllable to the rest at time T, there exists a control such that

$$e^{At}y_0 + (Lu)(t) = e^{i\omega t}y_0 + (Lu)(t)$$

has support in $[0,T]$, and so the left hand side of (20) is zero for $t > T$. Hence, $\langle y_0, Py_0\rangle = 0$. As $P = P^* \geq 0$, we see that $Py_0 = 0$, as wanted.

Now we extend this property to every element of the Jordan chain of y_0, using an induction argument. Let y_N be a generalized eigenvector of this chain and let us assume that $Py_k = 0$ for $k < N$. We prove that $Py_N = 0$ too.

Using formula (18), we see that the induction hypothesis implies

$$Pq_N(t) = 0$$

and combining (LOI), (19) and (18) we get

$$2\Re e\, e^{i\omega t}\,\langle Py_N, (Lu)(t)\rangle + \left\{\int_0^t \|u(s)\|^2\,\mathrm{d}s + \langle (Lu)(t), P(Lu)(t)\rangle\right\} \geq 0\,.$$

As above, the part which is linear in y_N has to be zero, i.e., $\langle Py_N, (Lu)(t)\rangle = 0$, so that

$$\langle y_N, Py(t;y_N,u)\rangle = \left\langle y_N, P\left(e^{At}y_N + (Lu)(t)\right)\right\rangle = \left\langle Py_N, e^{i\omega t}y_N\right\rangle\,.$$

We then control y_N to the rest using a suitable control u and, as above, we get $Py_N = 0$. This ends the proof. ∎

3. Examples and applications

Here we provide applications of our result to boundary control of parabolic coupled equations considered in [10] and to delay systems. The problem of finding explicit conditions on the minimal energy for large times is discussed in some details as well.

3.1. **Parabolic coupled system.** We will establish *conditions under which the control system of* [10] *is NCVE.* We start from establishing some properties of the system.

The system is linear and of the form:

$$\begin{cases} y_t - y_{xx} = A_0 y \quad \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = B_0 u, \quad y(\pi,0) = 0 \ \text{ in } (0,T), \\ y(\cdot,0) = y_0 \text{ in } (0,\pi), \end{cases} \tag{21}$$

where A_0 is a 2×2 real matrix and $B_0 \in \mathbb{R}^2$, $u \in L^2(0,T)$ is the control function and $y = \operatorname{col}\left(\ y_1, y_2\ \right)$ is the state variable. As in [10] we denote by

$$L^2(0,\pi)^2$$

the space $(L^2(0,\pi))^2$ and use similar notation for other function spaces as well.

In [10] the interval $(0,\pi)$ is replaced by $(0,1)$ and the initial datum $y_0 \in H^{-1}(0,\pi)^2$ (the dual space of the space $H^1_0(0,\pi)^2 = H^1_0(0,\pi;\mathbb{R}^2)$) but in the sequel we will assume that $y_0 \in L^2(0,\pi)^2$.

Let $y_0 \in L^2(0,\pi)^2$ and $u \in L^2(0,T)$, in [10] it is proved that there exists a unique solution $y \in L^2(Q)^2$ to (21). This is defined by transposition, i.e., requiring that, for each $g \in L^2(Q)^2$ one has

$$\int_Q y\cdot g\,dxdt = \langle y_0, \phi(\cdot,0)\rangle + \int_0^T B_0\cdot\phi_x(0,t)u(t)dt, \tag{22}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(0,\pi)^2$ and

$$\phi \in L^2(0,T;H^2(0,\pi)^2)\cap C^0([0,T];H^1_0(0,\pi)^2)$$

is the (unique) strong solution to the following equation involving g

$$\begin{cases} -\phi_t - \phi_{xx} = A_0^*\phi + g \quad \text{in } Q = (0,\pi)\times(0,T), \\ \phi(0,\cdot) = 0, \quad \phi(\pi,0) = 0 \ \text{ in } (0,T), \\ \phi(\cdot,T) = 0 \text{ in } (0,\pi). \end{cases}$$

It is not difficult to see that *the previous solution y can be written in the form* (4). In addition we see that Hypotheses 1 and 8 hold in this case. In fact:

Proposition 20. *The system* (21) *can be written in the form* (1) *and in addition Hypotheses 1 and 8 hold.*

Proof. We fix $H = L^2(0,\pi;\mathbb{C})^2$ and define the linear operator $A : \operatorname{dom} A \subset H \to H$,

$$\operatorname{dom} A = H^2(0,\pi;\mathbb{C})^2 \cap H^1_0(0,\pi;\mathbb{C})^2, \quad Af = f_{xx} + A_0 f, \quad f \in \operatorname{dom} A\,.$$

Since the operator $A_1 = D_{xx}$ with $\operatorname{dom} A_1 = \operatorname{dom} A$ generates a compact holomorphic semigroup on H the same happens for A (even if, in general, e^{At} is no more symmetric). Indeed A_0 is a bounded perturbation and we can apply well-know results in [38, Section 3.1].

In particular the resolvent operator of A is compact and the spectrum $\sigma(A)$ consists entirely of eigenvalues with finite algebraic multiplicity; this also shows that Assumption 8 can be used.

We show that when $y_0 \in L^2(0,\pi)^2$ the unique solution of (21)

$$y \in L^2(Q)^2 = L^2(0,T;L^2(0,\pi)^2)$$

defined by transposition can be written as in (4) and so it coincides with the solution to (1).

Note that at least for regular real functions $y_0 \in C^1([0,\pi])^2$ and $u \in C^1([0,T])$ one checks directly that the solution $y = y^{y_0,u}$ to (21) is given by

$$y(t) = e^{At}y_0 - A\int_0^t e^{A(t-s)}Cu(s)\,\mathrm{d}s, \tag{23}$$

where, $C : \mathbb{C} \to C^2([0,\pi])^2$, $h(x) = (Ca)(x)$, $x \in [0,\pi]$, $a \in \mathbb{C}$, is the unique solution to

$$\begin{cases} h'' + A_0 h = 0 & \text{in } (0,\pi), \\ h(0) = aB_0, & h(\pi) = 0, \end{cases}$$

for any $a \in \mathbb{C}$.

Let us introduce the control operator B. Setting $U = \mathbb{C}$, we define $B = -\mathcal{A}C$, where $\mathcal{A}$ is the extension of A to $(\operatorname{dom} A^*)'$, i.e., $\mathcal{A} : H \to (\operatorname{dom} A^*)'$. Hence, we can rewrite (23) as

$$y(t) = e^{At}y_0 + \int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s. \tag{24}$$

In order to check Assumption 1 we first clarify that, for any $u \in L^2(0,T;\mathbb{C})$, the mapping:

$$t \mapsto -A\int_0^t e^{A(t-s)}Cu(s)\,\mathrm{d}s = \int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s \ \text{ belongs to } L^2(0,T;H). \tag{25}$$

Since e^{At} is holomorphic, for any $\theta \in (0,1/4)$, for any f which belongs to the interpolation space

$$(H, \operatorname{dom} A)_{\theta,\infty} = (H, \operatorname{dom} A_1)_{\theta,\infty}$$

(see, for instance, [4, pag. 148]) we have $\|Ae^{At}f\|_H \le \frac{C_T}{t^{1-\theta}}\|f\|_{(H,\operatorname{dom} A)_{\theta,\infty}}$, $t \in (0,T)$. On the other hand, it is well known that

$$\operatorname{dom}[(-A_1)^\theta] = (H, \operatorname{dom} A_1)_{\theta,\infty} = H^{2\theta}(0,\pi;\mathbb{C})^2$$

(with equivalence of norms). Since, in particular, the operator

$$C : \mathbb{R} \to H^{2\theta}(0,\pi;\mathbb{C})^2$$

is continuous, it follows that $\|Ae^{At}Ca\|_H \le \frac{C_T\,|a|}{t^{1-\theta}}$, $t \in (0,T)$, $a \in \mathbb{C}$. Now, by the Young inequality for convolution we deduce easily (25). Moreover, we also obtain that the transformation $u \mapsto \int_0^t e^{\mathcal{A}(t-s)}Bu(s)\,\mathrm{d}s$ is linear and bounded from $L^2(0,T;\mathbb{C})$ into $L^2(0,T;H)$ and so Assumption 1 is satisfied.

It remains to show that for $y_0 \in L^2(0,\pi)^2$ and $u \in L^2(0,T)$, the function given in (24) is the weak solution to (21).

By (25) we know that $y = y^{y_0,u} \in L^2(0,T;L^2(0,\pi)^2)$. Choosing regular (z_n) and (u_n) converging to y_0 and u respectively in $L^2(0,\pi)^2$ and $L^2(0,T)$ we note that the solutions $y_n = y^{z_n,u_n}$ verify the identity (22).

Since y_n converges to y in $L^2(0,T;L^2(0,\pi)^2)$, passing to the limit as $n \to \infty$ in (22) we deduce that y verifies (22). ∎

3.2. **Null controllability with vanishing energy.** In [10] it is proved that (21) is null controllable at any time $T > 0$ if and only if

$$\operatorname{rank}[A_0|B_0] = \operatorname{rank}[B_0, A_0B_0] = 2 \quad \text{and} \quad \mu_1 - \mu_2 \neq j^2 - k^2, \;\; \forall k, j \in \mathbb{N}, \tag{26}$$

with $j \neq k$, where μ_1 and μ_2 are the eigenvalues of A_0.

We can prove the following additional result.

Theorem 21. *Assume that the control system* (21) *is null controllable (i.e., that condition (26) hold). Then the system* (21) *is NCVE if and only if*

$$\Re e\,(\mu_i) \leq 1, \quad i = 1, 2. \tag{27}$$

Proof. By combining Theorems 6 and 9 (see also Corollary 10) we see that (21) is NCVE if and only if

$$\sigma(A) \subset \{\Re e\,\lambda \leq 0\}.$$

Thus (27) follows easily if we show that

$$\sigma(A) = \{\lambda \in \mathbb{C} \;:\; \lambda = \mu_i - k^2, \quad i = 1, 2, \;\; k \in \mathbb{N}, \; k \geq 1\}. \tag{28}$$

To check (28) we first recall that $\sigma(A)$ consists entirely of eigenvalues (see the proof of Proposition 20).

Moreover, we will use that $A_D = D_{xx}$ with Dirichlet boundary condition is self-adjoint on $L^2(0, \pi)$ with $\sigma(A_D) = \{-k^2\}_{k\geq 1}$ and

$$A_D e_k = -k^2 e_k, \quad e_k(x) = \frac{\sqrt{2}}{\sqrt{\pi}} \sin(kx), \quad x \in [0, \pi]. \tag{29}$$

In order to characterize $\sigma(A)$, we fix an eigenvalue $\lambda \in \mathbb{C}$ and consider a corresponding eigenfunction $u \in \operatorname{dom} A$, i.e.,

$$u_{xx} + A_0 u = \lambda u, \quad u \neq 0. \tag{30}$$

We note that if A_0 is diagonalizable with only one (repeated) eigenvalue, then $\operatorname{rank}[A_0|B_0] = 1$. So, controllability implies that we have to examine only the following two cases:

(i) A_0 has a unique (real) eigenvalue $\mu = \mu_1 = \mu_2$ with $\dim(\operatorname{Ker}(A_0 - \mu)) = 1$.

We introduce a non-singular 2×2 real matrix P such that

$$P^{-1} A_0 P = J = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$$

and consider the real function $v = P^{-1}u \in \operatorname{dom} A$. We find $Pv_{xx} + A_0 P v = \lambda P v$, and so we can concentrate on the problem

$$v_{xx} + Jv = \lambda v.$$

If $v = \operatorname{col}(v^{(1)}, v^{(2)})$ we find

$$\begin{cases} v^{(1)}_{xx} + \mu v^{(1)} + v^{(2)} = \lambda v^{(1)}. \\ v^{(2)}_{xx} + \mu v^{(2)} = \lambda v^{(2)}. \end{cases}$$

Using (29), we deduce that $\lambda - \mu = -k^2$ for some $k \geq 1$. Moreover $u = Pv$ where $v = \operatorname{col}(e_k, 0)$ is an eigenfunction corresponding to $\lambda = \mu - k^2$.

The eigenvalue $\lambda = \mu - k^2$ has a Jordan chain of length 2 and the generalized eigenvalue has both the components equal to e_k. Hence, the Jordan chain of $\lambda_k = \mu - k^2$ has the following elements (with $c = \frac{\sqrt{2}}{\sqrt{\pi}}$):

$$\begin{pmatrix} c\sin(kx) \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} c\sin(kx) \\ c\sin(kx) \end{pmatrix}$$

and so the Jordan chains span the state space, in spite of the fact that the operator is not selfadjoint.

(ii) A_0 has distinct eigenvalues μ_1 and μ_2, which might be complex (conjugate).

We consider a non-singular 2×2 matrix P (possibly complex) such that

$$P^{-1}A_0P = J = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Introducing the complex function $v = P^{-1}u$, we find $v_{xx} + Jv = \lambda v$. If $v = \mathrm{col}\,(v^{(1)}, v^{(2)})$ we get

$$\begin{cases} v^{(1)}_{xx} + \mu_1 v^{(1)} = \lambda v^{(1)} \\ v^{(2)}_{xx} + \mu_2 v^{(2)} = \lambda v^{(2)}\,. \end{cases}$$

Using (29), we deduce that, for $k, n \geq 1$,

$$\text{either } \lambda - \mu_1 = -k^2 \text{ or } \lambda - \mu_2 = -n^2.$$

Moreover, if $\lambda = \mu - k^2$, then an eigenfunction is $u = Pv$ where $v = \mathrm{col}\,(e_k, 0)$. If $\lambda = \mu_2 - n^2$, then an eigenfunction is $u = Pv$ where $v = \mathrm{col}(0, e_n)$ (the functions e_n are defined in (29)). Hence, there is an orthonormal basis of eigenvectors of the state space in this case, whose elements are

$$\begin{pmatrix} c\sin(nx) \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ c\sin(kx) \end{pmatrix} \tag{31}$$

where $n \in \mathbb{N}$ and $k \in \mathbb{N}$ are independent.

The proof is complete. ∎

3.3. **An explicit computation of the control energy.** Theorem 21 provides necessary and sufficient conditions under which system (21) is NCVE. Recall that if NCVE holds then, for every $\epsilon > 0$, there exist controls steering to the rest any initial condition and whose norm is less then ϵ. Thus if the system is NCVE it might be of interest to estimate the control energy at time T , i.e.

$$Z_T(y_0) = \inf \int_0^T |u(s)|^2 ds$$

(the infimum is computed on the controls steering $y_0 \in H$ to the rest in time $T > 0$) and show directly that this converges to 0 as $T \to +\infty$. For an example of such computations in the case of the wave equation with boundary controls, see [19]; see also [39] for a related time optimal control problem in the distributed control case. We cite also [49, Sect. 4.3] for a general discussion on parabolic systems.

We are going to show that explicit estimates on the control energy are indeed possible in the case of the example (21), of course at the expenses of some more computations.

We confine ourselves to consider the case that A_0 is diagonalizable and $s(A) \leq 0$. Moreover, we put ourselves in the critical case that 0 is an eigenvalue, so that, after a coordinate transformation,

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

with $\mu \leq 1$ and both $\mu \neq 1$ and $\mu \neq 1 - (j^2 - k^2)$ for every j and $k \in \mathbb{N}$, in order to have controllability. Furthermore, it is not restrictive that we assume

$$B_0 = \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

($\beta \neq 0$ is required by controllability). Let, for $n \geq 1$,

$$y(x,t) = \begin{pmatrix} v(x,t) \\ w(x,t) \end{pmatrix}, \qquad \begin{cases} v_n(t) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi v(x,t) \sin nx \, \mathrm{d}x \,, \\ w_n(t) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\pi w(x,t) \sin nx \, \mathrm{d}x \,. \end{cases}$$

An integration by parts shows that $v_n(t)$ and $w_n(t)$ solve the following equations

$$\begin{cases} v_n'(t) &= (1-n^2)v_n(t) + nu(t), \\ w_k'(t) &= (\mu - k^2)w_k(t) + k\beta u(t). \end{cases} \tag{32}$$

Note that n and k here are independent, since an orthonormal basis of the state space is (31). So, the control $u(t)$ steers $v(x,0) = v_0(x)$ and $w(x,0) = w_0(x)$ to the rest in time T when $f(t) = u(T-t)$ solves the moment problem (with n, $k \geq 1$)

$$\begin{cases} \int_0^T e^{-(n^2-1)s} f(s) \, \mathrm{d}s &= \left[\frac{1}{n} e^{-(n^2-1)T}\right] v_{0,n} \,, \\ \int_0^T e^{-(k^2-\mu)s} f(s) \, \mathrm{d}s &= \left[\frac{1}{\beta k} e^{-(k^2-\mu)T}\right] w_{0,k} \,. \end{cases} \tag{33}$$

Here $v_{0,n}$ and $w_{0,k}$, $n, k \geq 1$, are the Fourier coefficients in the sine expansion of the initial conditions $v_0(x)$ and $w_0(x)$.

Let us define the following sequence $\{\Phi_n(t)\}_{n\geq 1}$:

$$\Phi_{2r-1}(t) = e^{-(r^2-1)t} \,, \qquad \Phi_{2r}(t) = e^{-(r^2-\mu)t}$$

for every natural number $r \geq 1$. Set also

$$\lambda_{2r-1} = r^2 - 1 \,, \qquad \lambda_{2r} = r^2 - \mu$$

i.e., for $n \geq 1$,

$$\lambda_n = \begin{cases} \frac{1}{4}(n+1)^2 - 1 = \frac{1}{4}n^2 + \left(\frac{1}{2}n - \frac{3}{4}\right), & n \text{ odd}, \\ \frac{1}{4}n^2 - \mu, & n \text{ even}. \end{cases} \tag{34}$$

So, $\Phi_1(t) = 1$ corresponds to the eigenvalue $\lambda_1 = 0$ of A. Properties of this sequence have been studied in [43] on the space $L^2(0, +\infty)$ and in $L^2(0,T)$ (the fact that Φ_1 is not square integrable is easily adjusted, see also below, in the proof of Lemma 22). It is proved that the sequence $\{\Phi_n(t)\}$ has a biorthogonal sequences in $L^2(0,T)$. Furthermore, from [2], $f(t)$ is given by

$$f(t) = \Psi_1^T(t) v_{0,1} + h^T(t) \,,$$
$$h^T(t) = \sum_{r\geq 2} \Psi_{2r-1}^T(t) \frac{1}{r} e^{-(r^2-1)T} v_{0,r} + \sum_{r \geq 1} \Psi_{2r}^T(t) \frac{1}{\beta r} e^{-(r^2-\mu)T} w_{0,r} \,. \tag{35}$$

where $\{\Psi_n^T\}_{n\geq 1}$ is any biorthogonal sequence such that the series converges in $L^2(0,T)$. The existence of this sequence is consequence of the following lemma, proved at the end of this section:

Lemma 22. *For every $\epsilon > 0$ there exists a biorthogonal sequence $\{\Psi_n^T\}_{n\geq 1}$ and a number $K(\epsilon)$, independent of T, such that*

$$|\Psi_n^T|_{L^2(0,T)} < K(\epsilon)e^{\epsilon\lambda_n}$$

We assume this lemma and we prove convergence of the series in (35). Furthermore we give an estimate for $|h^T|_{L^2(0,T)}$. We prove convergence for $T > 2$ since this is all that we need for the asymptotics of the energy. Convergence for $T \in (0, 2]$ is proved analogously.

We consider the first series, which can be treated as follows. We fix $\epsilon = 1/2$ and we denote $K = K(1/2)$. Then we have:

$$\left|\sum_{r\geq 2}\Psi_{2r-1}^T(t)\frac{1}{r}e^{-(r^2-1)T}v_{0,r}\right|_{L^2(0,T)} \leq K\sum_{r\geq 2}\frac{1}{r}e^{-(r^2-1)(T-1/2)}|v_{0,r}|$$
$$\leq Ke^{-(1/2)(T-1/2)}\left[\sum_{r\geq 2}\frac{1}{r}e^{-(r^2-3/2)(T-1/2)}|v_{0,r}|\right]$$
$$\leq Ke^{-(1/2)(T-1/2)}\left[\sum_{r\geq 2}\frac{1}{r}e^{-(r^2-3/2)}|v_{0,r}|\right]$$

(since $T > 2$). These inequalities prove in one shot convergence of the series and furthermore they prove that

$$\left|\sum_{r\geq 2}\Psi_{2r-1}^T(t)\frac{1}{r}e^{-(r^2-1)T}v_{0,r}\right|_{L^2(0,T)} \to 0$$

exponentially fast for $T \to +\infty$.

The second series can be treated analogously.

Hence, *if $v_{0,1} = 0$ (or if $\lambda = 0$ is not an eigenvalue) then $Z_T(y_0)$ decays exponentially for $T \to +\infty$.*

We consider now $Z_T(y_0)$ when $v_{0,1} \neq 0$. We keep the same elements Ψ_n^T as above for $n > 1$, so that the exponential estimate on the series is not affected, and we choose a suitable element Ψ_1.

Note that we can confine ourselves to give an estimate for $Z_T(y_0)$ when $T = N$, a positive integer.

Let $\{\Psi_n^1(t)\}$ be the biorthogonal sequence when $T = 1$ and let us consider its first element $\Psi_1^1(t)$. It satisfies

$$\int_0^1 \Psi_1^1(t)\Phi_n(t)\,\mathrm{d}t = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \qquad \text{otherwise}\,. \end{cases}$$

Let now γ be any positive number. For every $\Psi \in L^2(0, N)$ we have

$$\int_0^N \Psi(t)e^{-\gamma t}\,\mathrm{d}t = \sum_{m=0}^{N-1} e^{-\gamma m}\int_0^1 \Psi(m+t)e^{-\gamma t}\,\mathrm{d}t\,. \tag{36}$$

Let us define $\Psi_1^N(t)$ on $[0, N]$,

$$\Psi_1^N(t) = \Psi_1^1(t-m) \qquad \text{if } m \leq t < m+1, \qquad 0 \leq m \leq N-1\,.$$

Then, (36) shows that $\Psi_1^N(t)$ is orthogonal to $\Phi_n(t)$ on $[0, N]$, for every $n > 1$, and

$$\|\Psi_1^N\|_{L^2(0,N)} = \sqrt{N}\|\Psi_1^1\|_{L^2(0,1)}\,.$$

Now we consider the control function $f(t)$ on $[0,N]$ given by

$$f(t)=\alpha\Psi_1^N(t)+h^N(t)\,,$$
$$h^N(t)=\sum_{r\geq 2}\Psi_{2r-1}^N(t)\frac{1}{r}e^{-(r^2-1)N}v_{0,r}+\sum_{r\geq 1}\Psi_{2r}^N(t)\frac{1}{\beta r}e^{-(r^2-\mu)N}w_{0,r}\,,$$

for a suitable constant α to be fixed.

Note that $h^N(t)$ is the same as in (35), with $T=N$. Thanks to the fact that Ψ_1^N is orthogonal to every $\Phi_k(t)$, $k>1$, we see that this function $f(t)$ solves the moment problem (33) (with $T=N$) if

$$v_{0,1}-\int_0^N h^N(s)\,\mathrm{d}s=\alpha\int_0^N\Psi_1^N(t)\,\mathrm{d}t=N\alpha\int_0^1\Psi_1^1(t)\,\mathrm{d}t=N\alpha\,.$$

We have seen that the $L^2(0,N)$-norms of $h^N(t)$ decay exponentially: $\|h^N\|_{L^2(0,N)}\leq e^{-\sigma N}$ ($\sigma>0$). This shows that $\alpha\asymp 1/N$ and we have

$$\|\alpha\Psi_1^N\|_{L^2(0,N)}\asymp\frac{\text{const}}{\sqrt{N}}\qquad i.e.\qquad Z_N(y_0)\asymp\frac{\text{const}}{\sqrt{N}}\,.$$

This is the required estimate for the energy at time T. This estimate implies, in a less direct way, that the system is NCVE.

Proof. [Proof of Lemma 22] The proof consists on finding a relation between the norm of biorthogonals in $L^2(0,T)$ and in $L^2(0,+\infty)$. But, we note that $\Phi_1(t)=1$ is not square integrable. This is adjusted replacing $\Phi_n(t)$ with $e^{-t}\Phi_n(t)$, λ_n in (34) with λ_n+1 and then $f(t)$ with $e^tf(t)$. Once this has been understood, we go on using the notations λ_n and $\Phi_n(t)$ for these (modified) sequences.

The sequence of the exponents λ_n (now all positive) satisfies the conditions in [10, Lemma 3.1] (it does not satisfy the more stringent conditions in [8]). Hence (see [10, Lemma 3.1]), it admits a biorthogonal sequence $\{\Psi_n(t)\}$ in $L^2(0,+\infty)$ with the following property: for every $\epsilon>0$ there is a constant $K(\epsilon)$ such that

$$|\Psi_n(t)|_{L^2(0,+\infty)}\leq K(\epsilon)e^{\epsilon\lambda_n}\,.$$

The sequence λ_n in particular satisfies

$$\sum_{n\geq 1}\frac{1}{\lambda_n}<+\infty$$

so that $\{e^{-\lambda_n t}\}$ spans a closed *proper* subspace of $L^2(0,+\infty)$. Let $E(\infty)$ and $E(T)$ be the closed linear spans of the set $\{e^{-\lambda_n t}\}$ respectively in $L^2(0,+\infty)$ and $L^2(0,T)$. Let P_T be the linear operator which assigns to any element of $E(\infty)$ its restriction to $(0,T)$ (which is an element of $E(T)$). Clearly, P_T is linear and continuous (norm equal 1) and, from [43, p. 55], it is boundedly invertible,

$$\|P_T^{-1}\|\leq M_T$$

(the inverse is defined on $E(T)$). It follows from [43, p. 55] that for arbitrary positive T there exists a constant $C(T)$ such that for arbitrary real numbers a_n and arbitrary natural number N

$$\left\|\sum_{n=1}^N a_ne^{-\lambda_n t}\right\|_{L^2(0,+\infty)}\leq C(T)\left\|\sum_{n=1}^N a_ne^{-\lambda_n t}\right\|_{L^2(0,T)}\,.\tag{37}$$

Of course, $T \to C(T)$ is decreasing so that for $T > 1$ we have

$$\left\| \sum_{n=1}^{N} a_n e^{-\lambda_n t} \right\|_{L^2(0,T)} \le \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n t} \right\|_{L^2(0,+\infty)} \le C(1) \left\| \sum_{n=1}^{N} a_n e^{-\lambda_n t} \right\|_{L^2(0,T)} .$$

Hence we have

$$\|P_T\| \le 1 , \qquad \|P_T^{-1}\| \le C(1) .$$

This implies, for $T > 1$:

$$|\Psi_n|_{L^2(0,T)} \le C(1)|\Psi_n|_{L^2(0,+\infty)} \le C(1)K(\epsilon)e^{\epsilon\lambda_n}$$

as wanted. ∎

Finally, we cite [45] for different conditions on sequences of exponentials, which lead to (delicate) estimates on the solution of the corresponding moment problem.

3.4. **NCVE for delay systems.** Now we discuss a controlled delay system also considered in [40]. In this case a direct computation of the control energy at time T as it is done in the previous example seems to be difficult (see the explanation below). On the other hand, it is possible to apply Corollary 10 to deduce that the system is NCVE.

Let us consider a retarded system with state delays,

$$\dot{x} = \sum_{k=0}^{M} A_k x(t - k\tau) + Bu(t) \tag{38}$$

where $\tau > 0$, $x \in \mathbb{R}^n$, A_i are $n \times n$ constant matrix and $u \in \mathbb{R}^m$ (so that the constant matrix B is $n \times m$). We introduce $H = M\tau$.

Eq. (38) is a model of a semigroup system in $M^2 = \mathbb{R}^n \times L^2(-H, 0; \mathbb{R}^n)$, the state of the system being the couple $(x(t), x(t-s))$, with $s \in [-H, 0]$. See [4] for details.

It turns out that:

- the semigroup is compact for $t > H$, so that we are in the framework of Corollary 10.
- the spectrum of the generator is not empty (it might be finite in special cases) and (see [15]) its elements are the zeros of the holomorphic function

$$\det \left[\lambda I - \sum_{k=0}^{M} A_k e^{-\lambda k\tau} \right] \tag{39}$$

  (I is the $n \times n$ identity matrix);
- the system is null controllable if and only if

$$\operatorname{rank} \left[\begin{array}{cc} \lambda I - \sum_{k=0}^{M} A_k e^{-\lambda k\tau} & B \end{array} \right] = n \tag{40}$$

  for every λ (see [34]).

So, we can state that this system is NCVE when condition (40) holds and the holomorphic function in (39) has no zero with *positive* real part.

Controllability can often be easily checked while conditions for nonpositivity of the real parts of eigenvalues have been widely studied (see for example [3]).

Null controllability can be reduced to a moment problem of course, but arguments as those in Section 3.3 for the computation of the control energy seems difficult to apply, since now the eigenvalues are distributed on a (finite number) of sequences, each one of which has the following asymptotics:

$$\lambda_n = x_n + iy_n\,, \qquad x_n = m\big(\alpha - \log(\beta m)\big) + \mathrm{o}(1)\,, \quad y = \beta m + \mathrm{o}(1)$$

(α and β are suitable constants, see [3, Theorem 12.8]).

The sequence $\{-\lambda_n\}$ does not satisfy neither the conditions in [8, Lemma 3.1] neither those in [10, Lemma 3.1]. Furthermore, since the retarded systems have smoothing property, the sequence of the exponentials $\{e^{-\lambda_n t}\}$ is not a Riesz sequence in $L^2(0,T)$. So, it seems difficult in this case to compute explicitly the energy of the control at time T.

As a simple specific example we consider

$$\dot{x} = -ax(t) - bx(t-\tau) + u(t)$$

(which is clearly null controllable). Using [15, page 135] we see that this system is NCVE for every value of r if and only if $|b| \le a$. For a specific $r > 0$ the set of the (a,b) plane in which NCVE holds is represented in [15, Fig. 5.1].

Appendix A. Quadratic regulator problem with stability: the proof of Theorem 17

The proof of Theorem 17 requires some preliminary lemmas.

We first note that the functional I defined in (12) verifies $I(\lambda y_0) = |\lambda|^2 I(y_0)$, $\lambda \in \mathbb{C}$, $y_0 \in H$. An obvious consequence is

$$\begin{cases} I(x) = I(-x) & \text{and} \qquad I(0) = 0\,, \\ |\alpha| = |\beta| & \text{implies} \quad I(\alpha x) = I(\beta x), \quad \alpha,\beta \in \mathbb{C},\ x \in H. \end{cases} \tag{41}$$

Then we give a representation of $I(y_0)$.

Lemma 23. *Let Assumption 1 hold and suppose that system (1) is null controllable in time T. Then there exists an operator P defined on H such that*

$$I(y_0) = \langle y_0, P y_0\rangle\,. \tag{42}$$

The operator P has the following properties:

- **a)** $\langle Px, \xi\rangle = \langle x, P\xi\rangle \qquad \forall x\,,\ \xi \in H$;
- **b)** $P(x+\xi) = Px + P\xi \qquad \forall x\,,\ \xi \in H$;
- **c)** *the equality $P(qx) = qPx$ holds for all $x \in H$ and every complex number q with* rational *real and imaginary parts.*
- **d)** $\langle y_0, P y_0\rangle \ge 0$ *for all $y_0 \in H$.*

Proof. The proof uses [13, Sect. 9.2] (adapted to complex Hilbert spaces) and it is an adaptation of the proof of [9, Theorem 5].

Recall that $\mathcal{U}(y_0)$ is not empty since system (1) is null controllable. So, null controllability implies that $I(y_0)$ is finite for every $y_0 \in H$.

Let us fix x_0 and ξ_0 in $\mathcal{D} = \operatorname{dom} A$ and controls $u \in \mathcal{U}(x_0)$ and $v \in \mathcal{U}(\xi_0)$ Then we have

$$y(t; x_0 \pm \xi_0, u \pm v) = y(t; x_0, u) \pm y(t; \xi_0, v)$$

and J satisfies the parallelogram identity

$$J(x_0 + \xi_0; u + v) + J(x_0 - \xi_0; u - v) = 2\left[J(x_0; u) + J(\xi_0; v)\right]\,.$$

We must prove that $I(x)$ satisfies the parallelogram identity too. This part of the proof is the same as that in [9, Theorem 5] and it is reported for completeness.

We fix x and ξ and $\epsilon > 0$ and we choose u_x and u_ξ, corresponding to the initial conditions x and ξ, such that

$$J(x; u_x) < I(x) + \epsilon/2\,, \qquad J(\xi; u_\xi) < I(\xi) + \epsilon/2.$$

Then

$$\begin{aligned} &J(x+\xi; u_x + u_\xi) + J(x-\xi; u_x - u_\xi) \\ &= 2J(x; u_x) + 2J(\xi; u_\xi) < 2\left[I(x) + I(\xi)\right] + 2\epsilon. \end{aligned}$$

This proves the inequality

$$I(x+\xi) + I(x-\xi) \le 2\left[I(x) + I(\xi)\right]\,. \tag{43}$$

We prove that the inequality cannot be strict; i.e., we prove that if ϵ satisfy

$$I(x+\xi) + I(x-\xi) \le 2\left[I(x) + I(\xi)\right] - \epsilon \tag{44}$$

then $\epsilon = 0$.

If (44) holds then we can find $\tilde{u}$ and $\tilde{v}$, corresponding to the initial states $x+\xi$ and $x-\xi$, such that

$$J(x+\xi; \tilde{u}) + J(x-\xi; \tilde{v}) \le 2\left[I(x) + I(\xi)\right] - \epsilon/2\,.$$

For the initial conditions x, ξ we apply, respectively, controls

$$u_0 = \frac{\tilde{u}+\tilde{v}}{2}\,, \qquad v_0 = \frac{\tilde{u}-\tilde{v}}{2}.$$

Then

$$2\left[J(x; u_0) + J(\xi; v_0)\right] = J(x+\xi; u_0+v_0) + J(x-\xi; u_0-v_0) \le 2\left[I(x) + I(\xi)\right] - \epsilon/2\,.$$

We have also

$$I(x) + I(\xi) \le J(x; u_0) + J(\xi; v_0) \le \left[I(x) + I(\xi)\right] - \epsilon/4\,. \tag{45}$$

This shows $\epsilon = 0$ so that parallelogram identity holds.

The operator P is now constructed by polarization (compare with [14]),

$$\langle x, P\xi\rangle = I\left(\frac{1}{2}(x+\xi)\right) - I\left(\frac{1}{2}(x-\xi)\right) + i\left[I\left(\frac{1}{2}(x+i\xi)\right) - I\left(\frac{1}{2}(x-i\xi)\right)\right]. \tag{46}$$

The property $I(x) = \langle x, Px\rangle$ is a routine computation, using (41).

We prove property **a)**. Using (41) we see that the right hand side of (46) is equal to:

$$\begin{aligned} &I\left(\frac{1}{2}(\xi+x)\right) - I\left(\frac{1}{2}(\xi-x)\right) + iI\left(\frac{1}{2}(i\xi+x)\right) - iI\left(\frac{1}{2}(i\xi-x)\right) \\ &\quad = I\left(\frac{1}{2}(\xi+x)\right) - I\left(\frac{1}{2}(\xi-x)\right) + iI\left(\frac{1}{2}(\xi-ix)\right) - iI\left(\frac{1}{2}(\xi+ix)\right) \\ &\qquad = \overline{\langle \xi, Px\rangle} = \langle Px, \xi\rangle\,. \end{aligned}$$

In order to see property **b)** it is sufficient to prove additivity of the real part. In fact, using $4I(y_0) = I(2y_0)$, we check that

$$\Re e\ \left(4\langle y, P(\xi+x)\rangle\right) = \Re e\ \left(\langle 2y, P2(\xi+x)\rangle\right) = I(x+\xi+y) - I(x+\xi-y) \tag{47}$$

$$= 4\Re e\ \left(\langle y, P\xi\rangle + \langle y, Px\rangle\right) = I(\xi+y) - I(\xi-y) + I(x+y) - I(x-y)\,. \tag{48}$$

Using the parallelogram identity for $I(x)$, i.e., (43) with $=$ instead of $\leq$, and associating the terms of equal signs, we see that the right hand side of (48) is equal to

$$\frac{1}{2}\left[I(x+\xi+2y)+I(\xi-x)\right]-\frac{1}{2}\left[I(x+\xi-2y)+I(\xi-x)\right]$$
$$=\frac{1}{2}\left[I(x+\xi+2y)-I(x+\xi-2y)\right]$$
$$=\frac{1}{2}\left\{-I(x+\xi)+2\left[I(x+\xi+y)+I(y)\right]+I(x+\xi)-2\left[I(x+\xi-y)+I(y)\right]\right\}$$
$$=I(x+\xi+y)-I(x+\xi-y)$$

as wanted.

Property **c)** for q real rational is consequence of **b)**, as in [13, Sect. 9.2]. When $q=i$ equality follows since **a)** easily shows

$$\langle x,P(i\xi)\rangle=-i\langle x,P\xi\rangle=\langle x,iP\xi\rangle \qquad \text{i.e.,} \qquad P(i\xi)=iP\xi\,.$$

Hence, property **c)** holds also for iq with real rational q and then it holds for every complex number with rational real and imaginary parts.

Property **d)** is obvious. ∎

Now we prove:

Lemma 24. *Let Assumption 1 hold and suppose that system (1) is null controllable. Then, there exists a number M such that*

$$I(y)\leq M|y|^2,\quad y\in H.$$

Proof. It is sufficient to prove that $I(y)$ is bounded in a ball since $I(\lambda y_0)=|\lambda|^2I(y_0)$, $\lambda\in\mathbb{C}$, $y_0\in H$. This is known, see the second statement in Lemma 11. ∎

For the moment, we can't say that the operator P is linear, i.e., that $P(qx)=qPx$ for every *real* q. This will be proved below, as a consequence of this version of Schwarz inequality, which can be proved using solely the properties stated in Lemmas 23 and 24:

Lemma 25. *Let Assumption 1 hold and suppose that system (1) is null controllable. Then, we have*

$$|\langle Py,x\rangle|=|\langle y,Px\rangle|\leq M|y||x|,\quad x,y\in H. \tag{49}$$

Proof. The inequality is obvious if $\langle Py,x\rangle=0$. Otherwise, we note the following equality, which holds for every complex number λ which has rational real and imaginary parts:

$$0\leq\langle Px,x\rangle+2\Re e\,\left(\lambda\langle Py,x\rangle\right)+|\lambda|^2\langle Py,y\rangle\,.$$

This inequality is extended to every complex λ by continuity. The usual choice $\lambda=-\left(\langle Px,x\rangle\right)/\left(\langle Py,x\rangle\right)$ gives

$$|\langle Py,x\rangle|\leq\sqrt{\langle Px,x\rangle\langle Py,y\rangle}=\sqrt{I(x)I(y)}\leq M|x||y|\,. \;∎$$

Finally we can prove:

Lemma 26. *Let Assumption 1 hold and suppose that system (1) is null controllable. Then the operator P defined in Lemma 23 is linear and continuous on H. Hence, it is selfadjoint and non-negative.*

Proof. We first prove that for every complex q_0 and every $\xi \in H$ we have

$$P(q_0\xi) = q_0 P\xi\,.$$

Let $q_n \to q_0$ be a sequence with *rational* real and imaginary parts. Then we have (Lemma 25 is used in the second line)

$$\lim P(q_n\xi) = \lim q_n P\xi = q_0 P\xi\,,$$
$$|P(q_n\xi) - P(q_0\xi)| = |P(q_n - q_0)\xi| = \sup_{|y|=1} \langle y, P(q_n - q_0)\xi\rangle \le M|q_n - q_0|\,|\xi|.$$

So, $q_0 P\xi = \lim q_n P\xi = \lim P(q_n\xi) = P(q_0\xi)$. This gives linearity of the operator P which, from Lemma 23 is everywhere defined and symmetric. Continuity follows immediately from (49). ∎

An obvious but important observation is the following one: the time 0 as initial time has no special role and we can repeat the previous arguments, for every initial time $\tau \ge 0$ and $y_0 \in H$. Hence we can define $P_\tau : H \to H$ such that

$$\langle y_0, P_\tau y_0\rangle = \inf \int_\tau^{+\infty} |u(s)|^2 \,\mathrm{d}s, \tag{50}$$

where the infimum is computed on the set

$$\mathcal{U}(y_0, \tau) = \{u \in L^2(\tau, +\infty) \;:\; y(t; \tau, y_0, u) \in L^2(\tau, +\infty)\}\,.$$

We have a family P_τ of linear operators, and $P_0 = P$ is the operator defined in (15). The observation is:

Lemma 27. *Let Assumption 1 hold and suppose that system (1) is null controllable. Then the operator P_τ does not depend on τ:*

$$P_\tau = P_0 = P\,.$$

Proof. We observe the following equality, which holds for $t > \tau$:

$$y(t; \tau, y_0, u) = y(t - \tau; 0, y_0, v)\,, \qquad v(t) = u(t + \tau),$$

and both $v(t)$ and $y(t; 0, y_0, v)$, $t \ge 0$, are square integrable on $(0, +\infty)$ if $u(t)$ and $y(t; \tau, y_0, u)$ are square integrable on $(\tau, +\infty)$. Hence, the infimum of the functional in (50) is $\langle y_0, P_\tau y_0\rangle = \langle y_0, P y_0\rangle$, i.e., $P_\tau = P$. ∎

Proof. [Proof of Theorem 17] We write

$$\langle y_0, P y_0\rangle \le \int_0^{+\infty} |u(s)|^2 \,\mathrm{d}s = \int_0^\tau |u(s)|^2 \,\mathrm{d}s + \int_\tau^{+\infty} |u(s)|^2 \,\mathrm{d}s\,.$$

We choose a control u which is smooth on $[0, \tau]$ so that $y(\cdot; y_0, u)$ is continuous (cf. Lemma 2) and we keep the restriction of u to $[0, \tau]$ fixed. The vector $y(\tau; y_0, u)$ being controllable to the rest, we can use (42) with initial condition τ and we have

$$\langle y(\tau; y_0, u), P_\tau y(\tau; y_0, u)\rangle = \inf \int_\tau^{+\infty} |u(s)|^2 \,\mathrm{d}s$$

(as usual, the infimum is computed on those square integrable controls which produces a square integrable solution, on $[0, +\infty)$). So, we have

$$\langle y_0, P y_0\rangle \le \int_0^\tau |u(s)|^2 \,\mathrm{d}s + \langle y(\tau; y_0, u), P_\tau y(\tau; y_0, u)\rangle\,.$$

Using the fact that $P_\tau = P$ is independent of τ, we see that the following inequality holds for every control u which is of class C^1 on $[0,\tau]$, every $y_0 \in H$ and $t \geq 0$:

$$\text{(LOI)} \qquad \langle Py(t;y_0,u), y(t;y_0,u)\rangle - \langle Py_0, y_0\rangle + \int_0^t |u(s)|^2 \,\mathrm{d}s \geq 0\,. \qquad (51)$$

Finally a standard approximation argument shows that (LOI) holds even if $u \in L^2_{loc}(0,+\infty;U)$. ∎

Appendix B. The basic setting for boundary control

Here we present known facts about boundary control which are explained in [46, Sections 2.9 and 2.10]. Useful references are [12, Chapter 2], [4, Chapter 3] and [23, Chapter 1]. However, it seems to us that detailed arguments are not completely presented in standard control references and so we write this appendix for the reader's convenience.

A warning is needed: terms and some settings change in different books. For example [16] uses the same term, adjoint, for Banach space and Hilbert space adjoints, while it is convenient for us to use different terms. More important, the dual spaces and the Banach space adjoints are defined in terms either of linear forms or sesquilinear forms. The use of sesquilinear forms as in [16, 46] is the most convenient for us.

We must introduce few notations. As before, $\langle\cdot,\cdot\rangle$ will be used to denote the inner product in Hilbert spaces (if needed, the spaces are specified with an index; no index is present for the inner product in H).

If V is a complex Banach space (possibly Hilbert), V' denotes its topological dual (the Banach space of the continuous *linear* functionals defined on V). Thus, if $\omega \in V'$ we can compute $\omega(v)$ for every $v \in V$. We shall use the notation

$$_{V'}(\omega, v)_V$$

in order to denote the *sesquilinear* pairing of V and V', i.e., $_{V'}(\omega,v)_V$ is antilinear in ω and linear in v.

To give an example, we note that the concrete spaces encountered in control theory are complexification of spaces of real functions; i.e., if V_R is a linear space over $\mathbb{R}$, the elements of the corresponding complexified space V have the form

$$v = f + ig\,, \qquad f\,,\; g \in V_R\,.$$

The space V is a linear space on $\mathbb{C}$ and it is simple to construct sesquilinear forms on V, using elements of V': let ω a complex valued linear functional on V, i.e., $\omega \in V'$. The associated sesquilinear form is

$$_{V'}(\omega, v)_V = _{V'}(\omega, f+ig)_V = \overline{\omega(f-ig)}\,.$$

We shall use both the Hilbert space adjoint and the dual of an operator in the sense of Banach spaces. The Hilbert space adjoint is defined for densely defined operators A by

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \qquad \forall x \in \operatorname{dom} A\,, \quad \forall y \in \operatorname{dom} A^*$$

(this equality implicitly defines $\operatorname{dom} A^*$ as the set of those $y \in H$ such that $x \to \langle Ax, y\rangle$ is continuous).

This implies in particular that

$$\rho(A^*) = \overline{\rho(A)}\,.$$

The operator A^* is closed and if A is (densely defined) closed then A^* has dense domain too.

The Banach space dual of an operator A: $\operatorname{dom} A \subset V \to W$ (here V and W are Banach spaces) will be denoted A'. It is a linear operator from W' to V'. It is (uniquely) defined for densely defined operators A and

$$\operatorname{dom} A' = \{\omega \in W' : \quad v \mapsto \ {}_{W'}(\omega, Av)_W \text{ is continuous. }\}.$$

By definition,

$${}_{V'}(A'\omega, v)_V = \ {}_{W'}(\omega, Av)_W \,.$$

Sesquilinearity of the pairing implies that

$$\rho(A') = \overline{\rho(A)}$$

(see [16, pg. 184]). Hence, the conjugate of multiplication by λ is multiplication by $\bar{\lambda}$ (if instead the conjugate is defined in terms of bilinear forms then the resolvent is not changed). Moreover, A' has dense domain if A has dense domain and it is closed, provided that V is reflexive, in particular if it is a Hilbert space.

If $W = V$ and if A is the infinitesimal generator of a C_0 semigroup on V then it might be that A' is not a generator on V'. It happens that A' is the infinitesimal generator of a C_0-semigroup on V' if V is reflexive, in particular if it is a Hilbert space. In this case $e^{A't} = \left(e^{At}\right)'$. As for the Hilbert space adjoint A^*, it generates e^{A^*t} (see [38, Section 1.10]).

With these notations and preliminary information, we can now give the details of the setting used in the analysis of boundary control systems.

B.1. **The operators A and $\mathcal{A} = (A^*)'$.** Let A be the generator of a strongly continuous semigroup e^{At} on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $| \cdot |$.

We shall identify its topological dual H' with H using the Riesz isomorphism, which we denote R: $H \to H'$, defined as:

$$(Rv)(h) = \ {}_{H'}(Rv, h)_H = \langle h, v \rangle.$$

In practice, R is not explicitly written, hidden behind the equality $H = H'$ but in this appendix the distinction is needed for clarity.

Using the Riesz map R: $H \mapsto H'$ and the definition of A', we see that $\operatorname{dom} A' = R\,(\operatorname{dom} A^*)$. In fact

$${}_{H'}(Rh, Ak)_H = \langle Ak, h \rangle$$

and the right hand side is a continuous function of k if and only if the same holds for the left side.

For every $h \in \operatorname{dom} A^*$ and every $k \in \operatorname{dom} A$ we have:

$${}_{H'}\left(A'Rh, k\right)_H = \ {}_{H'}\left(R\left(R^{-1}A'Rh\right), k\right)_H = \langle k, R^{-1}A'Rh \rangle \,.$$

The definition of A' is

$${}_{H'}\left(A'Rh, k\right)_H = \ {}_{H'}\left(Rh, Ak\right)_H = \langle k, A^*h \rangle \,.$$

Hence (see [12, Sect. II.7]) we have

$$A^* = R^{-1}A'R \,. \tag{52}$$

The same relation holds for the semigroups

$$e^{A^*t} = R^{-1}e^{A't}R \,.$$

In the sequel we denote by V the Hilbert space $\mathrm{dom}\, A^*$, with inner product $\langle h, v\rangle_* = \langle h, v\rangle + \langle A^*h, A^*v\rangle$, $h, v \in \mathrm{dom}\, A^*$. We have

$$\mathrm{dom}\, A^* = V \subset H \overset{R}{\simeq} H' \subset V' \tag{53}$$

with dense and continuous injections.

Let j be the injection of V into H, $j\, v = v \in H$, for $v \in V$. Then, the definition of $j\,' : H' \to V'$ is

$$\langle j\, v, h\rangle = {}_{H'}(Rh, j\, v)_H = {}_{V'}(j\,'Rh, v)_V$$

and this shows that $j\,'Rh$ is the restriction of Rh (acting on H) to the subspace $V \subset H$.

As $A^* \in \mathcal{L}(V, H)$ we have $(A^*)' : H' \to V'$ belongs to $\mathcal{L}(H', V')$.

We denote $(A^*)'$ by $\mathcal{A}$, so that $\mathrm{dom}\, \mathcal{A} = H'$ (or, as usually written when H and H' are identified, $\mathrm{dom}\, \mathcal{A} = H$). The crucial property used in control theory is expressed by stating that $\mathcal{A}$ *extends* A. The precise statement is:

Lemma 28. *If $x \in \mathrm{dom}\, A$ then we have:*

$$Ax = R^{-1}(j\,')^{-1}\mathcal{A}\, Rx\, .$$

Proof. Indeed, if $x \in \mathrm{dom}\, A$, $v \in V = \mathrm{dom}\, A^*$, then

$$\begin{aligned} {}_{V'}(\mathcal{A}\, Rx, v)_V = {}_{V'}((A^*)'\, Rx, v)_V &= {}_{H'}(Rx, A^*v)_H = \langle A^*v, x\rangle \\ &= \langle v, Ax\rangle = {}_{H'}(RAx, j\, v)_H = {}_{V'}(j\,'RAx, v)_V \end{aligned}$$

and so $\mathcal{A}R = j\,'RA$. When $j\,'$ and R are not explicitly written, as usual, we get $\mathcal{A}x = Ax$. ∎

The second property that we want to prove is that V' is an extrapolation space generated by A. This means that we can see V' as the completion of H when endowed with the norm $|(\lambda I - A)^{-1} \cdot|$, for any $\lambda \in \rho(A)$. In order to see this, we fix any $\lambda \in \rho(A)$ and we prove that $|\cdot|_{V'}$ restricted to H is equivalent to $|(\lambda I - A)^{-1} \cdot|$, i.e., we prove:

Lemma 29. *On H, the norms of V' (more precisely, $h \mapsto |j\,'Rh|_{V'}$) and the norm $|(\lambda I - A)^{-1} \cdot|$ are equivalent.*

Proof. Let I denote the identity in H and also on V.

Let $\lambda \in \rho(A)$ and $h \in H$. In order to compute $|j\,'Rh|_{V'}$, we proceed as follows (recall that $j\,'Rh$ is the restriction of Rh to V).

$$\begin{aligned} |j\,'Rh|_{V'} &= \sup_{|v|_V \le 1} |{}_{V'}(j\,'Rh, v)_V| = \sup_{|v|_V \le 1} |\langle v, h\rangle| \\ &= \sup_{|v|_V \le 1} |\langle \left(\bar\lambda I - A^*\right)^{-1} \left(\bar\lambda I - A^*\right) v, h\rangle| \\ &= \sup_{|v|_V \le 1} |\langle \left(\bar\lambda I - A^*\right) v, (\lambda I - A)^{-1} h\rangle| \le C \left|(\lambda I - A)^{-1} h\right|, \end{aligned}$$

with $C = \sup_{|v|_V \le 1} |(\bar\lambda I - A^*)v|$. On the other hand,

$$\left|(\lambda I - A)^{-1} h\right| \le C_0 |Rh|_{H'} = C_0 |(j\,')^{-1} j\,'Rh|_{H'} \le C_1 |j\,'Rh|_{V'}.$$

The proof is complete. ∎

B.2. **The parabolic case and fractional powers of** $(\omega I - A^*)$**.** Our goal here is to show that if A (and so A^*) *generates a holomorphic semigroup* e^{At} (respectively e^{A^*t}) and in addition

$$B \in \mathcal{L}\left(U, (\text{dom } (\omega - A^*)^\gamma)'\right), \tag{54}$$

for some $\gamma \in [0, 1)$ and $\omega \in \rho(A)$, then we have the crucial estimate (6), i.e.,

$$\|e^{\mathcal{A}t}B\|_{\mathcal{L}(U,H)} \le \frac{Me^{\omega_1 t}}{t^\gamma}, \quad t > 0.$$

First recall that, since A is holomorphic, there always exists $\omega \in \rho(A)$ such that, for any $\gamma \in (0, 1)$, $(\omega I - A^*)^\gamma$ is a well defined closed operator with domain $V_\gamma \subset H$ (cf. [38, Section 2.6]). Then note that $\mathcal{A}$ generates the dual semigroup $e^{\mathcal{A}t}$ which is holomorphic on V'.

Arguing as in (53), we have

$$V_\gamma \subset H \simeq H' \subset V_\gamma'$$

and we denote by $\mathcal{A}_\gamma$ the operator $[(\omega - A^*)^\gamma]' : H \to V_\gamma'$. Since $B \in \mathcal{L}(U, V_\gamma')$, we may consider $B : U \to V'$, since $V_\gamma' \subset V'$ with dense and continuous injections. Thus we also have $B \in \mathcal{L}(U, V')$.

Moreover, since $B \in \mathcal{L}(U, V_\gamma')$, we have $B' \in \mathcal{L}(V_\gamma, U')$ and so, for $t > 0$:

$$\|B'e^{A^*t}\|_{\mathcal{L}(H,U')} = \|B'(\omega I - A^*)^{-\gamma}\,(\omega I - A^*)^\gamma e^{A^*t}\|_{\mathcal{L}(H,U')}$$

$$\le \|B'(\omega I - A^*)^{-\gamma}\|_{\mathcal{L}(H,U')}\,\|(\omega I - A^*)^\gamma e^{A^*t}\|_{\mathcal{L}(H,H)} \le \frac{Me^{\omega_1 t}}{t^\gamma}, \ \ t > 0 \tag{55}$$

(in the last line we have used a well known estimate for holomorphic semigroups). Next *we compute the dual operator of* $B'e^{A^*t}$ *and show that it is* $R^{-1}e^{\mathcal{A}t}B$*,* usually written as $e^{\mathcal{A}t}B$ when H and H' are identified.

We have, for any $x \in H$, $u \in U$, $t > 0$,

$${}_{U'}(B'e^{A^*t}x, u)_U = {}_{V_\gamma}(e^{A^*t}x, Bu)_{V_\gamma'} = {}_V(e^{A^*t}x, Bu)_{V'},$$

since $Bu \in V_\gamma' \subset V'$ (recall that $V = \text{dom}\, A^*$) and $e^{A^*t}x \in V$, $t > 0$. It follows that

$${}_{U'}(B'e^{A^*t}x, u)_U = {}_{H'}(e^{\mathcal{A}t}Bu, x)_H = \langle x, R^{-1}e^{\mathcal{A}t}Bu\rangle\,.$$

and so the claim follows.

The previous assertion implies the identity

$$\|B'e^{A^*t}\|_{\mathcal{L}(H,U')} = \|R^{-1}e^{\mathcal{A}t}B\|_{\mathcal{L}(U,H)}, \ \ t > 0, \tag{56}$$

which together with (55) implies the estimate (6) which has been used in Subsection 1.1.

Acknowledgement. The authors would like to thank the referees for their useful remarks and suggestions which helped to improve the paper.

References

[1] G. Avalos and I. Lasiecka, Optimal blowup rates for the minimal energy null control of the strongly damped abstract wave equation, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), pp. 601-616.

[2] S. A. Avdonin and S. A. Ivanov, Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems. Cambridge University Press, New York, 1995.

[3] R. Bellman, K.L. Cooke, Differential-difference equations. Academic Press, New York, 1963.

[4] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, Representation and control of infinite-dimensional systems. Second edition, Systems & Control: Foundations & Applications, Birkhäuser, Boston, 2007.

[5] F. Bucci and I. Lasiecka, Singular estimates and Riccati theory for thermoelastic plate models with boundary thermal control, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 11 (2004), pp. 545-568.

[6] R. F. Curtain, H. J. Zwartz, An introduction to Infinite-Dimensional Linear Systems Theory, Text in Applied Mathematics 21, Springer Verlag, 1995.

[7] K. J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations. Springer-Verlag, 2000.

[8] H. O. Fattorini, D. L. Russell, Exact Controllability Theorems for Linear Parabolic Equations in One Space Dimension. Arch. Rational Mech. Anal., 43 (1971), pp. 272-292.

[9] A. Favini, L. Pandolfi, A quadratic regulator problem related to identification problems and singular systems, SIAM J. Control Optim., 47 (2008), pp. 768-790.

[10] E. Fernández-Cara, M. Gonzáles-Burgos, L. de Teresa, Boundary controllability of parabolic coupled equations, J. Funct. Analysis, 259 (2012) pp. 1720-1758.

[11] P. A. Fuhrmann, On weak and strong reachability and controllability of infinite dimensional linear systems, J. Optimization Theory and Appl., 9 (1972) pp. 77-89.

[12] S. Goldberg, Unbounded linear operators. Theory and applications, McGraw-Hill Book Company, New York, 1966.

[13] D. Greub, Linear Algebra, Die Grundlehren der Mathematischen Wissenschaften, Band 97, Springer-Verlag, Berlin, 1967.

[14] P. Halmos, Introduction to Hilbert spaces and the theory of spectral multiplicity, Chelsea Publishing Company, New York, 1957.

[15] J. K. Hale, S. M. Verduyn Lunel, Introduction to functional-differential equations, Springer-Verlag, New York, 1993.

[16] T. Kato, Perturbation Theory of Linear Operators, Springer-Verlag, Berlin, 1976.

[17] A. Ichikawa, Null controllability with a vanishing energy for discrete-time systems in Hilbert space, SIAM J. Control Optim., 46 (2007), pp. 683-693.

[18] A. Ichikawa, Null controllability with vanishing energy for discrete-time systems, Systems Control Lett., 57 (2008), pp. 34-38.

[19] S. Ivanov, Control norms for large control times. ESAIM Control Optim. Calc. Var. 4 (1999), pp. 405-418.

[20] I. Lasiecka and R. Triggiani, A Cosine operator approach to medelling $L_2(0,T;L_2(\Gamma))$-Dirichlet boundary terms, Appl. Math. Optim. 10, 7 (1981), pp. 35-83.

[21] I. Lasiecka and R. Triggiani, Regularity of hyperbolic equations under $L_2(0,T;L_2(\Gamma))$-Dirichlet boundary terms, Appl. Math. Optim., 10 (1983), pp. 275-286.

[22] I. Lasiecka R. Triggiani, Exact boundary controllability on $L_2(\Omega)\times H^{-1}(\Omega)$ of the wave equation with Dirichlet boundary control acting on a portion of the boundary $\partial\Omega$ and related problems, Appl Mat Optim., 18 (1988), pp. 241-277.

[23] I. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories. I. Abstract parabolic systems, Cambridge University Press, Cambridge, 2000.

[24] I. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories. II. Abstract hyperbolic-like systems over a finite time horizon, Encyclopedia of Mathematics and its Applications, 75, Cambridge University Press, Cambridge, 2000.

[25] I. Lasiecka, Optimal control problems and Riccati equations for systems with unbounded controls and partially analytic generators-applications to boundary and point control problems, in Functional analytic methods for evolution equations, M. Iannelli, R. Nagel and S. Piazzera, ed., Lecture Notes in Math., 1855, Springer, Berlin, 2004, pp. 313-369.

[26] W. Li, X. Zhang, Controllability of parabolic and hyperbolic equations: toward a unified theory, in Control theory of partial differential equations, Lect. Notes Pure Appl. Math., 242, Chapman & Hall/CRC, Boca Raton, FL, 2005, pp. 157174.

[27] J. L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systemes distribués. Tome 1, Recherches en Mathematiques Appliquees [Research in Applied Mathematics], 8, Masson, Paris, 1988.

[28] J. C. Louis, D. Wexler, The Hilbert space regulator problem and operator Riccati equation under stabilizability, Ann. Soc. Sci. Bruxelles, 105 (1991), pp. 137-165.

[29] V. Komornik, Exact controllability and stabilization. The multiplier method, RAM: Research in Applied Mathematics, Masson, Paris, John Wiley & Sons, Ltd., Chichester, 1994.

[30] V. Komornik and P. Loreti, Fourier Series in Control Theory, Springer Monographs in Mathematics, Springer-Verlag, New York, 2005.

[31] L. Miller, A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups, Discrete Continuous Dynamical Systems, 14 (2010), pp. 1465-1485.

[32] J. M. A. M. Van Neerven, Null controllability and the algebric Riccati equation in Banach spaces, SIAM J. Control Optimization, 43 (2005), pp. 1313-1327.

[33] L. Pandolfi, A frequency domain approach to the boundary control problem for parabolic equations, in Modelling and optimization of distributed parameter systems (Warsaw, 1995), Chapman& Hall, New York, 1996, pp. 149-158.

[34] A. W. Olbrot, L. Pandolfi, Null controllability of a class of functional-differential systems, Internat. J. Control, 47 (1988), pp. 193-208.

[35] L. Pandolfi, The Kalman-Popov-Yakubovich theorem: an overview and new results for hyperbolic control systems, Nonlinear Anal., 30 (1997), pp. 735-745.

[36] L. Pandolfi, The Kalman-Yakubovich-Popov theorem for stabilizable hyperbolic boundary control systems, Integral Equations Operator Theory, 34 (1999), pp. 478-493.

[37] L. Pandolfi, Dissipativity and the Lur'e problem for parabolic boundary control systems, SIAM J. Control Optim., 36 (1998), pp. 2061-2081.

[38] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[39] K. D. Phung, G. Wang, X. Zhang, On the existence of time optimal controls for linear evolution equations, Discrete Contin. Dyn. Syst. Ser. B, 8 (2007), no. 4, pp. 925-941.

[40] E. Priola, J. Zabczyk, Null controllability with vanishing energy, SIAM J. Control Optim., 42 (2003), pp. 1013-1032.

[41] E. Priola, J. Zabczyk, Liouvile Theorems for non-local operators, Journal of Functional Analysis, 216 (2004), pp. 455-490.

[42] S. Rolewicz, On universal time for the controllability of time-depending linear control systems, Studia Math., 59 (1976/77), pp. 133-138.

[43] L. Schwartz, Etude des sommes d'exponentielles, Hermann, Paris 1959.

[44] J. A. Soriano, Exact controllability of the generalized telegraph equation, Rev. Mat. Univ. Complut. Madrid 8 (1995), pp. 459-493.

[45] G. Tenenbaum, M. Tucsnak, On the null controllability of diffusion equations, ESAIM: COCV, DOI:10.1051/cocv/2010035, 2010.

[46] M. Tucsnak, G. Weiss, Observation and control for operator semigroups. Birkhäuser Verlag, Basel, 2009.

[47] J. Zabczyk, Mathematical Control Theory: an introduction, Birkhäuser, Boston (Second printing, with corrections), 1995.

[48] X. Zhang, A unified controllability/observability theory of some stochastic and deterministic partial differential equation, Proc. Internat. Congress of Mathematicians, Hydebarad, India, 2010.

[49] E. Zuazua, Controllability and observability of partial differential equations: some results and open problems, in Handbook of differential equations: evolutionary equations. Vol. III, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007, pp. 527-621.