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# Closed trajectories of the conformal arclength functional

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**Abstract.** The purpose of this report is to give a brief overview of some unpublished results about the geometry of closed critical curves of a conformally invariant functional for space curves.

## 1. Introduction

Conformal geometry of curves is a well studied subject in classical differential geometry [4, 15, 16, 17]. In more recent times, the topic has been considered in connection with the regularization of the Kepler problem [6, 9, 14], within the theory of integrable systems and in the topology of knots [3, 1, 2, 5, 8, 10, 11]. In a previous paper, published several years ago [12], I studied the variational problem defined by the *conformal density* of a space curve, improperly called *conformal arc-element*. In that paper I computed the critical curves in terms of elliptic integrals. This was not much surprising because the trajectories of the variational problem can be obtained as projections of the integral curves of a collective completely integrable contact Hamiltonian system, defined on a 13-dimensional momentum space. Recently, my work was taken up and its results have been generalized to curves immersed in Euclidean spaces of arbitrary dimension [13]. The problem that I will consider in this report is the existence and the explicit determination of the *closed trajectories* of the variational problem. A more complete and exhaustive analysis will be the subject of a forthcoming paper.

## 2. Conformal geometry of space curves and the period map

Consider an oriented, complete bi-regular curve  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ , parameterized by the Euclidean arc-length. A vertex is a point  $\gamma(s)$  such that  $\dot{\kappa}(s)^2 + \kappa^2(s)\tau(s)^2 = 0$ , where  $\kappa$  and  $\tau$  are the curvature and the torsion respectively. This notion is invariant by conformal transformations of  $\mathbb{R}^3$ . The *conformal arc element* (or more appropriately the *conformal density*) of  $\gamma$  is the 1-form  $\zeta = \sqrt[4]{\dot{\kappa}^2 + \kappa^2\tau^2} ds$ . Assuming that the curve is generic (i.e., bi-regular and without vertices), then the conformal density is nowhere vanishing and we can define the two conformal curvatures

$$\begin{aligned} k_1 &= r^5 (\kappa^2 \tau^3 + \kappa \dot{\kappa} \dot{\tau} + \tau(2\dot{\kappa}^2 - \kappa \ddot{\kappa})), \\ k_2 &= \frac{1}{2} (\dot{r}^2 - 2r\ddot{r} - r^2 \kappa^2). \end{aligned} \tag{1}$$

where  $r = (\dot{\kappa}^2 + \kappa^2\tau^2)^{-1/4}$ . The conformal density and the two curvatures determine the shape of the curve, up to conformal transformations. This means that if two curves  $\gamma$  and  $\tilde{\gamma}$  have the same invariants, then there is a conformal transformation of  $\mathbb{R}^3$  that sends one trajectory onto the other. The conformal symmetry group of a curve  $\gamma$  is the group  $G_\gamma$  made up of all conformal transformations which leave unchanged its trajectory. From now on we will assume that the curves are bi-regular and without vertices.

**Definiton.** The critical curves of the functional defined by the integral of the conformal density are called *conformal geodesics*. For simplicity, a closed conformal geodesic with non-constant conformal curvatures is referred to as a *conformal string*.

The critical curves with periodic curvature functions are characterized by the Euler-Lagrange equations [12]

$$dk_1 + (k_1^2 - a)(k_1^2 - b)\zeta = 0, \quad k_2 = -\frac{3}{2}k_1 + \frac{a+b}{a}, \quad (2)$$

where  $a$  and  $b$  are two real constants, the *natural parameters*, such that  $a > 0$ ,  $b \neq 0$  and  $a > b$ . An analysis of the momentum map shows that the natural parameters of a string belong to the *admissible region*

$$\Sigma = \{(a, b) : a > 1, a^{-1} < b < a\}.$$

Using the conservation laws it can be checked (see [12]) that, up to conformal transformations, the parametrization of a critical curve with natural parameters  $(a, b) \in \Sigma$  is given by

$$\begin{aligned} x(t) &= \frac{\sqrt{2}}{r(t)}\mu\sqrt{k(t)^2 - v^2} \cos(\Theta_2(t)), \\ y(t) &= \frac{\sqrt{2}}{r(t)}\mu\sqrt{k(t)^2 - v^2} \sin(\Theta_2(t)), \\ z(t) &= \frac{\sqrt{2}}{r(t)}v\sqrt{\mu^2 - k(t)^2} \sin(\Theta_1(t)), \end{aligned} \quad (3)$$

where  $k(t)$  is the Jacobi elliptic function  $\sqrt{a} \cdot dn(\sqrt{at}, a^{-1}(a-b))$ ,  $\mu$  and  $v$  are the real constants

$$\mu = \sqrt{\frac{a+b + \sqrt{4+(a-b)^2}}{2}}, \quad v = \sqrt{\frac{a+b - \sqrt{4+(a-b)^2}}{2}}, \quad (4)$$

$\Theta_1(t)$  and  $\Theta_2(t)$  are the incomplete elliptic integrals of the third kind

$$\Theta_1(t) = \int_0^t \frac{\mu}{\mu^2 - k(u)^2} du, \quad \Theta_2(t) = \int_0^t \frac{v}{v^2 - k(u)^2} du.$$

and  $r(t)$  is the radial function

$$r(t) = \sqrt{\mu^2 - v^2k(t)} + v\sqrt{\mu^2 - k(t)^2} \cos(\Theta_1(t)).$$

The curve defined by the formula (3) is said to be the *symmetrical configuration* of the critical curve with natural parameters  $(a, b)$ . We denote by  $\Pi(m, n)$ ,  $-1 < n, m < 1$ , the complete elliptic integral of the third kind

$$\Pi(m, n) = \int_0^{\pi/2} \frac{dt}{(1 - n \sin^2(t))\sqrt{1 - m \sin^2(t)}},$$

and, for every  $(a, b) \in \Sigma$  we set

$$\begin{aligned}\Phi_1(a, b) &= -\frac{\mu(a, b)}{\pi\sqrt{a}(a - \mu(a, b)^2)} \Pi\left(\frac{a - b}{a - \mu(a, b)^2}, \frac{a - b}{a}\right), \\ \Phi_2(a, b) &= \frac{v(a, b)}{\pi\sqrt{a}(a - v(a, b)^2)} \Pi\left(\frac{a - b}{a - v(a, b)^2}, \frac{a - b}{a}\right).\end{aligned}\tag{5}$$

where  $\mu(a, b)$  and  $v(a, b)$  are defined as in (4). The real-analytic function

$$\Phi = (a, b) \in \Sigma \rightarrow (\Phi_1(a, b), \Phi_2(a, b)) \in \mathbb{R}^2$$

is called the *period map* of the conformal arc-length functional. With a thorough analysis of the behavior of the period map it can be proved that  $\Phi$  is a real-analytic diffeomorphism of  $\Sigma$  onto the circular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 1/2 < x < 1/\sqrt{2}, y > 0, x^2 + y^2 < 1/2\}.$$

### 3. Closed trajectories and their phenomenological invariants

From the previous discussion it follows that a critical curve with natural parameters  $(a, b) \in \Sigma$  is closed if and only if  $\Phi(a, b)$  is a rational point of the domain  $\Omega$ . This shows that, up to conformal transformations, the conformal strings are in one to one correspondence with the pairs  $(q_1, q_2)$  of rational numbers belonging to  $\Omega$ . More precisely, for every rational point  $q$  of the domain  $\Omega$  there is a unique conformal string in its symmetrical configuration, with natural parameters  $(a, b) \in \Sigma$  such that  $q = \Phi(a, b)$ . The rational points of  $\Omega$  are called the *modules of the conformal strings*. To clarify the geometric meaning of the modules we must say a few words about the maximal tori of the conformal group. The conformal group of the Euclidean space is isomorphic to the pseudo-orthogonal group  $SO(4, 1)$ . Its maximal tori are conjugates to the subgroup  $SO(2) \times SO(2) \subset SO(4, 1)$ . The infinitesimal generators for the action of  $SO(2) \times SO(2)$  on  $\mathbb{R}^3$  are

$$\xi_1 = -y\partial_x + x\partial_y, \quad \xi_2 = -\frac{xz}{\sqrt{2}}\partial_x - \frac{yz}{\sqrt{2}}\partial_y + \frac{x^2 + y^2 - z^2 - 2}{2\sqrt{2}}\partial_z.$$

The first vector field generates the rotations around the  $Oz$ -axis, while the second one generates the toroidal rotations around the *Clifford circle*, that is the circle of the  $Oxy$ -plane, centered at the origin, with radius  $\sqrt{2}$ . The geometric meaning of the modules is clarified by the following Theorem.

**Theorem.** *Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  be a conformal string in its symmetrical configuration. Denote by  $q_1 = m_1/n_1$  and  $q_2 = m_2/n_2$  the modules of  $\gamma$  and assume that  $\gcd(m_1, n_1) = \gcd(m_2, n_2) = 1$ . Let  $n$  be the least common multiple of  $n_1$  and  $n_2$ . Then, the conformal symmetry group of  $\gamma$  is the cyclic subgroup  $G_\gamma \subset SO(2) \times SO(2)$  of order  $n$  generated by the composition of a rotation  $R_1(2\pi q_1)$  of an angle  $2\pi q_1$  around the  $Oz$ -axis and a toroidal rotation  $R_2(2\pi q_2)$  of an angle  $2\pi q_2$  around the Clifford circle. Moreover, if we denote by  $h_1$  and  $h_2$  the coprime integers  $n/n_1$  and  $n/n_2$ , then  $m_1 h_1$  is the linking number of  $\gamma$  with the Clifford circle and  $m_2 h_2$  is the linking number of  $\gamma$  with the  $Oz$ -axis.*

This result implies that, up to the action of the conformal group, the conformal strings are uniquely determined by three numerical invariants: the order of the conformal symmetry group and the linking numbers with the  $Oz$ -axis and the Clifford circle.

The inverse of the period map can be calculated by numerical methods. Therefore, once that we know the three phenomenological invariants (order of symmetry and linking numbers), the

symmetrical configuration of the string can be determined explicitly. The numerical experiments suggest that the strings are simple curves. However, we do not have a mathematically rigorous proof of this fact. Another interesting problem is to find an asymptotic estimate of the cardinality  $\rho(n)$  of the set of the equivalence classes of conformal strings with order of symmetry  $n$ . Our expectation is that the growth of  $\rho(n)$  is quadratic.

#### 4. Prospectives

It is likely that the techniques and the results obtained so far can be extended to time-like curves in the conformal completion of the Minkowski space (and more generally, in any conformally flat Lorentzian 4-manifold). In this framework, it is important to analyze the physical meaning of the scattering data of the critical curves with non-periodic conformal curvatures.

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