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Model Predictive Control of stochastic LPV Systems via Random Convex Programs

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***Abstract*— This paper considers the problem of stabilization of stochastic Linear Parameter Varying (LPV) discrete time systems in the presence of convex state and input constraints. By using a randomization approach, a convex finite horizon optimal control problem is derived, even when the dependence of the system's matrices on the time-varying parameters is nonlinear. This convex problem can be solved efficiently, and its solution is a-priori guaranteed to be probabilistically robust, up to a user-defined probability level p. Then, a novel receding horizon control strategy that involves, at each time step, the solution of a finite-horizon scenario-based control problem, is proposed. It is shown that the resulting closed loop scheme drives the state to a terminal set in finite time, either deterministically, or with probability no less than p. The features of the approach are shown through a numerical example.**

I. INTRODUCTION

In the last decade, several approaches have been proposed for the design of Model Predictive Control (MPC) laws for Linear Parameter Varying (LPV) systems, see, e.g., [1], [2], [3], [4], [5], [6], [7]. The existing techniques have the following common features: they are deterministic algorithms, in the sense that for given state value x and parameter value θ they provide always the same optimal control sequence; they guarantee robust stability and satisfaction of constraints; finally they assume convexity of the sets Σ containing the time-varying system matrices $A(\theta)$, $B(\theta)$ and affine dependence of the matrices on the parameter θ. Some approaches are also able to reduce conservativeness when a bound on the rate of variation of the parameters is available, see, e.g., [1], [5].

However, there might well be control problems in which Σ is not convex, and $A(\theta)$, $B(\theta)$ do not depend affinely on θ. In these cases, the existing approaches cannot be applied directly (they may possibly be applied indirectly, by first overbounding Σ with its convex hull, at the cost of potentially introducing conservatism). In order to cope with this issue, we recently proposed [8] a novel approach for the design of MPC laws for LPV systems. Following an idea common to stochastic MPC techniques (see, e.g., [9], [10]), we assume that the time-varying parameters θ have known stochastic nature, and exploit this knowledge in the control design. The characterization of θ is quite general, since not only bounds on its rate of variation, but also complex nonlinear models of its time-evolution can be accounted for. Yet, the problem to be solved at each time step is still convex and of manageable size, and constraint satisfaction and convergence of the state to a terminal set are still achieved, with at least a user-defined probability p. The key point to achieve these features is a shift of paradigm, from a *deterministic* algorithm to a *randomized* one, i.e. an algorithm that relies on some random choices. In particular, we rely on the solution of a scenario Finite Horizon Optimal Control Problem (FHOCP), in which we do not consider all possible outcomes of parameter values, but only a finite number M of randomly chosen instances of them, named the "scenarios". By exploiting recent results in Random Convex Programming (RCP) [11], [12], [13], [14], we provide a precise guideline on how to choose M in order to have the guarantee that the probability of success is indeed at least p. Moreover, we propose a novel receding horizon implementation of the scenario FHOCP, named MPCR (MPC via Random convex programs), and prove its constraint satisfaction and convergence properties. Randomized approaches for MPC have been already presented in the literature (see, e.g., [15], [16]), however, they either result to be very computationally demanding and can not handle in a straightforward way the presence of state constraints, or do not give any theoretical guarantee on the actual probability of constraint satisfaction. This paper briefly resumes the main results of [8] and presents the application of the approach in a numerical example.

II. PROBLEM SETTING AND ASSUMPTIONS

Consider the following uncertain, discrete time LPV system:

$$x_{t+1} = A(\theta_t)x_t + B(\theta_t)u_t \tag{1}$$

where $t \in \mathbb{Z}$ is the discrete time variable, $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, $\theta_t \in \Theta_t \subseteq \mathbb{R}^g$ is the vector of uncertain parameters, and $A(\theta)$, $B(\theta)$ are matrices of suitable dimensions. The (generally time varying) sets Θ_t, containing the values of parameter θ_t at time t, are subsets of a time invariant set $\overline{\Theta}$. Let us consider the following assumptions.

Assumption 1: (Model set) The set $\Sigma \doteq \{A(\theta), B(\theta) : \theta \in \overline{\Theta}\}$ is bounded.

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Assumption 2: (Time varying parameters) We assume that the parameter θ_t is measured at each time step t. Also, $\{\theta_t\}_{t=\dots,-1,0,1,\dots}$ is assumed to be a strict-sense stationary stochastic process and, for any time instant τ, we denote with $\mathbb{P}_\tau$ the conditional distribution of the forward sequence $\delta_\tau = (\theta_{\tau+1|\tau}, \dots, \theta_{\tau+N-1|\tau})$, given the past sequence $\mathcal{P}(\tau) \doteq \{\theta_t\}_{t\leq\tau}$, where N is some given integer, and we let Δ_τ be the support set of $\mathbb{P}_\tau$, that is, the set containing the conditional values of δ_τ, given $\mathcal{P}(\tau)$. We assume further that it is possible to sample values of δ_τ according to $\mathbb{P}_\tau$.

Note that we make no specific assumptions on $\mathbb{P}_t$ and on the support sets Δ_t, which may be unbounded and of any form, as long as Assumption 1 holds. The probability measure $\mathbb{P}_t$ itself can be also not known explicitly, as long as there is some mechanism to obtain samples of δ_t.

The control problem is to regulate the system state to the origin, subject to the following (possibly time varying) input and output constraints:

$$x_t \in \mathbb{X}(\theta_t),\ u_t \in \mathbb{U}(\theta_t),\ \forall t \tag{2}$$

We assume the constraint sets to be convex in x and u:

Assumption 3: (Convexity of the constraint sets) For any $\theta_t \in \Theta_t$ and any t, $\mathbb{X}(\theta_t) \subset \mathbb{R}^n$ and $\mathbb{U}(\theta_t) \subset \mathbb{R}^m$ are convex; they contain the origin in their interiors and they are representable by:

$$\begin{array}{rcl} \mathbb{X}(\theta_t) & = & \{x \in \mathbb{R}^n : f_X(x,\theta_t) \preceq 0\} \\ \mathbb{U}(\theta_t) & = & \{u \in \mathbb{R}^m : f_U(u,\theta_t) \preceq 0\}, \end{array} \tag{3}$$

where $\preceq$ denotes element-wise inequalities, each entry of the functions $f_X : \mathbb{R}^n \times \Theta_t \to \mathbb{R}^r$, $f_U : \mathbb{R}^m \times \Theta_t \to \mathbb{R}^q$ is convex in x and u, respectively, and r, q are suitable integers.

We finally assume that there exists a convex positively invariant terminal set $\mathbb{X}_f$ and an associated affine state-feedback control law $u = K(\theta_t)\,x$, possibly depending on the parameter θ_t, that renders the origin of (1) robustly asymptotically stable, while robustly satisfying input and state constraints:

Assumption 4: (Terminal set and terminal control law) A set $\mathbb{X}_f$, containing the origin in its interior, and a linear state feedback terminal control law $u = K_f(\theta_t)\,x$, $K_f \in \mathbb{R}^{m\times n}$, exist for system (1), such that:

$$\mathbb{X}_f \quad = \quad \{x : f_{X_f}(x) \preceq 0\};$$

where $f_{X_f} : \mathbb{R}^n \to \mathbb{R}^l$ has convex components and l is a suitable integer, and:

$$\begin{gathered} \forall\theta_t \in \Theta_t,\ \forall x_t \in \mathbb{X}_f,\ \forall t, \\ A(\theta_t)x_t + B(\theta_t)K_f(\theta_t)\,x_t \in \mathbb{X}_f; \\ f_X(x_t,\theta_t) \preceq 0,\ f_U(K_f(\theta_t)\,x_t,\theta_t) \preceq 0. \end{gathered}$$

The origin of the closed loop system with the feedback law $u = K_f(\theta_t)$ is asymptotically stable.

III. MPC for LPV systems via Random Convex Programs

Let $N \in \mathbb{N}$ be a finite control horizon, chosen by the control designer, let $t \geq 0$ be the current time instant and let x_t be the system state observed at time t. We consider the predicted evolution of (1) for N steps forward, under a control law determined at the current time t:

$$u_{j|t} \doteq K_f(\theta_{j|t})x_{j|t} + v_{j|t},\ j = 0,\dots,N-1, \tag{4}$$

where $u_{j|t}$ is the predicted input at time $t+j$ computed at time t, $x_{0|t} = x_t$, $\theta_{0|t} = \theta_t$ and, for $j = 1,\dots,N$,

$$x_{j|t} = A_{cl}(\theta_{j|t})x_{j-1|t} + B(\theta_{j|t})v_{j-1|t}, \tag{5}$$

and $A_{cl}(\theta_{j|t}) = A(\theta_{j|t}) + B(\theta_{j|t})K_f(\theta_{j|t})$. Here, $v_{j|t}$, $j = 0,\dots,N-1$, are control corrections at time steps $t+j$, computed at time t. Closed-loop prediction of state and input trajectories is quite common in the context of robust MPC and MPC for LPV systems, since it allows significant improvements in feasibility, see e.g. [1], [17].
The recursion (5) implies

$$x_{t+j|t} = A_{cl}^j(\theta_{j|t})x_t + \Phi_j(\theta_{j|t})\mathcal{V}_t, \tag{6}$$

where

$$\begin{array}{rcl} \Phi_j(\theta) & = & [A_{cl}^{j-1}(\theta)B(\theta)\ \cdots\ A_{cl}(\theta)B(\theta)\ B(\theta)\ 0\ \cdots\ 0], \\ \mathcal{V}_t & = & [v_{0|t}^\top\ \cdots\ v_{N-1|t}^\top]^\top \in \mathbb{R}^{Nm}, \end{array}$$

and $^\top$ denotes the matrix transpose operation. The predicted state and input trajectories, for a given initial state x_t and sequence of control correction $\mathcal{V}_t$, are random, since they depend on the sequence δ_t of random variables $\theta_{j|t}$, $j = 1,\dots,N-1$. As a consequence of Assumption 2, we have that the random quantity δ_t belongs to the set Δ_t, and events related to δ_t are measured by $\mathbb{P}_t$. Finally, let us define the following (stochastic) cost function:

$$J(x_t,\delta_t;\mathcal{V}_t) \doteq \sum_{j=0}^{N-1} x_{t+j|t}^\top Q x_{t+j|t} + u_{t+j|t}^\top R u_{t+j|t} \tag{7}$$

where $Q = Q^\top \succ 0$, $R = R^\top \succ 0$ are weighting matrices chosen by the control designer.

Without additional assumptions, like linearity of the time-varying matrices w.r.t. to θ_t and convexity Θ_t, it is not possible in general to enforce robust constraint satisfaction, as done e.g. in [4], [5]. In our approach we deal with this issue by considering a discrete set of predicted state and input trajectories, obtained for a number M of randomly extracted *scenarios* of δ_t at time t, i.e. $\delta_t^{(1)},\dots,\delta_t^{(M)}$. Each scenario has the probability distribution $\mathbb{P}_t$ according to Assumption 2, and the quantity $\omega_t \doteq (\delta_t^{(1)},\dots,\delta_t^{(M)})$ is named the "multisample" of scenario extractions at time t. The probability distribution of $\omega_t \in \Delta_t^M$ is given by $\mathbb{P}_t^M$. Based on the random scenarios, we obtain M different state and input predictions from (6), namely, for $i = 1, \ldots, M$,

$$\begin{array}{rcl} x^{(i)}_{0|t} & = & x_t \\ x^{(i)}_{j|t} & = & A^j_{\rm cl}(\theta^{(i)}_{j|t})x_t + \Phi_j(\theta^{(i)}_{j|t})\mathcal{V}_t, \quad j = 1, \ldots, N, \\ u^{(i)}_{j|t} & = & K_f(\theta^{(i)}_{j|t})x^{(i)}_{j|t} + v_{j|t}, \quad j = 0, \ldots, N-1, \end{array} \tag{8}$$

where the sequence $\left\{\theta^{(i)}_{j|t}\right\}_{j=0}^{N-1} = \delta^{(i)}_t$. In the scenario optimization approach, we will minimize the worst case cost with respect to the M values of δ_t in the multisample, subject to the corresponding state and input constraints. In order to guarantee feasibility of the scenario optimization problem, we transform the hard constraints of Assumption 3 into soft ones, by introducing a slack variable $q_t \in \mathbb{R}$, $q_t \geq 0$. Then, the scenario-based FHOCP is defined as follows:

$$\mathcal{P}(x_t, \omega_t) : \min_{\mathcal{V}_t, z_t, q_t} \; z_t + \alpha q_t \tag{9a}$$

subject to

$$J(x_t, \delta^{(i)}_t; \mathcal{V}_t) \leq z_t; \;\; i = 1, \ldots, M \tag{9b}$$

$$f_X(x^{(i)}_{j|t}, \theta^{(i)}_{j|t}) - \mathbf{1}q_t \preceq 0; \;\; j = 1, \ldots, N-1, \; i = 1, \ldots, M \tag{9c}$$

$$f_U(u^{(i)}_{j|t}, \theta^{(i)}_{j|t}) - \mathbf{1}q_t \preceq 0; \;\; j = 0, \ldots, N-1, \; i = 1, \ldots, M \tag{9d}$$

$$f_{X_f}(x^{(i)}_{t+N|t}) - \mathbf{1}q_t \preceq 0; \;\; i = 1, \ldots, M. \tag{9e}$$

$$q_t \geq 0 \tag{9f}$$

In (9a), the weighting scalar $\alpha > 0$ is chosen by the control designer, and $\mathbf{1}$ denotes a column vector of appropriate length, containing all ones. We denote with $\mathcal{V}^*_t(x_t, \omega_t) = \{v^*_{0|t}, \ldots, v^*_{N-1|t}\}$, $z^*_t(x_t, \omega_t)$ and $q^*_t(x_t, \omega_t)$ an optimal solution to problem (9).

We note that, once the multisample ω_t has been extracted, all the constraints (9b)-(9e) are convex in the decision variables, hence the scenario FHOCP is a convex optimization problem, which can be solved efficiently also with a large number M of samples, even when the system's matrices and the constraints are not convex w.r.t. to the time varying parameters θ_t. This is the main advantage of using the scenario approach, since it allows to treat, in a straightforward way, problems where the assumption of convexity of the model set is not met. At the same time, the scenario approach still yields guarantees, in a probabilistic sense, on constraint satisfaction and convergence to the terminal set. Before introducing these properties, let us re-write problem $\mathcal{P}(x_t, \omega_t)$ in a more compact form. By collecting the optimization variables $(\mathcal{V}_t, z_t, q_t)$ in vector $s_t \in \mathbb{R}^{mN+2}$, the cost can be expressed as $z_t + \alpha q_t = c^\top s_t$, where $c = [0, \ldots, 0, 1, \alpha]^\top$. Moreover, the constraints (9b)-(9f) can be expressed compactly as $h(s_t, x_t, \delta^{(i)}_t) \leq 0$, for all $i = i, \ldots, M$, where $h : \mathbb{R}^{mN+2} \times \mathbb{R}^n \times \Delta_t \to \mathbb{R}$ is defined as:

$$h(s_t, x_t, \delta^{(i)}_t) \doteq \max\left\{ \max_{j=0,\ldots,N-1} \left\{ J(x_t, \delta^{(i)}_t; \mathcal{V}_t) - z_t, f_X(x^{(i)}_{j|t}, \theta^{(i)}_{j|t}) - \mathbf{1}q_t, f_U(u^{(i)}_{j|t}, \theta^{(i)}_{j|t}) - \mathbf{1}q_t \right\}, f_{X_f}(x^{(i)}_{t+N|t}) - \mathbf{1}q_t, -q_t, \right\}.$$

Notice that $h(s_t, x_t, \delta^{(i)}_t)$ is convex in both s_t and x_t, since it is the point-wise maximum of convex functions. The scenario FHOCP can hence be rewritten as

$$\mathcal{P}(x_t, \omega_t) : \quad \min_{s_t} c^\top s_t \quad \text{subject to:} \quad h(s_t, x_t, \delta^{(i)}_t) \leq 0, \; i = 1, \ldots, M. \tag{10}$$

We denote with $s^*_t(x_t, \omega_t) = (\mathcal{V}^*_t, z^*_t, q^*_t)$ an optimal solution of $\mathcal{P}(x_t, \omega_t)$. Notice that, due to the way it has been defined, problem $\mathcal{P}(x_t, \omega_t)$ is *always* feasible. We further assume that this problem always attains a unique optimal solution.

Since the scenario FHOCP accounts only for a finite number M of values of $\delta^{(i)}_t$, $i = 1, \ldots, M$, a crucial aspect to be considered is the probability with which the solution s^*_t is able to satisfy constraints also for another, previously unseen scenario $\delta_t \in \Delta_t$. This aspect is formalized by the notion of *reliability* Γ of the scenario-FHOCP:

$$\Gamma \doteq \mathbb{P}_t\{\delta_t : h(s^*_t, x_t, \delta_t) \leq 0\}, \tag{11}$$

where we notice that h is now evaluated at the optimal scenario solution s^*_t, and the state and input trajectories that enter the definition of h are the "actual," uncertain, ones, obtained at a random δ_t.

In order to provide an explicit link between Γ and the number of scenarios M considered in (9), we exploit the fact that problem $\mathcal{P}(x_t, \omega_t)$ belongs to the class of so-called Random Convex Programs (RCP) (see e.g. [11], [12], [14]) and, in particular, the results of [11], [14] apply to our context. Denote with $d = mN + 2$ the number of decision variables in problem $\mathcal{P}(x_t, \omega_t)$, let $p \in (0, 1)$ be a given desired reliability level, let $\beta \in (0, 1)$ be a given small probability level (say, $\beta = 10^{-9}$), and let M be an integer such that

$$\Phi(p, d, M) \leq \beta, \tag{12}$$

with $\Phi(p, d, M) \doteq \sum_{j=0}^{d-1} \binom{M}{j} (1-p)^j p^{M-j}$. Then, it holds that (see [11], [14])

$$\mathbb{P}^M_t \{\omega_t : \Gamma(\omega_t) \geq p\} \geq 1 - \beta. \tag{13}$$

The practical importance of the result (13) stems from the fact that the number M of scenarios necessary to fulfill condition (12) grows at most logarithmically with β^{-1} (see e.g. [11]). Tighter values of M for given β and p can be obtained by inverting numerically (12). Hence, the parameter β may be fixed by the designer to a very low level, say $\beta = 10^{-9}$, and still the number M of scenarios necessary to guarantee (13) remains manageable, as we show in the example of Section IV. With such a small value of β, a "certainty equivalence" principle can be adopted, by which, to all practical engineering purposes, the event $\{\Gamma(\omega_t) \geq p\}$ in (13) is the "certain" event. In other words, the possibility that $\{\Gamma(\omega_t) \geq p\}$ is not satisfied by the scenario problem is so remote that, before having any concern about it, the designer should better verify the validity of many other assumptions and approximations in the model. We will adopt this certainty equivalence principle henceforth in this

paper, essentially "eliminating" from consideration the outer probability level in (13), since with practical certainty (the expression "with practical certainty" shall be used in the rest of this note as a synonym of "with probability larger than $1-\beta$," where $\beta > 0$ is some extremely small value) the inequality $\{\Gamma(\omega_t) \geq p\}$ holds true. This simplifies greatly the practical application of scenario techniques, and makes the whole approach more clear and understandable by both theoreticians and control practitioners.
On the basis of (13), we can state a first result on the scenario FHOCP.

Proposition 3.1: (Finite horizon robustness) Given the state x_t of system (1) at time t, a desired reliability level $p \in (0,\, 1)$, and a very small $\beta \in (0, 1)$, let the number M of scenarios in problem $\mathcal{P}(x_t, \omega_t)$ be chosen so to satisfy (12), and let $(\mathcal{V}_t^*, z_t^*, q_t^*)$ be the solution of the scenario problem $\mathcal{P}(x_t, \omega_t)$. Then, with practical certainty (i.e. with probability larger than $1-\beta$) it holds that the computed control sequence $\mathcal{V}_t^*$:
a) steers the state of system (1) to the terminal set $\mathbb{X}_f$ in N steps, with probability at least p and constraint violation q_t^*, i.e.: $\mathbb{P}_t\{\delta_t : \; f_{X_f}(x_{t+N}, \delta_t) - \mathbf{1}q_t^* \preceq 0\} \geq p$;
b) Satisfies all state constraints with probability at least p and constraint violation q_t^*, i.e.: $\mathbb{P}_t\{\delta_t : \; f_X(x_{t+j}, \delta_t) - \mathbf{1}q_t^* \preceq 0, \, \forall j \in [1, N]\} \geq p$;
c) Satisfies all input constraints with probability at least p and constraint violation q_t^*, i.e.: $\mathbb{P}_t\{\delta_t : \; f_U(u_{t+j}^*, \delta_t) - \mathbf{1}q_t^* \preceq 0, \, \forall j \in [0, N-1]\} \geq p$.

The proof of this result follows immediately from eq. (13), which states that, with practical certainty, the optimal solution s_t^* of the scenario problem satisfies $h(s_t^*, x_t, \delta_t) \leq 0$ with probability at least p, which indeed implies that points **a)**-**c)** in the proposition hold.

Proposition 3.1 provides a sufficient condition on M, such that the reliability of the solution s_t^* is at least equal to a desired value p. This result can be regarded to as an "open-loop", or finite horizon, approach, since the corrective actions $v_{j|t}^*$ are computed at time instant t and then applied forward in time, without observing the actual evolution of the system from t onwards. In the next section we present the main contribution of this paper, that is the description and convergence proof of a new receding-horizon algorithm that embeds the scenario problem and allows to take advantage of the measure of the state x_t and of the parameter θ_t, at each time step.

In the following, the notation is set as follows: "$*$" variables, such as $z_t^*, q_t^*, \mathcal{V}_t^* = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\}$, denote the optimal solution of the scenario optimization problem $\mathcal{P}(x_t, \omega_t)$ at time t, given x_t; "$\sim$" variables, $\tilde{z}_t, \tilde{q}_t, \tilde{\mathcal{V}}_t$, denote, respectively, two scalar values and a sequence of N vectors of dimension m, as defined in the algorithm below; finally plain variables, $z_t, q_t, \mathcal{V}_t$, denote the running values of the variables z, q and of the sequence $\mathcal{V} = \{v_{0|t}, \ldots, v_{N-1|t}\}$ in the algorithm. The first entry in $\mathcal{V}_t$, namely $v_{0|t}$, is the actual control correction that is applied to the system (1) at time t.

MPCR Algorithm
(Initialization) Choose a desired reliability level $p \in (0, 1)$ and "certainty equivalence" level $\beta \in (0, 1)$ (say, $\beta = 10^{-9}$, or $\beta = 10^{-12}$). Let M be an integer satisfying (12). Choose $\varepsilon \in (0, 1]$ (see Remark 3.1 below for the meaning of ε and for guidelines on its choice). Given an initial state x_0, extract ω_0 according to $\mathbb{P}_0^M$, solve problem $\mathcal{P}_M(x_0, \omega_0)$ and obtain the optimal control sequence $\mathcal{V}_0^* = \{v_{0|0}^*, v_{1|0}^* \ldots, v_{N-1|0}^*\}$, and the optimal objective z_0^* and constraint violation q_0^*. Set $z_0 = z_0^*, q_0 = q_0^*$, $\mathcal{V}_0 = \mathcal{V}_0^*$, and apply to the system the control action $u_0 = K_f x_0 + v_{0|0}$.
(1) Let $t := t + 1$, observe x_t, and set

$$\begin{aligned}\tilde{\mathcal{V}}_t &= \{v_{1|t-1}, \ldots, v_{N-1|t-1}, 0\} \\ \tilde{z}_t &= \max\left(0, z_{t-1} - \varepsilon x_{t-1}^\top Q x_{t-1}\right) \\ \tilde{q}_t &= q_{t-1}\end{aligned}$$

(2) Extract the multisample ω_t according to $\mathbb{P}_t^M$, and solve problem $\mathcal{P}_M(x_t, \omega_t)$. Let $(\mathcal{V}_t^*, z_t^*, q_t^*)$ be the obtained optimal solution.
(3) Evaluate the following collectively exhaustive and mutually exclusive cases:
(3.a) If $z_t^* > \tilde{z}_t$ and $\tilde{z}_t < x_t^\top Q x_t$, then set $\mathcal{V}_t = \tilde{\mathcal{V}}_t$; $\quad z_t = 0$; $\quad q_t = \tilde{q}_t$;
(3.b) If $z_t^* > \tilde{z}_t$ and $\tilde{z}_t \geq x_t^\top Q x_t$, then set $\mathcal{V}_t = \tilde{\mathcal{V}}_t$; $\quad z_t = \tilde{z}_t$; $\quad q_t = \tilde{q}_t$;
(3.c) If $z_t^* \leq \tilde{z}_t$, then set $\mathcal{V}_t = \mathcal{V}_t^*$; $\quad z_t = z_t^*$; $\quad q_t = q_t^*$;
(4) Apply the control input $u_t = K_f\, x_t + v_{0|t}$, then go to 1).

Remark 3.1: The inequality $z_t^* \leq \tilde{z}_t$, checked at step (3) of the MPCR Algorithm, can be interpreted as a verification of a required minimum improvement, in terms of worst-case cost, achieved by the newly computed optimal solution $(\mathcal{V}_t^*, z_t^*, q_t^*)$ of the scenario problem at time step t, with respect to the previous step. The user-defined parameter $\varepsilon \in (0, 1]$ influences such a requirement: the closer the value of ε is set to 0, the more likely it is that case $z_t^* \leq \tilde{z}_t$ is met, so that the MPCR algorithm relies, at each time step, on the newly computed optimal solution. Vice-versa, the closer is the value of ε to 1, the more likely it is that the complementary condition $z_t^* > \tilde{z}_t$ is detected, so that the MPCR algorithm employs the previously computed solution.

The next result is concerned with the guaranteed properties of the closed loop system obtained by applying Algorithm 3.1. We note that, by virtue of Assumption 4, under the terminal control law $u = K_f\, x$ the origin of system (1) is robustly asymptotically stable with region of attraction equal to $\mathbb{X}_f$, and constraints are robustly satisfied for all $x \in \mathbb{X}_f$. Therefore, only the convergence of the state trajectories to $\mathbb{X}_f$ and the satisfaction of constraints for $x \notin \mathbb{X}_f$ are of interest here.

Theorem 3.1: (Properties of Scenario MPC) Let Assumptions 2-4 be satisfied and let $p \in (0,1)$ be a chosen reliability level. Let $v_{0|t}$, $t = 0,1,\ldots$ denote the sequence of control actions produced by the MPCR Algorithm, and consider the closed loop system obtained by applying to (1) the control law $u_t = K_f x_t + v_{0|t}$. Let $x_0 \notin \mathbb{X}_f$. Then:
(a) With practical certainty, at all time steps $t = 0,1,\ldots$, the probability that the state and input constraints are satisfied with constraint violation q_t is at least p, that is $\mathbb{P}_t\{\delta_t : f_X(x_{t+1},\delta_t) - \mathbf{1}q_t \preceq 0 \cap f_U(u_t,\delta_t) - \mathbf{1}q_t \preceq 0\} \geq p, \quad t = 0,1,\ldots$
(b) The MPCR Algorithm either: **(i)** makes the state trajectory converge to the terminal set in finite time, i.e. $x_{t+\overline{N}} \in \mathbb{X}_f$, for some $\overline{N} < \infty$, or **(ii)** there exists a finite time t^* such that, with practical certainty, the forward control sequence $\{v_{0|t^*}, v_{0|t^*+1}, \ldots v_{0|t^*+N-1}\}$ drives the state of the closed-loop system to the terminal set at time $t^* + N - 1$, with probability at least p and constraint violation q_{t^*}.

Proof: See [8]. ■

IV. Numerical example

We consider the system (1) with

$$\begin{array}{rcl} A(\theta_t) & = & \begin{bmatrix} \theta_{t,1}\log(\theta_{t,2}) & \theta_{t,3} + e^{\theta_{t,4}}\sin(\theta_{t,5}) \\ 0 & \theta_{t,6} \end{bmatrix} \\ B(\theta) & = & \begin{bmatrix} \theta_{t,3} + e^{\theta_{t,4}}\cos(\theta_{t,5}) \\ 0.5\theta_{t,6} \end{bmatrix}, \end{array} \quad (14)$$

where $\theta_{t,i}$ is the $i-$th component of the parameter vector θ. We also consider the following parameter-dependent constraints on the input and state variables:

$$\begin{array}{l} \mathbb{X}(\theta_t) = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0.1\theta_{t,8} \\ 0.1\theta_{t,8} & 1 \\ -1 & -0.1\theta_{t,8} \\ -0.1\theta_{t,8} & -1 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}. \\ \mathbb{U}(\theta_t) = \{u \in \mathbb{R} : |u| \leq 2 + 0.1\theta_{t,7}\} \end{array} \quad (15)$$

The parameters $\theta_{t,i}$, $i = 1,\ldots,8$, are independent and uniformly distributed in the ranges [0.9,1.1], [2,3], [.95,1.3], [-10,-2], [0,2π], [0.8,1.2], [-1,1], [-1,1], respectively. We employ the following terminal control law and terminal set satisfying Assumption 4:

$$\begin{array}{rcl} K_f & = & [-0.8057 \ -0.8543] \\ \mathbb{X}_f & = & \{x \in \mathbb{R}^2 : x^T Q_f x \leq 1\} \\ Q_f & = & \begin{bmatrix} 1.5452 & 0.1865 \\ 0.1865 & 0.9792 \end{bmatrix}. \end{array}$$

We designed the MPCR law with $N = 10$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$ and $\alpha = 10^4$, $\beta = 10^{-9}$ and $\varepsilon = 0.01$. By setting a desired guaranteed probability $p = 0.95$ for the design, we obtained a value $M = 840$ from (12). It has to be noted that the value of M does not depend on the dimension of the state variable or of the uncertainty/disturbance variables; it only depends on the chosen probability levels p, β and on the number of decision variables in the scenario FHOCP, i.e. the number m of inputs multiplied by the control horizon N, plus the slack variables z and q. However, the number of constraints embedded in $h(s, x_t, \delta_t)$ depends linearly on n, m and N. In conclusion, for a fixed value of β, the growth of the overall number of constraints in the scenario problem is $\sim (n \cdot m^2 \cdot N^2/(1-p))$, i.e. quadratic in the horizon N for fixed value of p, regardless of the number of time-varying parameters. We carried out $N_{\text{trials}} = 50,000$ Monte Carlo simulations, starting from the state value $x_0 = [-1, -2.5]^\top$, which is outside the state constraints and whose corresponding uncorrected input, i.e. $K_f x_0 = 2.94$, is also outside the input constraint set. Indeed this initial condition is not feasible for the deterministic counterpart of the scenario problem, hence for some extractions of ω_0 the constraint violation q_0^* is not negligible. In the Monte Carlo simulations, the probability of success $\hat{p}$ has been estimated as $\hat{p} = \dfrac{N_{\text{trials}} - N_{\text{failures}}}{N_{\text{trials}}}$, where N_{failures} is the number of simulations in which some of the constraints were not satisfied. The results of the Monte Carlo simulations, with

TABLE I

Numerical example. Empirical probabilities $\hat{p}$ of constraint satisfaction and convergence to the terminal set, for two different values of p and $\beta = 10^{-9}$.

	p=0.5 (M=77)	p=0.95 (M=840)
Finite Horizon	0.891	0.961
MPCR algorithm	0.975	0.996

either the finite horizon solution or the MPCR algorithm, are reported in Table I for $p = 0.5$ and $p = 0.95$. It can be noted that all of the empirical probabilities satisfy the theoretical bounds, and confirm that the receding horizon implementation, by re-optimizing the control corrections at each time step, yields higher probabilities w.r.t. the finite horizon solution.

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