Symmetry, Integrability and Geometry: Methods and Applications SIGMA 8 (2012), 030, 20 pages

Motions of Curves in the Projective Plane Inducing the Kaup–Kupershmidt Hierarchy*

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Received February 08, 2012, in final form May 11, 2012; Published online May 24, 2012
http://dx.doi.org/10.3842/SIGMA.2012.030

Abstract. The equation of a motion of curves in the projective plane is deduced. Local flows are defined in terms of polynomial differential functions. A family of local flows inducing the Kaup–Kupershmidt hierarchy is constructed. The integration of the congruence curves is discussed. Local motions defined by the traveling wave cnoidal solutions of the fifth-order Kaup–Kupershmidt equation are described.

Key words: local motion of curves; integrable evolution equations; Kaup–Kupershmidt hierarchy; geometric variational problems; projective differential geometry

2010 Mathematics Subject Classification: 53A20; 53A55; 33E05; 35Q53; 37K10

1 Introduction

The interrelations between hierarchies of integrable non linear evolution equations and motions of curves have been widely investigated in the last decades, both in geometry and mathematical physics. In the seminal papers [12, 13, 29], Goldstein, Petrich and Nakayama, Segur, Wadati, showed that the mKdV hierarchy can be deduced from local motions of curves in the Euclidean plane. Later, this result was extended to other 2-dimensional geometries [4, 5, 6, 7, 8, 32, 33] or to higher-dimensional homogeneous spaces [1, 2, 15, 16, 19, 21, 27]. The invariant curve flows related to integrable hierarchies are induced by infinite-dimensional Hamiltonian systems defined by invariant functionals and geometric Poisson brackets on the space of differential invariants of parameterized curves [22, 23]. Another feature is the existence of finite-dimensional reductions leading to Liouville-integrable Hamiltonian systems. Typically, these reductions correspond to curves which evolve by congruences of the ambient space. They have both a variational and a Hamiltonian description: as extremals of a geometric variational problem defined by the conserved densities of the hierarchy, and as solutions of a finite-dimensional integrable contact Hamiltonian systems. Examples include local motions of curves in two-dimensional Riemannian space-forms [25], local motions of star-shaped curves in centro-affine geometry [26, 32] and local motions of null curves in 3-dimensional pseudo-Riemannian space forms [27]. In [4, 5, 6, 7], K.-S. Chu and C. Qu gave a rather complete account of the integrable hierarchies originated by local motions of curves in 2-dimensional Klein geometries. In particular, they showed that the fifth-order Kaup–Kupershmidt equation is induced by a local motion in centro-affine geometry and that modified versions of the fifth- and seventh-order Kaup–Kupershmidt equations can be related to local motions of curves in the projective plane. Our goal is to demonstrate the existence of a sequence of local motions of curves in projective plane inducing the entire Kaup–Kupershmidt hierarchy. We analyze the congruence curves of the flows and we investigate in more details the congruence curves associated to the cnoidal traveling wave solutions [41] of the

*This paper is a contribution to the Special Issue "Symmetries of Differential Equations: Frames, Invariants and Applications". The full collection is available at http://www.emis.de/journals/SIGMA/SDE2012.html fifth-order Kaup–Kupershmidt equation. In particular, we show that the critical curves of the projective invariant functional [3, 28] are congruence curves of the first flow of the hierarchy.

The material is organized into three sections. In the first section we collect basic facts about the Kaup–Kupershmidt hierarchy from the existing literature [10, 11, 18, 24, 34, 37, 39] and we discuss an alternative form of the equations of the hierarchy. We examine the cnoidal traveling wave solutions of the fifth-order Kaup–Kupershmidt equation [41] and we exhibit new traveling wave solutions in terms of Weierstrass elliptic functions. In the second section we recall the construction of the projective frame along a plane curve without inflection or sextatic points and we introduce the projective line element and the projective curvature [3, 14, 30, 40]. Subsequently, we use the Cartan's canonical frame [3, 28] to study the equations of a motion of curves in the projective plane. Consequently, we construct local motions in terms of differential polynomial functions and we deduce the existence of a sequence of local flows inducing the Kaup–Kupershmidt hierarchy. In the third section we focus on congruence motions of the flows. In particular, we explicitly implement the general process of integration to determine the motions of the critical curves of the projective invariant functional [3, 28]. The symbolical and numerical computations as well the graphics have been worked out with the software *Mathematica 8*. We adopt [20] as a reference for elliptic functions and integrals.

2 The Kaup–Kupershmidt hierarchy

2.1 Preliminaries and notations

Let $J(\mathbb{R},\mathbb{R})$ be the jet space of smooth functions $u:\mathbb{R}\to\mathbb{R}$, equipped with the usual coordinates $(s,u_{(0)},u_{(1)},\dots,u_{(h)},\dots)$. The prolongation of a smooth function u is denoted by $j(u)$. Similarly, if $u(s,t)$ is a function of the variables s and t, its partial prolongation with respect to the s-variable will be denoted by $j_s(u)$. A map $\mathfrak{p}: J(\mathbb{R},\mathbb{R})\to\mathbb{R}$ is said to be a *polynomial differential function* if

$$\mathfrak{p}(\mathbf{u}) = P(u_{(0)},u_{(1)},\dots,u_{(h)}) \qquad \forall\,\mathbf{u}\in J(\mathbb{R},\mathbb{R}),$$

where P is a polynomial in $h+1$ variables. The algebra of polynomial differential functions, $J[\mathbf{u}]$, is endowed with the *total derivative*

$$D\mathfrak{p} = \sum_{p=0}^{\infty}\frac{\partial p}{\partial u_{(p)}}u_{(p+1)}$$

and the *Euler operator*

$$\mathcal{E}(\mathfrak{p}) = \sum_{\ell=0}^{\infty}(-1)^{\ell}D^{\ell}\left(\frac{\partial\mathfrak{p}}{\partial u_{(\ell)}}\right).$$

For each $\mathfrak{p}\in\operatorname{Ker}(\mathcal{E})$ there is a unique $D^{-1}\mathfrak{p}\in J[\mathbf{u}]$ such that

$$\mathfrak{p} = D\big(D^{-1}(\mathfrak{p})\big), \qquad D^{-1}\mathfrak{p}|_0 = 0.$$

We consider the linear subspace

$$P[\mathbf{u}] = \big\{\mathfrak{p}\in J[\mathbf{u}]:\ \mathcal{E}\big(u_{(0)}D^3\mathfrak{p} + 4u_{(0)}^2 D\mathfrak{p}\big) = 0\big\}. \tag{1}$$

The image of $P[\mathbf{u}]$ by the total derivative is denoted by $P'[\mathbf{u}]\subset J[\mathbf{u}]$. Next we consider the integro-differential operators

$$\Delta(\phi,u) = \phi_{3s} + 2u\phi_s + u_s\phi,$$

$$\Xi(\phi,u)=\phi_{3s}+8u\phi_s+7u_s\phi+2\big(u_{2s}+4u^2\big)\int_0^s\phi dr+2\int_0^s\big(u\phi_{2s}+4u^2\phi\big)dr,$$

$$\begin{aligned}\Theta(\phi,u)&=\frac{u_s}{9}\int_0^s\big(u\phi_{3s}+4u^2\phi_s\big)dr+\frac{1}{9}\big(20u^2u_s+25u_su_{2s}+10uu_{3s}+u_{5s}\big)\phi\\&+\left(\frac{3}{2}+\frac{16}{9}u^3+\frac{71}{18}u_s^2+\frac{41}{9}uu_{2s}+\frac{13}{18}u_{4s}\right)\phi_s+\left(\frac{59}{9}uu_s+\frac{35}{18}u_{3s}\right)\phi_{2s}\\&+\left(2u^2+\frac{49}{18}u_{2s}\right)\phi_{3s}-2u_s\phi_{4s}+\frac{2}{3}u\phi_{5s}+\frac{1}{18}\phi_{7s}\end{aligned}$$

and the linear operators

$$\mathcal{D}:\ J[\mathbf{u}]\to J[\mathbf{u}],\qquad \mathcal{J}:\ P'[\mathbf{u}]\to J[\mathbf{u}],\qquad \mathcal{S}:\ P[\mathbf{u}]\to J[\mathbf{u}]$$

defined by

$$\begin{aligned}\mathcal{D}(\mathfrak{w})&=D^3\mathfrak{w}+2u_{(0)}D\mathfrak{w}+u_{(1)}\mathfrak{w},\\ \mathcal{J}(\mathfrak{q})&=D^3\mathfrak{q}+8u_{(0)}D\mathfrak{q}+7u_{(1)}\mathfrak{q}+2\big(u_{(1)}+4u_{(0)}^2\big)D^{-1}\mathfrak{q}+2D^{-1}\big(u_{(0)}D^2\mathfrak{q}+4u_{(0)}^2\mathfrak{q}\big),\\ \mathcal{S}(\mathfrak{p})&=\frac{1}{9}u_{(1)}D^{-1}\big(u_{(0)}D^3\mathfrak{p}+4u_{(0)}^2D\mathfrak{p}\big)+\frac{1}{9}\big(20u_{(0)}^2u_{(1)}+25u_{(1)}u_{(2)}+10u_{(0)}u_{(3)}+u_{(5)}\big)\mathfrak{p}\\&+\left(\frac{3}{2}+\frac{16}{9}u_{(0)}^3+\frac{71}{18}u_{(1)}^2+\frac{41}{9}u_{(0)}u_{(2)}+\frac{13}{18}u_{(4)}\right)D\mathfrak{p}+\left(\frac{59}{9}u_{(0)}u_{(1)}+\frac{35}{18}u_{(3)}\right)D^2\mathfrak{p}\\&+\left(2u_{(0)}^2+\frac{49}{18}u_{(2)}\right)D^3\mathfrak{p}-2u_{(1)}D^4\mathfrak{p}+\frac{2}{3}u_{(0)}D^5\mathfrak{p}+\frac{1}{18}D^7\mathfrak{p}.\end{aligned}$$

From the definition of the operators is clear that

$$\mathcal{D}(\mathfrak{w})|_{j(u)}=\Delta(\mathfrak{w}|_{j(u)},u),\qquad \mathcal{S}(\mathfrak{p})|_{j(u)}=\Theta(\mathfrak{p}|_{j(u)},u),\qquad \mathcal{J}(\mathfrak{q})|_{j(u)}=\Xi(\mathfrak{q}|_{j(u)},u),$$

for every $\mathfrak{w}\in j[\mathbf{u}]$, $\mathfrak{p}\in P[\mathbf{u}]$, $\mathfrak{q}\in P'[\mathbf{u}]$ and every $u\in C^\infty(\mathbb{R},\mathbb{R})$.

Lemma 1. *The operators $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{S}$ satisfy*

$$\mathcal{D}\mathcal{J}(\mathfrak{p})=18\mathcal{S}\big(D^{-1}\mathfrak{p}\big)-27\mathfrak{p}\qquad\forall\,\mathfrak{p}\in P'[\mathbf{u}].$$

Proof. A direct computation shows that

$$\Delta(\Xi(\phi,u),u)=18\Theta\left(\int_0^s\phi dr,u\right)-27\phi\qquad\forall\,u,\phi\in C^\infty(\mathbb{R},\mathbb{R}).$$

This implies

$$\begin{aligned}\mathcal{D}\mathcal{J}(\mathfrak{p})|_{j(u)}&=\Delta(\Xi(\mathfrak{p}|_{j(u)},u),u)=18\Theta(D^{-1}(\mathfrak{p})|_{j(u)},u)-27\mathfrak{p}|_{j(u)}\\&=18\mathcal{S}(D^{-1}(\mathfrak{p}))|_{j(u)}-27\mathfrak{p}|_{j(u)},\end{aligned}$$

for every $\mathfrak{p}\in P[\mathbf{u}]$ and every $u\in C^\infty(\mathbb{R},\mathbb{R})$. This yields the required result. ■

2.2 Construction of the hierarchy

According to [11, 34, 39] there are three sequences

$$\{\mathfrak{h}_n\}_{n\in\mathbf{N}}\subset\mathrm{Im}(\mathcal{E}),\qquad \{\mathfrak{q}_n\}_{n\in\mathbf{N}}\subset P[\mathbf{u}],\qquad \{\mathfrak{p}_n\}_{n\in\mathbf{N}}\subset J[\mathbf{u}]$$

of polynomial differential functions defined by the recursion formulae

$$\mathfrak{h}_{n+2}=\mathcal{J}\mathcal{D}(\mathfrak{h}_n),\qquad \mathcal{D}(\mathfrak{h}_n)=D(\mathfrak{q}_n),\qquad \mathfrak{h}_n=\mathcal{E}(\mathfrak{p}_n)\tag{2}$$

and by the initial data

$$\mathfrak{h}_0=1,\qquad \mathfrak{h}_1=u_{(2)}+4u_{(0)}^2.$$

Definition 1. The *Kaup–Kupershmidt hierarchy* $\{\mathcal{K}_n\}$ is the sequence of evolution equations defined by

$$\mathcal{K}_n: \ u_t + \mathcal{D}\mathfrak{h}_n|_{j(u)} = 0.$$

In view of (2), the equations of the hierarchy can be written either in the Hamiltonian form

$$\mathcal{K}_n: \ u_t + \mathcal{D}\mathcal{E}(\mathfrak{p}_n)|_{j(u)} = 0,$$

or else in the conservation form

$$\mathcal{K}_n: \ u_t + D\mathfrak{q}_n = 0.$$

Remark 1. The polynomial differential functions $\mathfrak{h}_n$, $\mathfrak{q}_n$, $\mathfrak{p}_n$ and the equations of the hierarchy can be computed with any software of symbolic calculus (see Appendix A.1). For $n = 1, 2$ we find

$$\begin{aligned}
\mathfrak{h}_1(\mathbf{u}) &= u_{(2)} + 4u_{(0)}^2,\\
\mathfrak{h}_2(\mathbf{u}) &= 12u_{(2)}u_{(0)} + 6u_{(1)}^2 + u_{(4)} + \frac{32}{3}u_{(0)}^3,\\
\mathfrak{q}_1(\mathbf{u}) &= \frac{20}{3}u_{(0)}^3 + \frac{15}{2}u_{(1)}^2 + 10u_{(0)}u_{(2)} + u_{(4)},\\
\mathfrak{q}_2(\mathbf{u}) &= \frac{56}{3}u_{(0)}^4 + 70u_{(0)}u_{(1)}^2 + 56u_{(0)}^2u_{(2)} + \frac{49}{2}u_{(2)}^2 + 35u_{(1)}u_{(3)} + 14u_{(0)}u_{(4)} + u_{(6)},\\
\mathfrak{p}_1(\mathbf{u}) &= \frac{1}{2}u_{(0)}2u_{(2)} + \frac{4}{3}u_{(0)}^2,\\
\mathfrak{p}_2(\mathbf{u}) &= \frac{1}{2}u_{(0)}u_{(4)} + 4u_{(0)}^2u_{(2)} + 2u_{(0)}u_{(1)}^2 + \frac{8}{3}u_{(0)}^4.
\end{aligned}$$

Consequently, the first two equations of the hierarchy are

$$\begin{aligned}
&u_t + 10uu_{3s} + 25u_su_{2s} + 20u^2u_s + u_{5s} = 0,\\
&u_t + u_{7s} + 14uu_{5s} + 49u_su_{4s} + 84u_{2s}u_{3s} + 252uu_su_{2s} + 70u_s^3 + 56u^2u_{3s} + \frac{224}{3}u^3u_s = 0.
\end{aligned}$$

Definition 2. Denoting by $[r]$ the integer part of r, we set

$$\ell_n = \left[\frac{n}{2}\right] - \frac{1}{2}(1 + (-1)^n), \qquad \lambda_n = \frac{1}{2}(1 + (-1)^n))(-27)^{\left[\frac{n}{2}\right]}$$

and we define $\{\mathfrak{v}_n\}_{n\in\mathbb{N}} \subset P[\mathbf{u}]$ by

$$\mathfrak{v}_n = 18\sum_{h=0}^{\ell_n}(-27)^h\mathfrak{w}_{n-2h}, \qquad n = 1, \dots,$$

where $\mathfrak{w}_0 = 0$, $\mathfrak{w}_1 = 1/2$ and $\mathfrak{w}_n = \mathfrak{q}_{n-2}$, $n > 1$.

Proposition 1. *The equations of the hierarchy can be written in the form*

$$\mathcal{K}_n: \ \partial_t u + \mathcal{S}(\mathfrak{v}_n)|_{j_s(u)} + \lambda_n u_s = 0. \tag{3}$$

Proof. For $n = 1, 2$ the proposition can be checked by a direct computation. We prove (3) when n is odd. By induction, suppose that (3) is true for $n = 2p - 1$. Note that

$$\lambda_{2p-1} = \lambda_{2p+1} = 0, \qquad \ell_{2p+1} = \ell_{2p-1} + 1.$$

By the inductive hypothesis we have

$$\mathcal{K}_{2p-1}(u) = u_t + D(\mathfrak{q}_{2p-1})|_{j_s(u)} = u_t + \mathcal{S}(\mathfrak{v}_{2p-1})|_{j_s(u)} = u_t + \Theta(\mathfrak{v}_{2p-1}|_{j_s(u)}, u),$$

which implies

$$D(\mathfrak{q}_{2p-1})|_{j_s(u)} = \Theta(\mathfrak{v}_{2p-1}|_{j_s(u)}, u).$$

Using Lemma 1 we find

$$\begin{aligned}\mathcal{K}_{2p+1}(u) &= u_t + \mathcal{D}\mathcal{J}(D\mathfrak{q}_{2p-1})|_{j_s(u)} = u_t + 18\mathcal{S}(\mathfrak{q}_{2p-1})|_{j_s(u)} - 27D\mathfrak{q}_{2p-1}|_{j_s(u)}\\ &= u_t + 18\Theta(\mathfrak{q}_{2p-1}|_{j_s(u)}, u) - 27\Theta(\mathfrak{v}_{2p-1}|_{j_s(u)}, u)\\ &= u_t + \Theta((18\mathfrak{w}_{2p+1} - 27\mathfrak{v}_{2p-1})|_{j_s(u)}, u).\end{aligned}$$

Using

$$\begin{aligned}18\mathfrak{w}_{2p+1} - 27\mathfrak{v}_{2p-1} &= 18\left(\mathfrak{w}_{2p+1} - 27\sum_{h=0}^{\ell_{2p-1}}(-27)^h\mathfrak{w}_{2p-1-2h}\right)\\ &= 18\left(\mathfrak{w}_{2p+1} - \sum_{h=0}^{\ell_{2p+1}-1}(-27)^{h+1}\mathfrak{w}_{2p-1-2(h+1)}\right)\\ &= 18\left(\mathfrak{w}_{2p+1} - \sum_{h=1}^{\ell_{2p+1}}(-27)^h\mathfrak{w}_{2p+1-2h}\right)\\ &= 18\sum_{h=0}^{\ell_{2p+1}}(-27)^h\mathfrak{w}_{2p+1-2h} = \mathfrak{v}_{2p+1}\end{aligned}$$

we obtain

$$\mathcal{K}_{2p+1}(u) = u_t + \Theta(\mathfrak{v}_{2p+1}|_{j_s(u)}, u) = u_t + \mathcal{S}(\mathfrak{v}_{2p+1})|_{j_s(u)} + \lambda_{2p+1}u_s.$$

Next we prove (3) when n is even. By induction, suppose that (3) is true for $n = 2p$. Note that

$$\lambda_{2p+2} = (-27)^{p+1} = -27\lambda_{2p}, \qquad \ell_{2p+2} = p = \ell_{2p} + 1.$$

By the inductive hypothesis we have

$$\mathcal{K}_{2p}(u) = u_t + D(\mathfrak{q}_{2p})|_{j_s(u)} = u_t + \mathcal{S}(\mathfrak{v}_{2p})|_{j_s(u)} - \lambda_{2p}u_s = u_t + \Theta(\mathfrak{v}_{2p}|_{j_s(u)}, u) + \lambda_{2p}u_s,$$

which implies

$$D(\mathfrak{q}_{2p})|_{j_s(u)} = \Theta(\mathfrak{v}_{2p}|_{j_s(u)}, u) + \lambda_{2p}u_s.$$

From Lemma 1 we have

$$\begin{aligned}\mathcal{K}_{2p+2}(u) &= u_t + \mathcal{D}\mathcal{J}(D\mathfrak{q}_{2p})|_{j_s(u)} = u_t + 18\mathcal{S}(\mathfrak{q}_{2p})|_{j_s(u)} - 27D\mathfrak{q}_{2p}|_{j_s(u)}\\ &= u_t + 18\Theta(\mathfrak{q}_{2p}|_{j_s(u)}, u) - 27(\Theta(\mathfrak{v}_{2p}|_{j_s(u)}, u) + \lambda_{2p}u_s)\\ &= u_t + \Theta((18\mathfrak{w}_{2p+2} - 27\mathfrak{v}_{2p})|_{j_s(u)}, u) + \lambda_{2p+2}u_s.\end{aligned}$$

Using

$$18\mathfrak{w}_{2p+2} - 27\mathfrak{v}_{2p} = 18\left(\mathfrak{w}_{2p+2} - 27\sum_{h=0}^{\ell_{2p}}(-27)^h\mathfrak{w}_{2p-2h}\right)$$

$$= 18\left(\mathfrak{w}_{2p+2} - \sum_{h=0}^{\ell_{2p+2}-1} (-27)^{h+1}\mathfrak{w}_{2p-2h}\right) = 18\left(\mathfrak{w}_{2p+2} - \sum_{h=1}^{\ell_{2p+2}} (-27)^{h}\mathfrak{w}_{2p+2-2h}\right)$$
$$= 18\sum_{h=0}^{\ell_{2p+2}} (-27)^{h}\mathfrak{w}_{2p+2-2h} = \mathfrak{v}_{2p+2}$$

we find

$$\mathcal{K}_{2p+2}(u) = u_t + \Theta(\mathfrak{v}_{2p+1}|_{j_s(u)}, u)) + \lambda_{2p+2}u_s = u_t + \mathcal{S}(\mathfrak{v}_{2p+2})|_{j_s(u)} + \lambda_{2p+2}u_s. \qquad \blacksquare$$

2.3 Traveling waves of the fifth-order Kaup–Kupershmidt equation

Several classes of traveling wave solutions of the fifth-order equation $\mathcal{K}_1$ have been considered in the literature [17, 38, 41]. In this section we generalize the elliptic families examined in [41]. The hyperbolic traveling waves found in [38] can be obtained as limiting cases, when the parameter of the elliptic functions tends to 1. First consider the third-order ODE

$$k''' + 8kk' = 0, \tag{4}$$

where k', k'' and k''' denote the first-, second- and third-order derivatives of a real-valued function k with respect to the independent variable. The same notation will be used for vector-valued functions. Integrating twice (4) we find

$$k'^2 = -\frac{8}{3}k^3 + \frac{3}{2}g_2k - \frac{9}{4}g_3, \tag{5}$$

where g_2 and g_3 are real constants. Every k satisfying (5) generates the traveling wave solution

$$u(s,t) = k\left(s - \frac{3}{4}g_2t\right)$$

of the first equation of the hierarchy. Clearly, (5) can be integrated in terms of the Weierstrass $\wp$ functions, namely: if $\Delta(g_2, g_3) = -g_2^3 + 27g_3^2 > 0$, then

$$k(s) = -\frac{3}{2}\wp(s+c), \qquad s \in (2n\omega_1 - c, (2n+1)\omega_1 - c), \qquad n \in \mathbb{Z},$$

where ω_1 is the real half period and c is a real constant. If $\Delta(g_2, g_3) < 0$, then there are two types of solutions:

$$k(s) = -\frac{3}{2}\wp(s+c), \qquad s \in (2n\omega_1 - c, (2n+1)\omega_1 - c), \qquad n \in \mathbb{Z},$$
$$k(s) = -\frac{3}{2}\wp(s+\omega_3+c), \qquad s \in \mathbb{R},$$

where ω_1 and ω_3 are the real and the purely imaginary half periods. When $\Delta(g_2, g_3) < 0$, the Weierstrass functions can be written in terms of Jacobi elliptic functions and we get

$$k(s) = \frac{1}{2}(1+m) - \frac{3}{2}\mathrm{ns}(s+c|m)^2, \qquad k(s) = \frac{1}{2}\left(1 - 2m + 3m\,\mathrm{cn}(s+c|m)^2\right).$$

The parameter $m \in (0,1)$ is $e_3 - e_2$, where $e_1 > e_2 > e_3$ are the three real roots of the cubic polynomial $4t^3 - g_2t - g_3$. The functions of the second type are periodic, with minimal period $2K(m)$, where K is complete elliptic integral of the first kind. The velocity of the traveling waves originated by these functions is $v_m = -(1+m(m-1))$. The first family of [41] consists of the traveling waves of the second type. When $m \to 0$ we obtain

$$k(t) = \frac{1}{2} + \frac{3}{2}\operatorname{csc}(s+c)^2,$$

and, when $m \to 1$, we find

$$k(t) = 1 - \frac{3}{2}\coth(s+c)^2, \qquad k(t) = -\frac{1}{2} + \frac{3}{2}\operatorname{sech}(s+c)^2,$$

which coincide with the solutions (67) and (69) of [38]. Next we consider the third-order equation

$$k''' + kk' = 0. \tag{6}$$

Again, integrating twice, we find

$$k'^2 = -\frac{1}{3}k^3 - 12g_2k - 144g_3. \tag{7}$$

Each function satisfying (7) generates the traveling wave solution

$$u(s,t) = k(s - 132g_2t)$$

of the fifth-order Kaup–Kupershmidt equation. As in the previous case, the solutions of (7) can be expressed in terms of Weiertsrass $\wp$-functions and Jacobi elliptic functions: if $\Delta(g_2, g_3) = -g_2^3 + 27g_3^2 > 0$, then

$$k(s) = -12\wp(s+c), \qquad s \in (2n\omega_1 - c, (2n+1)\omega_1 - c), \qquad n \in \mathbb{Z},$$

and, if $\Delta(g_2, g_3) < 0$, we obtain

$$k(s) = 4(1+m) - 12\operatorname{ns}(s+c|m)^2, \qquad k(s) = 4(1-2m) + 12m\operatorname{cn}(s+c|m)^2.$$

The velocity of the traveling waves originated by these functions is $v_m = -176(1+m(m-1))$. The second family of [41] consists of the traveling waves of the second type. When $m \to 0$ we obtain

$$k(t) = 4 - 12\operatorname{csc}(s+c)^2,$$

and, when $m \to 1$, we find

$$k(t) = 8 - 12\coth(s+c)^2, \qquad k(t) = -4 + 12\operatorname{sech}(s+c)^2,$$

which coincide with the solutions (68) and (70) of [38].

3 Motion of curves in projective plane

3.1 Curves in projective plane and their adapted frames

Consider a smooth parameterized curve $\gamma : I \to \mathbb{RP}^2$, defined on some open interval $I \subset \mathbb{R}$ and let $G : I \to \mathbb{R}^3 \setminus \{(0,0,0)\}$ be any lift of γ. We say that $\gamma(t)$ is an *inflection point* if $\operatorname{Span}(G(t), G'(t), G''(t))$ has dimension ≤ 2. From now on we will consider only curves without points of inflection. Then,

$$\Gamma = \operatorname{Det}(G, G', G'')^{-1/3}G$$

is the unique lift such that

$$\mathrm{Det}(\Gamma, \Gamma', \Gamma'') = 1. \tag{8}$$

Differentiating (8) we see that there exist smooth functions $a, b : I \to \mathbb{R}$ such that

$$\Gamma''' = a\Gamma + b\Gamma'.$$

The *projective speed* v and the *projective arc-element* σ are defined by

$$v = \left(a - b'/2\right)^{1/3}, \qquad \sigma = v dt.$$

The primitives $s : I \to \mathbb{R}$ of the projective arc-element are the *projective parameters* and the zeroes of σ are the *sextatic points*. A curve without inflection or sextatic points is said to be *generic*. Obviously, every generic curves can be parameterized by the projective parameter.

Remark 2. For every point $t \in I$, there is a unique non-degenerate conic $\mathcal{C}_t$, the *osculating conic*, having fourth-order analytic contact with γ at $\gamma(t)$. The osculating conic is defined by the equation $x_1^2 - 2x_0x_2 = 0$ with respect to the homogenous coordinates of the projective frame

$$\left(\Gamma|_t, \Gamma'|_t, \Gamma''|_t - \frac{b(t)}{2}\Gamma|_t\right)$$

and, identifying $\mathbb{RP}^5$ with the space of plane conics, we define the osculating curve by

$$\mathcal{C} : \ t \in I \to \mathcal{C}|_t \in \mathbb{RP}^5.$$

The sextatic points are critical points of the osculating curve. The assumption on the non-existence of inflection and sextatic points is rather strong from a global viewpoint. For instance, any simple closed curve in $\mathbb{RP}^2$ possesses flex or sextatic points and a simple convex curve has at least six sextatic points [35, 36].

The *canonical projective frame field* [3] along a generic curve γ is the $SL(3, \mathbb{R})$-valued map defined by

$$F_0 = v\Gamma, \qquad F_1 = \frac{v'}{v}\Gamma + \Gamma', \qquad F_2 = \frac{1}{2v}\left(\frac{v'^2}{v^2} - b\right)\Gamma + \frac{v'}{v^2}\Gamma + \frac{1}{v}\Gamma''.$$

The canonical frame is invariant with respect to changes of the parameter and projective transformations. Furthermore, it satisfies the *projective Frenet system*

$$F' = F \cdot \begin{pmatrix} 0 & -k & 1 \\ 1 & 0 & -k \\ 0 & 1 & 0 \end{pmatrix} v. \tag{9}$$

The function $k : I \to \mathbb{R}$ is the *projective curvature*, whose explicit expression [30] is

$$k = -\frac{1}{v}\left(S(s) + \frac{b}{2}\right),$$

where s is a projective parameter function and S is the Schwarzian derivative

$$S(f) = \left(\frac{f_{tt}}{f_t}\right)_t - \frac{1}{2}\left(\frac{f_{tt}}{f_t}\right)^2.$$

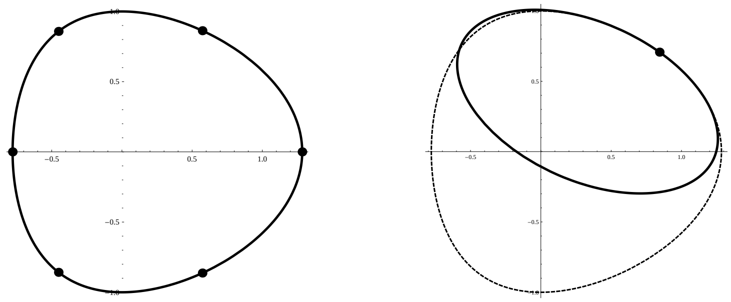

Figure 1. The sextatic points and the osculating conic $\mathcal{C}(\pi/4)$.

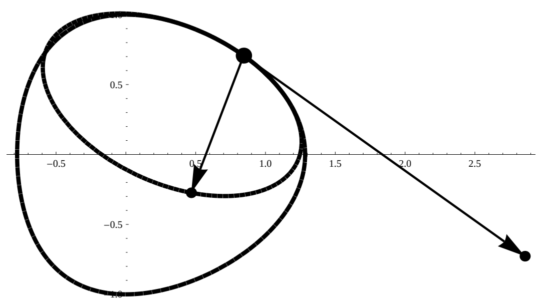

Figure 2. The projective frame and the osculating conic.

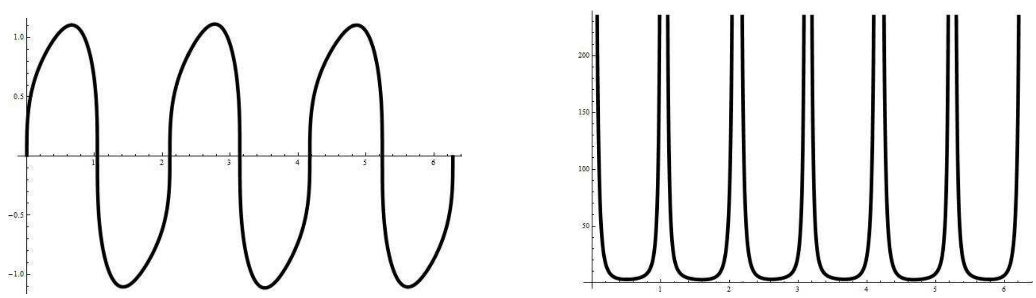

Figure 3. The speed and the projective curvature.

Remark 3. The construction of the canonical frame involves only algebraic manipulations and differentiations. So, it can be implemented in any software of symbolic calculus. In addition, the canonical frame can be constructed using the "invariantization" method of Fels–Olver [9]. In other words, there is a $SL(3,\mathbb{R})$-equivariant map $\mathfrak{F} : J^*_h(\mathbb{R},\mathbb{RP}^2) \to SL(3,\mathbb{R})$, defined on the fifth-order jet space of generic curves such that $\mathfrak{F} \circ j^{(5)}(\gamma)$ is the projective frame along γ, for every non-degenerate γ.

Example 1. The convex simple curve

$$\gamma : \ t \in \mathbb{R} \to \left[\left(\cos(t), \sin(t), e^{\frac{-\cos(t)}{4}}\right)\right] \in \mathbb{RP}^2$$

has exactly six sextatic points, attained at

$$\tau_1 = 0, \qquad \tau_2 \approx 1.0412803807424216, \qquad \tau_3 \approx 2.109976014903134,$$
$$\tau_4 = \pi, \qquad \tau_5 \approx 4.173209292276453, \qquad \tau_6 \approx 5.241904926437165.$$

Figs. 1 and 2 reproduce the curve, the sextatic points, the osculating conic and the projective frame at $t = \pi/4$. Fig. 3 reproduces the projective speed and the projective curvature. The speed vanishes at the sextatic points and the curvature becomes infinite at these points.

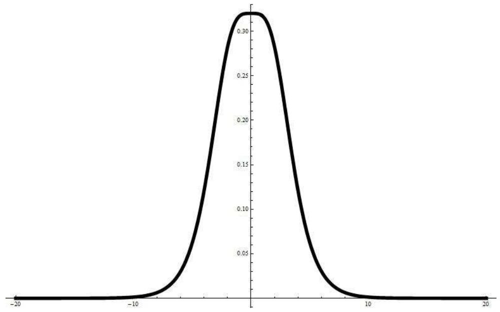

Figure 4. The projective curvature.

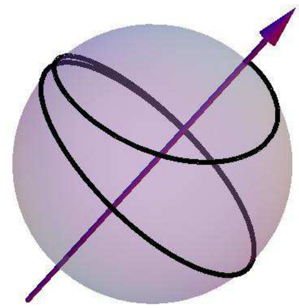

Figure 5. The spherical lift of the corresponding curve.

Remark 4. The curve is uniquely determined, up to projective congruences, by the speed and the curvature. If we assign smooth functions $\upsilon > 0$ and k, the Frenet system (9) can be integrated with standard numerical routines (see Appendix A.2). For instance, taking $\upsilon = 1$ and choosing the "anomalous" 1-soliton solution of the $\mathcal{K}_1$-equation [17, 31]

$$k(s) = \frac{2m^2\left(1 + 2\cosh(m(s - m^4t)\right)}{2\left(2 + \cosh(m(s - m^4t))^2\right)}, \qquad m = 0.8,$$

as projective curvature (see Fig. 4), the numerical solution of the linear system (9) gives rise to the curve whose spherical lift is reproduced in Fig. 5.

3.2 The equation of a motion of curves

A *motion* is a smooth one-parameter family $\gamma(s,t)$ of projective curves such that

$$\gamma_{[t]}:\ s \in \mathbb{R} \to \gamma(s,t) \in \mathbb{RP}^2,$$

is generic and parameterized by the projective parameter, for every $t \in I$. Denoting by $F_{[t]}$: $\mathbb{R} \to SL(3,\mathbb{R})$ and $k_{[t]} : \mathbb{R} \to \mathbb{R}$ the projective frame and the projective curvature of $\gamma_{[t]}$ we consider the *projective frame* and the *projective curvature* of the motion, defined by

$$\mathcal{F}:\ (s,t) \in \mathbb{R} \times I \to F_{[t]}(s) \in SL(3,\mathbb{R})$$

and

$$\kappa:\ (s,t) \in \mathbb{R} \times I \to k_{[t]}(s) \in \mathbb{R}.$$

The projective frame satisfies

$$\mathcal{F}^{-1}d\mathcal{F} = \mathcal{K}(s,t)ds + \Phi(s,t)dt, \tag{10}$$

where

$$\mathcal{K} = \begin{pmatrix} 0 & -\kappa & 1 \\ 1 & 0 & -\kappa \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \phi_0^0 & \phi_1^0 & \phi_2^0 \\ \phi_0^1 & \phi_1^1 & \phi_2^1 \\ \phi_0^2 & \phi_1^2 & -(\phi_0^0 + \phi_1^1) \end{pmatrix}. \tag{11}$$

The coefficient ϕ_0^2 is said to be the *normal velocity of the motion* and it will be denoted by υ.

Proposition 2. *The curvature of a motion of projective curves with normal velocity υ satisfies*

$$\partial_t\kappa = \Theta(\upsilon,\kappa) + \lambda\kappa_s, \tag{12}$$

where λ is a real constant, the internal parameter. Conversely, if κ is a solution of (12) then there is a motion γ with normal speed υ, internal parameter λ and curvature κ. Moreover, γ is unique up to projective transformations.

Proof. Differentiating (10) we obtain

$$\partial_s\Phi - \partial_t\mathcal{K} + [\mathcal{K}, \Phi] = 0, \tag{13}$$

which implies

$$\begin{aligned}
&(\phi_0^0)_s - \kappa\phi_0^1 + \phi_0^2 - \phi_1^0 = 0,\\
&(\phi_2^0)_s - 2\phi_0^0 - \phi_1^1 + \kappa(\phi_1^0 - \phi_2^1) = 0,\\
&(\phi_0^1)_s + \phi_0^0 - \kappa\phi_0^2 - \phi_1^1 = 0,\\
&(\phi_1^1)_s + \phi_1^0 + \kappa(\phi_0^1 - \phi_1^2) - \phi_2^1 = 0,\\
&(\phi_0^2)_s - \phi_1^2 = 0,\\
&(\phi_1^2)_s + \phi_0^0 + \kappa\phi_0^2 + 2\phi_1^1 = 0,\\
&(\phi_1^0)_s - (\phi_2^1)_s - 3\kappa\phi_1^1 + \phi_1^2 + \phi_0^1 - 2\phi_2^0 = 0,
\end{aligned} \tag{14}$$

and

$$\partial_t\kappa + (\phi_2^1)_s - \phi_0^1 + \kappa(\phi_0^0 + 2\phi_1^1) + \phi_2^0 = 0. \tag{15}$$

If we set $\upsilon = \phi_0^2$, then (14) gives

$$\begin{aligned}
\phi_0^0 &= \left(\frac{1}{3}\kappa + \frac{8}{9}\kappa\kappa_s + \frac{1}{9}\kappa_{3s}\right)\upsilon + \left(\frac{1}{2}\kappa_s + \frac{8}{9}\kappa^2\right)\upsilon_s + \left(\frac{1}{6} + \frac{5}{6}\kappa_s\right)\upsilon_{2s} + \frac{5}{9}\kappa\upsilon_{3s} + \frac{1}{18}\upsilon_{5s},\\
\phi_0^1 &= \lambda - \frac{1}{9}\int_0^s \left(\kappa\upsilon_{3s} + 4\kappa^2\upsilon_s\right)ds - \left(\frac{1}{9}\kappa_{2s} + \frac{4}{9}\kappa^2\right)\upsilon - \left(\frac{1}{2} + \frac{7}{18}\kappa_s\right)\upsilon_s - \frac{4}{9}\kappa\upsilon_{2s} - \frac{1}{18}\upsilon_{4s},\\
\phi_1^0 &= -\lambda\kappa + \frac{\kappa}{9}\int_0^s \left(\kappa\upsilon_{3s} + 4\kappa^2\upsilon_s\right)ds + \left(1 + \frac{4}{9}\kappa^3 + \frac{1}{3}\kappa_s + \frac{8}{9}\kappa_s^2 + \kappa\kappa_{2s} + \frac{1}{9}\kappa_{4s}\right)\upsilon\\
&\quad + \left(\frac{5}{6}\kappa + \frac{55}{18}\kappa\kappa_s + \frac{11}{8}\kappa_{3s}\right)\upsilon_s + \frac{4}{3}\left(\kappa^2 + \kappa_{2s}\right)\upsilon_{2s} + \left(\frac{1}{6} + \frac{25}{18}\kappa_s\right)\upsilon_{3s} + \frac{11}{8}\kappa\upsilon_{4s} + \frac{1}{18}\upsilon_{6s},
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\phi_1^1 &= -\frac{2}{3}\kappa\upsilon - \frac{1}{3}\upsilon_{2s},\\
\phi_1^2 &= \lambda - \frac{1}{9}\int_0^s \left(\kappa\upsilon_{3s} + 4\kappa^2\upsilon_s\right)ds - \left(\frac{4}{9}\kappa^2 + \frac{1}{9}\kappa_{2s}\right)\upsilon + \left(\frac{1}{2} - \frac{7}{18}\kappa_s\right)\upsilon_s - \frac{4}{9}\kappa\upsilon_{2s} - \frac{1}{18}\upsilon_{4s},\\
\phi_2^0 &= \lambda - \frac{1}{9}\int_0^s \left(\kappa\upsilon_{3s} + 4\kappa^2\upsilon_s\right)ds + \left(\frac{5}{9}\kappa^2 + \frac{2}{9}\kappa_{2s}\right)\upsilon + \frac{7}{9}\kappa_s\upsilon_s + \frac{8}{9}\kappa\upsilon_{2s} + \frac{1}{9}\upsilon_{4s},\\
\phi_2^1 &= -\lambda\kappa + \frac{\kappa}{9}\int_0^s \left(\kappa\upsilon_{3s} + 4\kappa^2\upsilon_s\right)ds + \left(1 + \frac{4}{9}\kappa^3 - \frac{1}{3}\kappa_s + \frac{8}{9}\kappa_s^2 + \kappa\kappa_{2s} + \frac{1}{9}\kappa_{4s}\right)\upsilon\\
&\quad + \left(-\frac{5}{6}\kappa + \frac{55}{18}\kappa\kappa_s + \frac{11}{18}\kappa_{3s}\right)\upsilon_s + \frac{4}{3}\left(\kappa^2 + \kappa_{2s}\right)\upsilon_{2s} - \left(\frac{1}{6} - \frac{25}{18}\kappa_s\right)\upsilon_{3s} + \frac{11}{18}\kappa\upsilon_{4s} + \frac{1}{18}\upsilon_{6s},
\end{aligned} \tag{17}$$

where λ is a real constant. From (16) and (17) we deduce that (15) is satisfied if and only if

$$\partial_t\kappa = \Theta(\upsilon, \kappa) + \lambda\kappa_s.$$

Conversely, if κ is a solution of (12) and if we define ϕ_j^i, $\mathcal{K}$ and Φ as in (11), (16) and (17), then K and Φ satisfy (13). Using Frobenius theorem we deduce the existence of a smooth map

$$\mathcal{F}:\ \mathbb{R}\times I \to SL(3,\mathbb{R})$$

such that $\mathcal{F}^{-1}d\mathcal{F} = \mathcal{K}ds + \Phi dt$. The map $\mathcal{F}$ is unique up to left multiplication by an element of $SL(3,\mathbb{R})$. Setting $\gamma(s,t) = [F_0(s,t)]$ we have a motion of projective curves with curvature κ, normal velocity υ, internal parameter λ and projective frame $\mathcal{F}$. This yields the required result. ∎

3.3 Local motions

From the proof of Proposition 2 we see that the Φ-matrix of a motion of projective curves can be written as

$$\Phi = \widetilde{\Phi}(\upsilon,\kappa) + \lambda\mathcal{K}(\kappa),$$

where the coefficients of $\widetilde{\Phi}(\upsilon,\kappa)$ are the integro-differential operators

$$\begin{aligned}
\tilde{\phi}_0^2(\upsilon,\kappa) &= \upsilon,\\
\tilde{\phi}_0^0(\upsilon,\kappa) &= \left(\frac{1}{3}\kappa+\frac{8}{9}\kappa\kappa_s+\frac{1}{9}\kappa_{3s}\right)\upsilon+\left(\frac{1}{2}\kappa_s+\frac{8}{9}\kappa^2\right)\upsilon_s+\left(\frac{1}{6}+\frac{5}{6}\kappa_s\right)\upsilon_{2s}+\frac{5}{9}\kappa\upsilon_{3s}+\frac{1}{18}\upsilon_{5s},\\
\tilde{\phi}_0^1(\upsilon,\kappa) &= -\frac{1}{9}\int_0^s\left(\kappa\upsilon_{3s}+4\kappa^2\upsilon_s\right)ds-\left(\frac{1}{9}\kappa_{2s}+\frac{4}{9}\kappa^2\right)\upsilon-\left(\frac{1}{2}+\frac{7}{18}\kappa_s\right)\upsilon_s-\frac{4}{9}\kappa\upsilon_{2s}-\frac{1}{18}\upsilon_{4s},\\
\tilde{\phi}_1^0(\upsilon,\kappa) &= \frac{\kappa}{9}\int_0^s\left(\kappa\upsilon_{3s}+4\kappa^2\upsilon_s\right)ds+\left(1+\frac{4}{9}\kappa^3+\frac{1}{3}\kappa_s+\frac{8}{9}\kappa_s^2+\kappa\kappa_{2s}+\frac{1}{9}\kappa_{4s}\right)\upsilon\\
&\quad+\left(\frac{5}{6}\kappa+\frac{55}{18}\kappa\kappa_s+\frac{11}{8}\kappa_{3s}\right)\upsilon_s+\frac{4}{3}\left(\kappa^2+\kappa_{2s}\right)\upsilon_{2s}+\left(\frac{1}{6}+\frac{25}{18}\kappa_s\right)\upsilon_{3s}\\
&\quad+\frac{11}{8}\kappa\upsilon_{4s}+\frac{1}{18}\upsilon_{6s},\\
\tilde{\phi}_1^1(\upsilon,\kappa) &= -\frac{2}{3}\kappa\upsilon-\frac{1}{3}\upsilon_{2s},\\
\tilde{\phi}_1^2(\upsilon,\kappa) &= -\frac{1}{9}\int_0^s\left(\kappa\upsilon_{3s}+4\kappa^2\upsilon_s\right)ds-\left(\frac{4}{9}\kappa^2+\frac{1}{9}\kappa_{2s}\right)\upsilon+\left(\frac{1}{2}-\frac{7}{18}\kappa_s\right)\upsilon_s-\frac{4}{9}\kappa\upsilon_{2s}-\frac{1}{18}\upsilon_{4s},\\
\tilde{\phi}_2^0(\upsilon,\kappa) &= -\frac{1}{9}\int_0^s\left(\kappa\upsilon_{3s}+4\kappa^2\upsilon_s\right)ds+\left(\frac{5}{9}\kappa^2+\frac{2}{9}\kappa_{2s}\right)\upsilon+\frac{7}{9}\kappa_s\upsilon_s+\frac{8}{9}\kappa\upsilon_{2s}+\frac{1}{9}\upsilon_{4s},\\
\tilde{\phi}_2^1(\upsilon,\kappa) &= \frac{\kappa}{9}\int_0^s\left(\kappa\upsilon_{3s}+4\kappa^2\upsilon_s\right)ds+\left(1+\frac{4}{9}\kappa^3-\frac{1}{3}\kappa_s+\frac{8}{9}\kappa_s^2+\kappa\kappa_{2s}+\frac{1}{9}\kappa_{4s}\right)\upsilon\\
&\quad+\left(-\frac{5}{6}\kappa+\frac{55}{18}\kappa\kappa_s+\frac{11}{18}\kappa_{3s}\right)\upsilon_s+\frac{4}{3}\left(\kappa^2+\kappa_{2s}\right)\upsilon_{2s}-\left(\frac{1}{6}-\frac{25}{18}\kappa_s\right)\upsilon_{3s}\\
&\quad+\frac{11}{18}\kappa\upsilon_{4s}+\frac{1}{18}\upsilon_{6s}.
\end{aligned}$$

Bearing in mind the definition (1) of the linear subspace $P[\mathbf{u}]$, we deduce the existence of linear operators $\mathcal{M}_j^i : P[\mathbf{u}] \to J[\mathbf{u}]$ such that

$$\widetilde{\phi}_j^i(\mathfrak{p}|_{j_s(\kappa)},\kappa) = \mathcal{M}_j^i(\mathfrak{p})|_{j_s(\kappa)},$$

for every $\mathfrak{p} \in P[\mathbf{u}]$. This implies the following corollary.

Corollary 1. *If $\mathfrak{p}$ belongs to $P[\mathbf{u}]$ and if κ is solution of the evolution equation*

$$\partial_t\kappa = \mathcal{S}(\mathfrak{p})|_{j_s(\kappa)} + \lambda\kappa_s$$

then, there is a motion γ, uniquely defined up to projective transformations, with curvature κ and normal velocity $\mathfrak{p}|_{j_s(\kappa)}$. Motions of this type are said to be local.

Remark 5. Local motions are the integral curves of *local vector fields* on the infinite-dimensional space $\mathcal{P}$ of unit-speed generic curves of $\mathbb{RP}^2$. More precisely, if we take any $\mathfrak{p} \in P[\mathbf{u}]$ and any real constant λ then there is a unique vector field $X_{\mathfrak{p},\lambda}$ on $\mathcal{P}$ whose integral curve through $\gamma_{[0]} \in \mathcal{P}$ is the local motion γ such that:

- $\gamma(s,0) = \gamma_{[0]}(s)$;
- its curvature κ is the solution of the Cauchy problem

$$\kappa_t + \mathcal{S}(\mathfrak{p})|_{j_s(\kappa)} + \lambda\kappa_s = 0, \qquad \kappa(s,0) = k_{[0]}(s);$$

- its normal speed is $\mathfrak{p}|_{j_s(\kappa)}$.

Definition 3. We say that $X_{\mathfrak{p},\lambda}$ is the *local vector field* with potential $\mathfrak{p}$ and spectral parameter λ. The dynamics of a local vector field is governed by the *induced evolution equation*

$$\kappa_t + \mathcal{S}(\mathfrak{p})|_{j_s(\kappa)} + \lambda\kappa_s = 0.$$

From these observations and using Proposition 1 we have the following result.

Theorem 1. *For every $n \in \mathbb{N}$ the local vector field $X_{\mathfrak{v}_n,\lambda_n}$ defined by the polynomial differential function $\mathfrak{v}_n \in P[\mathbf{u}]$ and by the spectral parameter λ_n induces the n-th equation of the Kaup–Kupershmidt hierarchy.*

4 Congruence motions

Consider a local dynamics with potential $\mathfrak{p}$ and internal parameter λ. A curve $\tilde{\gamma}$ which evolves without changing its shape (by projective transformations) is said to be a *congruence curve* of the flow. Denote by $\tilde{k}$ the curvature of $\tilde{\gamma}$ and by $\kappa(s,t)$ the curvature of the evolution $\gamma(s,t)$ of $\tilde{\gamma}(s)$. If $\tilde{k}$ is non constant, then

$$\kappa(s,t) = \tilde{k}(s+vt),$$

for some constant v. So, κ is a traveling wave solution of the induced evolution equation and $\tilde{k}$ satisfies the ordinary differential equation

$$\Theta(\mathfrak{p}|_{j_s(u)}, u) + (\lambda + v)u_s = 0. \tag{18}$$

Unit-speed generic curves whose curvature satisfies (4) or (6) are examples of congruence curves of the first flow of the hierarchy. On the other hand, (4) is the Euler–Lagrange equation of the invariant functional defined by the integral of the projective arc-element σ [3, 28]. This implies the following corollary.

Corollary 2. *Every critical curve of the functional*

$$\gamma \to \int_\gamma \sigma$$

is a congruence curve of the first flow of the Kaup–Kupershmidt hierarchy.

The projective frame $F(s,t)$ satisfies

$$F^{-1}dF = \tilde{K}(s+vt)dt + \tilde{\Phi}(s+vt)ds, \tag{19}$$

where the $\mathfrak{sl}(3,\mathbb{R})$-valued functions $\tilde{K}$ and $\tilde{\Phi}$ are defined as in (11), (16) and (17), with normal speed $\mathfrak{p}|_{j_s(\tilde{k})}$. We define the *Hamiltonian* by

$$H = \tilde{\Phi} - v\tilde{K} : \ \mathbb{R} \to \mathfrak{sl}(3,\mathbb{R}).$$

The integrability condition of (19) is the Lax equation

$$H' = [H, \tilde{K}]$$

which implies the *conservation law*

$$\tilde{F} \cdot H \cdot \tilde{F}^{-1} = \xi,$$

where $\tilde{F}$ is the projective frame of $\tilde{\gamma}$ and ξ is a fixed element of $\mathfrak{sl}(3,\mathbb{R})$, the *momentum* of the congruence curve $\tilde{\gamma}$. In particular, H and ξ have the same spectrum. From now on we assume that $F(0) = \mathrm{Id}_{3\times3}$.

Proposition 3. *The motion of a congruence curve $\tilde{\gamma}$ is given by*

$$\gamma(s,t) = \mathrm{Exp}(t\xi) \cdot \tilde{\gamma}(s + vt). \tag{20}$$

Proof. Define $\gamma(s,t)$ as in (20) and set

$$F(s,t) = \mathrm{Exp}(t\xi) \cdot \widetilde{F}(s + vt) \qquad \forall\, (s,t) \in \mathbb{R} \times I.$$

Since F is a lift of $\gamma(s,t)$, it suffices to prove that F satisfies (19). From the definition we deduce

$$F^{-1}\partial_s F|_{(s,t)} = \tilde{K}(s + vt)$$

and

$$\begin{aligned} F\partial_t F|_{(s,t)} &= (\tilde{F}(s+vt)^{-1} \cdot \mathrm{Exp}(-t\xi)) \cdot (\mathrm{Exp}(t\xi)\xi \cdot \widetilde{F}(s+vt) + v\mathrm{Exp}(t\xi) \cdot \partial_s \widetilde{F}|_{s+vt}) \\ &= \tilde{F}(s+vt)^{-1} \cdot \xi \cdot \widetilde{F}(s+vt) + v\tilde{K}(s+vt) = H(s+vt) + v\tilde{K}(s+vt) \\ &= \tilde{\Phi}(s+vt). \end{aligned}$$

This implies the required result. ■

We now prove the following proposition.

Proposition 4. *If $\tilde{k}$ is a non-constant real-analytic solution of (18) and if ξ has three distinct eigenvalues then the corresponding congruence curve can be found by quadratures.*

Proof. It suffices to show that the solution of the linear system

$$\tilde{F}_s = \tilde{F} \cdot \tilde{K}, \qquad \tilde{F}(0) = \mathrm{Id}_{3\times3}$$

can be constructed from $\tilde{k}$ by algebraic manipulations, differentiations and integrations of functions involving $\tilde{k}$ and its derivatives. This can be shown with the following reasoning:

Step I. The Hamiltonian H can be directly constructed from the prolongation $j(\tilde{k})$ of the curvature and from the potential $\mathfrak{p}$. Then, we compute the momentum $\xi = H(0)$ and its eigenvalues τ_0, τ_1 and τ_2. For each τ_j we choose row vectors H^{j_1} and H^{j_2} of H such that

$$S_j = \left({}^t H^{j_1} - \tau_j \epsilon_{j_1}\right) \times \left({}^t H^{j_2} - \tau_j \epsilon_{j_2}\right) \neq 0,$$

where $(\epsilon_0, \epsilon_1, \epsilon_2)$ is the standard basis of $\mathbb{C}^3$.

Step II. Next, we define the S-matrix

$$S = (S_1, S_2, S_3) : \ I \to \mathfrak{gl}(3,\mathbb{C}).$$

This is a real-analytic map which can be computed in terms of $\tilde{k}$ and its derivatives. Subsequently, we define the Σ-matrix

$$\Sigma = F \cdot S : \ \mathbb{R} \to \mathfrak{gl}(3, \mathbb{C}). \tag{21}$$

The columns $\Sigma_j(s)$ are eigenvectors of the momentum ξ, with eigenvalue τ_j, for each $j = 0, 1, 2$ and every s. In particular, Σ_j has constant direction. Hence, there exist real-analytic complex valued functions r_j such that

$$\Sigma_j' = r_j \Sigma_j, \qquad j = 0, 1, 2. \tag{22}$$

Differentiating $\Sigma = F \cdot S$ and using the structure equations satisfied by F we deduce

$$S' + \tilde{K} \cdot S = S \cdot \Delta(r_0, r_1, r_2),$$

where $\Delta(r_0, r_1, r_2)$ is the diagonal matrix with elements r_0, r_1 and r_2. This shows that r_0, r_1 and r_2 can be computed in terms of $\tilde{k}$ and its derivatives.

Step III. We compute the *integrating factors*

$$\rho_j(s) = \mathrm{Exp}\left(\int_0^s r_j(u) du \right), \qquad j = 0, 1, 2.$$

Equation (22) implies

$$\Sigma_j = \rho_j C_j, \qquad j = 0, 1, 2,$$

where C_0, C_1 and C_3 are constant vectors of $\mathbb{C}^3$. We then have

$$\Sigma = C \cdot \Delta(\rho_0, \rho_1, \rho_2), \tag{23}$$

where C is a fixed element of $GL(3, \mathbb{C})$. Substituting (21) into (23) we obtain

$$F = M(0)^{-1} \cdot M,$$

where the M-matrix is defined by

$$M \cdot S = \Delta(\rho_0, \rho_1, \rho_2).$$

All the steps involve only linear algebra manipulations, differentiations and the quadratures of the functions r_0, r_1 and r_2, as claimed. ■

Example 2. We illustrate the integration of the congruence curves with projective curvature

$$k_m(s) = \frac{1}{2}\left(1 - 2m + 3m\mathrm{cn}(s + c|m)^2\right), \qquad m \in (0, 1).$$

Computing the H-matrix we obtain

$$H = \begin{pmatrix} h_1^1 & h_2^1 & h_3^1 \\ 0 & -2h_1^1 & h_1^2 \\ 9 & 0 & h_1^1 \end{pmatrix},$$

where the coefficients are given by

$$h_1^1 = \frac{3}{2}\left(3\mathrm{dn}(s|m)^2 + m - 2\right), \qquad h_2^1 = 9 - 9m\mathrm{cn}(s|m)\mathrm{dn}(s|m)\mathrm{sn}(s|m),$$

$$h_1^3 = \frac{9}{4}\left(-3\mathrm{dn}(s|m)^4 - 2(m-2)\mathrm{dn}(s|m)^2 + m^2\right),$$
$$h_1^2 = 9m\mathrm{cn}(s|m)\mathrm{dn}(s|m)\mathrm{sn}(s|m) + 9.$$

Next we compute the momentum and we get

$$\xi = \begin{pmatrix} \frac{3}{2}(m+1) & 9 & \frac{9}{4}(m-1)^2 \\ 0 & -3(m+1) & 9 \\ 9 & 0 & \frac{3}{2}(m+1) \end{pmatrix}.$$

The discriminant of its characteristic polynomial is

$$\delta = -31 + m(6 + m(7 + (-6+m)m)).$$

From now on we assume $\delta \neq 0$. If $\delta < 0$ the momentum has one real eigenvalue and two complex conjugate eigenvalues, otherwise the momentum has three distinct real eigenvalues. The eigenvalues are:

$$\tau_0 = -3\frac{\sqrt[3]{2}(n_1+n_2)^{2/3} + 2n_3}{\sqrt[3]{4}(n_1+n_2)^{1/3}}, \qquad \tau_1 = 3\frac{\sqrt[3]{2}(1-i\sqrt{3})(n_1+n_2)^{2/3} + 2(1+i\sqrt{3})n_3}{2\sqrt[3]{4}(n_1+n_2)^{1/3}},$$
$$\tau_2 = 3\frac{\sqrt[3]{2}(1+i\sqrt{3})(n_1+n_2)^{2/3} + 2(1-i\sqrt{3})n_3}{2\sqrt[3]{4}(n_1+n_2)^{1/3}},$$

where

$$n_1 = 3\left(\sqrt{-3(-31 + m(6 + m(7 + (-6+m)m)))} - 9\right),$$
$$n_2 = (2-m)(m+1)(2m-1), \qquad n_3 = ((m-1)m+1).$$

We set

$$S_j = ({}^tH^2 - \tau_j\epsilon_2) \times ({}^tH^3 - \tau_j\epsilon_3), \qquad j = 0,1,2,$$

so that

$$S_j^1 = -\frac{1}{2}\left(9\mathrm{dn}(s|m)^2 + 3(m-2) - 2\tau_j\right)\left(9\mathrm{dn}(s|m)^2 + 3(m-2) + \tau_j\right),$$
$$S_j^2 = 81\left(m\mathrm{cn}(s|m)\mathrm{dn}(s|m)\mathrm{sn}(s|m) + 1\right), \qquad S_j^3 = 9\left(9\mathrm{dn}(s|m)^2 + 3(m-2)q^2 + \tau_j\right)$$

and

$$r_j = \frac{\dot{S}_j^3}{S_j^3} + \frac{S_j^2}{S_j^3} = \frac{9 - 9m\mathrm{cn}(s|m)\mathrm{dn}(s|m)\mathrm{sn}(s|m)}{9\mathrm{dn}(s|m)^2 + 3(m-2) + \tau_j},$$

The quadratures can be carried out in terms of elliptic integrals of the third kind and we obtain

$$\rho_j = \sqrt{2}\sqrt{9\mathrm{dn}(s|m)^2 + 3(m-2) + \tau_j} \cdot e^{9\frac{\Pi\left(\frac{9m}{3(m+1)+\tau_j};\mathrm{am}(s|m)|m\right)}{\left(3(m+1)+\tau_j\right)}}, \tag{24}$$

where

$$\Pi(\zeta;\phi|m) = \int_0^\phi \frac{d\theta}{(1-\zeta\sin^2(\theta))\sqrt{1-m\sin^2(\theta)}}$$

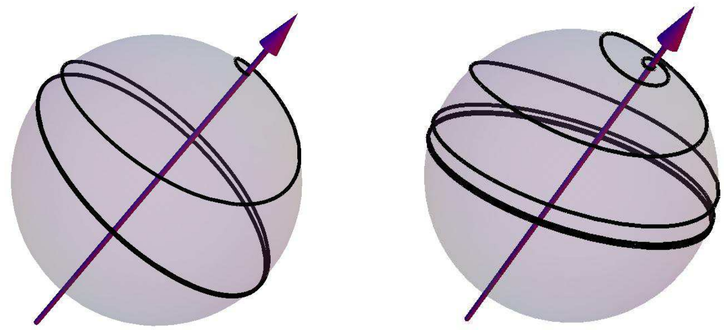

Figure 6. Congruence curves with parameter $m = 0.3$ and spectra σ_1 and σ_2 respectively.

is the incomplete integral of the third kind and $\mathrm{am}(u|m)$ is the amplitude of the Jacobi elliptic functions. Note that on the right hand side of (24) we have a smooth real branch of a multi-valued analytic function. The first column vector of S^{-1} is the transpose of

$$\left(\frac{1}{(\tau_0-\tau_1)(\tau_0-\tau_2)}, \frac{1}{(\tau_1-\tau_0)(\tau_1-\tau_2)}, \frac{1}{(\tau_2-\tau_0)(\tau_2-\tau_1)}\right)$$

and the coefficients $\tilde{M}_i^j$ of $M(0)^{-1}$ are

$$\tilde{M}_j^1 = \frac{-9(m+1)^2 + 3(m+1)\tau_1 + 2\tau_j^2}{2\sqrt{6(m+1)+2\tau_j}},$$
$$\tilde{M}_j^2 = \frac{81}{\sqrt{6(m+1)+2\tau_j}}, \qquad \tilde{M}_j^3 = \frac{9\sqrt{3(m+1)+\tau_j}}{\sqrt{2}}.$$

Then, the homogeneous components of a congruence curve with projective curvature $\tilde{k}_m$ are

$$\tilde{x}^j(s) = \sum_{k=0}^{2} \tilde{M}_k^j \frac{\rho_k(s)}{\prod_{h\neq k}(\tau_k - \tau_h)}, \qquad j = 0, 1, 2,$$

and the evolution of the congruence curve is given by

$$x^j(s,t) = \sum_{k=0}^{2} \mathrm{Exp}(t\xi)_k^j \tilde{x}^k(s - (1 + m(m-1))t), \qquad j = 0, 1, 2.$$

Remark 6. These curves have a spiral behavior and a gnomonic growth (i.e. made of successive self-congruent parts). Fig. 6 reproduces the spherical lifts of the congruence curves with parameter $m = 0.3$ and spectra

$$\sigma_1 = (-6.00233 + 6.06928i, -6.00233 - 6.06928i, 12.0047),$$
$$\sigma_2 = (-9.94554, -9.94554, 19.8911),$$

respectively. Fig. 7 reproduces the spherical lifts of the trajectories $\gamma(-,t)$ of the motion of a congruence curves with parameter $m = 0.7$, spectrum

$$\sigma = (-5.45852 + 6.40263i, 10.917, -5.45852 - 6.40263i)$$

and $t = 0, 0.5, 1, 1.5$ respectively.

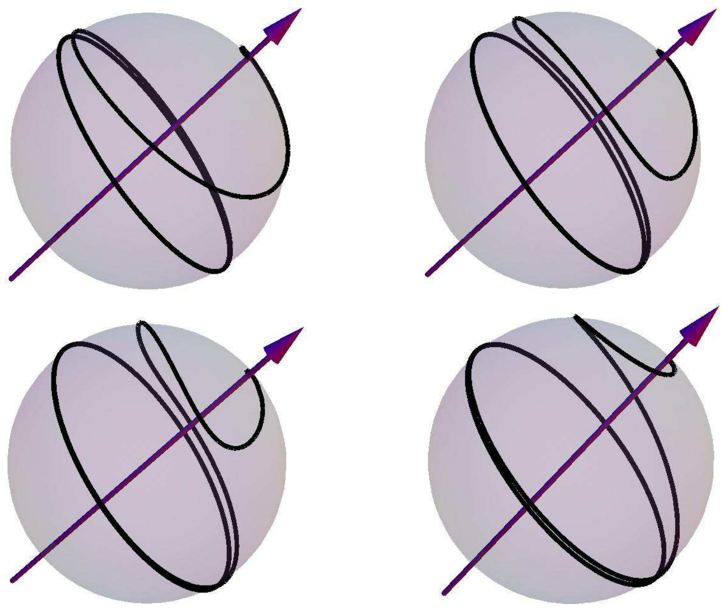

Figure 7. Trajectories $\gamma(-,t)$ of the congruence curve with parameter $m = 0.3$, spectrum σ for $t = 0, 0.5, 1$ and 1.5 respectively.

A Appendix

A.1 Code to compute the equations of the Kaup–Kupershmidt hierarchy

```
J1[h_, v_]:=D[h, {s, 3}];
J2[h_, v_]:=2 * D[v * Integrate[h, s], {s, 2}];
J3[h_, v_]:=8 * v^2 * Integrate[h, s] + 3(v * D[h, s] + D[v * h, s]);
J4[h_, v_]:=2 * Integrate[v * D[h, {s, 2}] + 4 * v^2 * h, s];
𝒥[h_, v_]:=Expand[FullSimplify[J1[h, v] + J2[h, v] + J3[h, v] + J4[h, v]]];
𝒟[h_, v_]:=D[h, {s, 3}] + v * D[h, s] + D[v * h, s];
H[0][v_]:=1; H[1][v_]:=D[v, {s, 2}] + 4 * v^2; H[n_][v_]:=𝒥[𝒟[H[n − 2][v], v], v];
𝔥[n_]:=Expand[H[n][u[s, t]]];
𝔮[n_]:=Expand[Integrate[𝒟[𝔥[n], u[s, t]], s]];
𝔭[n_]:=Expand[Integrate[H[n][ϵ * u[s, t]] * u[s, t], {ϵ, 0, 1}]]
KK[n_]:=Expand[D[u[s, t], t] + 𝒟[H[n][u[s, t]], u[s, t]]];
```

A.2 Code to solve numerically the projective Frenet system

Step I: define the speed, the curvature and domain of definition

```
m:=0.8;   t:=0;   v[s_]:=1;   a:=− 20;   b:=20;
```

$$k[\mathrm{s\_}]{:=}\frac{3m^2(1+2\mathrm{Cosh}[m*(s-m\verb|^|4*t)])}{2(2+\mathrm{Cosh}[m*(s-m\verb|^|4*t)])^2};$$

Step II: the routine to integrate the linear system

```
sol[1]:=NDSolve [{x'[t] == y[t], x[0] == 1, y'[t] == −k[t] * x[t] + z[t], y[0] == 0,
z'[t] == x[t] − k[t] * y[t], z[0] == 0}, {x, y, z}, {t, a, b}];
sol[2]:=NDSolve [{x'[t] == y[t], x[0] == 0, y'[t] == −k[t] * x[t] + z[t], y[0] == 1,
z'[t] == x[t] − k[t] * y[t], z[0] == 0}, {x, y, z}, {t, a, b}];
sol[3]:=NDSolve [{x'[t] == y[t], x[0] == 0, y'[t] == −k[t] * x[t] + z[t], y[0] == 0,
z'[t] == x[t] − k[t] * y[t], z[0] == 1}, {x, y, z}, {t, a, b}];
S[1][t_]:=Evaluate[{x[t], y[t], z[t]}/.sol[1]];
```

$S[2][t\_]$:=Evaluate[$\{x[t], y[t], z[t]\}$/.sol[2]];
$S[3][t\_]$:=Evaluate[$\{x[t], y[t], z[t]\}$/.sol[3]];
$\Gamma[t\_]$:=$\{S[1][t][[1]][[1]], S[2][t][[1]][[1]], S[3][t][[1]][[1]]\}$;
$\gamma[t\_]$:=$\frac{1}{\sqrt{\Gamma[t].\Gamma[t]}}\Gamma[t]$;

Acknowledgements

The work was partially supported by MIUR project: *Metriche riemanniane e varietà differenziabili*; by the GNSAGA of INDAM and by TTPU University in Tashkent. The author would like to thank the referees and G. Marí Beffa for their useful comments and suggestions.

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