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Analysis of composite plates through cell-based smoothed finite element and 4-noded mixed interpolation of tensorial components techniques

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Abstract

In this work, static bending and free vibration analysis of composite plates is performed under Carrera's Unified Formulation (CUF). With the objective of eliminating the shear locking phenomena that may occur in the Finite Element Method (FEM) based-analysis, it is presented a technique that combines the cell-based smoothed finite element method (CSFEM) and the 4-noded mixed interpolation of tensorial components approach (MITC4). The smoothing method is used for the approximation of the bending strains. The mixed interpolation allows the calculation of the shear transverse stress in a different manner. In several examples, accurate results are obtained, proving the efficiency of CSFEM-MITC4 methodology.

1 Introduction

Increasingly complex composite structures implies complex and effective means of analysis. Different approaches can be used in the study of laminated composite strutures [26,20,11,13]. In recent years, two-dimensional (2D) theories

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using higher-order displacement functions had proven to be a true alternative to the computationally very expensive 3D models. The theories can be equivalent-single-layer (ESL) or layerwise [30]. For the general description of 2D formulations for multilayered plates and shells, a Unified Formulation was derived by Carrera CUF) [7,21,5]. This formulation is a powerful tool to implement in a single software a large number of 2D models theories, ranging from ESL models to higher layerwise descriptions. The CUF can be used in the Finite Element Method (FEM) environment or with meshless methods [9].

Nevertheless, even with the very useful CUF, there is an important shortcoming of the FEM. For thin structures, the inclusion of both bending and shear stiffness in a unique rotational degree of freedom cause the locking of the finite element solution, leading to inaccurate numerical results. This shear locking phenomena can be alleviated by the use of some techniques: taking optimal rules of integration [10]; using the assumed strain method [1,32]; using field redistributed shape functions [22]; using the mixed interpolation of tensorial components (MITC) technique; incorporating the strain smoothing technique (SFEM) [24,2,18,17,23,16,25].

Another approach for the elimination of the shear locking phenomena is a combination of the previous remedies. Therefore, in this work a combination of the cell-based finite element method (CSFEM) with the 4-noded quadrilateral mixed interpolation of tensorial components technique (MITC4) and selective integration rule, is considered in the static bending and free vibration analysis of laminated composites, under the CUF framework. The main idea is to profit from each technique best and formulate an efficient and effective methodology for the free-locking analysis. The displacements are approximated through a sinusoidal deformation theory, and a complete study of the influence of the models parameters is performed.

The paper is organized as follows. Section 2 introduces the cell-based smoothed finite element method. In section 3 is present the shear strain field according to the 4-noded mixed interpolation tensorial components technique and it is emphasized the separation between bending and shear contributions for the stiffness matrix. The fundamental nuclei of the Carrera's unified formulation are given in detail. The shear locking phenomena is discussed in section 4. The present approach is compared with results available in the literature in section 5, concerning with the static bending and free vibration analysis.

2 Cell-Based Finite Element Method

In the Cell-Based Finite Element Method (CSFEM) each element is subdivided into smoothing domains, which are subcells where is applied a smoothing technique to the in-plane strains by a divergence estimation via a spatial averaging of the strain fields. The strain field, $\tilde{\varepsilon}_{ij}^{h}$ is computed by a weighted average of the standard strain field ε_{ij}^{h}. At a point $\mathbf{x}_C$ in an element Ω^h, the smoothed strain field is given by:

$$\tilde{\varepsilon}_{ij}^{h} = \int\limits_{\Omega^h} \varepsilon_{ij}^{h}(\mathbf{x})\Phi(\mathbf{x} - \mathbf{x}_C)\ \mathrm{d}\mathbf{x} \tag{1}$$

where Φ is a smoothing function and is chosen to be:

$$\Phi = \begin{cases} \frac{1}{A_C} & \mathbf{x}_C \in \Omega_C \\ 0 & \mathbf{x}_C \notin \Omega_C \end{cases} \tag{2}$$

being A_C is the area of the subcell.

The domain integrals are transformed into boundary integrals and so it is unnecessary to compute the gradient of shape functions to obtain the element bending stiffness matrix. For more detailed discussion see the references [19,17,18,25]

3 MITC4 under Carrera's Unified Formulation

The transverse shear strains, interpolated according to the 4-noded mixed interpolation of tensorial components (MITC4) technique, assume the following shear strain field:

$$\{\varepsilon_s\} = \begin{Bmatrix} \{\varepsilon_{xz}\} \\ \\ \{\varepsilon_{yz}\} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}(1+\xi)\varepsilon_{xz}^{N} + \frac{1}{2}(1-\xi)\varepsilon_{xz}^{Q} \\ \\ \frac{1}{2}(1+\eta)\varepsilon_{yz}^{P} + \frac{1}{2}(1-\eta)\varepsilon_{yz}^{M} \end{Bmatrix} \tag{3}$$

where M,N,P and Q are sample points in the element.

The stiffness matrix K is cleaved in two contributions, bending and shear:

$$
\begin{aligned}
[K] = & \ [K_b] + [K_s] \\
[K_b] = & < [B_b]^T [Q_b][B_b] >; \quad [K_s] =< [B_s]^T [Q_s][B_s] >
\end{aligned}
\tag{4}
$$

with the following notation:

$$
< \ldots >= \sum_{k=1}^{ns} \int_{V_k} (\ldots) dV_k \tag{5}
$$

The numerical code reflects this separation between bending and shear strains, under the framework of Carrera's Unified Formulation (CUF). According to this formulation , the governing equations that are derived based on the *Principle of Virtual Displacements*, can be written in terms of a few fundamental nuclei, which are simple matrices representing the basic element from which the stiffness matrices of the whole structure can be computed. The MITC4 technique is introduced within this procedure, and the corresponding governing equation is given by:

$$
\delta L_{int}^k = \delta \boldsymbol{q}_{\tau i}^{kT} \boldsymbol{K}^{k\tau sij} \boldsymbol{q}_{sj}^k \tag{6}
$$

The explicit expression of the fundamental nucleo $\boldsymbol{K}^{k\tau sij}$ is given below. More details can be found in [6]. Introducing the following notation:

$$
\triangleleft \ldots \triangleright_\Omega = \int_\Omega \ldots d\Omega \tag{7}
$$

the fundamental nucleo can be written as:

$$
\begin{aligned}
K_{xx}^{k\tau sij} =& C_{55}^{k} N_{iN} N_{jN} \lhd N_a N_a \rhd_{\Omega} F_{s,z} F_{\tau,z} + C_{55}^{k} N_{jQ} N_{iN} \lhd N_b N_a \rhd_{\Omega} F_{s,z} F_{\tau,z} + \\
& C_{55}^{k} N_{iN} N_{jQ} \lhd N_a N_b \rhd_{\Omega} F_{s,z} F_{\tau,z} + C_{55}^{k} N_{iQ} N_{jQ} \lhd N_b N_b \rhd_{\Omega} F_{s,z} F_{\tau,z} + \\
& C_{11}^{k} \lhd N_{i,x} N_{j,x} \rhd_{\Omega} F_s F_{\tau} + C_{16}^{k} \lhd N_{i,y} N_{j,x} \rhd_{\Omega} F_s F_{\tau} + \\
& C_{16}^{k} \lhd N_{i,x} N_{j,y} \rhd_{\Omega} F_s F_{\tau} + C_{66}^{k} \lhd N_{i,y} N_{j,y} \rhd_{\Omega} F_s F_{\tau} \\
\\
K_{yx}^{k\tau sij} =& C_{45}^{k} N_{iM} N_{jN} \lhd N_d N_a \rhd_{\Omega} F_{s,z} F_{\tau,z} + C_{45}^{k} N_{iP} N_{jN} \lhd N_c N_a \rhd_{\Omega} F_{s,z} F_{\tau,z} + \\
& C_{45}^{k} N_{iM} N_{jQ} \lhd N_d N_b \rhd_{\Omega} F_{s,z} F_{\tau,z} + C_{45}^{k} N_{iP} N_{jQ} \lhd N_c N_b \rhd_{\Omega} F_{s,z} F_{\tau,z} + \\
& C_{16}^{k} \lhd N_{i,x} N_{j,x} \rhd_{\Omega} F_s F_{\tau} + C_{12}^{k} \lhd N_{i,y} N_{j,x} \rhd_{\Omega} F_s F_{\tau} + \\
& C_{66}^{k} \lhd N_{i,x} N_{j,y} \rhd_{\Omega} F_s F_{\tau} + C_{26}^{k} \lhd N_{i,y} N_{j,y} \rhd_{\Omega} F_s F_{\tau} \\
\\
K_{zx}^{k\tau sij} =& C_{45}^{k} N_{i,yM} N_{jN} \lhd N_d N_a \rhd_{\Omega} F_{s,z} F_{\tau} + C_{45}^{k} N_{i,yM} N_{jQ} \lhd N_d N_b \rhd_{\Omega} F_{s,z} F_{\tau} + \\
& C_{55}^{k} N_{i,xN} N_{jN} \lhd N_a N_a \rhd_{\Omega} F_{s,z} F_{\tau} + C_{55}^{k} N_{i,xN} N_{jQ} \lhd N_a N_b \rhd_{\Omega} F_{s,z} F_{\tau} + \\
& C_{45}^{k} N_{i,yP} N_{jN} \lhd N_c N_a \rhd_{\Omega} F_{s,z} F_{\tau} + C_{45}^{k} N_{i,yP} N_{jQ} \lhd N_c N_b \rhd_{\Omega} F_{s,z} F_{\tau} + \\
& C_{55}^{k} N_{i,xQ} N_{jN} \lhd N_b N_a \rhd_{\Omega} F_{s,z} F_{\tau} + C_{55}^{k} N_{i,xQ} N_{jQ} \lhd N_b N_b \rhd_{\Omega} F_{s,z} F_{\tau} + \\
& C_{13}^{k} \lhd N_i N_{j,x} \rhd_{\Omega} F_s F_{\tau,z} + C_{36}^{k} \lhd N_i N_{j,y} \rhd_{\Omega} F_s F_{\tau,z} \\
\\
K_{xy}^{k\tau sij} =& C_{45}^{k} N_{iN} N_{jM} \lhd N_a N_d \rhd_{\Omega} F_{s,z} F_{\tau,z} + C_{45}^{k} N_{iQ} N_{jM} \lhd N_b N_d \rhd_{\Omega} F_{s,z} F_{\tau,z} + \\
& C_{45}^{k} N_{iN} N_{jP} \lhd N_a N_c \rhd_{\Omega} F_{s,z} F_{\tau,z} + C_{45}^{k} N_{iQ} N_{jP} \lhd N_b N_c \rhd_{\Omega} F_{s,z} F_{\tau,z} + \\
& C_{16}^{k} \lhd N_{i,x} N_{j,x} \rhd_{\Omega} F_s F_{\tau} + C_{66}^{k} \lhd N_{i,y} N_{j,x} \rhd_{\Omega} F_s F_{\tau} + \\
& C_{12}^{k} \lhd N_{i,x} N_{j,y} \rhd_{\Omega} F_s F_{\tau} + C_{26}^{k} \lhd N_{i,y} N_{j,y} \rhd_{\Omega} F_s F_{\tau}
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
K_{yy}^{k\tau sij} =& C_{44}^k N_{iM} N_{jM} \lhd N_d N_d \rhd_\Omega F_{s,z} F_{\tau,z} + C_{44}^k N_{iP} N_{jM} \lhd N_c N_d \rhd_\Omega F_{s,z} F_{\tau,z} + \\
& C_{44}^k N_{iM} N_{jP} \lhd N_d N_c \rhd_\Omega F_{s,z} F_{\tau,z} + C_{44}^k N_{iP} N_{jP} \lhd N_c N_c \rhd_\Omega F_{s,z} F_{\tau,z} + \\
& C_{66}^k \lhd N_{i,x} N_{j,x} \rhd_\Omega F_s F_\tau + C_{26}^k \lhd N_{i,y} N_{j,x} \rhd_\Omega F_s F_\tau + \\
& C_{26}^k \lhd N_{i,x} N_{j,y} \rhd_\Omega F_s F_\tau + C_{22}^k \lhd N_{i,y} N_{j,y} \rhd_\Omega F_s F_\tau \\
\\
K_{zy}^{k\tau sij} =& C_{44}^k N_{i,yM} N_{jM} \lhd N_d N_d \rhd_\Omega F_{s,z} F_\tau + C_{44}^k N_{i,yM} N_{jP} \lhd N_d N_c \rhd_\Omega F_{s,z} F_\tau + \\
& C_{45}^k N_{i,xN} N_{jM} \lhd N_a N_d \rhd_\Omega F_{s,z} F_\tau + C_{45}^k N_{i,xN} N_{jP} \lhd N_a N_c \rhd_\Omega F_{s,z} F_\tau + \\
& C_{44}^k N_{i,yP} N_{jM} \lhd N_c N_d \rhd_\Omega F_{s,z} F_\tau + C_{44}^k N_{i,yP} N_{jP} \lhd N_c N_c \rhd_\Omega F_{s,z} F_\tau + \\
& C_{45}^k N_{i,xQ} N_{jM} \lhd N_b N_d \rhd_\Omega F_{s,z} F_\tau + C_{45}^k N_{i,xQ} N_{jP} \lhd N_b N_c \rhd_\Omega F_{s,z} F_\tau + \\
& C_{36}^k \lhd N_i N_{j,x} \rhd_\Omega F_s F_{\tau,z} + C_{23}^k \lhd N_i N_{j,y} \rhd_\Omega F_s F_{\tau,z} \\
\\
K_{xz}^{k\tau sij} =& C_{45}^k N_{iN} N_{j,yM} \lhd N_a N_d \rhd_\Omega F_s F_{\tau,z} + C_{45}^k N_{iQ} N_{j,yM} \lhd N_b N_d \rhd_\Omega F_s F_{\tau,z} + \\
& C_{55}^k N_{iN} N_{j,xN} \lhd N_a N_a \rhd_\Omega F_s F_{\tau,z} + C_{55}^k N_{iQ} N_{j,xN} \lhd N_b N_a \rhd_\Omega F_s F_{\tau,z} + \\
& C_{45}^k N_{iN} N_{j,yP} \lhd N_a N_c \rhd_\Omega F_s F_{\tau,z} + C_{45}^k N_{iQ} N_{j,yP} \lhd N_b N_c \rhd_\Omega F_s F_{\tau,z} + \\
& C_{55}^k N_{iN} N_{j,xQ} \lhd N_a N_b \rhd_\Omega F_s F_{\tau,z} + C_{55}^k N_{iQ} N_{j,xQ} \lhd N_b N_b \rhd_\Omega F_s F_{\tau,z} + \\
& C_{13}^k \lhd N_{i,x} N_j \rhd_\Omega F_{s,z} F_\tau + C_{36}^k \lhd N_{i,y} N_j \rhd_\Omega F_{s,z} F_\tau \\
\\
K_{yz}^{k\tau sij} =& C_{44}^k N_{iM} N_{j,yM} \lhd N_d N_d \rhd_\Omega F_s F_{\tau,z} + C_{44}^k N_{iP} N_{j,yM} \lhd N_c N_d \rhd_\Omega F_s F_{\tau,z} + \\
& C_{45}^k N_{iM} N_{j,xN} \lhd N_d N_a \rhd_\Omega F_s F_{\tau,z} + C_{45}^k N_{iP} N_{j,xN} \lhd N_c N_a \rhd_\Omega F_s F_{\tau,z} + \\
& C_{44}^k N_{iM} N_{j,yP} \lhd N_d N_c \rhd_\Omega F_s F_{\tau,z} + C_{44}^k N_{iP} N_{j,yP} \lhd N_c N_c \rhd_\Omega F_s F_{\tau,z} + \\
& C_{45}^k N_{iM} N_{j,xQ} \lhd N_d N_b \rhd_\Omega F_s F_{\tau,z} + C_{45}^k N_{iP} N_{j,xQ} \lhd N_c N_b \rhd_\Omega F_s F_{\tau,z} + \\
& C_{36}^k \lhd N_{i,x} N_j \rhd_\Omega F_{s,z} F_\tau + C_{23}^k \lhd N_{i,y} N_j \rhd_\Omega F_{s,z} F_\tau \\
\\
K_{zz}^{k\tau sij} =& C_{44}^k N_{i,yM} N_{j,yM} \lhd N_d N_d \rhd_\Omega F_s F_\tau + C_{45}^k N_{i,xN} N_{j,yM} \lhd N_a N_d \rhd_\Omega F_s F_\tau + \\
& C_{44}^k N_{i,yP} N_{j,yM} \lhd N_c N_d \rhd_\Omega F_s F_\tau + C_{45}^k N_{i,xQ} N_{j,yM} \lhd N_b N_d \rhd_\Omega F_s F_\tau + \\
& C_{45}^k N_{i,yM} N_{j,xN} \lhd N_d N_a \rhd_\Omega F_s F_\tau + C_{55}^k N_{i,xN} N_{j,xN} \lhd N_a N_a \rhd_\Omega F_s F_\tau + \\
& C_{45}^k N_{i,yP} N_{j,xN} \lhd N_c N_a \rhd_\Omega F_s F_\tau + C_{55}^k N_{i,xQ} N_{j,xN} \lhd N_b N_a \rhd_\Omega F_s F_\tau + \\
& C_{44}^k N_{i,yM} N_{j,yP} \lhd N_d N_c \rhd_\Omega F_s F_\tau + C_{45}^k N_{i,xN} N_{j,yP} \lhd N_a N_c \rhd_\Omega F_s F_\tau + \\
& C_{44}^k N_{i,yP} N_{j,yP} \lhd N_c N_c \rhd_\Omega F_s F_\tau + C_{45}^k N_{i,xQ} N_{j,yP} \lhd N_b N_c \rhd_\Omega F_s F_\tau + \\
& C_{45}^k N_{i,yM} N_{j,xQ} \lhd N_d N_b \rhd_\Omega F_s F_\tau + C_{55}^k N_{i,xN} N_{j,xQ} \lhd N_a N_b \rhd_\Omega F_s F_\tau + \\
& C_{45}^k N_{i,yP} N_{j,xQ} \lhd N_c N_b \rhd_\Omega F_s F_\tau + C_{55}^k N_{i,xQ} N_{j,xQ} \lhd N_b N_b \rhd_\Omega F_s F_\tau + \\
& C_{33}^k \lhd N_i N_j \rhd_\Omega F_{s,z} F_{\tau,z}
\end{aligned} \tag{9}
$$

4 Shear locking phenomena

For thin structures, the inclusion of both bending and shear stiffness in a unique rotational degree of freedom, may cause the locking of the finite element, with oscillations in shear and membrane strains. There are some remedies for the locking phenomena: use an optimal rule of integration [10]; use the assumed strain method [1,32]; use field redistributed shape functions [22]. In this study, three procedures are combined to eliminate the locking: the cell-based smoothing technique (CSFEM), and the 4-noded mixed interpolation tensorial component (MITC4) technique that calculates the transverse shear stresses σ_{xz} and σ_{yz} in a different manner from other tensorial components. For the approximation of the bending strains it is considered the CSFEM. In the case of the shear strains the methodology uses MITC4 approach. If the thickness-to-side ratio of the structure is bigger than 0.1 a normal integration scheme (2×2 Gauss points) is used. It should be noted that the MITC4 technique by itself doesn't require any kind of selective integration in order to overcome the shear locking phenomena. In this paper, due to the combination with CSFEM technique, it was chosen a selective rule of integration providing some stiffness overestimation to compensate the inclusion of CSFEM technique, and that led to accurate solutions in less computational time, even thought some spurious mode appeared.

5 Numerical examples

Static bending and free vibration analysis of composite laminate plate is performed as follows. The in-plane displacements u, v and the transverse displacement w are expressed by sinusoidal shear deformation theory denoted by SINUS:

$$\begin{aligned} u &= u_o + zu_1 + \sin\left(\frac{\pi z}{h}\right) u_2 \\ v &= v_o + zv_1 + \sin\left(\frac{\pi z}{h}\right) v_2 \\ w &= w_o + zw_1 + \sin\left(\frac{\pi z}{h}\right) w_2 \end{aligned} \tag{10}$$

where u_o, v_o and w_o are translations of a point at the middle-surface of the plate [33].

In this study a 20×20 structured quadrilateral mesh is considered for the pretended comparison with benchmark results. The present results are denoted by CSFEM-MITC4. Concerning to the shear strains, the performed integration rule depends on the thickness-to-side ratio, as mentioned above.

5.1 *Static bending*

In this section the static bending analysis is made for cross-ply laminated plates with three and four layers under following sinusoidal load:

$$p_z = P_o \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{11}$$

where P_o is the amplitude of the mechanical load.

5.1.1 *Four layer (0°/90°)$_\mathrm{s}$ square cross-ply laminated plate under sinusoidal load*

A square simply supported laminate plate of thickness-to-side ratio h/a, composed of four equally thick layers oriented at $(0°/90°)_\mathrm{s}$ is considered. The plate is subjected to a vertical pressure given by eqn:r[illegible]hload. The material properties are as follows: $E_1 = 25E_2$; $G_{12} = G_{13} = 0.5E_2$; $G_{23} = 0.2E_2$; $\nu_{12} = 0.25$. In Table 1, we present results for the SINUS theory with the combined CSFEM-MITC4 approach. We compare the results with higher order plate theories [28,8], first order theory [29], an exact solution [27], and the standard (FEM Q4) and smoothed (CS-FEM Q4) 4-noded element with field consistent approach [22]. It can be seen that the results from the CSFEM-MITC4 formulation show very good agreement with those in the literature and is insensitive to shear locking with the selective rule of integration.

Method	$a/h = 4$	$a/h = 10$	$a/h = 100$
HSDT [28]	1.8937	0.7147	0.4343
FSDT [29]	1.7100	0.6628	0.4337
Elasticity [27]	1.9540	0.7430	0.4347
RBF [8]	1.9783	0.7325	0.4307
FEM Q4 [22]	1.8949	0.7135	0.4302
CS-FEM Q4 (4 subcells) [22]	1.9089	0.7195	0.4304
Present (CSFEM-MITC4)	1.9086	0.7201	0.4304

Table 1
Normalized central deflection $\overline{w} = w(a/2, a/2, 0)\frac{100E_2h^3}{Pa^4}$ of a simply supported cross-ply laminated square plate $[0°/90°/90°/0°]$, with $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, ν_{12} =0.25. Quadrilateral mesh with 20 × 20 elements for the present formulation

5.1.2 *Three layer (0°/90°/0°) square cross ply laminated plate under sinusoidal load*

A square laminate plate of thickness-to-side ratio h/a, composed of three equally thick layers oriented at $(0°/90°/0°)$ is considered. It is simply supported on all edges and subjected to a vertical pressure of the form (11). The material properties are: E_1 =132.38 GPa, $E_2 = E_3$ =10.756 GPa, G_{12} =3.606 GPa, $G_{13} = G_{23} = 5.6537$ GPa, $\nu_{12} = \nu_{13} = 0.24$, $\nu_{23} = 0.49$. In Table 2, we present results for the SINUS theory with the present CSFEM-MITC4 approach. The results from the present approach are compared with an analytical solution [3,4], results from MITC4 formulation [6], and results from the standard (FEM Q4) and smoothed (CS-FEM Q4) 4-noded element with field consistent approach [22]. The numerical results from the present formulation are precise and agree with the existing solutions, being insensitive to shear locking, as the plate gets thinner.

$\overline{w}$	a/h				
	10	50	100	500	1000
Analytical (ESL-2) [3,4]	0.9249	0.7767	0.7720	0.7705	0.7704
MITC4 [6]	0.9195	0.7713	0.7666	0.7650	0.7650
FEM Q4 [22]	0.9152	0.7700	0.7651	0.7636	0.7635
CS-FEM Q4 (4 subcells) [22]	0.9235	0.7703	0.7655	0.7639	0.7639
Present (CSFEM-MITC4)	0.9238	0.7704	0.7655	0.7639	0.7639

Table 2
Transverse displacement $\overline{w} = w(a/2, a/2, h/2)$ at the center of a multilayered plate $[0°/90°/0°]$ with E_1 = 132.38 GPa, $E_2 = E_3$ = 10.756 GPa, G_{12} = 3.606 GPa, $G_{13} = G_{23}$ = 5.6537 GPa, $\nu_{12} = \nu_{13} = 0.24$, $\nu_{23} = 0.49$. Quadrilateral mesh with 20 × 20 elements for the present formulation

5.2 *Free vibration - cross-ply laminated plates*

Consider a simply supported square plate of the cross-ply lamination $(0°/90°)_s$ where all layers are assumed to be of the same thickness, density and made up of the same linear elastic material. The following material properties are considered for each layer

$$\frac{E_1}{E_2} = 10,20,30, \text{ or } \;\; 40; \quad G_{12} = G_{13} = 0.6E_2;$$
$$G_{23} = 0.5E_2; \nu_{12} = 0.25.$$

The subscripts 1 and 2 denote the directions normal and the transverse to the fiber direction in a lamina, which may be oriented at an angle to the plate axes. The ply angle of each layer is measure from the global $x-$axis to the fiber direction. The thickness-to-side ratio is $h/a = 0.2$.

Method	Mesh	subcell(s)	E_1/E_2			
			10	20	30	40
Liew [15]			8.2924	9.5613	10.3200	10.8490
Reddy, Khdeir [14]			8.2982	9.5671	10.3260	10.8540
FSDT [9]	21×21		8.2982	9.5671	10.3258	10.8540
HSDT [9] ($\nu_{23} = 0.18$)	21×21		8.2999	9.5411	10.2687	10.7652
FEM Q4 [22]	20×20		8.3651	9.5801	10.2980	10.7894
CS-FEM Q4 [22]	20×20	4	8.3639	9.5790	10.2970	10.7883
Present (CSFEM-MITC4)	20×20	4	8.3775	9.5857	10.3001	10.7892

Table 3
Normalized fundamental frequency $\Omega = \omega a^2/h\sqrt{\rho/E_2}$ of a simply supported cross-ply laminated square plate $(0°/90°)_s$ with $h/a = 0.2$, $\frac{E_1}{E_2} = 10$, 20, 30 or 40, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = 0.25$.

Table 3 lists the fundamental frequency for different ratio of Young's modulus, E_1/E_2. The results from the present CSFEM-MITC4 formulation are compared with the meshfree results of Liew *et al.* [15], the results based on higher order theory [14], the results based on FSDT and higher order theories with radial basis functions [9] and the results using the standard (FEM Q4) and smoothed (CS-FEM Q4) 4-noded element with field consistent approach [22]. It can be observed that the present numerical procedure provides accurate results and similar to those in the literature.

Method	a/h					
	2	4	10	20	50	100
FSDT [34]	5.4998	9.3949	15.1426	17.6596	18.6742	18.8362
Model-1 (12dofs) [12]	5.4033	9.2870	15.1048	17.6470	18.6720	18.8357
Model-2 (9dofs) [12]	5.3929	9.2710	15.0949	17.6434	18.6713	18.8355
HSDT [28]	5.5065	9.3235	15.1073	17.6457	18.6718	18.8356
HSDT [31]	6.0017	10.2032	15.9405	17.9938	18.7381	18.8526
FEM Q4 [22]	5.4029	9.3005	15.1790	17.7578	18.7993	18.9657
CS-FEM Q4 (4 subcells) [22]	5.4026	9.2998	15.1766	17.7540	18.7947	18.9611
Present (CSFEM-MITC4)	5.3986	9.2975	15.1674	17.7471	18.7895	18.9561

Table 4
Variation of fundamental frequencies, $\Omega = \omega a^2/h\sqrt{\rho/E_2}$ with a/h for a simply supported square laminated plate $[0°/90°/90°/0°]$, with $E_1/E_2 = 40$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. Quadrilateral mesh with 20×20 elements for the present formulation

In Table 4 is exhibited the effect of the thickness-to-side ratio of a simply supported cross-ply laminated square plate on the fundamental frequency, for Young's modulus $E_1/E_2 = 40$. The results from the present CSFEM-MITC4 formulation are compared with the results based on first order theory [34], analytical solutions [12], results from higher order theories [28,31], and results using the standard (FEM Q4) and smoothed (CS-FEM Q4) 4-noded element with field consistent approach [22]. It can be seen that the present results are in a good agreement with the results available in the literature and they are accurate even for thin plates, which proves that the present methodology serves its propose of eliminating the shear locking.

6 Conclusion

In this work a technique that combines the cell-based smoothed finite element method (CSFEM) and the 4-noded mixed interpolation of tensorial components approach (MITC4) was presented for the static bending and free vibration analysis of composite plates and performed under Carrera's Unified Formulation (CUF). Throughout a set of benchmark examples it proved to be an efficient methodology, providing accurate results due the elimination of the shear locking phenomena. The CSFEM-MITC4 procedure has the potential to be successful in future works on the analysis of multilayered structures.

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