CHERN-FLAT AND RICCI-FLAT INVARIANT ALMOST HERMITIAN STRUCTURES

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Abstract. We study left-invariant almost Hermitian structures on homogeneous spaces having either flat Chern connection or flat Ricci-Chern form. Many examples are carefully described and a classification is given in low dimensions.

1. introduction

The Chern connection is a classical object in almost Hermitian and almost Kähler geometry. It is defined as the unique Hermitian connection whose torsion has vanishing $(1,1)$-component, and it was firstly introduced by Ehresmann and Libermann in [12]. In the complex case it coincides with the connection used by Chern in [7] to compute representatives of characteristic classes (hence the name). In the nearly Kähler case this connection has parallel and skew-symmetric torsion (see e.g. [14]), while in the quasi-Kähler (or $(1,2)$-symplectic) case it coincides with the so called second canonical Hermitian connection (see [15]).

Let (M,g,J) be an almost Hermitian manifold and let ∇ be its Chern connection and R the curvature tensor corresponding to ∇. Let $\mathrm{Ric}(J) := \frac{1}{2}\mathrm{tr}_\omega \mathrm{R}(X,Y,\cdot,\cdot)$ be the Ricci form of the Chern connection, where ω is the fundamental form associated to (g,J).

Definition 1.1. *For the purpose of this article, the almost Hermitian structure (g,J) is called*

- Ricci-flat *if* $\mathrm{Ric}(J) = 0$*;*
- Chern-flat *if* $\mathrm{R} = 0$*.*

The starting point of this paper was the problem of determining all the invariant Ricci-flat and Chern-flat almost complex structures on the Iwasawa manifold. The latter is a classical example of a 6-dimensional compact manifold admitting both complex structures and symplectic structures but no Kähler structure. It is defined as a compact quotient of the complex Heisenberg group and the geometry of its almost complex structures was thoroughly investigated in [2]. Let (M,g) be the Iwasawa manifold equipped with the standard metric and denote by $\mathcal{Z}_g$ the space of left-invariant almost complex structures compatible with g and a fixed orientation. Then $\mathcal{Z}_g$ is canonically identified with $\mathbb{CP}^3$ and has the four distinguished points J_0, J_1, J_2 , J_3 which correspond to vertices of a tetrahedron [3, p.155] and [21]. We will show that J_0 and J_3 are the only Chern-flat structures and that all $J \in \mathcal{Z}_g$ are Ricci-flat.

In [10] the authors proved that the holonomy group of the Chern connection of a quasi-Kähler structure on a compact manifold M is trivial if and only if M is a 2-step nilmanifold whose associated Lie algebra satisfies some relations. One of these relations is the following

$$[JX, JY] = -[X,Y] \tag{1.1}$$

for all X, Y in the Lie algebra associated to M. In the present paper we call an invariant almost complex structure satisfying (1.1) an *anti-abelian* almost complex structure. One of the goals of this paper is to show that for a nilmanifold the vanishing of the curvature tensor of the Chern connection implies that the holonomy group of the Chern connection is trivial, i.e. the induced representation of the first fundamental group is trivial. Indeed this is a corollary of the following result

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second **Theorem 1.2.** *Let M be manifold whose universal covering is a Lie group G and let $\pi\colon G\to M$ be the covering map. Let (g,J) be an almost Hermitian structure on M such that $\pi^*g\,,\pi^*J$ are left-invariant.*

(i) *If π^*J is anti-abelian then J is Chern-flat.*

(ii) *If G is nilpotent, then π^*J is anti-abelian if and only if J is Chern-flat.*

(iii) *There exists a compact solvmanifold (M,g,J) such that J is Chern-flat but π^*J is not anti-abelian.*

cortes **Corollary 1.3.** *Let $(M = \Gamma\backslash G, J, g)$ be a nilmanifold with a left-invariant almost Hermitian structure. The following facts are equivalent:*

(i) *J is anti-abelian;*
(ii) *the holonomy group of the Chern connection associated to (g, J) is trivial;*
(iii) *(g, J) is Chern-flat.*

The anti-abelian condition (1.1) is "opposite" to the more familiar abelian condition $[JX, JY] = [X, Y]$. Abelian complex structures were introduced in [5] and were intensely studied in [4, 6, 11, 8, 19]. Observe that J is anti-abelian and integrable if and only if it is bi-invariant, i.e. $J[X, Y] = [JX, Y]$. It follows from Corollary 1.3 that studying Chern-flat structures on nilmanifolds is equivalent to studying anti-abelian almost complex structures on nilpotent Lie algebras (nilalgebras).

According to published classifications, there are just 3 isomorphism classes of 4-dimensional nilalgebras and there are 34 isomorphism classes of 6-dimensional nilpotent Lie algebra [20, 16]. These isomorphism classes can be described by using an adequate choice of the basis as in [22] (see Table 1).

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension 4 or 6 and let $\{e_1, e_2, \cdots, e_n\}$ be a basis as in Table 1. Declaring the vectors e_i to be orthonormal determines an invariant Riemannian metric g and an orientation $\sigma = e_1 \wedge \cdots \wedge e_n$. We denote by $\mathcal{Z}_g$ the set of invariant positively-oriented orthogonal almost complex structures (OACS's).

For low dimensions we have the following result.

secondmain **Theorem 1.4.** *If a 4-dimensional Lie algebra $\mathfrak{g}$ admits an anti-abelian almost complex structure J then $\mathfrak{g}$ is either an abelian Lie algebra or $\mathfrak{g}$ is the complexification of a 2-dimensional solvable Lie algebra.*

In dimension 6 the nilalgebras admitting an anti-abelian OACS are given by the table

$\mathfrak{g}$	*Structure equations*	*Anti-abelian structures in $\mathcal{Z}_g$*
$\mathfrak{g}_{27}$	$0,0,0,0,13+42,14+23$	$J_0\,,J_3$
$\mathfrak{g}_{28}$	$0,0,0,0,12,14+23$	$J_0'\,,J_3'$
$\mathfrak{g}_{29}$	$0,0,0,0,0,12+34$	$J_0'\,,J_3'$
$\mathfrak{g}_{34}$	$0,0,0,0,0,0$	$\mathcal{Z}_g$

where $J_0\,, J_3\,, J_0'\,, J_3'$ are the OACS's given by the forms

$$\omega_0 = e^{12} + e^{34} + e^{56}\,, \quad \omega_3 = -e^{12} - e^{34} + e^{56}\,,$$
$$\omega_0' = e^{13} + e^{24} + e^{56}\,, \quad \omega_3' = -e^{13} - e^{24} + e^{56}\,.$$

In the above list: $\mathfrak{g}_{27}$ and $\mathfrak{g}_{28}$ are the unique 6-dimensional irreducible nilalgebras having first Betti number equal to 4; $\mathfrak{g}_{27}$ is the Lie algebra associated to the 3-dimensional complex Heisenberg group (and to the Iwasawa manifold); $\mathfrak{g}_{29}$ is reducible and is the direct sum of $\mathbb{R}$ with the Lie algebra $\mathfrak{g} = (0,0,0,0,12+34)$ associated to the 5-dimensional real Heisenberg group; $\mathfrak{g}_{34}$ is merely the 6-dimensional abelian Lie algebra.

For Ricci-flat almost Hermitian structures, we obtain the following results in dimension 4 and 6.

main4 **Theorem 1.5.** *For 4-dimensional nilagebras we have the table*

$\mathfrak{g}$	*Structure equations*	*Ricci-Flat structures*	AK *Ricci-flat structures*
$\mathbb{R}^4$	$0,0,0,0$	$\mathcal{Z}_g$	$\mathcal{Z}_g$
$\mathfrak{h}_3 \oplus \mathbb{R}$	$0,0,12,0$	$\mathcal{Z}_g$	$Je_3 \in \langle e_1, e_2\rangle$
$\mathfrak{n}_4$	$0,0,12,23$	$J \in \mathcal{Z}_g \ : \ Je_3 \in \langle e_1, e_3, e_4\rangle$	$\pm(e^{13} + e^{24})$

where AK *means that (g, J) is almost Kähler (i.e. the fundamental form of (g, J) is closed).*

Let $\mathfrak{g}$ be a 6-dimensional nilalgebra having $b_1 \geq 4$. Then

1. *$\mathfrak{g}$ belongs to the list*

$$(0,0,0,0,12,14+23)\,,\quad (0,0,0,0,12,34)\,,\quad (0,0,0,0,13+42,14+23)$$
$$(0,0,0,0,0,12+34)\,,\quad (0,0,0,0,12,13)\,,\quad (0,0,0,0,0,12)\,,\quad (0,0,0,0,0,0),$$

and any $J \in \mathcal{Z}_g$ is Ricci-flat.

2. *$\mathfrak{g}$ is one of the Lie algebras*

$$(0,0,0,0,12,15+34)\,,\ (0,0,0,0,12,15)\,,\ (0,0,0,0,12,14+25)$$

then $J \in \mathcal{Z}_g$ is Ricci-flat if and only if $[Je_6, e_6] = 0$ and one of the following two conditions holds:

i. $[e_5, Je_5] = 0\,;$

ii. *$J\langle e_5, e_6\rangle$ is contained in $\langle e_1, e_2, e_3, e_4\rangle$.*

In the above list $\mathfrak{h}_3 \oplus \mathbb{R}$ is the Lie algebra associated to the so called Kodaira-Thurston manifold (see e.g. [3, 25]). Moreover, we remark that almost Kähler Ricci-flat structures can be seen as special Hermitian-symplectic structures (see [18, 23, 26]).

As a consequence of Theorem 1.5 we have the following

Corollary 1.6. *Let (M, g) be the Iwasawa manifold with its standard metric. Then any $J \in \mathcal{Z}_g$ is Ricci-flat. Moreover $J \in \mathcal{Z}_g$ is Chern-flat if and only if $J = J_0$ or $J = J_3$.*

The paper is organized as follows:
In § 2 we recall some basic facts about almost complex structures and the Chern connection. In § 3 we prove theorems 1.2 and 1.4. In § 4 we study left-invariant Ricci-flat almost Hermitian structures on some 4-dimensional and 6-dimensional nilmanifolds proving Theorem 1.5.

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2. Preliminaries

Let M^{2n} be a $2n$-dimensional smooth manifold. An *almost Hermitian structure* on M is a pair (g, J), where J is an almost complex structure ($J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{I}$) and g is a Riemannian metric such that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. Any almost Hermitian structure induces the so-called *fundamental form* (or *Kähler form*) $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$. Moreover, the almost complex structure J allows us to decompose the complexified tangent bundle in

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}\,,$$

where

$$T^{1,0}_x = \{v \in T_xM \otimes \mathbb{C} \;:\; Jv = \mathrm{i}\,v\}\,;$$
$$T^{0,1}_x = \{v \in T_xM \otimes \mathbb{C} \;:\; Jv = -\,\mathrm{i}\,v\}\,;$$

and, consequently, the vector bundle $\Lambda^r_{\mathbb{C}}M$ of complex r-forms on M splits as

$$\Lambda^r_{\mathbb{C}}M = \bigoplus_{p+q=r} \Lambda^{p,q}\,.$$

An almost complex structure is called *integrable* (or a *complex structure*) if the associated Nijenhuis tensor

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

vanishes everywhere. Given an almost Hermitian structure (g, J) we denote by D the Levi-Civita connection of g.

An almost Hermitian structure (g, J) is called

- *almost Kähler* if $d\omega = 0$;
- *nearly Kähler* if DJ is a skew-symmetric tensor;
- *Kähler* if $N_J = 0$ and $d\omega = 0$ or, equivalently, if $DJ = 0$.

Given an almost Hermitian manifold (M^{2n}, g, J) there exists a unique connection ∇ on M satisfying

$$\nabla g = \nabla J = 0\,, \quad \mathrm{Tor}(\nabla)^{1,1} = 0\,,$$

where $\mathrm{Tor}(\nabla)^{1,1}$ denotes the $(1,1)$-part of the torsion of ∇. Condition $\mathrm{Tor}(\nabla)^{1,1} = 0$ means that $\mathrm{Tor}(\nabla)(Z_1, \overline{Z}_2) = 0$ for any pair Z_1, Z_2 of vector fields of type $(1,0)$. ∇ is usually called the *Chern connection*. In terms of the Levi-Civita connection D of g, ∇ is described by the following formula

nabla

$$g(\nabla_X Z, Y) = g(D_X Z, Y) + \frac{1}{4} g((D_{JY}J + JD_Y J)X, Z) - \frac{1}{4} g((D_{JZ}J + JD_Z J)X, Y)\,. \tag{2.1}$$

Let R be the curvature tensor of ∇. Then it is defined the Ricci form of ∇

$$\mathrm{Ric}(J)(X, Y) = \sum_{i=1}^{n} \mathrm{R}(X, Y, Z_i, Z_{\bar{i}})\,,$$

where $\{Z_1, \ldots, Z_n\}$ is an arbitrary unitary frame. $\mathrm{Ric}(J)$ is a closed 2-form and $\frac{\mathrm{i}}{2\pi}\mathrm{Ric}(J)$ represents the first Chern class of J. Moreover, we can locally write $\mathrm{Ric}(J) = d\theta$, where θ is the 1-form

$$\theta(X) = \sum_{i=1}^{n} g(\nabla_X Z_i, Z_{\bar{i}})\,,$$

$\{Z_1, \ldots, Z_n\}$ being a locally defined unitary frame (see the appendix at the end of the paper for a proof of this well-known result). An almost Hermitian structure is called *Ricci-flat* if $\mathrm{Ric}(J)$ vanishes.

In the last part of the paper we study Ricci-flat structures on Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. An almost Hermitian structure on $\mathfrak{g}$ is a pair (g, J), where J is an isomorphism of $\mathfrak{g}$ such that $J^2 = -\mathrm{Id}$ and g is a J-compatible inner product. If we have a Lie group G (or more generally a quotient of a Lie group by a lattice), then giving a left-invariant almost Hermitian structure is equivalent to assigning an almost Hermitian structure on the Lie algebra of G. Hence the study of left-invariant almost Hermitian structures reduces to the study of almost Hermitian structure on Lie algebras.

Now we describe the notation we will use for Lie algebras:
Let $\mathfrak{g}$ be a Lie algebra; then $\mathfrak{g}$ can be described in terms of the so-called *structures equations.* Indeed, $\mathfrak{g}$ is completely determined by the exterior derivatives of an arbitrary coframe and this allows us to describe $\mathfrak{g}$ by using a list of numbers. For example, if we write

$$\mathfrak{g} = (0,0,0,0,13+42,14+23)\,,$$

we mean that there exists a coframe $\{e^1,\dots,e^6\}$ of $\mathfrak{g}$ satisfying the following equations

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = 0\,,\\ de^5 = e^{13} + e^{42}\,,\\ de^6 = e^{14} + e^{23}\,, \end{cases}$$

where $e^{i_1\cdots i_r} := e^{i_1}\wedge\cdots\wedge e^{i_r}$. Notice that different structure equations could define the same Lie algebra. Moreover, if some structure equations are fixed, then any associated coframe $E := \{e^1,\dots,e^n\}$, induces the metric $g = \sum e^i \otimes e^i$. If E, E' are two frames associated to the same structure equations, then the induced metric are isometric, this means that there exists an isomorphism of Lie algebras $L\colon \mathfrak{g}\to\mathfrak{g}$ such that $L^*(g') = g$. Thus if some structure equations on a Lie algebra $\mathfrak{g}$ are fixed, then we have a metric g on $\mathfrak{g}$ which is unique up to isomorphisms of $\mathfrak{g}$.

3. Chern-flat structures

In this section we study left-invariant Chern-flat almost complex structures and we prove Theorem 1.2 and Theorem 1.4.

3.1. **Left-invariant flat connections.** Let G be a simply-connected Lie group endowed with a left-invariant flat connection ∇ (i.e. $\mathrm{R}^\nabla = 0$). Fix a left-invariant frame $E := \{e_1,\dots,e_n\}$, where $n = dim_{\mathbb{R}}(G)$, and let $X := \{X_1,\dots,X_n\}$ be a ∇-parallel frame such that $e_i(e) = X_i(e)$ for $i = 1,\dots,n$, where e denotes the identity of G. Then there exists a map $\rho : G \to \mathrm{GL}(n)$, $\rho(g) = (\rho(g)_{ij})$:

$$X_g = (X_1\cdots X_n) = (e_1\cdots e_n)\begin{pmatrix} \rho_{11}(g) & \cdots & \rho_{1n}(g)\\ \vdots & \vdots & \vdots\\ \rho_{n1}(g) & \cdots & \rho_{nn}(g)\end{pmatrix} = E_g\,\rho(g)\,.$$

Note that $\rho(e) = \mathrm{Id}$. Let $L_h : G \to G$ be a left multiplication by $h \in G$. Then $L_hX = (L_hX_1\cdots L_hX_n)$ is also a ∇-parallel frame, where $(L_h(X))_g = dL_h(X_{h^{-1}g})$, since ∇ is a left-invariant connection. So there exists a matrix $M(h) \in \mathrm{GL}(n)$ such that

$$L_hX = (L_hX_1\cdots L_hX_n) = X\,M(h)\,.$$

On the other side, since $(L_h(X))_g = dL_h(X_{h^{-1}g})$, we have that $L_hX = E\rho(h^{-1}g)$. Then we get $L_hX = X\,M(h) = E\,\rho(g)\,M(h) = E\rho(h^{-1}g)$ which implies

$$\rho(g)M(h) = \rho(h^{-1}g)\ .$$

Evaluation at $g = e$ gives $M(h) = \rho(h^{-1})$ and this shows that ρ^t is a representation of G, where ρ^t denotes the transpose of ρ.

Proposition 3.1. *Let G be an n-dimensional simply-connected Lie group. There is a correspondence between flat left-invariant connections on G and representations of G on $\mathbb{R}^n$.*

Assume now that (g,J) is a left-invariant almost Hermitian structure on G and that ∇ is a left-invariant flat connection such that

$$\begin{cases} \nabla g = 0\\ \nabla J = 0\,. \end{cases}$$

Let $X := (X_1\,JX_1\;X_2\,JX_2\;\cdots\;X_m\,JX_m)$, $2m = n$, be a ∇-parallel unitary frame and let $E := (e_1\,Je_1\;e_2\,Je_2\;\cdots\;e_m\,Je_m)$ be the left-invariant unitary frame such that $X(e) = E(e)$. Then the above map ρ takes values in the unitary group $\mathrm{U}(m)$, i.e. $\rho : G \to \mathrm{U}(m) \subset \mathrm{GL}(n)$. Let $Z_k = \frac{X_k - \mathrm{i}JX_k}{\sqrt{2}}$ be the $(1,0)$ ∇-parallel section of $T^{1,0}G$ induced by X_k for $k = 1,\cdots,m$. Then $Z := (Z_1\,Z_2\cdots Z_m)$ is a ∇-parallel unitary $(1,0)$-frame. Similarly, the real left-invariant frame E produces a $(1,0)$ left-invariant unitary frame $S := (\frac{e_1-\mathrm{i}Je_1}{\sqrt{2}}\cdots\frac{e_m-\mathrm{i}Je_m}{\sqrt{2}})$. Then

$$Z = S\rho(g)$$

where $\rho(g) \in \mathrm{U}(m)$ is a $m \times m$ matrix with complex entries.

3.2. **Flat left-invariant almost Hermitian structures on nilpotent Lie Groups.** We have the following

main **Proposition 3.2.** *Let (g, J) be a left-invariant almost Hermitian structure on a $2m$-dimensional nilpotent simply-connected Lie group G. Then the Chern connection associated to (g, J) is flat if and only if the almost complex Lie algebra $(\mathfrak{g}, J)$ is anti-abelian, i.e. $[J\cdot, J\cdot] = -[\cdot, \cdot]$.*

Proof. Let ∇ be the Chern connection associated to (g, J) and assume that the curvature of ∇ vanishes. Then there exists a parallel global unitary $(1,0)$-frame $Z = (Z_1 \cdots Z_m)$ on G. Fix a left-invariant complex unitary $(1,0)$-frame $S = (w_1 \cdots w_m)$ satisfying $S(e) = Z(e)$. As explained in the previous subsection, there exists a representation $\rho^t : G \to \mathrm{U}(m)$ satisfying

$$Z(g) = S(g)\rho(g)\,.$$

Since G is nilpotent and $\mathrm{U}(m)$ is compact, it follows that $\rho(G)$ is completely reducible and abelian; hence we can assume, by a change of frames, that $\rho(g)$ is diagonal. Namely, there exist smooth maps $\chi_1, \cdots, \chi_m : G \to \mathrm{U}(1)$ satisfying

$$\begin{cases} Z_1(g) = \chi_1(g) w_1(g), \\ \vdots \\ Z_m(g) = \chi_m(g) w_m(g)\,. \end{cases}$$

Now using $\mathrm{Tor}(\nabla)^{1,1} = 0$ we get

$$0 = [Z_i, \overline{Z}_j] = [\chi_i w_i, \overline{\chi}_j \overline{w}_j] = \chi_i(\overline{\chi}_j)_i \overline{w}_j - \overline{\chi}_j(\chi_i)_{\overline{j}} w_i + \chi_i \overline{\chi}_j [w_i, \overline{w}_j]\,.$$

So the assumption that ∇ is Chern-flat implies

$$[w_i, \overline{w}_j] = \frac{(\chi_i)_{\overline{j}}}{\chi_i} w_i - \frac{(\overline{\chi}_j)_i}{\overline{\chi}_j} \overline{w}_j\,.$$

Since G is nilpotent, there exists an integer $k \geq 0$ such that

$$\begin{cases} 0 = [w_i, [\cdots, [w_i, [w_i, \overline{w}_j]] = (\frac{(\overline{\chi}_j)_i}{\overline{\chi}_j})^k [w_i, \overline{w}_j]\,, \\ 0 = [\overline{w}_j, [\cdots, [\overline{w}_j, [w_i, \overline{w}_j]] = (\frac{(\chi_i)_{\overline{j}}}{\chi_i})^k [w_i, \overline{w}_j] \end{cases}$$

which implies $[w_i, \overline{w}_j] = 0$ for $1 \leq i, j, \leq m$. This shows that $\mathfrak{g}$ is anti-abelian.

On the other side assume that the Lie algebra $\mathfrak{g}$ of G is anti-abelian and let $\tilde{\nabla}$ the unique connection on G whose parallel vector fields are left-invariant vector fields. Since

$$[Z, \overline{W}] = 0$$

for any $Z\,, W \in \mathfrak{g}^{1,0}$, then the torsion of $\tilde{\nabla}$ has vanishing $(1,1)$-component. Clearly, $\tilde{\nabla} g = \tilde{\nabla} J = 0$, hence, by uniqueness, $\tilde{\nabla}$ is the Chern connection. □

anti-flat **Remark 3.3.** Note that for the "if part" of the above proof we don't use that G is nilpotent and simply-connected.

Notice that a Lie algebra $\mathfrak{g}$ is anti-abelian, i.e. $[Jv, Jw] = -[v, w]$, if and only if for all v

$$[Jv, v] = 0\ .$$

Indeed, $[Jv, Jw] = -[v, w]$ is equivalent to $[Jv, w] = [v, Jw]$ which is the bilinear form associated by polarization to the quadratic form $[Jv, v]$. Moreover, if J is anti-abelian, then it preserves the center of $\mathfrak{g}$. This implies that if a Lie algebra admits an anti-abelian structure, then its center has even dimension. Observe also that an anti-abelian almost complex structure J on $\mathfrak{g}$ is integrable if and only if $\mathfrak{g}$ is complex Lie algebra, i.e $[JX, Y] = J[X, Y]$.

3.3. **Proof of Theorem 1.2.** Now are ready to prove Theorem 1.2

Proof. Claim (i) follows from Proposition 3.2 and Remark 3.3. For (ii) let (M, g, J) be an infra-nilmanifold (i.e. a Riemannian manifold whose universal covering is a nilpotent Lie group) endowed with an almost Hermitian structure such that π^*g, π^*J are left-invariant. Assume that the curvature tensor of the Chern connection associated to (g, J) vanishes. Then the universal covering of M is a simply-connected nilpotent Lie group N with an almost Hermitian Chern-flat left-invariant structure. So the claim follows from Proposition 3.2. For (iii) we observe that a flat Kähler torus can be *presented* as a compact quotient of a solvable Lie group with a non anti-abelian OACS. Indeed, let $\mathfrak{s} = \langle e_1, e_2, e_3, e_4\rangle$ be the solvable Lie algebra whose Lie bracket is given by the relation

$$[e_1, e_2] = -e_3 \quad [e_1, e_3] = e_2\,.$$

In terms of structure equations we can write $\mathfrak{s} = (0, -13, 12, 0)$. Here is a representation of $\mathfrak{s}$:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\ e_3 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Let S be a Lie group whose Lie algebra is $\mathfrak{s}$. Consider on S the left-invariant almost Hermitian structure defined by (g, J), where J is the almost complex structure determined by the relations

$$Je_1 = e_4\,, \quad Je_2 = e_3$$

and $g = \sum_{i=1}^{4} e^i \otimes e^i$, $\{e^1, \dots, e^4\}$ being the dual frame to $\{e_1, \dots, e_4\}$. The only non-zero covariant derivatives with respect to the Levi-Civita connection are:

$$D_{e_1}e_2 = -e_3\ \ ,\ \ D_{e_1}e_3 = e_2\ \ .$$

So $\mathfrak{s}$ splits $\mathfrak{s} = \langle e_1, e_4\rangle \oplus \langle e_2, e_3\rangle$ where the factors are D-parallel distributions. This implies that (S, g, J) is a flat Kähler manifold and so the Chern connection of (S, g, J) is flat. On the other hand $(\mathfrak{s}, J)$ is not anti-abelian since

$$[Je_1, Je_2] = [e_4, e_3] = 0 \neq -[e_1, e_2] = e_3\,.$$

The Lie algebra $\mathfrak{s} = \langle e_1, e_2, e_3, e_4\rangle$ can be identified with the Lie algebra of the group $S := \mathrm{Iso}(\mathbb{R}^2) \times S^1$ where $\mathrm{Iso}(\mathbb{R}^2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ is the group of (orientation preserving) isometries of the Euclidean plane $\mathbb{R}^2$. □

Now we are ready to prove Corollary 1.3.

Proof of Corollary 1.3. $(i) \implies (ii)$ is a direct consequence of Theorem 1.2, since left-invariant vector fields on G can be pushed-down to the quotient $M = \Gamma\backslash G$.

$(ii) \implies (iii)$ it is obvious.

$(iii) \implies (i)$ comes from the item (ii) of Theorem 1.2. □

Rcortes

Remark 3.4. We do not know if Corollary 1.3 is true when G is not nilpotent, e.g. for solvmanifolds. Since there exist flat compact Kähler manifolds with non trivial holonomy group (see [9]) it follows that $(iii) \to (ii)$ of Corollary 1.3 does not hold for infra-nilmanifolds, i.e. Riemannian manifolds whose universal covering is a nilpotent Lie group .

3.4. **Proof of Theorem 1.4.** Now we are going to prove Theorem 1.4.

Assume that $\mathfrak{g}$ is 4-dimensional and let $\mathfrak{g}_{\mathbb{C}}$ its complexification. Let J be an anti-abelian structure. Then split $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{1,0} \oplus \mathfrak{g}_{\mathbb{C}}^{0,1}$ as usual and note that the anti-abelian condition is equivalent to $[\mathfrak{g}_{\mathbb{C}}^{1,0}, \mathfrak{g}_{\mathbb{C}}^{0,1}] = 0$. Let Z_1, Z_2 be a base of $\mathfrak{g}_{\mathbb{C}}^{1,0}$ and assume $[Z_1, Z_2] = aZ_1 + bZ_2 + c\overline{Z}_1 + d\overline{Z}_2 \neq 0$. Then from the Jacobi identity we get

$$\begin{cases} 0 = [[Z_1, Z_2], \overline{Z}_1] = d[\overline{Z}_2, \overline{Z}_1] \\ 0 = [[Z_1, Z_2], \overline{Z}_2] = c[\overline{Z}_1, \overline{Z}_2]\,. \end{cases}$$

So $d = c = 0$ and then $\mathfrak{g}_{\mathbb{C}}^{1,0}$ is a Lie algebra, i.e $[\mathfrak{g}_{\mathbb{C}}^{1,0}, \mathfrak{g}_{\mathbb{C}}^{1,0}] \subset \mathfrak{g}_{\mathbb{C}}^{1,0}$. Now is clear that $\mathfrak{g}$ is the complexification of the 2-dimensional solvable Lie algebra.

For the dimension 6 case we need the classification of of 6-dimensional nilpotent Lie algebras showed in table 1.

A first obstruction to the existence of an anti-abelian almost complex structure on a Lie algebra is given to the dimension of its center; namely, table

Table 1. Six dimensional nilpotent Lie algebras

Structure equations	Dimension of the center
$\mathfrak{g}_1 = (0,0,12,13,14+23,34+52)$	1
$\mathfrak{g}_2 = (0,0,12,13,14,34+52)$	1
$\mathfrak{g}_3 = (0,0,12,13,14,15)$	1
$\mathfrak{g}_4 = (0,0,12,13,14,23+15)$	1
$\mathfrak{g}_5 = (0,0,0,12,14,24)$	1
$\mathfrak{g}_6 = (0,0,12,13,23,14+25)$	1
$\mathfrak{g}_7 = (0,0,12,13,23,14-25)$	1
$\mathfrak{g}_8 = (0,0,12,13,14+23,24+15)$	1
$\mathfrak{g}_9 = (0,0,0,12,14,15+23)$	1
$\mathfrak{g}_{10} = (0,0,0,12,14-23,15+34)$	1
$\mathfrak{g}_{11} = (0,0,0,12,23,14+35)$	1
$\mathfrak{g}_{12} = (0,0,0,12,23,14-35)$	1
$\mathfrak{g}_{13} = (0,0,0,12,13,14+35)$,	1
$\mathfrak{g}_{14} = (0,0,0,12,14,15+23+24)$	1
$\mathfrak{g}_{15} = (0,0,0,0,12,15+34)$	1
$\mathfrak{g}_{16} = (0,0,0,12,13+42,14+23)$	2
$\mathfrak{g}_{17} = (0,0,0,12,14,13+42)$	2
$\mathfrak{g}_{18} = (0,0,0,12,13+14,24)$	2
$\mathfrak{g}_{19} = (0,0,0,12,13,14)$	2
$\mathfrak{g}_{20} = (0,0,12,13,23,14)$	2
$\mathfrak{g}_{21} = (0,0,0,12,14,15)$	2
$\mathfrak{g}_{22} = (0,0,0,12,13,14+23)$	2
$\mathfrak{g}_{23} = (0,0,0,12,13,24)$	2
$\mathfrak{g}_{24} = (0,0,0,12,14,15+24)$	2
$\mathfrak{g}_{25} = (0,0,0,0,12,14+23)$	2
$\mathfrak{g}_{26} = (0,0,0,0,12,34)$	2
$\mathfrak{g}_{27} = (0,0,0,0,13+42,14+23)$	2
$\mathfrak{g}_{28} = (0,0,0,0,12,14+25)$	2
$\mathfrak{g}_{29} = (0,0,0,0,0,12+34)$	2
$\mathfrak{g}_{30} = (0,0,0,12,13,23)$	3
$\mathfrak{g}_{31} = (0,0,0,0,12,13)$	3
$\mathfrak{g}_{32} = (0,0,0,0,12,15)$	3
$\mathfrak{g}_{33} = (0,0,0,0,0,12)$	4
$\mathfrak{g}_{34} = (0,0,0,0,0,0)$	6

Lemma 3.5. *Let $\mathfrak{g}$ be a 6-dimensional nilpotent Lie algebra admitting an anti-abelian structure, then the dimension of the center of $\mathfrak{g}$ is equal to 0, 2 or 6.*

Proof. Assume that there exists an anti-abelian almost complex structure J on $\mathfrak{g}$. Since the center $\mathfrak{z}$ of $\mathfrak{g}$ is J-invariant it is even-dimensional. It remains to prove that the dimension of the center of $\mathfrak{g}$ cannot be equal to 4. Since J preserves $\mathfrak{z}$, there exists a J-invariant complement $W = \langle w_1, w_2\rangle$ of $\mathfrak{z}$ in $\mathfrak{g}$. We can write

$$J(w_1) = Aw_1 + Bw_2\,, \quad J(w_2) = A'w_1 + B'w_2\,.$$

Now

$$[Jw_1, w_1] = 0 \Longrightarrow B = 0$$

which is a contradiction. □

Corollary 3.6. *The Lie algebras $\{\mathfrak{g}_i\}_{i=1,\dots 14}, \mathfrak{g}_{30}, \mathfrak{g}_{31}, \mathfrak{g}_{32}, \mathfrak{g}_{33}$, do not admit any anti-abelian almost complex structure.*

Before the next result, we need an easy technical lemma:

tec

Lemma 3.7. *Let (V, J) be a complex vector space and let $W \subseteq V$ be a J-invariant subspace. Let $v \in V \backslash W$ such that $Jv \in \langle v\rangle \oplus W$. Then $v = 0$.*

Proof. Assume $Jv = a\,v + w$, for some $w \in W$. Then

$$-v = J^2 v = a\,Jv + Jw = a^2\,v + w + Jw\,.$$

This readily implies $v = 0$, as required. □

Lemma 3.8. *Let $\mathfrak{g}$ be a 6-dimensional nilpotent Lie algebra with the center of dimension 2. Assume that there exists $v \in \mathfrak{g}$ such that, if $V := \langle v \rangle$, then $\dim[V, \mathfrak{g}] = 3$. Then $\mathfrak{g}$ does not admit any anti-abelian structure.*

Proof. Let J be an anti-abelian structure. Using $[Jv, v] = 0$, we immediately get that $Jv \in \langle v \rangle \oplus \mathfrak{z}$. Since J preserves $\mathfrak{z}$, Lemma 3.7 gives a contradiction. □

Corollary 3.9. *The Lie algebras $\mathfrak{g}_{16}\,, \mathfrak{g}_{17}\,, \mathfrak{g}_{19}\,, \mathfrak{g}_{20}\,, \mathfrak{g}_{21}\,, \mathfrak{g}_{22}\,, \mathfrak{g}_{24}$ do not admit anti-abelian almost complex structures.*

It remains to prove that $\mathfrak{g}_{18}$, $\mathfrak{g}_{23}$, $\mathfrak{g}_{25}$ and $\mathfrak{g}_{26}$ don't have anti-abelian almost complex structures. Here we work case by case.

Lemma 3.10. *The Lie algebra $\mathfrak{g}_{18} = (0, 0, 0, 12, 13 + 14, 24)$ does not admit any anti-abelian almost complex structure.*

Proof. Assume that there exists an anti-abelian almost complex structure J on $\mathfrak{g}_{23}$. We can write

$$\begin{aligned} Je_1 &= Ae_1 + Be_2 + Ce_3 + De_4 + Ee_5 + Fe_6\,, \\ Je_2 &= A'e_1 + B'e_2 + C'e_3 + D'e_4 + E'e_5 + F'e_6\,. \end{aligned}$$

Then

$$\begin{aligned} [Je_1, e_1] = 0 &\Longrightarrow Je_1 = Ae_1 + Ce_3 - Ce_4 + Ee_5 + Fe_6\,, \\ [Je_2, e_2] = 0 &\Longrightarrow Je_2 = B'e_2 + C'e_3 + E'e_5 + F'e_6\,. \end{aligned}$$

Moreover,

$$[Je_1, Je_2] = -AB'e_4 - CB'e_6 - AC'e_5\,.$$

Hence $[Je_1, Je_2] = -[e_1, e_2]$ implies

$$\begin{cases} AB' = 1 \\ CB' = 0 \\ AC' = 0\,. \end{cases}$$

Applying Lemma 3.7, we get that the above system simples $A = B' = 0$, $AB' = 0$, which is a contradiction. □

Lemma 3.11. *The Lie algebra $\mathfrak{g}_{23} = (0, 0, 0, 12, 13, 24)$ does not admit any anti-abelian almost complex structure.*

Proof. Assume that there exists an anti-abelian almost complex structure J on $\mathfrak{g}_{23}$. We can write

$$\begin{aligned} Je_1 &= Ae_1 + Be_2 + Ce_3 + De_4 + Ee_5 + Fe_6\,, \\ Je_2 &= A'e_1 + B'e_2 + C'e_3 + D'e_4 + E'e_5 + F'e_6\,. \end{aligned}$$

Then

$$\begin{aligned} [Je_1, e_1] = 0 &\Longrightarrow Je_1 = Ae_1 + De_4 + Ee_5 + Fe_6\,, \\ [Je_2, e_2] = 0 &\Longrightarrow Je_2 = B'e_2 + C'e_3 + E'e_5 + F'e_6\,. \end{aligned}$$

Moreover,

$$[Je_1, Je_2] = -AB'e_4 + DB'e_6 - AC'e_5\,.$$

Hence $[Je_1, Je_2] = -[e_1, e_2]$ implies

$$\begin{cases} AB' = 1 \\ DB' = 0 \\ AC' = 0\,. \end{cases}$$

Applying again Lemma 3.7, we get $A = B' = 0$, $AB' = 0$, which is a contradiction. □

Lemma 3.12. *The Lie algebra $\mathfrak{g}_{25} = (0, 0, 0, 0, 12, 14 + 23)$ does not admit any anti-abelian almost complex structure.*

Proof. Assume that there exists an anti-abelian almost complex structure J on $\mathfrak{g}_{24}$. We can write

$$Je_1 = Ae_1 + Be_2 + Ce_3 + De_4 + Ee_5 + Fe_6\,,$$
$$Je_2 = A'e_1 + B'e_2 + C'e_3 + D'e_4 + E'e_5 + F'e_6\,.$$

Then

$$[Je_1, e_1] = 0 \Longrightarrow Je_1 = Ae_1 + Ce_3 + Ee_5 + Fe_6\,,$$
$$[Je_2, e_2] = 0 \Longrightarrow Je_2 = B'e_2 + D'e_4 + E'e_5 + F'e_6\,.$$

Moreover,

$$[Je_1, Je_2] = -AB'e_5 + CB'e_6 - AD'e_6\,.$$

Hence $[Je_1, Je_2] = -[e_1, e_2]$ implies

$$\begin{cases} AB' = 0\,, \\ CB' - AD' = 1\,. \end{cases}$$

Hence it has to be $A = 0$ or $B = 0$. Assume $A = 0$, then

$$Je_1 = Ce_3 + Ee_5 + Fe_6\,,$$
$$Je_2 = \frac{1}{C}\,e_2 + D'e_4 + E'e_5 + F'e_6\,.$$

Using $J^2 = -\mathrm{Id}$, we get

$$Je_3 = -\frac{1}{C}\,e_3 + v\,, \quad v \in \mathfrak{z}\,.$$

Hence

$$[Je_2, Je_3] = [\frac{1}{C}e_2 + D'e_4, -\frac{1}{C}\,e_3] = \frac{1}{C^2}\,e_6\,.$$

Since $[e_2, e_3] = -[Je_2, Je_3]$ and $[e_2, e_3] = -e_6$, we get a contradiction.

Assume now $B' = 0$. Then

$$Je_1 = Ae_1 + Ce_3 + Ee_5 + Fe_6\,,$$
$$Je_2 = -\frac{1}{A}e_4 + E'e_5 + F'e_6\,.$$

Using $J^2 = -\mathrm{Id}$, we get

$$Je_4 = Ae_2 + w\,, \quad w \in \mathfrak{z}\,.$$

Then

$$[Je_1, Je_4] = [Ae_1 + Ce_3, Ae_2] = -A^2e_5 + CAe_6$$

and $[Je_1, Je_4] = -[e_1, e_4]$ implies $A = 0$ which is a contradiction, again. □

Lemma 3.13. *The Lie algebra* $\mathfrak{g}_{26} = (0, 0, 0, 0, 12, 34)$ *does not admit any anti-abelian almost complex structure.*

Proof. The idea of the proof is to show that the matrix of an anti-abelian complex structure J must have a real eigenvalue which is a contradiction.

Let J be a anti-abelian complex structure. Since J preserves the center $\mathfrak{z} = \langle e_5, e_6\rangle$ we have

$$\begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} & 0 & 0 \\ J_{21} & J_{22} & J_{23} & J_{24} & 0 & 0 \\ J_{31} & J_{32} & J_{33} & J_{34} & 0 & 0 \\ J_{41} & J_{42} & J_{43} & J_{44} & 0 & 0 \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{pmatrix}.$$

Condition $[Je_1, e_1] = 0$ implies $J_{21} = 0$. Analogously the other conditions $[Je_i, e_i] = 0$ implies

$$J_{12} = J_{43} = J_{34} = 0\,.$$

Hence J reduces to

$$\begin{pmatrix} J_{11} & 0 & J_{13} & J_{14} & 0 & 0 \\ 0 & J_{22} & J_{23} & J_{24} & 0 & 0 \\ J_{31} & J_{32} & J_{33} & 0 & 0 & 0 \\ J_{41} & J_{42} & 0 & J_{44} & 0 & 0 \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{pmatrix}.$$

Now, relations

$$
\begin{aligned}
[Je_1, e_3] &= J_{41}e_6\,,\\
[e_1, Je_3] &= -J_{23}e_5\,,\\
[Je_1, e_4] &= -J_{31}e_6\,,\\
[e_1, Je_4] &= -J_{24}e_5
\end{aligned}
$$

imply

$$J_{31} = J_{41} = 0\,.$$

Since $J^2 = -\mathrm{Id}$ it turns out that $J_{11}^2 = -1$, which is a contradiction. □

3.4.1. *Anti-abelian structure in $\mathcal{Z}_g$.* Now we end the proof of Theorem 1.4 classifying the anti-abelian structures in $\mathcal{Z}_g$. We start form the Lie algebra $\mathfrak{g}_{27} = (0, 0, 0, 0, 13+42, 14+23)$ associated to the Iwasawa manifold.

Lemma 3.14. *The only Chern-flat almost complex structures in $\mathcal{Z}_g(\mathfrak{g}_{27})$ are the structure J_0 and J_3 having as fundamental forms*

$$\omega_0 = e^{12} + e^{34} + e^{56}\,, \quad \omega_3 = -e^{12} - e^{34} + e^{56}\,,$$

respectively.

Proof. Let J in $\mathcal{Z}_g(\mathfrak{g}_{27})$ be a Chern-flat structure. Then Theorem 1.2 implies that $(\mathfrak{g}, J)$ must be anti-abelian, i.e. $[Jv, w] = [v, Jw]$ for all $v, w \in \mathfrak{g}$. In particular J preserves the center $\mathfrak{z} = \langle e_5, e_6\rangle$ of g and its orthogonal complement $\mathfrak{z}^t = \langle e_1, e_2, e_3, e_4\rangle$. Then $Je_5 = \pm e_6$. Now

$$Je_1 = ae_1 + be_2 + ce_3 + de_4$$

and since

$$[Je_1, e_1] = 0$$

we get

$$0 = [Je_1, e_1] = a[e_1, e_1] + b[e_2, e_1] + c[e_3, e_1] + d[e_4, e_1] = -ce_5 + de_6\ .$$

Then $c = d = 0$ and we get $Je_1 = \pm e_2$. In a similar way we get from $[Je_3, e_3] = 0$ that $Je_3 = \pm e_4$. Let now $\epsilon_1, \epsilon_3, \epsilon_5 \in \{1, -1\}$ be such that

$$
\begin{cases}
Je_1 = \epsilon_1 e_2\,,\\
Je_3 = \epsilon_3 e_4\,,\\
Je_5 = \epsilon_5 e_6
\end{cases}
$$

and the product $\epsilon_1\epsilon_3\epsilon_5 = 1$. Now

$$[Je_1, e_3] = [e_1, Je_3]$$

and we get

$$[Je_1, e_3] = \epsilon_1[e_2, e_3] = \epsilon_1 e_6 = [e_1, Je_3] = \epsilon_3[e_1, e_4] = \epsilon_3 e_6$$

which implies $\epsilon_1 = \epsilon_3$ and $\epsilon_5 = 1$. This show that J is either J_0 or J_3. □

Analogously we have that following lemma whose proof is omitted

Lemma 3.15. *The only Chern-flat almost complex structures in $\mathcal{Z}_g(\mathfrak{g}_{28})$ and in $\mathcal{Z}_g(\mathfrak{g}_{29})$ are the structure J_0' and J_3' having as fundamental forms*

$$\omega_0' = e^{13} + e^{24} + e^{56}\,, \quad \omega_3' = -e^{13} - e^{24} + e^{56}\,.$$

4. Ricci-flat structures

In this section we prove Theorem 1.5.

Let (M, g, J) be an almost Hermitian manifold, D be the Levi-Civita connection of g and ∇ be the Chern connection. We recall the definition of the Ricci form:

$$\mathrm{Ric}(J)(X, Y) = \frac{1}{2}\mathrm{tr}_\omega \mathrm{R}(X, Y, \cdot, \cdot) = \sum_{i=1}^{n} \mathrm{R}(X, Y, Z_i, Z_{\bar{i}})\,,$$

where $\{Z_1, \dots, Z_n\}$ is an arbitrary unitary frame on M. Using equation (2.1), we obtain

theta

$$
\begin{aligned}
\theta(X) &= \sum_{i=1}^{n}\left\{g(D_X Z_i, Z_{\bar{i}}) - \frac{1}{2}g(D_{Z_i}X + \mathrm{i}\, D_{Z_i}JX, Z_{\bar{i}}) + \frac{1}{2}g(D_{Z_{\bar{i}}}X - \mathrm{i}\, D_{Z_{\bar{i}}}JX, Z_i)\right\}\\
&= \sum_{i=1}^{n}\left\{g(D_X Z_i, Z_{\bar{i}}) - g(D_{Z_i}X^{0,1}, Z_{\bar{i}}) + g(D_{Z_{\bar{i}}}X^{1,0}, Z_i)\right\}\ ,
\end{aligned}
\tag{4.1}
$$

or, in terms of brackets,

theta2 (4.2)
$$\theta(X)=\sum_{i=1}^n\left\{g([X^{0,1},Z_i],Z_{\bar{i}})-g([X^{1,0},Z_{\bar{i}}],Z_i)\right\}=2\,\mathrm{i}\sum_{i=1}^n\mathfrak{Im}\left\{g([X^{0,1},Z_i],Z_{\bar{i}})\right\}$$

(see the appendix at the end of the paper for a proof of this last formula). Now we consider the homogenous case. Let $M=\Gamma\backslash G$ be a quotient of a simply-connected Lie group G by a lattice Γ and let g be a fixed left-invariant metric on M. We denote by $\mathcal{Z}_g$ the space of left-invariant almost complex structures on M compatible to g and a fixed orientation. In this section we study the problem of classifying Ricci-flat almost complex structures in $\mathcal{Z}_g$. This problem is purely algebraic and can be directly studied on the Lie algebra $\mathfrak{g}$ associated to G. So g can be regarded as a metric on $\mathfrak{g}$ and $\mathcal{Z}_g$ can be viewed as the space of almost complex structures on $\mathfrak{g}$ compatible to g and a fixed orientation. Note that for any J in $\mathcal{Z}_g$, $\mathrm{Ric}(J)=d\theta$, where θ is the 1-form on $\mathfrak{g}$ globally defined as in (4.2). So $J\in\ker d$ is Ricci-flat if and only if θ has no component on $\ker d$.

The metric g induces the so-called *musical isomorphims*

$$\natural\colon\mathfrak{g}\to\mathfrak{g}^*\,,\quad\flat\colon\mathfrak{g}^*\to\mathfrak{g}\,,$$

where $\natural(X)(Y)=g(X,Y)$ and $\flat=\natural^{-1}$. If A is a fixed subspace of $\mathfrak{g}^*$, we denote by $A^\flat$ the vector space of $\mathfrak{g}$ corresponding to A by $\flat$. Finally, we denote by $\mathfrak{z}$ the center of $\mathfrak{g}$. Moreover, for every $X\in\mathfrak{g}$, we denote by X^t its orthogonal complement.

4.1. **Some general results.** Now we prove some general result. First of all we consider the following

Proposition 4.1. *For any $J\in\mathcal{Z}_g$, we have that $\mathfrak{z}\cap J\mathfrak{z}\subseteq\ker\theta$.*

Proof. If $X\in\mathfrak{z}\cap J\mathfrak{z}$, then $X^{1,0}$ and $X^{1,0}$ belongs to $\mathfrak{z}\otimes\mathbb{C}$ and formula (4.2) implies the statement. □

The proposition above has the following easy-prove consequence

Corollary 4.2. *Let $J\in\mathcal{Z}_g$ and assume that $\left[(\ker d)^\flat\right]^t$ is contained in $\mathfrak{z}\cap J\mathfrak{z}$, then* $\mathrm{Ric}(J)=0$.

The following two results give a description of $\mathcal{Z}_g$ when $[(\ker d)^\flat]^t\subseteq\mathfrak{z}$.

prpar **Proposition 4.3.** *Assume that $\dim(\ker d)^t=1$ and $[(\ker d)^\flat]^t\subseteq\mathfrak{z}$. Then any $J\in\mathcal{Z}_g$ is Ricci-flat.*

Proof. Let e^1 be a fixed unitary generator of $(\ker d)^t$ and let $J\in\mathcal{Z}_g$. Then we can write the fundamental form of J as

$$\omega=\sum_{k=1}^n e^{2k-1}\wedge e^{2k}\,,$$

where $\{e^2,\dots,e^{2n}\}$ is a suitable frame of $\ker d$. Let $Z_k:=\frac{1}{\sqrt{2}}(e_{2k-1}-\mathrm{i}\,e_{2k})$; then we can write $[e_1^{0,1},Z_j]=\mathrm{i}\,\lambda_j e_1$, for some $\lambda_j\in\mathbb{R}$. Notice that

$$[e_1^{0,1},Z_{\bar{1}}]=\frac{1}{2\sqrt{2}}[e_1-\mathrm{i}\,e_2,e_1+\mathrm{i}\,e_2]=0\,,$$

i.e. $\lambda_1=0$, and

$$[\mathfrak{g},\mathfrak{g}]=\langle e_1\rangle\,.$$

Moreover

$$\theta(e_1)=2\,\mathrm{i}\sum_{i=1}^n\mathfrak{Im}\left\{g([e_1^{0,1},Z_i],Z_{\bar{i}})\right\}=\mathrm{i}\sum_{i=2}^n\mathfrak{Im}\left\{g(\lambda_i e_1,Z_{\bar{i}})\right\}=0\,.$$

Hence θ has no components on $(\ker d)^t$. Consequently $d\theta=0$, which is equivalent to $\mathrm{Ric}(J)=0$. □

prepa **Proposition 4.4.** *Let $(M=\Gamma\backslash G,g)$ be a nilmanifold equipped with a left-invariant metric. Assume that $\dim(\ker d)^t=2$ and $[(\ker d)^\flat]^t=\mathfrak{z}$. Then any $J\in\mathcal{Z}_g$ is Ricci-flat.*

Proof. Let $\{e^1,e^2\}$ be a fixed oriented orthonormal frame of $(\ker d)^t$ and let $J\in\mathcal{Z}_g$. Then the fundamental form ω associated to J can be written as

omega (4.3)
$$\omega=e^1\wedge(ae^2+bf^1)+(af^1-be^2)\wedge f^2+\sum_{i=2}^n f^{2i-1}\wedge f^{2i}\,,$$

where $\{f^1,\dots,f^{2n}\}$ is a suitable frame of $(\ker d)^t$ and $a^2+b^2=1$ (see [3]). Notice that

1. $\mathfrak{z}=\langle e_1,e_2\rangle$;
2. $[f_i,f_j]\in\mathfrak{z}$;
3. $J(e_1)=ae_2+bf_1\,,\quad J(e_2)=-ae_1-bf_2$;

4. $[\mathfrak{g},\mathfrak{g}]=\langle e_1,e_2\rangle$.

Let $\{Z_1,\dots,Z_n\}$ be the unitary frame

$$Z_1=\frac{1}{\sqrt{2}}(e_1-\mathrm{i}\,Je_1)\,,\quad Z_j=\frac{1}{\sqrt{2}}(f_{2j-1}+\mathrm{i}\,f_{2j})\,, j=2,\dots,n\,.$$

Then

$$\begin{aligned}\theta(e_1)&=2\,\mathrm{i}\sum_{i=1}^{n}\mathfrak{Im}\left\{g([e_1^{0,1},Z_i],Z_{\bar{i}})\right\}=2\,\mathrm{i}\,\mathfrak{Im}\left\{g([e_1^{0,1},Z_1],Z_{\bar{1}})+g([e_1^{0,1},Z_2],Z_{\bar{2}})\right\}\\&=2\,\mathrm{i}\,\mathfrak{Im}\left\{g([e_1^{0,1},Z_2],Z_{\bar{2}})\right\}=\mathrm{i}\,\mathfrak{Im}\left\{g([\mathrm{i}\,bf_1,-\mathrm{i}\,f_2],-be_2)\right\}=0\end{aligned}$$

and

$$\begin{aligned}\theta(e_2)&=2\,\mathrm{i}\sum_{i=1}^{n}\mathfrak{Im}\left\{g([e_2^{0,1},Z_i],Z_{\bar{i}})\right\}=2\,\mathrm{i}\,\mathfrak{Im}\left\{g([e_2^{0,1},Z_1],Z_{\bar{1}})+g([e_2^{0,1},Z_2],Z_{\bar{2}})\right\}\\&=\frac{1}{2}\,\mathrm{i}\,\mathfrak{Im}\left\{b^2g([f_1,f_2],e_1)\right\}=0\end{aligned}$$

imply that θ is closed and that J is Ricci-flat, as required. □

4.2. **The 4-dimensional case.** In this section we take into account 4-dimensional Lie algebras and we prove the first part of Theorem 1.5.

Let $(\mathfrak{g},[\cdot,\cdot],g)$ be a 4-dimensional Lie algebra with a fixed metric and an orientation. In order to prove Theorem 1.5, we have to parameterize almost complex structure in $\mathcal{Z}_g$. This can be done as follows: Let V be a fixed 2-dimensional subspace of $\mathfrak{g}^*$ and let $\{e^3,e^4\}$ be an oriented orthonormal frame of V; then the fundamental form of an arbitrary $J\in\mathcal{Z}_g$ can be written in terms of $\{e^3,e^4\}$ as

omega4

$$\omega=e^3\wedge(ae^4+bf^1)+(be^4-af^1)\wedge f^2\,,\tag{4.4}$$

where $\{f^1,f^2\}$ is a suitable frame of the orthogonal complement of V. Let $\{f_1,f_2,f_3,f_4\}$ be the dual basis to $\{f^1,f^2,e^3,e^4\}$; the the two vectors

Z4

$$Z_1=\frac{1}{\sqrt{2}}(e_3-\mathrm{i}(ae_4+bf_1))\,,\quad Z_2=\frac{1}{\sqrt{2}}(be_4-af_1-\mathrm{i}\,f_2)\tag{4.5}$$

defines an unitary frame with respect to (g,J).

Here we write down the 4-dimensional Lie algebras which will take into account:

- $\mathbb{R}^4=(0,0,0,0)$;
- $\mathfrak{h}_3(\mathbb{R})\oplus\mathbb{R}=(0,0,12,0)$;
- $\mathfrak{n}_4=(0,0,12,23)$;
- $\mathfrak{aff}(\mathbb{R})\oplus\mathbb{R}^2=(12,0,0,0)$;
- $\mathfrak{sol}_3\oplus\mathbb{R}=(0,0,13,-14)$.

4.2.1. *Ricci-flat structure on* $\mathbb{R}^4$. This case is trivial: any $J\in\mathcal{Z}_g$ is almost Kähler and Chern flat.

4.2.2. *Ricci-flat structure on* $\mathfrak{h}_3(\mathbb{R})\oplus\mathbb{R}$. This is the Lie algebra associated to the Kodaira-Thurston manifold (see [24, 1] for a detailed description). In this case we have $\ker d=\langle e^1,e^2,e^4\rangle$ and $(\ker d)^t=\langle e^3\rangle$. Moreover the center of $\mathfrak{g}$ is spanned by e_3,e_4 and $[(\ker d)^\flat]^t\subseteq\mathfrak{z}$. Hence Proposition 4.3 implies the following

K-T **Proposition 4.5.** *Any* $J\in\mathcal{Z}_g(\mathfrak{h}_3(\mathbb{R})\oplus\mathbb{R})$ *is Ricci-flat.*

We remark that from [25] it follows that the standard symplectic structure on $\mathfrak{h}_3(\mathbb{R})\oplus\mathbb{R}$ is Ricci-flat.

4dim3step

4.2.3. *Ricci-flat almost complex structures on* $\mathfrak{n}_4$. We recall the $\mathfrak{h}_3(\mathbb{R})\oplus\mathbb{R}$ and $\mathfrak{n}_4$ are the unique 4-dimensional non-abelian nilalgebras admitting almost Kähler structures (see [13]). $\mathfrak{n}_4$ has the following properties:

$$\xi=\langle e_4\rangle\,;\quad[\mathfrak{g},\mathfrak{g}]=\langle e_3,e_4\rangle\,.$$

prop4dim3step **Proposition 4.6.** *An almost complex structure* $J\in\mathcal{Z}_g(\mathfrak{n}_4)$ *is Ricci-flat if and only if* $Je_3\in\langle e_1,e_3,e_4\rangle$.

Proof. Let $J\in\mathcal{Z}_g(\mathfrak{n}_4)$, $V=\langle e_3,e_4\rangle$ and $\{f_1,f_2\}$ be an oriented orthonormal frame of $V^t=\langle e_1,e_2\rangle$. Then using (4.4) and (4.5), we have

$$\theta(e_3)==-\mathrm{i}\,\frac{1}{2}b\,g([e_3,f_2],e_4)$$

and

$$\theta(e_4) == \frac{1}{2}\,\mathrm{i}\,a^2 b g([e_3,f_1],e_4) - \frac{1}{2}a^2 b\ \mathrm{i}\,g([e_3,f_1],e_4) = 0\,.$$

So

$$\mathrm{Ric}(J) = -\,\mathrm{i}\,\frac{1}{2} b\, g([e_3,f_2],e_4)\, e^{23}$$

and J is Ricci-flat if and only if

$$bg([e_3,f_2],e_4) = 0\,.$$

This last equation is satisfied if and only if $[e_3, Je_3] = 0$ or, equivlently, if $Je_3 \in \langle e_1, e_3, e_4\rangle$. □

4.2.4. *Ricci-flat almost complex structures on* $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}^2$. Now we consider the case of $\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}^2$. It is well known that the simply-connected Lie group G associated to this algebra does not admit a compact quotient. Notice that in this case

$$\ker d = \langle e^2, e^3, e^4\rangle\,,\ \ (\ker d)^t = \langle e^1\rangle\,,\ \mathfrak{z} = \langle e_3, e_4\rangle\,,\ [\mathfrak{g},\mathfrak{g}] = \langle e_1\rangle\,.$$

Moreover, the fundamental form of an arbitrary $J \in \mathcal{Z}_g$ can be written as

omegaJ

$$\omega = e^1 \wedge (ae^2 + bf^1) + (be^2 - af^1) \wedge f^2\,, \tag{4.6}$$

where $\{f^1, f^2\}$ is a suitable frame of $\langle e^3, e^4\rangle$. We have the following

Proposition 4.7. *Let $J \in \mathcal{Z}_g(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}^2)$. Then J is Ricci-flat if and only if $J\langle e_1, e_2\rangle \subseteq \langle e_3, e_4\rangle$*

Proof. Let $J \in \mathcal{Z}_g$ with associated fundamental form given by (4.6). Then we have

$$\begin{aligned}\theta(e_1) =& \frac{1}{2}\,\mathrm{i}\,\mathfrak{Im}\Big\{g([e_1 + \mathrm{i}(ae_2 + bf_1), e_1 - \mathrm{i}(ae_2 + bf_1)], e_1 + \mathrm{i}(ae_2 + bf_1))\\ &+ g([e_1 + \mathrm{i}(ae_2 + bf_1), (be^2 - af^1) - \mathrm{i}\,f_2], (be^2 - af^1) + \mathrm{i}\,f_2)\Big\}\\ =& \frac{1}{2}\,\mathrm{i}\,\mathfrak{Im}\Big\{g([e_1 + \mathrm{i}\,ae_2, e_1 - \mathrm{i}\,ae_2], e_1)\Big\} = -\,\mathrm{i}\,a\, g([e_1,e_2],e_1) = \mathrm{i}\,a\,.\end{aligned}$$

Hence

$$\mathrm{Ric}(J) = \mathrm{i}\,a\, e^{12}$$

and J is Ricci-flat if and only if $a = 0$. Such a condition is equivalent to require

$$\omega = e^1 \wedge f^1 + e^2 \wedge f^2\,.$$

□

4.2.5. *The case of* $\mathfrak{sol}_3 \oplus \mathbb{R}$. Now we consider the case of the 4-dimensional solvable Lie algebra $\mathfrak{sol}_3 \oplus \mathbb{R} = (0, 0, 13, -14)$. We have the following relations

$$\xi = \langle e_2\rangle\,, \quad [\mathfrak{g},\mathfrak{g}] = \langle e_3, e_4\rangle\,,$$

Proposition 4.8. *An almost complex structure J in $\mathcal{Z}_g(\mathfrak{sol}_3 \oplus \mathbb{R})$ is Ricci-flat if and only if $Je_3 = \pm e_4$.*

Proof. Let $J \in \mathcal{Z}_g(\mathfrak{sol}_3 \oplus \mathbb{R})$ be arbitrary. Then the fundamental form of J can be written as

$$\omega = e^3 \wedge (ae^4 + bf^1) + (be^4 - af^1) \wedge f^2\,,$$

where $\{f^1, f^2\}$ is a suitable frame of $\langle e^1, e^2\rangle$. Moreover a direct computation gives

$$\theta(e_3) = -\,\mathrm{i}\,b\, g([e_3,f_1],e_3) + \frac{1}{2}\,\mathrm{i}\,b\, g([f_1,e_4],e_4)$$

and

$$\theta(e_4) = -\,\mathrm{i}\,bg([f_2,e_3],e_3) - \frac{3}{2}\,\mathrm{i}\,b\, g([f_2,e_4],e_4)\,.$$

So J is Ricci-flat if and only if

$$\begin{cases} 2b\, g([f_1,e_3],e_3) + b\, g([f_1,e_4],e_4) = 0\,,\\ 2bg([f_2,e_3],e_3) + 3b\, g([f_2,e_4],e_4) = 0\,,\end{cases}$$

and these equations can be solved if and only if $b = 0$. □

4.2.6. *Ricci-flat almost Kähler structures in the previous examples.* It is interesting to understand in the above examples the Ricci-flat almost Kähler structures.

- In the case of the *Kodaira-Thurston manifold* we have $de^1 = de^2 = 0$, $df^1, df^4 \in \langle e^1, e^2\rangle$. A direct computation gives that in this case J is almost Kähler and Ricci-flat if and only of $Je_3 \in \langle e_1, e_2\rangle$
- In the second example $\mathfrak{g} = (0, 0, 12, 34)$ the unique Ricci-flat almost Kähler structures in $\mathcal{Z}_g$ are the ones with associated fundamental form $\omega'_+ = e^{13} + e^{24}$, $\omega'_- = e^{13} - e^{24}$
- Also in the case $\mathfrak{g} = (12, 0, 0, 0)$ there are no almost Kähler Ricci-flat structures in $\mathcal{Z}_g$.
- In the the case $\mathfrak{g} = (0, 0, 13, -14)$ the unique almost Kähler Ricci-flat structures in $\mathcal{Z}_g$ are $\omega_+ = e^{12} + e^{34}$ and $\omega_- = -e^{12} - e^{34}$.

4.3. **The 6-dimensional case.** Let $(\mathfrak{g}, g)$ be a 6-dimensional Lie algebra with a fixed metric and an orientation. Let $\{e_1, \dots, e_6\}$ be an oriented orthonormal basis of $\mathfrak{g}$. Then the fundamental form of an arbitrary $J \in \mathcal{Z}_g$ can be written as

omega (4.7)
$$\omega = e^5 \wedge (ae^6 + bf^1) + (af^1 - be^6) \wedge f^2 + f^{34}\,,$$

where $a^2 + b^2 = 1$ and $\{f^1, f^2, f^4, f^4\}$ is a suitable frame of $\langle e_1, e_2, e_3, e_4\rangle^\natural$. Then in this case the three vectors

$$Z_1 = \frac{1}{\sqrt{2}}(e_5 - \mathrm{i}(ae_6 + bf_1))\,, \quad Z_2 = \frac{1}{\sqrt{2}}(af_1 - be_6 - \mathrm{i}\, f_2)\,, \quad Z_3 = \frac{1}{\sqrt{2}}(f_3 - \mathrm{i}\, f_4)$$

define an unitary frame.

The aim of this subsection is to study $\mathcal{Z}_g$ on 6-dimensional nilpotent Lie algebras having $b_1 \geq 4$. It is well known that 6-dimensional nilpotent Lie algebras having $b_1 \geq 4$ are classified by the following list

$$\begin{array}{llll} (0,0,0,0,12,15+34)\,, & (0,0,0,0,12,15)\,, & (0,0,0,0,12,14+25)\,, & \\ (0,0,0,0,12,14+23)\,, & (0,0,0,0,12,34)\,, & (0,0,0,0,13+42,14+23)\,, & \\ (0,0,0,0,0,12+34)\,, & (0,0,0,0,12,13)\,, & (0,0,0,0,0,12)\,, & (0,0,0,0,0,0)\,. \end{array}$$

4.3.1. *Two cases where Proposition 4.3 can be applied.* First of all we consider the following Lie algebras: $(0, 0, 0, 0, 0, 12)$, $(0, 0, 0, 0, 0, 12 + 34)$. This two algebras are associated to the groups

$$\mathbb{R}^3 \times H^3(\mathbb{R})\,, \quad \mathbb{R}^1 \times H^5(\mathbb{R})\,,$$

where $H^n(\mathbb{R})$ is the n-dimensional Heisenberg group. In this two cases Proposition 4.3 can be applied and any $J \in \mathcal{Z}_g$ is Ricci flat.

4.3.2. *Some cases where Proposition 4.4 can be applied.* Proposition 4.4 can be applied to the following Lie algebras:

$$(0,0,0,0,12,13)\,,\ (0,0,0,0,12,14+23)\,,\ (0,0,0,0,12,34)\,,\ (0,0,0,0,13+42,14+23)\,.$$

So in these cases any $J \in \mathcal{Z}_g$ is Ricci-flat.

The Ricci-flatness of the standard complex structure of $(0, 0, 0, 0, 13+42, 14+23)$ can be also deduced using Theorem 4 of [17].

4.3.3. *The final cases.* It remains to consider the following three cases:

$$(0,0,0,0,12,15+34)\,,\ (0,0,0,0,12,14+25)\,,\ (0,0,0,0,12,15)\,.$$

We have the following

Proposition 4.9. *In the above cases an almost complex structure $J \in \mathcal{Z}_g$ is Ricci-flat of and only if $[e_5, Je_6] = 0$ and one of the following two conditions is satisfied*

1. $[e_5, Je_5] = 0$;
2. $J\langle e_5, e_6\rangle$ *is contained in* $\langle e_1, e_2, e_3, e_4\rangle$.

Proof. Let $J \in \mathcal{Z}_g$. Then the we can write the fundamental form of J as

$$\omega = e^5 \wedge (ae^6 + bf^1) + (af^1 - be^6) \wedge f^2 + f^{34}\,,$$

where $\{f^1, f^2, f^3, f^4\}$ is a suitable frame of $\langle e^1, e^2, e^3, e^4\rangle$. A direct computation gives

$$\theta(e_5) = -\frac{1}{2}\,\mathrm{i}\, b\, g([e_5, f_2], e_6)$$

and

$$\theta(e_6) = -\,\mathrm{i}\, ab^2 g([e_5, f_1], e_6)\,.$$

Hence

$$\mathrm{Ric}(J) = -\frac{1}{2}\,\mathrm{i}\, b\, g([e_5, f_2], e_6)\, de^5 - \mathrm{i}\, ab^2 g([e_5, f_1], e_6)\, de^6\,.$$

Since de^5, de^6 are linear independent, $\mathrm{Ric}(J)=0$ if and only if one of the following conditions holds:

- $b=0$;
- $a=0$, $[e_5,f_2]=0$;
- $[e_5,f_2]=[e_5,f_1]=0$.

Now,

$$[e_5,Je_5]=b\,[e_5,f_1]\,,\quad [e_5,Je_6]=-b\,[e_5,f_2]$$

and $a=0$ is equivalent to require $J\langle e_5,e_6\rangle\subseteq\langle e_1,e_2,e_3,e_3\rangle$. □

4.3.4. *6-dimensional Ricci-flat almost Kähler structures.* It is interesting to understand in the above examples the Ricci-flat almost Kähler structures in $\mathcal{Z}_g$. Here as usual we write the fundamental form of $J\in\mathcal{Z}_g$ as

$$\omega=e^5\wedge(ae^6+bf^1)+(af^1-be^6)\wedge f^2+f^{34}\,.$$

Then in each case, we have

$\mathfrak{g}$	Almost Kähler Ricci-flat structures in $\mathcal{Z}_g$
$0,0,0,0,12,15+34$	$\emptyset$
$0,0,0,0,12,15$	$\emptyset$
$0,0,0,0,12,14+25$	$\emptyset$
$0,0,0,0,12,14+23$	$\emptyset$
$0,0,0,0,12,34$	$\langle e^{16},e^{26},e^{56},e^{35},e^{45}\rangle^t_{\mathcal{Z}_g}$
$0,0,0,0,13+42,14+23$	$e^6\wedge f^1-e^5\wedge f^2+f^{34}\ :\ de^6\wedge f^1=de^5\wedge f^2$
$0,0,0,0,0,12+34$	$\emptyset$
$0,0,0,0,12,13$	$e^6\wedge f^1-e^5\wedge f^2+f^{34}\ :\ e^{13}\wedge f^1=e^{12}\wedge f^2$
$0,0,0,0,0,12$	$\langle e^{36},e^{46},e^{56}\rangle^t_{\mathcal{Z}_g}$
$0,0,0,0,0,0$	$\mathcal{Z}_g$

4.4. **Generalizations to higher dimensions.** Using Proposition 4.3, we have the following result which generalizes the cases of the $\mathfrak{h}_3(\mathbb{R})\oplus\mathbb{R}$ and $\mathfrak{h}_3(\mathbb{C})=(0,0,0,0,13+42,14+23)$.

Proposition 4.10. *Let $\mathfrak{g}=\mathfrak{h}_n(\mathbb{R})\oplus\mathbb{R}$ be the $2n$-dimensional Lie algebra*

$$(0,\dots,0,12+34+\cdots+(2n-2)(2n-1),0)$$

obtained as the product of the Lie algebra associated to the $2n-1$-dimensional Heisenberg group $H_n(\mathbb{R})$ times $\mathbb{R}$. Then, we with respect to the standard metric g, any $J\in\mathcal{Z}_g$ is Ricci-flat.

Moreover, we have the following result which generalizes the case of the Iwasawa manifold.

Proposition 4.11. *Let $(\mathfrak{h}_n(\mathbb{C}),g)$ be the Lie algebra associated to the complex n-dimensional Heisenberg group with the standard metric and the standard orientation. Then any $J\in\mathcal{Z}_g$ is Ricci-flat.*

5. Appendix

5.1. **The Ricci form of a complex connection.** Let (M^n,J) be an almost complex manifold. A connection ∇ on M is called *complex* if $\nabla J=0$. Let ∇ be a complex connection and let R be its curvature tensor. Since $\nabla J=0$, ∇ induces a connection ∇^{T} on the canonical bundle $\mathrm{T}=\Lambda^{n,0}_J M$. Let σ be a local section of T. Since T has rank one, there exists a complex 1-form θ such that

$$\nabla^{\mathrm{T}}_X\sigma=\theta(X)\sigma\,,$$

for every complex vector field X.
Let $\mathrm{R}^{\mathrm{T}}(X,Y)$ be the curvature of ∇^{T}. Namely,

$$\mathrm{R}^{\mathrm{T}}(X,Y)\sigma:=\nabla^{\mathrm{T}}_X\nabla^{\mathrm{T}}_Y\sigma-\nabla^{\mathrm{T}}_Y\nabla^{\mathrm{T}}_X\sigma-\nabla^{\mathrm{T}}_{[X,Y]}\sigma\,,$$

where σ is a local section. Then a simple computation gives

$$\mathrm{R}^{\mathrm{T}}(X,Y)\sigma=d\theta(X,Y)\;\sigma\;.$$

Let $\{Z_1,Z_2,\cdots,Z_n\}$ be a local (1,0)-frame of $TM\otimes\mathbb{C}$ and let $\{\zeta^1,\dots,\zeta^n\}$ be the associated coframe. Then

$$\nabla^{\mathrm{T}}_X(\zeta^1\wedge\zeta^2\wedge\cdots\wedge\zeta^n)=\nabla_X\zeta^1\wedge\zeta^2\wedge\cdots\wedge\zeta^n+\cdots+\zeta^1\wedge\zeta^2\wedge\cdots\wedge\nabla_X\zeta^n$$

and then

$$\nabla^{\mathrm{T}}_Y\nabla^{\mathrm{T}}_X\sigma=\nabla_Y\nabla_X\zeta^1\wedge\zeta^2\wedge\cdots\wedge\zeta^n+\cdots+\zeta^1\wedge\zeta^2\wedge\cdots\wedge\nabla_X\nabla_Y\zeta^n$$

Thus, taking into that we may assume $[X,Y]=0$ because R and $R^{\rm T}$ are tensors and that some terms of the above sum simplify, we get

$$\mathrm{R}^{\rm T}_{X,Y}\sigma=\mathrm{R}(X,Y)\zeta^1\wedge\zeta^2\wedge\cdots\wedge\zeta^n+\cdots+\zeta^1\wedge\zeta^2\wedge\cdots\wedge\mathrm{R}(X,Y)\zeta^n\,.$$

Taking into account that $\mathrm{R}(X,Y)$ is an endomorphism of $T^{1,0}$, we get

$$\mathrm{R}^{\rm T}(X,Y)\sigma=\mathrm{tr}(\mathrm{R}(X,Y))\sigma$$

and then

$$\mathrm{tr}(\mathrm{R}(X,Y))=d\theta(X,Y)\,.$$

Hence if g is a J-Hermitian metric (not necessarily $\nabla g=0$) and $\{Z_1,\dots,Z_n\}$ is an unitary frame, then we get

$$\sum_{j=1}^n g(\mathrm{R}_{X,Y}Z_k,Z_{\overline{k}})=d\theta(X,Y)\,.$$

5.1.1. *A direct consequence.* Let now (M,J,g) be a almost Hermitian manifold and let ∇ be a Hermitian connection ($\nabla g=\nabla J=0$). Let us compute $d\theta(X,Y)$ directly. Keep in mind that we can assume $[X,Y]=0$, since $d\theta$ is a tensor. Then

$$\begin{aligned}d\theta(X,Y)&=X\theta(Y)-Y\theta(X)\\&=\sum_{r=1}^n Yg(\nabla_XZ_r,Z_{\overline{r}})-Xg(\nabla_YZ_r,Z_{\overline{r}})\\&=\sum_{r=1}^n g(\nabla_Y\nabla_XZ_r,Z_{\overline{r}})+g(\nabla_XZ_r,\nabla_YZ_{\overline{r}})-g(\nabla_YZ_r,\nabla_XZ_r)-g(\nabla_X\nabla_YZ_r,Z_{\overline{r}})\\&=\sum_{r=1}^n g(\mathrm{R}_{X,Y}Z_r,Z_{\overline{r}}))+\sum_{r=1}^n g(\nabla_XZ_r,\nabla_YZ_{\overline{r}})-g(\nabla_YZ_r,\nabla_XZ_{\overline{r}})\end{aligned}$$

and we get the following

Proposition 5.1. *Let (M,J,g) be an almost Hermitian manifold and let ∇ be an almost Hermitian connection i.e. $\nabla g=\nabla J=0$. Then if $\{Z_1,\dots,Z_r\}$ is a local unitary frame, then*

$$\sum_{r=1}^n g(\nabla_XZ_r,\nabla_YZ_{\overline{r}})=\sum_{r=1}^n g(\nabla_YZ_r,\nabla_XZ_{\overline{r}})\,.$$

5.2. **Proof of formula (4.2).** Now we show how to prove formula (4.2) starting from (2.1). The starting point is

$$\theta(X)=\sum_{i=1}^n\left\{g(D_XZ_i,Z_{\overline{i}})-g(D_{Z_i}X^{0,1},Z_{\overline{i}})+g(D_{Z_{\overline{i}}}X^{1,0},Z_i)\right\}$$

where $\{Z_i\}$ is a local unitary frame. Then using the Koszul formula

$$2g(D_XY,Z)=Xg(Y,Z)+Zg(X,Y)-Yg(Z,X)+g([X,Y],Z)+g([Z,X],Y)-g([Y,Z],X)$$

we have

$$\begin{aligned}2g(D_XZ_r,Z_{\overline{r}})=&Xg(Z_r,Z_{\overline{r}})+Z_{\overline{r}}g(X,Z_r)-Z_rg(Z_{\overline{r}},X)+g([X,Z_r],Z_{\overline{r}})+g([Z_{\overline{r}},X],Z_r)\\&-g([Z_r,Z_{\overline{r}}],X)\\=&Z_{\overline{r}}g(X,Z_r)-Z_rg(Z_{\overline{r}},X)+g([X,Z_r],Z_{\overline{r}})+g([Z_{\overline{r}},X],Z_r)-g([Z_r,Z_{\overline{r}}],X)\,.\end{aligned}$$

Analogously

$$2g(D_{Z_r}X^{0,1},Z_{\overline{r}})=Z_{\overline{r}}g(Z_r,X^{0,1})+g([Z_r,X^{0,1}],Z_{\overline{r}})+g([Z_{\overline{r}},Z_r],X^{0,1})-g([X^{0,1},Z_{\overline{r}}],Z_r)\,.$$

and

$$2g(D_{Z_{\overline{r}}}X^{1,0},Z_r)=Z_rg(Z_{\overline{r}},X^{1,0})+g([Z_{\overline{r}},X^{1,0}],Z_r)+g([Z_r,Z_{\overline{r}}],X^{1,0})-g([X^{1,0},Z_r],Z_{\overline{r}})\,.$$

Hence a direct computation yields

$$\begin{aligned}2\theta(X)=&2\sum_{r=1}^n\left\{g(D_XZ_r,Z_{\overline{r}})-g(D_{Z_r}X^{0,1},Z_{\overline{r}})+g(D_{Z_{\overline{r}}}X^{1,0},Z_i)\right\}\\=&\sum_{r=1}^n\Big\{g([2X^{0,1},Z_r],Z_{\overline{r}})-g([2X^{1,0},Z_{\overline{r}}],Z_r)\Big\}=4\,\mathrm{i}\sum_{r=1}^n\{\mathfrak{Im}\,g([X^{0,1},Z_r],Z_{\overline{r}})\}\,,\end{aligned}$$

i.e.

$$\theta(X) = 2\,\mathrm{i}\sum_{r=1}^{n}\{\Im\mathfrak{m}\, g([X^{0,1}, Z_r], Z_{\overline{r}})\}\,,$$

as required.

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