A BERGER TYPE NORMAL HOLONOMY THEOREM FOR COMPLEX SUBMANIFOLDS

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ABSTRACT. We prove Berger type theorems for the normal holonomy $\Phi^\perp$ (i.e., the holonomy group of the normal connection) of a full complete complex submanifold M both of $\mathbb{C}^n$ and of the complex projective space $\mathbb{C}P^n$. Namely,

(1) for $\mathbb{C}^n$, if M is irreducible, then $\Phi^\perp$ acts transitively on the unit sphere of the normal space;
(2) for $\mathbb{C}P^n$, if $\Phi^\perp$ does not act transitively, then M is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to 3.

The methods in the proofs rely heavily on the singular data of appropriate holonomy tubes (after lifting the submanifold to the complex Euclidean space, in the $\mathbb{C}P^n$ case) and basic facts of complex submanifolds.

Berger's Holonomy Theorem [3] is probably the most important general (local) result of Riemannian geometry: the restricted holonomy group of an irreducible Riemannian manifold acts transitively on the unit sphere of the tangent space except in the case that the manifold is a symmetric space of rank bigger or equal to two.

In submanifold geometry a prominent rôle is played by the holonomy group of the natural connection of the normal bundle, the so-called *normal holonomy group*.

For submanifolds of $\mathbb{R}^n$ or more generally of spaces of constant curvature, a fundamental result is the *Normal Holonomy Theorem* [17]. It asserts roughly that the non-trivial component of the action of the normal holonomy group on any normal space is the isotropy representation of a Riemannian symmetric space (called s-representation for short). The Normal Holonomy Theorem is a very important tool for the study of submanifold geometry, especially in the context of submanifolds with "simple extrinsic geometric invariants", like isoparametric and homogeneous submanifolds (see [4] for an introduction to this subject). In particular, in this extrinsic setting, some distinguished class of homogeneous submanifolds, the orbits of s-representations, play a similar rôle as symmetric spaces in intrinsic Riemannian geometry. Typically, requiring that a submanifold has "simple extrinsic geometric

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invariants" (e.g. "enough" parallel normal fields with respect to which the shape operator has constant eigenvalues) implies that the submanifold belongs to this class. Therefore, these methods based on the study of normal holonomy allowed to prove many results for submanifolds with "simple extrinsic geometric invariants" [4, 7, 8, 18, 19, 20, 24, 26, 27, 28]. But, actually, they turned out to be useful in (intrinsic) Riemannian geometry, as basic tools for a geometric proof of Berger's Theorem [21].

In [2] the normal holonomy group of complex submanifolds of a complex space form was studied. It was proven that if the normal holonomy group acts irreducibly on the normal space then it is linearly isomorphic to the holonomy group of an irreducible Hermitian symmetric space. Moreover the normal holonomy group acts irreducibly if the submanifold is full (that is, it is not contained in a totally geodesic proper complex submanifold) and the second fundamental form at some point has no nullity.

In the present paper, we prove Berger type theorems both for complex submanifolds of $\mathbb{C}^n$ and complex submanifolds of the complex projective space $\mathbb{C}P^n$.

Main Theorem 1. *The normal holonomy group of a complete irreducible and full immersed complex submanifold of $\mathbb{C}^n$ acts transitively on the unit sphere of the normal space. Indeed, $\Phi^\perp = U(k)$, where k is the codimension of the submanifold.*

Main Theorem 2. *Let M be a full and complete complex projective submanifold of $\mathbb{C}P^n$. Then the following are equivalent:*

1. *The normal holonomy is not transitive on the unit sphere of the normal space (i.e., different from $U(k)$, $k = codim(M)$, since it is an s-representation).*
2. *M is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to 3.*

Notice that (2) $\Rightarrow$ (1) was proved in [6]. It is well-known that *the complex orbit M, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space is extrinsic symmetric* or equivalently its second fundamental form is parallel [6]. Thus, *a full and complete complex submanifold $M \subset \mathbb{C}P^n$ whose normal holonomy group is not transitive on the unit sphere of the normal space has parallel second fundamental form.*

The proofs of the above results will be given in Sections 3 and 4 respectively.

The completeness assumption cannot be dropped either in Main Theorem 1 or in Main Theorem 2 (see Section 5).

The main tool is the study of the full holonomy tube (i.e., a holonomy tube with flat normal bundle, see § 1.7) of an Euclidean submanifold N whose normal holonomy group acts irreducibly and non-transitively on the unit sphere of the normal space. Let $M = N_\zeta$ be the full holonomy tube on N. We define a canonical foliation of M whose leaves are holonomy tubes of some focal manifold as well. It comes out there is a strong similarity with polar actions. Indeed, the orthogonal distribution to the holonomy tubes is integrable and its leaves behave like sections in a polar representation (Proposition 2). To show that the leaves of the canonical foliation are orbits of an isotropy representation (s-representation) we assume the horizontal distribution of a full holonomy tube is covered by kernels of shape operators. This implies that M and N are foliated by holonomy tubes around isotropy orbits (Theorem 1).

In order to apply this setting to a complete irreducible and full immersed complex submanifold M of the complex Euclidean space $\mathbb{C}^n$, we notice that if the normal holonomy does not act transitively on the unit sphere of the normal space, then there are abundantly many kernels of shape operators in order to cover the horizontal distribution of a full holonomy tube. Hence M is foliated by holonomy tubes around orbits of the isotropy representation of a Hermitian symmetric space (Theorem 2). Main Theorem 1 is then a consequence of this result and the fact that the normal holonomy group of a complex irreducible full submanifold of $\mathbb{C}^n$ acts irreducibly on the normal space [10].

Coming to complex submanifolds of $M \subset \mathbb{C}P^n$, we will lift M to a submanifold $\tilde{M}$ of $\mathbb{C}^{n+1}\backslash\{0\}$. A key point is then showing that the normal holonomy of $\tilde{M}$ is not transitive on the unit sphere of the normal space if this is the case for M. This will be done in Section 4.

1. Preliminaries

We begin recalling some basic facts, which are now well-known, and have been extensively used by the authors in their work on submanifold geometry. For most of the proofs we refer to [4].

1.1. General notation and basic facts.

Let $M \subset \mathbb{R}^n$ be a Euclidean submanifold, with induced metric $\langle\, ,\, \rangle$ and Levi-Civita connection ∇. We will use the notation ∇^E for the Levi-Civita connection in $\mathbb{R}^n$.

We will always denote by $\nu M := TM^\perp$ the normal bundle of M endowed with the normal connection $\nabla^\perp$. The maximal parallel and flat subbundle of νM will be written as $\nu_0 M$. Since we are working locally, all manifolds will be assumed to be simply connected. Hence $\nu_0 M$ is globally flat, that is, $\nu_0 M$ is spanned by the parallel normal fields to M. The normal curvature tensor will be denoted by $R^\perp$. We have that $R^\perp_{X,Y}\xi = 0$, for X, Y tangent fields and ξ normal field lying in $\nu_0 M$.

The second fundamental form (with respect to the ambient Euclidean space) will be denoted by α and the associated shape operator by A. These two tensors are related by the well known formula, for any X, Y tangent fields and ξ normal field, $\langle \alpha(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle$, which is symmetric in X, Y.

When there are several submanifolds involved, and it is not clear form the context, we add an upper script M, e.g. α^M or A^M.

The connection $\nabla \oplus \nabla^\perp$ of $TM \oplus \nu M$ will be denoted by $\bar{\nabla}$.

We recall the well-known formulae relating the basic objects in submanifolds geometry

(Gauss) $$\langle R_{X,Y}Z, W\rangle = \langle \alpha(X,W), \alpha(Y,Z)\rangle - \langle \alpha(X,Z), \alpha(Y,W)\rangle\,,$$

(Codazzi) $\langle (\bar{\nabla}_X A)_\xi Y, Z\rangle$, or equiv., $(\bar{\nabla}_X \alpha)(Y,Z)$ are symmetric in X, Y, Z ,

(Ricci) $$\langle R^\perp_{X,Y}\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle\,.$$

Let N be another submanifold such that $M \subset N \subset \mathbb{R}^n$. We say that M is *invariant under the shape operator* A^N, briefly M is A^N-invariant, if $A^N_\eta(T_x M) \subset T_x M$ for all $x \in M$, $\eta \in \nu_x M$. Equivalently, $\alpha^N(T_xM, \nu_xM \cap T_xN) = 0$. Observe that in this case $\nu N_{|M}$ is a parallel subbundle of νM.

A distribution $\mathcal{D}$ of N is called A^N-invariant if $A^N_\eta(\mathcal{D}_x) \subset \mathcal{D}_x$, for all $x \in N$, $\eta \in \nu_x N$.

The linear subspace of T_pM

$$\mathcal{N}_p = \bigcap_{\xi \in \nu_p M} \ker A_\xi = \{X_p \in T_pM : \alpha^M(\cdot, X) = 0\}$$

is called the *nullity space of M at p*. The collection of all these spaces is called the *nullity distribution* of M. Note that this is actually a distribution only on any connected component of a suitable dense and open subset of M.

The *normal exponential* of the Euclidean submanifold M, $\exp_\nu : \nu M \to \mathbb{R}^n$, is defined by $\exp_\nu(\xi_p) = p + \xi_p$. We set

$$\nu^r M = \{\xi \in \nu M : \ ||\xi|| < r\} \ , \qquad\qquad S_r\nu M = \{\xi \in \nu M : \ ||\xi|| = r\} \, .$$

If r is small, by making M possibly smaller around a point q, $\exp_\nu : \nu^r M \to \mathbb{R}^n$ is a diffeomorphism onto its image. In this case the so-called spherical ϵ-tube around M, denoted by $\exp_\nu(S_\varepsilon \nu M)$, is a submanifold of $\mathbb{R}^n$, for all $\varepsilon < r$.

The submanifold $M \subset \mathbb{R}^n$ is said to be *full* if it is not contained in any proper affine subspace of the ambient space. The submanifold M is said to be locally *reducible* if one can write locally $M = M_1 \times M_2$ where $M_1 \subset \mathbb{R}^k$, $M_2 \subset \mathbb{R}^{n-k}$ and $\mathbb{R}^n$ decomposes orthogonally as $\mathbb{R}^k \times \mathbb{R}^{n-k}$. We say that M is locally *irreducible* if it is not locally reducible. There are two very useful tools for deciding whether a submanifold M of Euclidean space is not full or reducible.

(1) M is not full if and only if there exists parallel normal field $\xi \neq 0$ such that $A_\xi \equiv 0$.
(2) *Moore's lemma.* M is locally reducible if and only if there exists a non trivial A-invariant parallel distribution of M.

Let $X^n = G/K$ be a simply connected complete symmetric space without Euclidean de Rham factor, where G is the connected component of the full group of isometries of X. The isotropy representation of K in the Euclidean space $T_{[e]}X \simeq \mathbb{R}^n$ is called an *s-representation.* Any principal orbit $M = K.v$ is an *isoparametric submanifold* of $\mathbb{R}^n$. Namely, νM is globally flat and A_ξ has constant eigenvalues for any parallel normal field ξ to M. More in general, if M is not necessarily a principal orbit then it has *constant principal curvatures* [13], i.e. the shape operator $A_{\xi(t)}$ has constant eigenvalues for any parallel normal field $\xi(t)$ along any curve.

Let M be a Euclidean submanifold. The so-called *normal holonomy group* $\Phi_p^\perp$ of M at p is the holonomy group of the normal connection of M at p. The *normal holonomy theorem* [17] states that the connected component of the normal holonomy group acts on the normal space, up to its fixed set, as an s-representation. Any s-representation acts *polarly* on the ambient space, i.e. there exists a subspace Σ that meets all orbits in an orthogonal way [24] (such a Σ is the normal space of any principal orbit). Conversely, given a polar representation there exists an s-representation with the same orbits [9].

A local group of isometries G of a Riemannian manifold X is said to act *locally polarly* if the distribution of normal spaces to maximal dimensional (local) orbits is integrable (or, equivalently, autoparallel; see [24]). If G acts locally polarly on X and $S \subset X$ is a locally G-invariant submanifold, then the restriction of G to S acts locally polarly on S (this follows form Corollary 3.2.5 and Proposition 3.2 of [4], though we will only need the special cases given by Proposition 3.2.9 of this reference and Lemma 2.6 in [22]).

The Normal Holonomy Theorem was extended to Riemannian submanifolds of the Lorentz space [22]. The conclusion, in this case, is that the normal holonomy acts polarly on the (Lorentzian type) normal space. This means that the normal spaces to any maximal dimensional time-like orbit meet any nearby orbit orthogonally. We will need to make use of this result, though we are only interested in Euclidean submanifolds.

1.2. **Complex submanifolds of $\mathbb{C}P^n$.** Recall that $\mathbb{C}P^n$ is obtained by $\mathbb{C}^{n+1}\backslash\{0\}$ identifying complex lines through the origin. Hence there is a canonical projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n$. Of course, one may also regard $\mathbb{C}P^n$ as a quotient of the unit $(2n+1)$-sphere in $\mathbb{C}^{n+1}$ under the action of $U(1)$, i.e., $\mathbb{C}P^n = S^{2n+1}/U(1)$ (this is because every line in $\mathbb{C}^{n+1}$ intersects the unit sphere in a circle; for $n = 1$ this construction yields the classical Hopf bundle). Thus one has a submersion $S^{2n+1} \to \mathbb{C}P^n$. The Fubini-Study metric $\langle\, , \rangle_{FS}$ is then characterized by requiring this submersion to be Riemannian.

Let $M \subset \mathbb{C}P^n$ be a full complex submanifold of the complex projective space.

Let us denote by $\widetilde{M}$ the lift of M to $\mathbb{C}^{n+1}\backslash\{0\}$, i.e. $\widetilde{M} := \pi^{-1}(M)$. Let $\mathcal{V}$ be the vertical distribution of the submersion $\pi : \widetilde{M} \to M$. It is standard to show that $\mathcal{V} \subset \mathcal{N}^{\widetilde{M}}$. If X is a tangent vector to M we will write $\widetilde{X}$ for its horizontal lift to $\mathbb{C}^{n+1}\backslash\{0\}$.

The submersion $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n$ is not Riemannian. Anyway, the following O'Neill's type formula holds

Proposition 1 (O'Neill's type formula)**.** *Let $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathbb{C}^{n+1}\backslash\{0\})$ be the horizontal lift of the vector fields $X, Y \in \Gamma(\mathbb{C}P^n)$. Then,*

$$(D_{\widetilde{X}}\widetilde{Y})_{\widetilde{p}} = (\widetilde{\nabla^{FS}_X Y})_{\widetilde{p}} + \mathcal{O}(\widetilde{X}, \widetilde{Y}) \tag{1.1}$$

where $\mathcal{O}(\widetilde{X}, \widetilde{Y}) \in \mathcal{V}_{\widetilde{p}}$ is vertical.

The proof is the same as the standard one [23]. Indeed, the restriction $d\pi : \mathcal{V}^\perp \to T\mathbb{C}P^n$ is a dilatation, i.e. $\pi^*\langle\, , \rangle_{FS} = \lambda^2\langle\, , \rangle$, where π^* is the pullback to the horizontal part. An important remark is that the function λ is constant on horizontal curves. Hence moving along horizontal curves one remains in the same sphere S^{2n+1} (of radius $1/\lambda$).

1.3. **Parallel normal fields.** Let $M \subset \mathbb{R}^n$ be a submanifold and let ξ be a non-umbilic parallel normal field to M. Since we are working locally, we may assume that the different eigenvalue functions of the shape operator A_ξ, $\lambda_1, \cdots, \lambda_g : M \to \mathbb{R}$ have constant multiplicities and so they are differentiable functions. Let $E_1 \cdots, E_g$ be their associated smooth eigendistributions, i.e., $TM = E_1 \oplus \cdots \oplus E_g$ and $A_{\xi\,|E_i} = \lambda_i \mathrm{Id}_{|E_i}$. The eigendistributions $E_1, \cdots, E_g$ are integrable, due to the Codazzi identity. Moreover, each eigendistribution is A-invariant. Indeed, since $\nabla^\perp\xi = 0$, $\langle R^\perp_{X,Y}\xi, \eta\rangle = 0$ and so, by the Ricci identity, $[A_\xi, A_\eta] = 0$, for all η normal field to M.

Assume that one of the eigenvalues, let us say λ_1 is constant. Using again the Codazzi identity, we get that the eigendistribution E_1 is not only integrable but also autoparallel in M (see [4]). Moreover, any (totally geodesic) integral manifold $S_1(x)$ of E_1 is not a full submanifold of $\mathbb{R}^n$. Indeed,

$$S(x) \subset x + T_x S(x) \oplus \nu_x M\,.$$

If the submanifold $M \subset \mathbb{R}^n$ has flat normal bundle, then all the shape operators commute and so they can be simultaneously diagonalized. Around a generic point, there are (unique, up to order) normal fields $\eta_1, \cdots, \eta_g$, the so-called *curvature normals*, and A-invariant eigendistributions $E_1, \cdots, E_g$ such that

$$TM = E_1 \oplus \cdots \oplus E_g$$

and $A_{\xi | E_i} = \langle \xi, \eta_i \rangle \mathrm{Id}_{E_i}$, for all normal sections ξ.

The integral manifolds of E_i are umbilical submanifolds of the ambient space (if $\dim E_i \geq 2$). If a curvature normal η_i is parallel (in the normal connection), then E_i is an autoparallel distribution of M. Moreover, any leaf $S_i(q)$ of E_i is (an open subset of) a sphere, which is totally geodesic in M. In fact, $S_i(q)$ is the sphere of the affine subspace $q + E_i(q) \oplus \eta_i(q)$ centered at $q + ||\eta_i||^{-2}\eta_i(q)$

Let now $M \subset N \subset \mathbb{R}^n$ be submanifolds with flat normal bundle and such that M is A^N-invariant. Observe that $\nu N_{| M}$ is a parallel (and flat) subbundle of νM. We relate the curvature normals in N with the ones in M:

Lemma 1. *Let $M \subset N \subset \mathbb{R}^n$ be submanifolds with flat normal bundle and such that M is A^N-invariant. Assume that η is a parallel curvature normal of N with associated autoparallel eigendistribution E. Suppose $\bar{E} := E_{| M}$ is contained in TM. Then $\bar{\eta} := \eta_{| M}$ is a parallel curvature normal with associated (autoparallel) eigendistribution $\bar{E}$.*

Proof. Let ξ be a parallel normal field to M which lies in $\nu N_{| M}$. Then $A^N_{\xi | TM} = A^M_\xi$ and so $A^M_{\xi | \bar{E}} = \langle \xi, \bar{\eta} \rangle \mathrm{Id}_{\bar{E}}$.
Let now ζ be a parallel normal field to M which is tangent to N. Since A^M_ζ commutes with all shape operators A^M_ξ, it commutes, in particular, with all A^M_ξ such that ξ lies in $\nu N_{| M}$. Thus A^M_ζ has to leave the common eigenspace $\bar{E}$ invariant. Let us compute $A^M_{\zeta | \bar{E}}$. Let X, Y be tangent fields to N which lie in E. Then

$$\langle A^M_\zeta(X), Y \rangle = -\langle \nabla^N_X \zeta, Y \rangle = \langle \zeta, \nabla^N_X Y \rangle = 0$$

since E is autoparallel. Then $A^M_{\zeta | E} = 0 = \langle \zeta, \bar{\eta} \rangle \mathrm{Id}_E$. This shows that $\bar{\eta}$ is a (parallel) curvature normal of M with associated eigendistribution $\bar{E}$. ☐

The same is true if M, N are Riemannian submanifolds of Lorentz space.

1.4. **Parallel and focal manifolds.** Let $M \subset \mathbb{R}^n$ be a submanifold and let $\xi \neq 0$ be a parallel normal field to M. Observe that this implies that $\nu_0 M$ is a non trivial subbundle of νM.

Assume that 1 is not an eigenvalue of $A_{\xi(x)}$, for any $x \in M$. The *parallel manifold* is defined by

$$M_\xi := \{x + \xi(x) : x \in M\}$$

and is a submanifold of $\mathbb{R}^n$. Note that the normal spaces $\nu_p M$ and $\nu_{p+\xi(p)} M_\xi$ identify since they are parallel (affine) spaces in $\mathbb{R}^n$.

If 1 is a constant eigenvalue of A_ξ with constant multiplicity then M_ξ is also a submanifold of Euclidean space, a so-called *focal (or parallel focal) manifold to* M, and

$$\pi : M \to M_\xi$$

is a submersion, where $\pi(x) = x + \xi(x)$ (but not in general a Riemannian submersion). The fibers $\pi^{-1}(\{\pi(x)\})$ are totally geodesic in M and A-invariant. Indeed, these fibers are the integral manifolds of the eigendistribution

$$\mathcal{V}^\pi := \ker(\mathrm{Id} - A_\xi)$$

associated to the constant eigenvalue 1.

There is an orthogonal decomposition

$$T_x M = T_{\pi(x)}(M_\xi) \oplus \mathcal{V}^\pi_x$$

and, by what remarked in the previous subsection,

$$\pi^{-1}(\{\pi(x)\}) \subset \pi(x) + \nu_{\pi(x)}(M_\xi)$$

that is, the fibers lie in the normal space of the focal manifold.

From the last two relations, there is the orthogonal splitting

$$\nu_{\pi(x)}(M_\xi) = \nu_x M \oplus \mathcal{V}^\pi_x\,.$$

The horizontal distribution $\mathcal{H}^\pi$ of M is the one perpendicular to $\mathcal{V}^\pi$. Observe that the horizontal distribution $\mathcal{H}^\pi$ is not in general integrable but it is A^M-invariant, since $\mathcal{V}^\pi$ is so. By the above relations one has, as subspaces,

$$\mathcal{H}^\pi_x = T_{\pi(x)}(M_\xi)\,.$$

1.5. **Parallel transport and shape operators of parallel (focal) manifolds.** The following discussion is similar to that in [13]. Let $c(t)$ be a curve in M_ξ and let $q \in \pi^{-1}(\{c(0)\})$. Then locally there is a unique horizontal lift $\tilde{c}(t)$ of $c(t)$ with $\tilde{c}(0) = q$, i.e. $\pi \circ \tilde{c} = c$ and $\tilde{c}'(t) \in \mathcal{H}^\pi_{\tilde{c}(t)}$. Then, $\eta(t) = \tilde{c}(t) - c(t)$ is a parallel normal field to M_ξ along $c(t)$, since its Euclidean derivative at t lies in $T_{c(t)}(M_\xi)$. Conversely, if $\eta(t)$ is the parallel normal field along $c(t)$, with $\eta(0) = q - c(0)$, then $\tilde{c}(t) := c(t) + \eta(t)$ is the horizontal lift of $c(t)$ with initial condition q. This implies the important fact that the $\nabla^\perp$-parallel transport along a curve c in M_ξ, joining p and q, $\tau_c^\perp : \nu_p(M_\xi) \to \nu_q(M_\xi)$, maps (locally) $\pi^{-1}(\{p\})$ into $\pi^{-1}(\{q\})$.

Let now $\tilde{\beta}(t)$ be a horizontal curve in M and let $\beta(t) = \pi(\tilde{\beta}(t))$. Let $\eta(t)$ be a parallel normal field to M along $\tilde{\beta}(t)$. Since $\nu_{\tilde{\beta}(t)} M \subset \nu_{\beta(t)}(M_\xi)$, then $\eta(t)$ may also be regarded as a normal field to M_ξ along $\beta(t)$. Moreover, $\eta(t)$ is also a parallel normal field to M_ξ along $\beta(t)$. Indeed, since $\eta(t)$ is parallel along $\tilde{\beta}(t)$, one has that

$$\frac{d}{dt}\eta(t) = -A^M_{\eta(t)}.\tilde{\beta}'(t) \subset \mathcal{H}^\pi_{\tilde{\beta}(t)} = T_{\beta(t)}(M_\xi) \tag{1.2}$$

by the A^M invariance of $\mathcal{H}^\pi$.

Using (1.2), since $\beta(t) = \pi(\tilde{\beta}(t)) = \tilde{\beta}(t) + \xi(\tilde{\beta}(t))$ one has that

$$\beta'(t) = \tilde{\beta}'(t) - A^M_{\xi(\tilde{\beta}(t))}.\tilde{\beta}'(t) = (\mathrm{Id} - A^M_{\xi(\tilde{\beta}(t))}).\tilde{\beta}'(t).$$

On the other hand, since $\eta(t)$ is a parallel normal field along $\beta(t)$,

$$\frac{d}{dt}\eta(t) = -A^{M_\xi}_{\eta(t)}.\beta'(t)\,. \tag{1.3}$$

Then, since the expressions (1.2) and (1.3) coincide, and $\tilde{\beta}(t)$ is an arbitrary horizontal curve, one gets the well-known formulae relating the shape operators of M and M_ξ, sometimes called "*tube formulae*" [4]

$$A^{M_\xi}_{\eta_x} = A^M_{\eta_x}(\mathrm{Id} - A^M_{\xi(x)})^{-1}_{|\mathcal{H}^\pi_x} \tag{1.4}$$

for all $\eta_x \in \nu_x M$

In a similar way we have

$$A^M_{\eta_x\,|\mathcal{H}^\pi} = A^{M_\xi}_{\eta_x}(\mathrm{Id} - A^{M_\xi}_{-\xi(x)})^{-1} \tag{1.5}$$

for all $\eta_x \in \nu_x M$.

1.6. **Parallel manifolds at infinity.** Let $M \subset \mathbb{R}^n$ be a submanifold with a parallel normal field ξ. In some cases, for our geometric study of M, there are not enough parallel manifolds $M_{\lambda\xi}$ to M in the Euclidean space, $\lambda \in \mathbb{R}$. So, it is convenient to regard M as a Riemannian submanifold of a Lorentz space, in which case the family of parallel manifolds to M is enlarged. This construction is worth while when 0 is an eigenvalue of A_ξ with constant multiplicities (otherwise, everything can be carried out in the original Euclidean space). In this case the integral manifolds of the A-invariant autoparallel distribution $E = \ker A_\xi$ are the fibers of the submersion defined by passing to a parallel focal manifold. Observe that E is not in general the nullity distribution, i.e. the one given by the intersection of the kernels of all shape operators.

For this purpose, let $L^{n+2} = (\mathbb{R}^{n+2}, \langle\ ,\ \rangle)$, where

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2}\,.$$

Recall that the hyperbolic space of radius r is given by

$$H^{n+1}(r) = \{x \in L^{n+2} : \langle x, x\rangle = -r^2,\ x_1 > 0\}\,.$$

In this way $H^{n+1}(r)$ is regarded as a totally umbilical (Riemannian) hypersurface of L^{n+2}. Indeed, the vector field $\eta(x) = -x$ is a parallel (time-like) normal field to L^{n+2} and $A^{L^{n+2}}_\eta = \mathrm{Id}$. Now regard $\mathbb{R}^n$ as a horosphere Q^n of the hyperbolic space, which is also a totally umbilical hypersurface. In this way one can regard M as a submanifold of Lorentz space. Now there is in M an extra, somewhat trivial, parallel normal field given by the restriction to M of the vector field of the hyperbolic space which we also call η (the normal vector field to the horosphere, in the hyperbolic space, gives no useful information). Now, in the Lorentz space, we have the family of parallel manifolds to M given by $M_{a\xi+b\eta}$, cf. [11].

There is no essentially new parallel manifold except the focal one $M_{\tilde\xi}$, where $\tilde\xi = a\xi + \eta$, a is small enough so that $\tilde\xi$ be time-like (it is convenient, for reasons related to the normal holonomy of the focal manifold, to choose a time-like parallel normal field for the focalization).

In this way

$$\pi : M \to M_{\tilde\xi}\,,$$

where $\pi(x) = x + \tilde\xi(x)$ is a submersion. The fibers $\pi^{-1}(\{\pi(x)\})$ are just the integral manifolds of $E = \mathcal{V}^\pi = \ker A_\xi = \ker(\mathrm{Id} - \tilde{A}_{\tilde\xi})$, where $\tilde{A}$ denotes the shape operator of M as a submanifold of the Lorentz space. Note that the focal manifold $M_{\tilde\xi}$ is contained in a de Sitter space of radius $|a|\|\xi\|$.

One can relate parallel transport and shape operators in parallel (focal) manifolds like in the previous subsection (see [22, 4]).

1.7. **Holonomy tubes.** Let M be a Euclidean submanifold or a Riemannian submanifold of the Lorentz space. Let $\xi_p \in \nu_p M$. If M is a submanifold of Lorentz space then ξ_p is assumed to be time-like. The *holonomy tube* around M through ξ_p is defined by

$$M_{\xi_p} = \{c(1) + \xi(1)\} = \{c(1) + \tau_c^{\perp}(\xi_p)\},$$

where $c : [0,1] \to M$ is an arbitrary curve starting at p and $\xi(t)$ is the parallel transport of ξ_p along $c(t)$. If 1 is not an eigenvalue of the shape operator A_{ξ_p}, then M_{ξ_p} is a submanifold of the ambient space, e.g. if ξ_p is near 0 (perhaps by making M smaller). One has a projection $\pi : M_{\xi_p} \to M$, defined by $\pi(c(1) + \xi(1)) = c(1)$. Moreover, $q \mapsto \eta(q) := \pi(q) - q$ is a parallel normal field to M_{ξ_p} and so we have that M is a parallel manifold (in general, focal) to its holonomy tube. Namely,

$$M = (M_{\xi_p})_\eta$$

Observe that $\eta(p + \xi_p) = -\xi_p$. Note that the fibers of π are given by the orbits of the normal holonomy group of M. Namely,

$$\pi^{-1}(\{\pi(p)\}) = \pi(p) + \Phi^{\perp}_{\pi(p)}.(p - \pi(p))$$

In the Lorentzian case this fiber is contained in the hyperbolic space of the normal space given by the time-like vector ξ_p, which is invariant under this holonomy action. Moreover, this action, restricted to this hyperbolic space is locally polar. Observe that in this Lorentzian case, the holonomy tube is a Riemannian submanifold, since M and the holonomy orbit $\Phi^{\perp}_{\pi(p)}.(p - \pi(p))$ are so.

On the other hand, let M be a submanifold with a parallel normal field η such that 1 is a constant eigenvalue, with constant multiplicity r, of A_η, $r < \dim(M)$. By § 1.5 (see also § 1.6 for the Lorentzian case), we have that

$$(M_\eta)_{-\eta(q)} \subset M$$

for all $q \in M$. That is, M is foliated by the holonomy tubes around the parallel manifold M_η (this foliation could be trivial, i.e., with only one leaf).

Let us observe that if the normal vector $\xi_p \in \nu_p M$ extends to a parallel normal field then the holonomy tube M_{ξ_p} is a parallel non-focal manifold to M. This is equivalent to the fact that ξ_p is fixed by the normal holonomy group of M.

If the orbit $\Phi^{\perp}_{\pi(p)}.(p - \pi(p))$ is maximal dimensional (and hence isoparametric in the normal space) then the holonomy tube M_{ξ_p} has flat normal bundle; see [13] for the Euclidean case. The Lorentzian case is similar since normal holonomy orbits, through principal time-like vectors, are isoparametric in a hyperbolic space (and also when regarded as Riemannian submanifolds of the normal space).

Conversely, if the holonomy tube M_{ξ_p} has flat normal bundle then the holonomy orbit must have flat normal bundle, in the normal space, and hence is maximal dimensional. In the Euclidean space this is well-known, since singular orbits of s-representation must have non-trivial normal holonomy [14]. In the Lorentzian space the polar actions are, orbit-like, essentially the same as in Euclidean space, up to some transitive factors in hyperbolic space or horospheres (see [22, Theorem 2.3]).

We shall call *full holonomy tube* a holonomy tube with flat normal bundle. By the above discussion we have

Lemma 2. *The holonomy tube M_{ξ_p} has flat normal bundle, i.e., it is a full holonomy tube if and only if the normal holonomy orbit $\Phi^{\perp}_{\pi(p)}.(p-\pi(p))$, with $\pi : M_{\xi_p} \to M$ the projection, is maximal dimensional (and hence an isoparametric submanifold of the normal space $p+\nu_p M$).*

Remark 1. Let M_{ξ_p} be a full holonomy tube and let $\bar{\eta}$ be a curvature normal, with associated autoparallel eigendistribution E of the isoparametric submanifold $p+\Phi^{\perp}_{\pi(p)}.(p-\pi(p))$ of the normal space $p+\nu_p M$ ($\bar{\eta}$ must be parallel in the normal connection). Then $\bar{\eta}$ is the restriction to $p+\Phi^{\perp}_{\pi(p)}.(p-\pi(p))$ of a parallel curvature normal η of M_{ξ_p}, whose associated eigendistribution, restricted to the holonomy orbit, coincides with E (cf. Lemma 1, § 1.3, page 6). Moreover, the restriction of η to any holonomy orbit is a curvature normal of this orbit.

2. Foliation by holonomy tubes

This section is the main core of this paper. We begin with a submanifold $M \subseteq \mathbb{R}^n$ endowed with a parallel normal field ξ such that 1 is an eigenvalue with constant multiplicity of A_ξ. As we know from the previous subsection, M is foliated by the holonomy tubes $H(x) := (M_\xi)_{x-\pi(x)} = (M_\xi)_{-\xi(x)}$, $x \in M$. We may assume, since we are working locally that all these holonomy tubes have the same dimension.

In § 2.1 we describe the properties of this foliation (Proposition 2). It comes out there is a strong similarity with polar actions. Indeed, the orthogonal distribution to $H(x)$ is integrable and its leaves $\Sigma(x)$ behave like sections in a polar representation. In § 2.2 we compare the eigendistributions of nearby parallel manifolds.

Then we introduce a canonical foliation for submanifolds of $\mathbb{R}^n$ whose normal holonomy group acts irreducibly and non-transitively on the unit sphere of the normal space. In § 2.3 we begin with a submanifold $N \subset \mathbb{R}^n$, take a full holonomy tube $N_{\zeta_p} =: M$ and we assume there is a parallel normal field ξ on M with $\ker A_\xi \neq \{0\}$. Then M is foliated by the holonomy tubes $H^\xi(x)$ around the focal manifold at infinity $M_{\tilde{\xi}} \subset L^{n+2}$. This may seem to depend on the choice of the parallel normal field ξ, but in § 2.4 we show it is not the case. Now, N can be regarded as a focal manifold of M, with projection $\pi : M \to N$. In § 2.5 we project down to N the canonical foliation on M. The homogeneity of this canonical foliation is finally proven in § 2.6 provided that the horizontal distribution of a full holonomy tube is covered by kernels of shape operators (Theorem 1).

2.1. Polar-like properties of the foliation by holonomy tubes.

Let M be submanifold of Euclidean space or, more generally, a Riemannian submanifold of Lorentzian space. Let ξ be a parallel normal field to M and assume that M_ξ is a parallel focal manifold to M, i.e 1 is an eigenvalue, with constant multiplicity, of the shape operator A_ξ. As we have observed in the previous subsection, M is foliated by the holonomy tubes $(M_\xi)_{x-\pi(x)} = (M_\xi)_{-\xi(x)}$, $x \in M$, that we assume are all of the same dimension.

Let $\tilde{\nu}$ be the distribution in M which is perpendicular to the tangent spaces of the holonomy tubes. Observe that the restriction of $\tilde{\nu}$ to any fiber $S(x) = \pi^{-1}(\pi(x))$ coincides with the distribution given by the normal spaces to the orbits of the normal holonomy group $\Phi^{\perp}_{\pi(x)}$ in $S(x)$. But this action must be locally polar (see the end of § 1.1). Then the normal spaces to the orbits is an autoparallel distribution. This shows that $\tilde{\nu}$ is autoparallel, since the fibers $S(x)$ are totally geodesic.

Let us examine the construction of the integral manifolds $\Sigma(q)$ of $\tilde{\nu}$ more closely. This construction is implicit in the proof of [Lemma 2.6]OW). The main point is that the restriction, to an invariant submanifold, of a locally polar action is again locally polar [Lemma 2.6]OW. Indeed,

$$\Sigma(q) = S(q) \ \cap \ q + \nu_{-\xi(q)}(\, \Phi^{\perp}_{\pi(q)}.(-\xi(q)\,)\,) \tag{2.1}$$

where $\Phi^{\perp}$ denotes the normal holonomy group of M_ξ and the normal space to the holonomy orbit is inside $\nu_{\pi(q)}(M_\xi)$. Observe that the above expression is independent of x in a given $\Sigma(q)$, and shows that $\Sigma(q)$ is totally geodesic.

If $x \in \Sigma(q)$ then, by (2.1), $(x - q)$ belongs to the normal space, in $\nu_{\pi(q)}(M_\xi)$, of the holonomy orbit $\Phi^{\perp}_{\pi(q)}.(-\xi(q)\,)$. This orbit has the same dimension as its nearby orbit $\Phi^{\perp}_{\pi(q)}.(-\xi(x)\,)$ (note that $\pi(x) = \pi(q)$). This implies that $(x - q)$ is a fixed vector of the slice representation of the isotropy subgroup $(\Phi^{\perp}_{\pi(q)})_{-\xi(q)}$ of $\Phi^{\perp}_{\pi(q)}$ at $-\xi(q)$. Hence the extension η of $(x-q)$ to a $\Phi^{\perp}_{\pi(q)}$-invariant normal field to $\Phi^{\perp}_{\pi(q)}(-\xi(q))$, where $\eta(q) = x - q$, is parallel in the normal connection of the orbit, regarded as a submanifold of $\nu_{\pi(q)}(M_\xi)$ (see Proposition 2.4 of [22] and Proposition 3.2.4 of [4]). Observe that this orbit could be non-principal in the ambient space. Since $x \in \Sigma(q)$ is arbitrary we obtain that

$$-q + \Sigma(q) \subset \nu_0(\Phi^{\perp}_{\pi(q)}(-\xi(q)))$$

(recall that ν_0 is the maximal parallel and flat subbundle of ν). By the above construction we have that the normal parallel transport, along any curve in $\Phi^{\perp}_{\pi(q)}(-\xi(q))$, from q to q', maps $-q + \Sigma(q)$ into $-q' + \Sigma(q')$. It is standard to prove and well-known (see [13]) that a parallel and $\Phi^{\perp}_{\pi(q)}$-invariant normal field to the holonomy orbit $\Phi^{\perp}_{\pi(q)}(-\xi(q))$ extends to parallel normal field η of the holonomy tube $H(q) := (M_\xi)_{-\xi(q)}$ (we require that η be both parallel and $\Phi^{\perp}_{\pi(q)}(\xi(q))$-invariant since the holonomy orbit could be non-full). Then,

$$-q + \Sigma(q) \subset \nu_0\,(H(q))\ .$$

Moreover, the sets $-x + \Sigma(x)$ move parallel with respect to the normal connection of $H(q)$, $x \in H(q)$. This implies that its tangent spaces give rise to a parallel and flat subbundle of the normal bundle $\nu(H(q))$ in the ambient space. That is, the restriction to $H(q)$ of $\tilde{\nu}$ is a parallel and flat subbundle of $\nu(H(q))$.

Let $x \in \Sigma(q)$ and let η be the parallel normal field to $H(q)$ with $\eta(q) = x - q$. Then observe that $H(x) = (H(q))_\eta$, i.e. the different holonomy tubes inside M are parallel manifolds.

We can now prove that $\tilde{\nu}$ is a A^M-invariant distribution of M. Indeed, let X, Y be vector fields on M, where X is tangent to the holonomy tubes and Y is perpendicular, i.e. Y lies in $\tilde{\nu}$. The Euclidean derivative $(\nabla^E_X Y)_x \in T_x H(x) \oplus \tilde{\nu}_x$, since $\tilde{\nu}_{|\,H(x)}$ is a parallel subbundle of the normal bundle in the ambient space. Then it has no normal component to M. Thus $\alpha^M(\tilde{\nu}, \tilde{\nu}^{\perp}) = 0$ and therefore $\tilde{\nu}$ is A^M-invariant.

We summarize what we have proven in the following:

Proposition 2. *Let M be a Euclidean submanifold or, more generally, a Riemannian submanifold of Lorentz space. Let ξ be a parallel normal field to M, with a constant eigenvalue 1 with constant multiplicity. For any $x \in M$, we denote by $H(x) \subset M$ the holonomy tube $(M_\xi)_{-\xi(x)}$ of the focal manifold M_ξ and we assume* *that all $H(x)$ have the same dimension. Let $\tilde{\nu}$ be the distribution in M which is perpendicular to the family of holonomy tubes. Then,*

(i) *$\tilde{\nu}$ is autoparallel and invariant under all shape operator of M. Moreover, if $\Sigma(x)$ is a leaf of $\tilde{\nu}$ through x, then*

$$\Sigma(x) = (x + \nu_x H(x)) \cap M \,.$$

(ii) *The leaves $\Sigma(q)$ are invariant under the parallel transport in the normal bundle of the focal manifold M_ξ. That is, if c is a curve in M_ξ from $\pi(x)$ to $\pi(y)$ then*

$$\tau_c^\perp(\Sigma(x)) = \Sigma(y) \,.$$

(iii) *The restriction of $\tilde{\nu}$ to any $H(x)$ is a parallel (and flat) subbundle of $\nu_0 H(x)$. Moreover,*

$$\Sigma(x) \subset x + (\nu_0 H(x))_x$$

and $\Sigma(y)$ moves parallel, in the normal connection of the holonomy tube $H(x)$. That is, if c is a curve in $H(x)$ from y to z, then

$$\tau_c^\perp(\Sigma(y)) = \Sigma(z) \,.$$

(iv) *Let $x \in \Sigma(q)$ and identify $(x-q)$ with the parallel normal field to $H(q)$ with this initial condition at q. Then $H(x) = (H(q))_{x-q}$.*

2.2. **Nearby parallel manifolds.** Let $M \subset \mathbb{R}^n$ be a submanifold with a parallel normal field ξ such that the eigenvalues of the shape operator A_ξ have constant multiplicities. Let $0, \lambda_1 < \cdots, < \lambda_g$ be the different eigenvalue functions with associated eigendistributions $E_0, \cdots, E_g$. Let η be another parallel normal field to M, such that 1 is not an eigenvalue of A_η. Consider the non-focal parallel manifold M_η.

Assume that the eigenvalue functions of $A_\xi^{M_\eta}$ are 0, $\bar{\lambda}_1 < \cdots, < \bar{\lambda}_g$, with associated eigendistributions $\bar{E}_0, \cdots, \bar{E}_g$, where $\dim(E_i) = \dim(\bar{E}_i)$, for all $i = 0, \cdots, g$ (and we assume that the same is true if we re-scale η by a real number $0 \leq t \leq 1$). Note that A_η must leave the eigendistributions $E_0, \cdots, E_g$ invariant, since it commutes with A_ξ. Thus, in this case, from the tube formulae relating shape operators of parallel manifolds, A_η has only one eigenvalue function, let us say β_i in each E_i, $i \geq 1$. Since we are assuming that η is small, we have that, for $i = 1, \cdots, g$

$$\bar{\lambda}_i \circ h = \frac{\lambda_i}{1 - \beta_i} \,,$$

where $h : M \to M_\eta$ is the parallel map, i.e. $h(q) = q + \eta(q)$.

Let J the subset of $\mathbb{R}$ which consists of the constant eigenvalues of A_ξ and let J^η the analogous subset with respect to $A_\xi^{M_\eta}$. Let now $a \in J \cap J^\eta$. Then, possibly by re-scaling η (actually, we are assuming that a belongs to $\cap_t J^{t\eta}$)) we must have that there is an index j such that $\lambda_j = \bar{\lambda}_j \circ h \equiv a$. This implies that $\beta_j \equiv 0$.
Let now

$$I = \{i : \lambda_i = \bar{\lambda}_i \circ h \text{ and it is constant}\}$$

and

$$E = \bigoplus_{j \in I} E_j$$

Observe that $A_{\eta\,|E} \equiv 0$. So, if $c(t)$ is a curve that lie in E, then $\frac{d}{dt}\eta(c(t)) \equiv 0$, since both tangential and normal part of the derivative vanish. So we have proved the following

Lemma 3. *The parallel normal field η is constant along E.*

2.3. **The canonical foliation of a full holonomy tube.** Let $N \subset \mathbb{R}^n$ be a submanifold of Euclidean space and let us consider a holonomy tube

$$M = N_{\zeta_q}$$

around N, where ζ_q is a principal vector for the normal holonomy group $\Phi_q^\perp$ of N at q, and 1 is not an eigenvalue of $A^N_{\zeta_q}$. Then M has flat normal bundle, i.e., it is a full holonomy tube. Let $\pi : M := N_{\zeta_q} \to N$ be the projection and let $\psi(p) = \pi(p) - p$. Hence, $N = M_\psi$, i.e. the manifold N is a parallel (focal, if N has non-flat normal bundle) manifold to its holonomy tube.

Let, for $p \in M$,

$$S(p) = \pi^{-1}(\pi(p)) = p + \Phi^\perp_{\pi(p)}.(p - \pi(p))\,.$$

Since we are working locally, we may assume that N is simply connected, so its normal holonomy group (and hence $S(p)$) is connected.

For a generic $p \in M$, the common eigenspaces of the shape operators of M define, in a neighbourhood U of p, C^∞ eigendistributions $E'_1, \cdots, E'_s$ of M, with associated C^∞ curvature normals $\eta_1, \cdots, \eta_s$. We assume $U = M$. Observe that $\ker(\mathrm{Id} - A^M_\psi) = \mathcal{V}$, where $\mathcal{V}$ is the vertical distribution of M, i.e. $\mathcal{V}_x = T_x S(x)$. We have that $\mathcal{V}$ is the direct sum of some of the eigendistributions $E'_1, \cdots, E'_s$ (see the remark at page 10 in § 1.7). Namely, those eigendistributions whose index i verify that $\langle \psi, \eta_i \rangle \equiv 1$. We may assume that

$$\mathcal{V} = E'_1 \oplus \cdots \oplus E'_l$$

where $l < s$. Observe that the curvature normals $\eta_1, \cdots, \eta_\ell$ are parallel, since they are the extensions of the curvature normals of any fiber $S(x)$, which is an isoparametric submanifold (see the remark at page 10 in § 1.7).

Let $\xi \neq 0$ be a parallel normal field to M and let, $\ker(A^M_\xi) = E^\xi_0,\ E^\xi_1, \cdots, E^\xi_r$ be the eigendistributions associated to the constant eigenvalues of the shape operator A^M_ξ (we may have to consider a smaller M). Observe that any E^ξ_k is the sum of some of the eigendistributions $E'_1, \cdots, E'_s$. Since $\langle \xi, \eta_k \rangle$ is a constant eigenvalue of A^M_ξ (for $0 \leq k \leq \ell$) we get that, if $1 \leq i \leq \ell$, then

$$E'_i \subset E^\xi_{j(i)}$$

for some $j(i) \in \{1, \cdots, r\}$. Observe that in general it could be that $\langle \xi, \eta_i \rangle \equiv \langle \xi, \eta_j \rangle$ for $i \neq j$ and so, in this case, $E_i \oplus E_j$ is contained in some eigendistribution of A^M_ξ.

Assumption. *In the sequel of this section we will suppose that the normal holonomy group $\Phi^\perp$ of N acts irreducibly and not transitively on the normal space.*

Therefore $S(x)$ *is an irreducible isoparametric submanifold of the normal space* $\nu_{\pi(x)}N$ (observe that N must be an irreducible and full submanifold).

Now, *assume further that* 0 *is a constant eigenvalue of* A^M_ξ*,i.e.* E^ξ_0 *is non-trivial.* We introduce a *canonical foliation* of M, starting from ξ, but we will later show it is independent on ξ (§ 2.4). Recall M is foliated by the holonomy tubes $H^\xi(x)$

around the focal manifold $M_{\xi} \subset L^{n+2}$, that we assume (possibly in a smaller M) are all of the same dimension (see § 1.6 and 1.7). To visualize the holonomy tube let us define the equivalence relation in M, $x \underset{\xi}{\sim} y$ if there is a curve in M from x to y and such that it is always perpendicular to the distribution E_0^{ξ}. Then, locally,

$$H^{\xi}(x) = \{y \in M : x \underset{\xi}{\sim} y\} .$$

From the Homogeneous Slice Theorem [13] (the local version follows from Theorem 3.1 in [22]) one has that, starting from $x \in M$ and moving perpendicularly to E_0^{ξ} one can reach any other point of $S(x)$. Indeed, let $\mathcal{D} = \mathcal{V} \cap E_0^{\xi}$. One has that $\mathcal{D} \neq \mathcal{V}$, otherwise $0 = A^M_{\xi|\mathcal{V}_x} = A^{S(x)}_{\xi(x)}$. Hence the restriction of ξ to $S(x)$ is a parallel normal field whose shape operator is null. Hence $S(x)$ is not full in the normal space $\nu_{\pi(x)}M$. A contradiction. Now observe that $\mathcal{D} = \ker(\mathrm{Id} - A^{S(x)}_{\vartheta})$, where $\vartheta = \psi_{|S(x)} - \xi_{|S(x)}$. Then, beginning with a point $x \in M$, moving perpendicularly to $\mathcal{D}$, but remaining inside $S(x)$, we reach any other point of $S(x)$, by the Homogeneous Slice Theorem. So, moving perpendicularly to E_0^{ξ}, starting at x, we reach any point in $S(x)$.

So we have proven that

$$S(x) \subset H^{\xi}(x) . \tag{2.2}$$

Let now $\tilde{\nu}^{\xi}$ be the normal space to the foliation of M by the holonomy tubes $H^{\xi}(x)$ and let us denote by $\Sigma_{\xi}(x)$ the totally geodesic leaves of $\tilde{\nu}^{\xi}$. By the Proposition 2 in § 2.1, if $x \in \Sigma_{\xi}(q)$,

$$H^{\xi}(x) = (H^{\xi}(q))_{\varsigma} ,$$

where ς is the parallel normal field to $H^{\xi}(q)$ with $\varsigma(q) = x - q$.

Remark 2. (i) Observe that $H^{\xi}(q)$ has flat normal bundle. Indeed, $\nu H^{\xi}(q) = \tilde{\nu}^{\xi}_{|H^{\xi}(q)} \oplus \nu M_{|H^{\xi}(q)}$ and both subbundles are parallel and flat (see Proposition 2 in § 2.1).
(ii) By (2.2), the restrictions of the parallel curvature normals $\eta_1, \cdots, \eta_\ell$ of M to any holonomy tube $H^{\xi}(q)$ are parallel curvature normals of this tube. The associated eigendistributions are just the restriction to $H^{\xi}(q)$ of the corresponding eigendistributions $E_1', \cdots, E_\ell'$ of M (see Lemma 1 in § 1.3).

We continue with the assumptions before the above remark.

Since $\tilde{\nu}^{\xi} \subset E_0^{\xi} = \ker(A^M_{\xi})$ we have that ξ is constant along $\Sigma_{\xi}(q)$, in the ambient space. So,

$$\xi(q) = \xi(x)$$

as vectors of the ambient space ($x \in \Sigma_{\xi}(q)$). The same is true for any point in $H^{\xi}(q)$, i.e.

$$\xi(q') = \xi(q' + \varsigma(q'))$$

for all $q' \in H^{\xi}(q)$.

We can now apply Lemma 3, since the shape operators $A_{\xi}^{(H^{\xi}(q))_{\varsigma}}$ and $A_{\xi}^{H^{\xi}(q)}$ share the same constant eigenvalues $\langle \xi, \eta_1 \rangle, \cdots, \langle \xi, \eta_\ell \rangle$. So we conclude that ς must be constant, in the ambient space, along any fiber $S(q')$, $q' \in H^{\xi}(q)$. Then, by Proposition 2 in § 2.1, we get that the sets $\Sigma_{\xi}(y)$ are constant (i.e., differ by a translation) if y moves in $S(q')$, for all $q' \in H^{\xi}(q)$ (of course locally).

2.4. **Independence of the foliation on the parallel normal field.** Let us decompose

$$\tilde{\nu}^{\xi} = \tilde{\nu}_1^{\xi} \oplus \cdots \oplus \tilde{\nu}_t^{\xi}$$

into different eigendistributions of the family of shape operators of M, restricted to $\tilde{\nu}^{\xi}$ (perhaps in smaller M). Let $\eta_{h(1)}, \cdots \eta_{h(t)}$ be the associated curvature normals, i.e. $A^M_{\mu \,|\, \tilde{\nu}_i^{\xi}} = \langle \mu, \eta_{h(i)} \rangle \mathrm{Id}_{\tilde{\nu}_i^{\xi}}$. Observe that the eigendistribution $E'_{h(i)}$ contains $\tilde{\nu}_i^{\xi}$ and there is no reason for the equality. Since $\tilde{\nu}^{\xi}$ is autoparallel and A^M-invariant, one has that the restriction $\eta_{h(i)|\,\Sigma_{\xi}(y)}$ is a curvature normal of $\Sigma_{\xi}(y)$, for all $y \in M$.

Since the integral manifolds $\Sigma_{\xi}(y)$ are constant along $S(x)$, $y \in S(x)$, we must have that $\eta_{h(i)|S(x)}$ is a constant normal vector field to $S(x)$ (regarded as a full submanifold of $\nu_{\pi(x)}N$). Then $\eta_{j(i)} = 0$ and so $\Sigma_{\xi}(x)$ is totally geodesic in the ambient space for all $x \in M$ (an hence an open subset of an affine subspace). This shows, since $\tilde{\nu}^{\xi}$ is A^M-invariant, that $\tilde{\nu}^{\xi}$ is contained in the nullity of the second fundamental form α^M. Or equivalently,

$$\tilde{\nu}^{\xi} \subset \bigcap_{\eta \in \nu M} \ker A_{\eta} \, .$$

In particular, if ξ' is any other given parallel field to M with 0 as constant eiegenvalue of $A^M_{\xi'}$ (possibly in a smaller M) making the same constructions for ξ', one has that

$$\tilde{\nu}^{\xi} \subset E_0^{\xi'} \, .$$

Since the distribution $(\tilde{\nu}^{\xi})^{\perp}$ is integrable (the integral manifolds are $H^{\xi}(x)$) one has that locally

$$H^{\xi'}(x) \subset \ H^{\xi}(x)$$

for all $x \in M$ (recall that $H^{\xi'}$ is obtained by moving perpendicularly to $E^{\xi'}$). But in the same way we must have the other inclusion. So, locally,

$$H^{\xi}(x) = \ H^{\xi'}(x)$$

or equivalently

$$\tilde{\nu}^{\xi} = \tilde{\nu}^{\xi'} \, .$$

Remark 3. $\tilde{\nu}^{\xi}$ is horizontal with respect to π, i.e. $\tilde{\nu}^{\xi} \subset \mathcal{V}^{\perp}$. This follows immediately from the fact that $S(x) \subset H^{\xi}(x)$.

2.5. **Projecting down the foliation.** Observe that $x' - x$ belongs to $\Sigma_{\xi}(x)$, for all $x' \in \Sigma_{\xi}(x)$ since this submanifold is totally geodesic in the ambient space.

In this way M can be locally written as the union of parallel manifolds to $H^{\xi}(x)$

$$M = \bigcup_{x' \in \Sigma_{\xi}(x)} (H^{\xi}(x))_{x'-x} \qquad \text{(locally)}$$

where $(x' - x)$ is identified with a parallel normal field along $H^{\xi}(x)$, with this initial condition at x.

It is standard to prove, since $\Sigma_{\xi}(x')$ is locally constant, for $x' \in S(x)$, that $\tilde{\nu}^{\xi}$ projects down to an autoparallel distribution $\pi(\tilde{\nu}^{\xi})$ of N which is contained in the nullity of the second fundamental form α^N of N. The integral manifolds are $\pi(\Sigma_{\xi}(x))$, which are open subsets of affine subspaces of the ambient space. The complementary distribution is integrable with A^N-invariant leaves given by $\pi(H^{\xi}(x))$. Moreover, the restriction of $\pi(\tilde{\nu}^{\xi})$ to $\pi(H^{\xi}(x))$ is a parallel and flat subbundle of the normal space $\nu(\pi(H^\xi(x)))$, in the ambient space. Namely, if $x \in \Sigma_\xi(q)$, $x - q$ can be extended to a parallel normal field η to $H^\xi(q)$ which we have seen to be constant on $S(q)$. Then it projects down to a parallel normal field of $\pi(H^\xi(q))$. We also obtain that

$$N = \bigcup_{y \in \pi(\Sigma_\xi(x))} (\pi(H^\xi(x)))_{y-\pi(x)} \qquad \text{(locally)}$$

Lemma 4. (i) *The normal holonomy $\Phi^\perp_H$ of $\pi(H^\xi(x))$ at $\pi(x)$, restricted to the invariant subspace $\nu_{\pi(x)}N$, coincides with the normal holonomy group $\Phi^\perp_N$ of N at $\pi(x)$.*
(ii) $\nu_0(\pi(H^\xi(x))) = (\pi(\tilde{\nu}^\xi))_{\,|\,\pi(H^\xi(x))}$.

Proof. The inclusion in part (i) of the first group into the second is clear. Let us prove the other inclusion. Since the distribution $\pi(\tilde{\nu}^\xi)$ of N is inside the nullity of α^N, by the Ricci identity, it is in the nullity of the normal curvature tensor $R^\perp$ of N and in particular $R^\perp_{X,Y} = 0$ if X lies in $\pi(\tilde{\nu}^\xi)$ and Y in the perpendicular (integrable) distribution. Then, if c is a curve in N the parallel transport $\tau^\perp_c$ coincides with $\tau^\perp_{c_2} \circ \tau^\perp_{c_1}$, where c_1 is a curve which lies in $\pi(H^\xi(x))$ and c_2 lies in $\pi(\Sigma_\xi(x))$ (see the lemma in the Appendix of [18]). Both curves c_1 and c_2 are loops, if c is short (because in our situation we have two integrable distribution). But $\tau^\perp_{c_2} = \mathrm{Id}$, since the normal space of N is constant along any curve in the nullity of α^N. This shows the other inclusion.

Part (ii) follows from the fact that

$$\nu(\pi(H^\xi(x))) = (\pi(\tilde{\nu}^\xi))_{\,|\,\pi(H^\xi(x))} \oplus (\nu N)_{\,|\,\pi(H^\xi(x))}$$

and that the first subbundle of this sum is parallel and flat (recall that the normal holonomy group of N acts irreducibly). ☐

2.6. **Homogeneity of the canonical foliation.** We come back to the foliation $x \mapsto H^\xi(x)$ in the principal holonomy tube $M = (N)_{\eta_p}$. Let ξ' be another parallel normal field to M. We have seen, perhaps in a smaller M, that

$$H^\xi(x) = H^{\xi'}(x)\,,$$

for all $x \in M$.

Let $\mathcal{H}$ be the distribution in M perpendicular to the vertical distribution (with respect to $\pi : M \to N$), i.e. the distribution perpendicular to the leaves

$$p \mapsto S(p) = p + \Phi^\perp_{\pi(p)}.(p - \pi(p))\,.$$

We are around a generic point such that $(\ker A^M_\xi + \ker A^M_{\xi'})$ is a distribution of M.

Proposition 3. *Assume that $\mathcal{H} \subset (\ker A^M_\xi + \ker A^M_{\xi'})$. Then, for all $x \in M$, $H^\xi(x) = H^{\xi'}(x)$ is an isoparametric submanifold of $\mathbb{R}^n$.*

Proof. Observe that $H^\xi(x) = H^{\xi'}(x)$ has flat normal bundle, for all $x \in M$. Indeed,

$$\nu(H^\xi(x)) = (\tilde{\nu}^\xi)_{\,|\,H^\xi(x)} \oplus (\nu M)_{\,|\,H^\xi(x)}$$

and both subbundles are parallel and flat (see part (iii) of Proposition 2 in § 2.1). From the assumptions, one obtains that any curvature normal $\bar{\eta}$ of $H^\xi(x) = H^{\xi'}(x)$ is obtained by one of the following (non-exclusive) possibilities:

(a) $\bar{\eta}$ is the extension of a curvature normal of a (isoparametric) fiber $S(x)$ of $\pi : M \to N$.
(b) $\bar{\eta}$ is the extension of a curvature normal of a (isoparametric) fiber of the focalization at infinity $\pi^{\xi} : M \to M_{\tilde{\xi}}$.
(c) $\bar{\eta}$ is the extension of a curvature normal of a (isoparametric) fiber of the focalization at infinity $\pi^{\xi'} : M \to M_{\tilde{\xi}'}$.

This shows that $\bar{\eta}$ is parallel in the normal connection and hence $H^{\xi}(x) = H^{\xi'}(x)$ is an isoparametric submanifold of $\mathbb{R}^n$ (see the remark at page 10 in § 1.7). □

Corollary 1. *There is a compact group of isometries of $\mathbb{R}^n$, which acts as the isotropy representation of a simple symmetric space such that (locally) $K.\pi(x) = \pi(H^{\xi}(x))$, for all $x \in M$.*

Proof. From the above proposition one obtains that $\pi(H^{\xi}(x))$ is a submanifold with constant principal curvatures. If $\pi(H^{\xi}(x))$ is reducible or non-full then N would be reducible or non full since

$$N = \bigcup_{y \in \pi(\Sigma_{\xi}(x))} (\pi(H^{\xi}(x)))_{y - \pi(x)} \qquad \text{(locally)}$$

and $\pi(\Sigma_{\xi}(x)) \underset{\text{locally}}{\sim} (\pi(\tilde{\nu}^{\xi}))_{\pi(x)} = \nu_0(\pi(H^{\xi}(x)))$ (see Lemma 4). But the normal holonomy of $\pi(H^{\xi}(x))$ is irreducible and non-transitive (in the orthogonal complement of the fixed point set). Then, by making use of Thorbergsson's Theorem [28], $\pi(H^{\xi}(x))$ is a focal manifold of a homogeneous isoparametric submanifold (we have used that an isoparametric submanifold is always contained in a complete one [24]). Then there exists a compact group of isometries K of Euclidean space, acting as the isotropy representation of a simple symmetric space, and such that (locally) $K.\pi(x) = \pi(H^{\xi}(x))$. This for a fixed x. But, for $x' \neq x$, $\pi(H^{\xi}(x'))$ is a parallel manifold, in the ambient space, to $\pi(H^{\xi}(x))$. Since the group K gives the parallel transport in $\nu_0(K.x)$ (see Proposition 3.2.4 in [4]), one has that $K.\pi(x) = \pi(H^{\xi}(x))$, for all $x \in M$. □

We summarize the main result in this section, which will be the key tool for the whole article, in the following

Theorem 1 (Main tool)**.** *Let $N \subset \mathbb{R}^n$ be a submanifold and assume that its normal holonomy group acts irreducibly and non-transitively on the normal space. Let $\eta_q \in \nu_q N$ be a principal vector for the normal holonomy action of $\Phi_q^{\perp}$ on $\nu_q N$. Let us consider the normal holonomy tube $M := N_{\eta_q}$, which has flat normal bundle (η_q short, in a neighbourhood of a generic q).*
Assume that there exist two non-trivial parallel normal fields ξ, ξ' to N_{η_q} such that

$$\mathcal{H} \subset \ker(A^M_{\xi}) + \ker(A^M_{\xi'}) \,,$$

where $\mathcal{H}$ is the horizontal distribution in the holonomy tube M (we are assuming, since we are working locally that the right hand side of the above inclusion, as well as both of its terms, is a C^{∞}-distribution).
Then there is a compact group K of isometries of $\mathbb{R}^n$, acting as the isotropy representation of an irreducible symmetric space such that, locally around q,

$$N = \bigcup_{v \in (\nu_0(K.q))_q} (K.q)_v$$

i.e., N is locally, the union of the parallel orbits to $K.q$.

Moreover, the nullity of the second fundamental form α^M at x is just $(\nu_0(K.x))_x$.

Remark 4. We are in the assumptions of the above theorem.
(i) The orbit $K.q$ cannot be isoparametric, otherwise $\mathbb{R}^n = \bigcup_{v\in\nu_q(K.q)} (K.q)_v = \bigcup_{v\in(\nu_0(K.q))_q} (K.q)_v = N$.
(ii) Observe that $\dim(\nu_0(K.x)) \geq 1$ (i.e. dimension of the standard fiber) since the position vector field, from the fixed point of K, gives a parallel normal field. So, the nullity is non-trivial. Note that $\bigcup_{v\in\nu_0(K.p)} (K.q)_v$ is globally never a submanifold since there are always focal parallel orbits. We will come back to this discussion, on the completeness of N, in the case that N is a complex submanifold of $\mathbb{C}^n$.

3. Complex submanifolds of $\mathbb{C}^n$ with non-transitive normal holonomy

Let $N \subset \mathbb{C}^n$ be a complex (not necessarily complete) submanifold which is irreducible and full. The standard complex structure of $\mathbb{C}^n$ is denoted, as usual, by J.

Then, by [10], the normal holonomy group acts irreducibly on the normal space. Assume furthermore, that *the normal holonomy group of N is non-transitive* on the normal sphere. Let $\Phi_q^\perp$ be the normal holonomy group at $q \in N$, which acts by complex transformations on $\nu_q N$. Choose $\xi_q^1 \in \nu_q N$ such that the orbit $\Phi_q^\perp.\xi_q^1$ projects down to the (unique) complex orbit in the (complex) projectivization of the normal space $P(\nu_q N)$ of $\nu_q N$ (see [5]). This implies that the orthogonal complement of ξ_q^1 in the normal space of the holonomy orbit, $(\xi_q^1)^\perp \cap \nu_{\xi_q^1}\Phi_q^\perp.\xi_q^1$ is a complex subspace of $\nu_q N$. Since the normal holonomy is not transitive on the sphere, $(\xi_q^1)^\perp \cap \nu_{\xi_q^1}\Phi_q^\perp.\xi_q^1$ is not a trivial subspace.

Now choose $\xi_q^2 \neq 0$ which lies in $(\xi_q^1)^\perp \cap \nu_{\xi_q^1}\Phi_q^\perp.\xi_q^1$.

Since $R^\perp_{X,Y}$ always lies in the holonomy algebra one gets that $0 = \langle R^\perp_{X,Y}\xi_q^1, \xi_q^2\rangle$. So, by the Ricci identity, $[A^N_{\xi_q^1}, A^N_{\xi_q^2}] = 0$. The same is true if we replace ξ_q^2 by $J\xi_q^2$. So, $A^N_{\xi_q^1}$ also commutes with $A^N_{J\xi_q^2}$.

By the well-known formulae of complex geometry $A^N_{J\xi_q^2} = -JA^N_{\xi_q^2}$ and J anti-commutes with all shape operators. So,

$$[A^N_{\xi_q^1}, A^N_{J\xi_q^2}] = J(A^N_{\xi_q^1}A^N_{\xi_q^2} + A^N_{\xi_q^2}A^N_{\xi_q^1}) = 0.$$

But

$$[A^N_{\xi_q^1}, A^N_{\xi_q^2}] = A^N_{\xi_q^1}A^N_{\xi_q^2} - A^N_{\xi_q^2}A^N_{\xi_q^1} = 0\,.$$

Then

$$A^N_{\xi_q^1}A^N_{\xi_q^2} = A^N_{\xi_q^2}A^N_{\xi_q^1} = 0\,. \tag{3.1}$$

We may assume that the slice representation orbit $(\Phi_q^\perp)_{\xi_q^1}.\xi_q^2$ is a principal one in the normal space to the holonomy orbit, where $(\Phi_q^\perp)_{\xi_q^1}$ is the isotropy subgroup at ξ_q^1. Observe that we can find such a ξ_q^2, since ξ_q^1 is a fixed point for the slice representation of $(\Phi_q^\perp)_{\xi_q^1}$.

Observe, by construction, that one has also that

$$A^N_{\tau_c^\perp(\xi_q^1)}A^N_{\tau_c^\perp(\xi_q^2)} = 0$$

where $\tau_c^\perp$ is the normal parallel transport along any arbitrary curve c in N which starts at q.

Consider the iterated holonomy tube

$$(N_{\xi_q^1})_{\xi_q^2},$$

which coincides with the full holonomy tube N_{ζ_q}, where $\zeta_q = \xi_q^1 + \xi_q^2$ (see the theorem in Appendix of [18]). Of course we have to choose ξ_q^1 short and after that ξ_q^2 short enough. The vector ξ_q^1 gives rise to a parallel normal field $\tilde{\xi}$ to the partial holonomy tube $N_{\xi_q^1}$ so that $(N_{\xi_q^1})_{\tilde{\xi}} = N$ (see § 1.7). This parallel normal field can be lifted to a parallel normal field ξ of $(N_{\xi_q^1})_{\xi_q^2} = N_{\zeta_q}$. We can do so, since $\tilde{\xi}(x)$ is fixed by the normal holonomy group of $N_{\xi_q^1}$ at x and hence it is perpendicular to any holonomy orbit. Similarly, ξ_q^2 gives rise to a parallel normal field ξ' in $(N_{\xi_q^1})_{\xi_q^2} = N_{\zeta_q}$.

By (3.1) and the tube formulae relating shape operators of parallel focal manifolds (see § 1.5) one obtains that

$$A_\xi^M A_{\xi'\,|\,\mathcal{H}}^M = 0$$

where $M := N_{\zeta_q}$ and $\mathcal{H}$ is the horizontal distribution on M. Clearly, by (3.1) we also have $A_{\xi'}^M A_{\xi\,|\,\mathcal{H}}^M = 0$. Therefore $A_{\xi\,|\,\mathcal{H}}^M$ and $A_{\xi'\,|\,\mathcal{H}}^M$ are simultaneously diagonalizable, so $\mathcal{H} \subset (\ker A_\xi^M + \ker A_{\xi'}^M)$.

By Theorem 1 in the previous section, one has the following

Theorem 2. *Let $N \subset \mathbb{C}^n$ be a complex irreducible and full submanifold such that the normal holonomy group (which must act irreducibly by* [10]*) is not transitive on the unit sphere of the normal space. Then there is a group K, acting as the isotropy representation of an irreducible Hermitian symmetric space, such that N is locally given, around a generic point q, as*

$$N = \bigcup_{v \in (\nu_0(K.q))_q} (K.q)_v\,.$$

Moreover the nullity space of N at p is $\mathcal{N}_p^N = (\nu_0(K.p))_p$.

Proof. It remains only to show that K is of Hermitian type. We may assume that the origin $0 \in \mathbb{C}^n$ is the fixed point of K. If $p \in N$, then the position vector $\overrightarrow{p}$, by the description given before, belongs to T_pN. So, $i\overrightarrow{p} \in T_pN$. Then the orbits of the S^1 action $(t,x) \mapsto e^{it}x$ on $\mathbb{C}^n$ are tangent to N at the points of N. This implies that N is (locally) S^1-invariant. Let now $\bar{K}$ be the subgroup of linear isometries of $\mathbb{C}^n$ generated by K and S^1. Then $\bar{K}.p \subset N$ and so $\bar{K}$ is not transitive in the sphere. By the Theorem of Simons [25, 17], since K acts irreducibly, one must have $\bar{K} = K$ and so K is of Hermitian type. □

As a corollary, we are now ready to prove the Berger type theorem for submanifolds of $\mathbb{C}^n$

Proof of Main Theorem 1. We are in the assumptions at the beginning of this section. If the normal holonomy group of N is not transitive, then, locally,

$$N = \bigcup_{v \in (\nu_0(K.q))_q} (K.q)_v$$

where K acts as in the previous theorem, and in particular it is irreducible (we assume that 0 is the fixed point of K). Recall we are assuming that N is complete (not necessarily immersed), so, if $p \in N$, since N is analytic, then the line $t \mapsto tp$ is contained in N (i.e. this line is the image, via the immersion, of a geodesic in N. In order to simplify the notation we avoid the immersion map). From the construction, for all t, $T_{tp}N = T_pN$, as subspaces of $\mathbb{C}^n$. So, the isotropy K_{tp} must leave this subspace invariant. A contradiction for $t = 0$, since K acts irreducibly. Thus the normal holonomy group must be transitive. □

4. Complex submanifolds of $\mathbb{C}P^n$ with non-transitive normal holonomy

Let $M \subset \mathbb{C}P^n$ be a full complex submanifold of the complex projective space. Let $\mathcal{N}_p^M = \{X_p \in T_pM : \alpha^M(\cdot, X) = 0\}$ be the nullity of the second fundamental form α^M at $p \in M$.

The goal of this section is to prove the following theorem.

Theorem 3. *Let $M \subset \mathbb{C}P^n$ be a complex, complete and full submanifold. If the normal holonomy group $\Phi_p^\perp$ does not act transitively on the unit sphere of the normal space $\nu_p(M)$ at $p \in M$ then M is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space.*

We start with the following

Lemma 5. *Let $M \subset \mathbb{C}P^n$ be a complex submanifold and let $\widetilde{M} \subset \mathbb{C}^{n+1}$ be its lift to $\mathbb{C}^{n+1}$. Assume that the tangent vector $\widetilde{v}_{\widetilde{p}} \in T_{\widetilde{p}}\widetilde{M}$ is not a complex multiple of the position vector $\widetilde{p}$. If $\widetilde{v}_{\widetilde{p}} \in \mathcal{N}^{\widetilde{M}}$ then its projection v_p to T_pM belongs to the nullity of the second fundamental form of M, i.e. $v_p \in \mathcal{N}^M$.*

Proof. We can assume that $\widetilde{v}_{\widetilde{p}} \in T_{\widetilde{p}}\widetilde{M}$ is horizontal with respect to the submersion $\pi : \widetilde{M} \to M$. Let $\widetilde{v} \in \Gamma(T\widetilde{M})$ be a horizontal and projectable vector field that extends v_p. Let $\widetilde{X} \in \Gamma(\widetilde{M})$ be an arbitrary horizontal and projectable vector field defined around $\widetilde{p} \in \widetilde{M}$. From equation (1.1) we get

$$(D_{\widetilde{X}}\widetilde{v})_{\widetilde{p}} = \widetilde{\nabla^{FS}_X v} + \mathcal{O}(\widetilde{X}, \widetilde{v}).$$

So

$$(\nabla^{\widetilde{M}}_{\widetilde{X}}\widetilde{v})_{\widetilde{p}} + \alpha^{\widetilde{M}}(\widetilde{X}, \widetilde{v}) = \widetilde{\nabla^M_X v} + \widetilde{\alpha^M(X, v)} + \mathcal{O}(\widetilde{X}, \widetilde{v}).$$

Taking normal and tangent components with respect to $\widetilde{M}$ we get $\alpha^{\widetilde{M}}(\widetilde{X}_{\widetilde{p}}, \widetilde{v}_{\widetilde{p}}) = \widetilde{\alpha^M(v_p, X_p)}$. Thus, if $\widetilde{v}_{\widetilde{p}} \in \mathcal{N}^{\widetilde{M}}$, then $v_p \in \mathcal{N}^M$. □

Lemma 6. *Assume that $M \subset \mathbb{C}P^n$ is full and its normal holonomy group does not act transitively on the normal space $\nu_p(M)$. Then the normal holonomy group of $\widetilde{M}$ does not act transitively on $\nu_{\widetilde{p}}(\widetilde{M})$, where $\pi(\widetilde{p}) = p$.*

Proof. Let $\widetilde{R}^\perp$ be the curvature tensor of the normal connection of $\widetilde{M}$. Notice that the Ricci equation implies $\widetilde{R}^\perp_{X,\cdot} = 0$ if $X \in \mathcal{N}^{\widetilde{M}}$. So we can use the Lemma in the appendix of [18]. Namely, any normal parallel transport $\tau^\perp_{\widetilde{\gamma}}$ along a loop $\widetilde{\gamma}(t)$ starting at $\widetilde{p}$ can be written as $\tau^\perp_{\widetilde{\gamma}} = \tau^\perp_v \circ \tau^\perp_{\widetilde{c}}$, where v is a vertical curve (i.e., $\frac{dv}{dt} \in \mathcal{N}^{\widetilde{M}}$) and $\widetilde{c}$ is a horizontal, that is to say, $\frac{d\widetilde{c}}{dt} \in (\mathcal{N}^{\widetilde{M}})^\perp$. Notice that τ_v is just an Euclidean translation, i.e., $\tau_v(\xi_q) = \xi_p$, where $\xi_p = \xi_q$ as vectors of $\mathbb{C}^{n+1}$. Observe also that the horizontal curve $\widetilde{c}$ is the lift (starting at $\widetilde{p} \in \widetilde{M}$) of a loop c starting at $p \in M$. Let $\widetilde{\xi}_{\widetilde{p}} \in \nu_{\widetilde{p}}(\widetilde{M})$ be a normal vector and let $\xi_p \in \nu_p(M)$ its projection to M. Let $\xi(t)$ be the normal parallel transport along the loop c. Then by using equation (1.1) it is not difficult to see that the horizontal lift $\widetilde{\xi}(t)$ is parallel with respect to the normal connection along the curve $\widetilde{c}$.

Observe that $\widetilde{c}(1)$ belongs to the sphere of radius $||\widetilde{c}(0)||$, and $\pi(\widetilde{c}(0)) = \pi(\widetilde{c}(1)) = p$. So, $\widetilde{c}(1) = e^{i\theta}$, for some $\theta \in [0, 2\pi)$. Since the isometry $x \mapsto e^{i\theta}x$ of $\widetilde{M}$ projects down to the identity of M, one has that $e^{i\theta}\tau^{\perp}_{\widetilde{c}} = e^{i\theta}\tau^{\perp}_{\widetilde{\gamma}}$, which coincides, via $d\pi_{|\widetilde{p}}$, with $\tau^{\perp}_{c}$. Since any $e^{i\theta}$ belongs to the normal holonomy group of $\widetilde{M}$ (recall that the normal holonomy acts as an s-representation; see [10, Remark 2.2]) we can conclude that the normal holonomy groups of M and $\widetilde{M}$ identify (via $d\pi_{|\widetilde{p}}$). Thus, if one of them does not act transitively on the unit sphere neither does the other. □

Remark 5. Notice that two orbit equivalent Hermitian s-representations are equivalent. Since the normal holonomy groups of M and $\widetilde{M}$ act as Hermitian s-representations, the above proof shows that the holonomy representations are indeed equivalent. Roughly speaking, the holonomy groups of M and $\widetilde{M}$ are equal.

Now we are ready to prove Theorem 3 and therefore Main Theorem 2. The main tool is Theorem 2.

Proof of Theorem 3. Notice that Lemma 6 allows us to apply Theorem 2 to $N = \widetilde{M}$. So we get that

$$\widetilde{M} = \bigcup_{v \in (\nu_0(K.q))_q} (K.q)_v \ ,$$

where K is the isotropy group of a irreducible Hermitian symmetric space. Observe also that $\nu_0(K.q)_q$ is a complex subspace since it is equal to the nullity of the second fundamental form of the complex submanifold N. Then Lemma 5 and Theorem 4 in the Appendix imply that $\dim_{\mathbb{C}}(\nu_0(K.q)_q) = 1$, otherwise the nullity of the second fundamental form of M would be not trivial. Since $\widetilde{M}$ is full we get that the unique fixed point of K is the origin $0 \in \mathbb{C}^{n+1}$. So the leaves of the nullity distribution $\mathcal{N}^{\widetilde{M}}$ are just the complex lines given by the fibers of the submersion $\pi : \widetilde{M} \to M$. Thus, K acts transitively on the complex submanifold $M \subset \mathbb{C}P^n$. Therefore, M is a complex orbit of the projectivization of an irreducible Hermitian s-representation (cf. [6]). □

5. Further comments

We now explain why the completeness assumption cannot be dropped either in Main Theorem 1 or in Main Theorem 2.

Let M be a submanifold of Euclidean space and let

$$N = \bigcup_{v \in \nu_0(M)_q} M_v$$

(defined locally around M, v short enough). It is standard to show that the normal holonomy of N at q coincides with the semisimple part of the normal holonomy of M at q. Let now K act on $\mathbb{C}^{n+1}$ as an irreducible Hermitian s-representation of rank r and let $v \in \mathbb{C}^{n+1}$ be such that $K.v$ projects down to a complex orbit of $\mathbb{C}P^n$.

Choose a short enough normal vector $\xi \neq 0$ to $K.v$ at v that it is perpendicular to v. Moreover, assume that the normal holonomy orbit of $\Phi^{\perp}.\xi = K_v.\xi$ is a complex submanifold of the projectivization of the semisimple normal space $\nu_0(K.v)^{\perp} = v^{\perp} \subset \nu(K.v)$. By [HO], the dimension over $M = K.(v+\xi)$ of $\nu_0(K.(v+\xi))$ is 2. Moreover, the semisimple part of the normal holonomy representation of $K.(v+\xi))$ at $v+\xi$ has rank $r-2$. From the above choice of v and ξ, it is not hard to see that this semisimple part of the normal holonomy representation is of Hermitian type. Defining N like at the beginning of this discussion one has that N is a complex submanifold of $\mathbb{C}^{n+1}$ with not transitive irreducible normal holonomy, if $r \geq 4$. Moreover, N projects down to the projective space $\mathbb{C}P^n$ as a complex submanifold $\bar{N}$ with non transitive holonomy (see Lemma 6 and its proof). Notice however that $\bar{N}$ cannot be extended to a complete complex submanifold. Indeed, the second fundamental form has nullity on an open set and so $\bar{N}$ cannot be homogeneous as it would follow from Main Theorem 1 (see Theorem 4 in the Appendix). This shows that the assumption of completeness cannot be dropped.

We would like also remark that our main results are far from being true for (non necessarily complex) submanifolds of Euclidean space $\mathbb{R}^n$. For example, a submanifold with flat normal bundle is not necessarily homogeneous. Anyway, it is an open problem if compact homogeneous submanifolds whose normal holonomy is not transitive on the unit sphere of the normal space, are orbits of s-representations (cf. [7, Conjecture 6.2.14, page 198]).

Appendix

The aim of this appendix is to prove the following

Theorem 4. *Let M be a complete full complex submanifold of $\mathbb{C}P^n$ whose normal holonomy is not transitive, then the second fundamental form has no nullity in an open subset of M.*

Before giving the proof, we note that the above result is also a consequence of the fact that complete complex submanifolds of the projective space have no nullity at some open subset (see [1, Theorem 3]). We include a proof for the sake of being self contained.

Proof. From Lemma 6 and Remark 5, we have that the normal holonomy groups of M and $\tilde{M}$ are equal. Then we can apply Theorem 2 to $\tilde{M}$. We will show that there are other singular points different from 0, if $K.q$ is not most singular (in the hypothesis and description of the above theorem, and 0 is the fixed point of K). Assume M to be complete and let $q \in M$ and let $\eta \in (\nu_0 K.q)_q$, not a multiple of the position vector $\overrightarrow{q}$. and identify η with a parallel normal field to $K.q$. The shape $A^{K.q}_{\eta}$ has constant eigenvalues $\lambda_1, \cdots, \lambda_g$. Let $E_1, \cdots, E_g$ be the associated eigendistributions on $K.q$. We may assume that all the eigenvalues are different from 0, by adding to η a small multiple of the position vector. The isotropy subgroup $K_i := K_{q+\lambda_i^{-1}\eta}$, which is bigger that K_q, must act transitively on the integral manifold $S_i(q) \subset K.q$ through q of the eigendistribution E_i, $i = 1, \cdots, g$. So, the subgroup $\bar{K}$ of K generated by the isotropy subgroups $K_1, \cdots, K_g$ acts transitively on $K.q$. But, from the description of Theorem 2, one has that $T_{q+t\eta}M = T_qM$ and so $K_iT_qM = T_qM$, since isotropy subgroups preserve tangent spaces of invariant submanifolds. But the tangent space of M do not change if one moves along $(\nu_0 M)_q$. Then

$$\bigcup_{x\in M} T_x M = \bar{K} T_q M = T_q M$$

A contradiction.

Then M is not smooth at some $q + \lambda_i^{-1}\eta \neq 0$. This singularity projects down to the projective space. Thus M would be not complete. □

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