Polynomial Chaos Helps Assessing Parameters Variations of PCB Lines

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Abstract—This paper presents an effective solution for the analysis of long PCB interconnects with the inclusion of uncertainties resulting from different sources of variation, like temperature or fabrication process, on both the structure and loading conditions. The proposed approach is based on the expansion of the well-known frequency-domain telegraph equations in terms of orthogonal polynomials. The method is validated by means of a systematic comparison with the results of Monte Carlo simulations, for an application example involving a coupled-microstrip interconnect on PCB.

Index Terms—Stochastic analysis, Polynomial chaos, Tolerance analysis, Uncertainty, Circuit modeling, Circuit simulation, Transmission lines.

I. INTRODUCTION

Nowadays, the early design phase of complex electronic equipments requires the assessment of system performance via simulation and verification tools. This is a fundamental step for discovering and correcting problems and avoiding very expensive refabrication. In this assessment, the availability of partial models for devices and of the uncertainties due to the fabrication process or operating temperature, unavoidably impacts on the accuracy of predictions of sensitive effects like crosstalk or noise margins. Owing to this, methods and tools for the simulation of a circuit, with the inclusion of parameters variability effects on its electrical behavior, are highly desirable.

The typical resource for collecting quantitative information on the statistical behavior of the circuit response is based on the well-known Monte Carlo (MC) method that is possibly combined with techniques for the optimal selection of the experiments, i.e., the subset of model parameters in the design space. Such methods, however, are computationally expensive since they require a large number of simulations, thus limiting their application to the analysis of complex realistic structures.

In the past few years, an effective solution that overcomes the previous limitation and that is based on the so-called polynomial chaos (PC) theory has been proposed [2], [3]. This methodology allowed a clever solutions of a lumped circuit with random parameters via the expansion of its modified nodal analysis (MNA) set of equations in terms of orthogonal polynomials [4]. Recently, the above technique has been extended to the modeling of long distributed interconnects described by transmission-line equations with uncertain electrical or geometric parameters of the structure [5], [6].

This paper further demonstrates the feasibility, flexibility and strength of this methodology by integrating the variability arising from boundary conditions (i.e., termination networks) into the stochastic simulation of an interconnect structure. Variations in the line parameters are thus combined with variations in source and load conditions. A realistic application example involving a coupled PCB line affected by both thermal and process variations concludes the paper.

II. POLYNOMIAL CHAOS PRIMER

This section provides a brief overview of the PC method. The idea underlying this technique is the spectral expansion of a stochastic function (intended as a given function of a random variable) in terms of a truncated series of orthogonal polynomials. Within this framework, any function H, carrying the effects of variability, can be approximated by means of the following truncated series

$$H(\xi) = \sum_{k=0}^{P} H_k \cdot \phi_k(\xi), \qquad (1)$$

where $\{\phi_k\}$ are suitable orthogonal polynomials expressed in terms of the random variable ξ. The above expression is defined by the class of the orthogonal bases, by the number of terms P (limited to the range $2 \div 20$ for practical applications, depending also on the number of random variables considered) and by the expansion coefficients H_k. The choice of the orthogonal basis relies on the distribution of the random variables being considered. Temperature variations as well as the uncertainties arising from fabrication tolerances turn out to be properly characterized in terms of Gaussian variability. Therefore, in this case, the most appropiate orthogonal functions for the expansion (1) are the Hermite polynomials, the first three being $\phi_0 = 1$, $\phi_1 = \xi$ and $\phi_2 = (\xi^2 - 1)$, where ξ is the standard normal random variable, with zero mean and unity standard deviation.

The orthogonality property of Hermite polynomials is expressed by

$$< \phi_k, \phi_j >=< \phi_k, \phi_k > \delta_{kj}, \qquad (2)$$

where δ_{kj} is the Kronecker delta and $< \cdot, \cdot >$ denotes the inner product in the Hilbert space of the variable ξ with Gaussian weighting function, i.e.,

$$\begin{cases} < \phi_k(\xi), \phi_j(\xi) >= \int_{-\infty}^{+\infty} \phi_k(\xi)\phi_j(\xi)W(\xi)d\xi \\ W(\xi) = \exp(-\xi^2/2)/(\sqrt{2\pi}). \end{cases} \quad (3)$$

With the above definitions, the expansion coefficients H_k of (1) are computed via the projection of H onto the orthogonal components ϕ_k, i.e., $H_k =< H(\xi), \phi_k(\xi) > / < \phi_k^2(\xi) >$. It is worth noting that relation (1), which is a known nonlinear function of the random variable ξ, can be used to predict the probability density function (PDF) of $H(\xi)$ via numerical simulation or analytical formulae [7].

For a more comprehensive and formal discussion of PC theory, with the extension to multiple variables, the reader is referred to [2], [3] and references therein.

III. PC APPLICATION TO STOCHASTIC TRANSMISION-LINE EQUATIONS

For conciseness, the discussion is based on a loaded three-conductor line, as the coupled microstrip structure shown in Fig. 1, in presence of a single random parameter possibly affecting both the line and the termination network. For notational convenience, the line is assumed lossless. However, the proposed method is general and can be readily extended to lossy lines affected by multiple random variables and having a larger number of conductors [6], [5]. In the following, the procedure for including the parameters variability effects on both the transmission-line equation as well as the boundary conditions is provided.
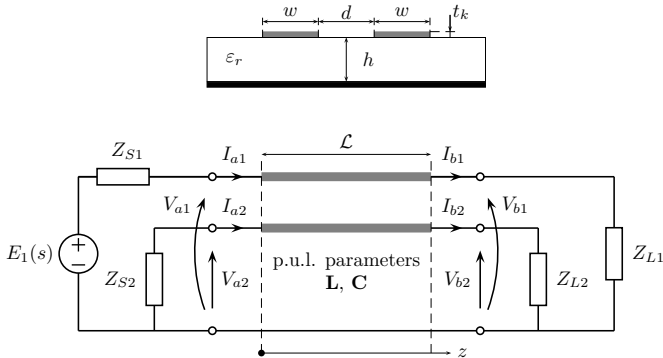

Fig. 1. Test structure considered to demonstrate the proposed approach. Top panel: transmission-line cross-section; bottom panel: simulation test case.

A. Stochastic Frequency-Domain Transmission-Line Model

If the interconnect structure of Fig. 1 is affected by variations parameterized by the random variable ξ (the typical examples are the uncertainties on the geometric and material parameters of the cross-section) the wave propagation along the line can be described by means of a stochastic version of the telegraphers equation in the Laplace domain [8]

$$\frac{d}{dz}\begin{bmatrix} \mathbf{V}(z,s,\xi) \\ \mathbf{I}(z,s,\xi) \end{bmatrix} = -s\begin{bmatrix} 0 & \mathbf{L}(\xi) \\ \mathbf{C}(\xi) & 0 \end{bmatrix}\begin{bmatrix} \mathbf{V}(z,s,\xi) \\ \mathbf{I}(z,s,\xi) \end{bmatrix}. \quad (4)$$

In the above equation, s is the Laplace variable, $\mathbf{V} = [V_1(z,s,\xi), V_2(z,s,\xi)]^T$ and $\mathbf{I} = [I_1(z,s,\xi), I_2(z,s,\xi)]^T$ are vectors collecting the voltage and current variables along the multiconductor line (z coordinate) and $\mathbf{C}(\xi)$ and $\mathbf{L}(\xi)$ are the p.u.l. capacitance and inductance matrices, depending on the geometric and material properties of the structure. It should be noted that the variations, here emphasized by ξ, affect the p.u.l. parameters thus leading to randomly-varing voltages and currents.

According to [5], the application of expansion (1) in terms of Hermite polynomials to the p.u.l parameters and to the unknown voltage and current variables, as well as the projection of (4) onto the polynomial basis, yield the following modified equation:

$$\frac{d}{dz}\begin{bmatrix} \tilde{\mathbf{V}}(z,s) \\ \tilde{\mathbf{I}}(z,s) \end{bmatrix} = -s\begin{bmatrix} 0 & \tilde{\mathbf{L}} \\ \tilde{\mathbf{C}} & 0 \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{V}}(z,s) \\ \tilde{\mathbf{I}}(z,s) \end{bmatrix} \quad (5)$$

where vectors $\tilde{\mathbf{V}} = [\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2, \ldots]^T$ and $\tilde{\mathbf{I}} = [\mathbf{I}_0, \mathbf{I}_1, \mathbf{I}_2, \ldots]^T$ collect the different coefficients of the polynomial chaos expansion for the voltage and current variables. The new p.u.l. matrix $\tilde{\mathbf{C}}$ contains the coefficient of the expansion of the p.u.l. capacitance in proper positions. For a second-order expansion, it becomes

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 & 2\mathbf{C}_2 \\ \mathbf{C}_1 & \mathbf{C}_0 + 2\mathbf{C}_2 & 2\mathbf{C}_1 \\ \mathbf{C}_2 & \mathbf{C}_1 & \mathbf{C}_0 + 4\mathbf{C}_2 \end{bmatrix} \quad (6)$$

and similar relation holds for matrix $\tilde{\mathbf{L}}$.

It is worth noting that (5) no longer depends on the random variable ξ, that is eliminated by the projection procedure. Therefore, (5) is a deterministic equation and plays the role of the set of equations of a multiconductor transmission line with a number of conductors that is $(P+1)$ times larger than those of the original line and whose unique solution gives the unknown coefficients for the voltage and current variables. The increment of the system size is not detrimental for the method, since for small values of P (as typically occurs in practice), the additional overhead in handling the augmented equations is much less than the time required to run a large number of MC simulations.

B. Boundary Conditions and Simulation

For the deterministic case, the standard procedure for the solution of a loaded transmission line like the one of Fig. 1 amounts to combining the port electrical relations of the terminal elements defining the source and load with the transmission-line equation, and solving the resulting system (cfr Ch.s 4 and 5 of [8]).

Similarly, when the problem becomes stochastic, the source and load equations are expanded and projected to account for their variabilities as already done above for the transmission-line equation. It is relevant to remark that the expansion of terminal equations must be consistent with (5), in order to easily allow their incorporation. This means that each variable in the problem must be expanded with respect to all the

random variables, whether they affect the termination network or the line, or both. For the example of Fig. 1, the Thevenin's equivalent port equations at the stochastic terminations become

$$\begin{cases} \mathbf{V}_a(s,\xi) &= \mathbf{V}_S(s,\xi) - \mathbf{Z}_S(s,\xi)\mathbf{I}_a(s,\xi) \\ \mathbf{V}_b(s,\xi) &= \mathbf{Z}_L(s,\xi)\mathbf{I}_b(s,\xi), \end{cases} \quad (7)$$

with $\mathbf{V}_S(s,\xi) = [E_1(s,\xi),0]^T$ and $\mathbf{Z}_{S,L}(s,\xi) = \text{diag}([Z_{S1,L1}(s,\xi),Z_{S2,L2}(s,\xi)])$. The second-order expansion of the first row of (7) leads to

$$\mathbf{V}_{a,0}\phi_0(\xi) + \mathbf{V}_{a,1}\phi_1(\xi) + \mathbf{V}_{a,2}\phi_2(\xi) = \mathbf{V}_{S,0}\phi_0(\xi) +$$
$$\mathbf{V}_{S,1}\phi_1(\xi) + \mathbf{V}_{S,2}\phi_2(\xi) + (\mathbf{Z}_{S,0}\phi_0(\xi) + \mathbf{Z}_{S,1}\phi_1(\xi) + \quad (8)$$
$$\mathbf{Z}_{S,2}\phi_2(\xi))(\mathbf{I}_{a,0}\phi_0(\xi) + \mathbf{I}_{a,1}\phi_1(\xi) + \mathbf{I}_{a,2}\phi_2(\xi)).$$

The projection of the previous equation and of its companion relation arising from the second row of (7) on Hermite polynomials, leads to the following augmented deterministic port equations

$$\begin{cases} \tilde{\mathbf{V}}_a(s) &= \tilde{\mathbf{V}}_S(s) - \tilde{\mathbf{Z}}_S(s)\tilde{\mathbf{I}}_a(s) \\ \tilde{\mathbf{V}}_c(s) &= \tilde{\mathbf{Z}}_L(s)\tilde{\mathbf{I}}_c(s), \end{cases} \quad (9)$$

where $\tilde{\mathbf{V}}_S(s)$ collects the expansion coefficients of $\mathbf{V}_S(s)$, while $\tilde{\mathbf{Z}}_S(s)$ and $\tilde{\mathbf{Z}}_L(s)$ have a structure similar to (6).

Once the unknown voltages and currents are computed, the quantitative information on the spreading of circuit responses can be readily obtained from the analytical expression of the unknowns. As an example, the frequency-domain solution of the terminal voltage V_{b2}, arising form (7) and (5) with $P = 2$, is

$$V_{b2}(j\omega) = V_{b2,0}(j\omega) + V_{b2,1}(j\omega)\xi + V_{b2,2}(j\omega)(\xi^2 - 1), \quad (10)$$

where the first numerical index denotes the conductor and the second one denotes the expansion term. The above relation can be used to compute the PDF of the output quantity (e.g., the magnitude $|V_{b2}(j\omega)|$) using the rules of random variable transformations given in [7].

IV. NUMERICAL RESULTS

This section provides an example of application in which the proposed technique is applied to the analysis of the effects of thermal and process variations on the response of the interconnect structure of Fig. 1. The coupled microstrip line is realized on FR4 substrate. The nominal parameters are $w = 100\,\mu m$, $d = 80\,\mu m$, $h = 60\,\mu m$, $t_k = 35\,\mu m$, $\varepsilon_r = 4.7$ and $\mathcal{L} = 5\,cm$. The series impedances of the Thevenin equivalents and of the line terminations are $Z_{S,1} = Z_{S,2} = R_S = 25\,\Omega$ and $Z_{L,1} = Z_{L,2} = 1/(j\omega C_L)$ (being $C_L = 10\,pF$), respectively. All these values are referred to a temperature of 50 °C.

According to the vendor information of the substrate and of standard SMD components, the following thermal coefficients are considered: $\alpha_R = 200$ ppm/°C, $\alpha_C = 200$ ppm/°C and $\alpha_\varepsilon = 500$ ppm/°C for the resistance, capacitance and substrate permittivity, respectively. The temperature variation ΔT is

modeled as a Gaussian random variable with zero mean and standard deviation $\sigma_T = 40\,°C$; hence, we can express it in terms of the normalized variable ξ_1 as follows

$$\Delta T = \sigma_T\xi_1 \Rightarrow \begin{cases} R_S &= R_S|_{\Delta T=0}(1 + \alpha_R\sigma_T\xi_1) \\ C_L &= C_L|_{\Delta T=0}(1 + \alpha_C\sigma_T\xi_1) \\ \varepsilon_r &= \varepsilon_r|_{\Delta T=0}(1 + \alpha_\varepsilon\sigma_T\xi_1) \end{cases} \quad (11)$$

Furthermore, an additional statistically-independent variation, due to the production process, is supposed to affect the permittivity with a relative standard deviation of $\sigma_\varepsilon = 0.032$. Therefore, we can introduce a second random variable ξ_2 and write the following complete expression for the permittivity:

$$\varepsilon_r = \bar{\varepsilon}_r(1 + \alpha_\varepsilon\sigma_T\xi_1 + \sigma_\varepsilon\xi_2), \quad (12)$$

where $\bar{\varepsilon}_r$ denotes the nominal value of permittivity at $T = 50\,°C$.

Figure 2 shows the frequency response of $|H_{b1}|$ (top panel) and $|H_{b2}|$ (bottom panel), having defined $H_{bj} = V_{bj}/E_1$. In both figures, the black thick line represents the response of the structure for the nominal values of its parameters, while the thinner black lines indicate the limits of the 3σ bound (σ being the standard deviation) determined from the results of the proposed technique. Figure 2 also includes a number of responses (see the gray lines) obtained via a limited set of MC simulations. Clearly, the parameter variations lead to a spread in the transfer functions, that is well predicted by the estimated 3σ limit in both cases.
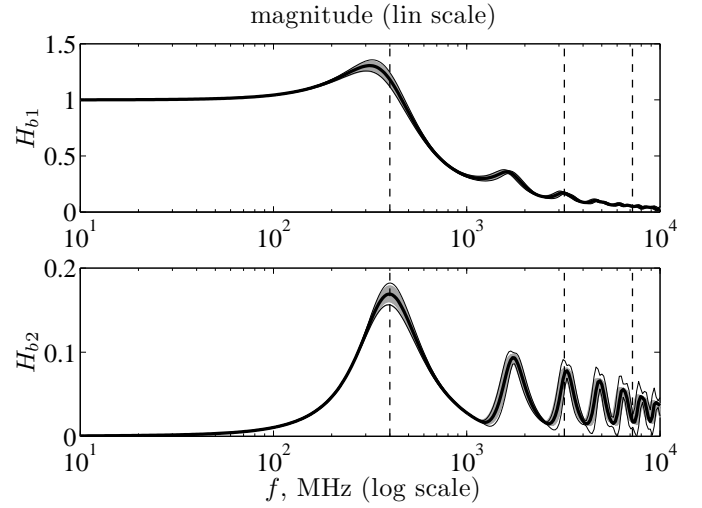

Fig. 2.   Magnitudes of $|H_{b1}|$ (top panel) and $|H_{b2}|$ (bottom panel). Solid black thick line: nominal response; solid black thin lines: 3σ limits of the fourth-order polynomial chaos expansion; gray lines: a sample of responses obtained by means of the MC method (limited to 100 curves, for graph readability).

Often the knowledge of the standard deviation represents a limited information, since the quantitative information about how the values are distributed is missing. Nonetheless, from the analytical PC model we can also obtain the probability density function of the system responses. Figures 3 and 4

compare the PDFs of $|H_{b1}|$ and $|H_{b2}|$ computed at three different frequencies by means of the PC technique with those generated after 40,000 MC simulations. The frequencies correspond to the dashed vertical lines shown in Fig. 2.

The good agreement between the PDFs obtained from the PC model and the corresponding set of MC simulations confirms the potential of the proposed method. It is also clear from this example that a PC expansion with four terms is already accurate enough to capture the dominant statistical information of the system response.
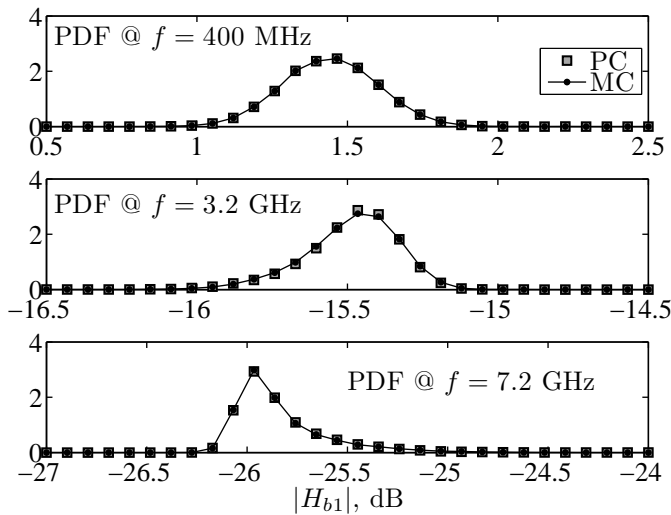
Fig. 3. Probability density function of $|H_{b1}|$ computed at three different frequencies. Of the two distributions, the one marked MC refers to 40,000 MC simulations, while the one marked PC refers to the response obtained via a fourth-order PC expansion.
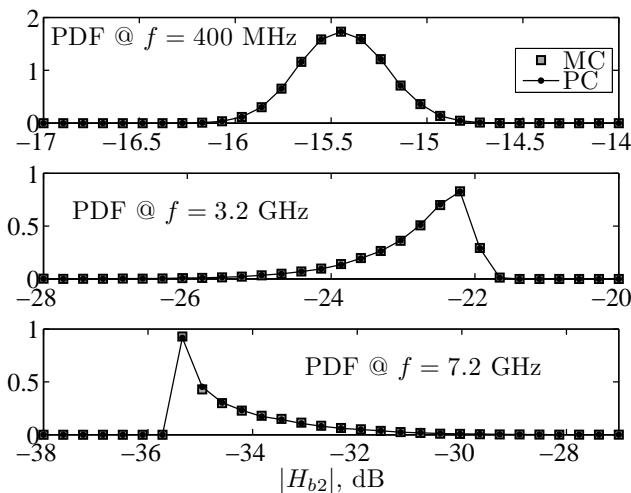
Fig. 4. Probability density function of $|H_{b2}|$. Same comments of Fig. 3 apply here.

Finally, Tab. I collects the main figures related to the efficiency of the proposed PC method vs. conventional MC. The PC model has an overhead due to the computation of the Hermite expansions as well as the creation of the augmented matrix system. Nevertheless, it turns out to be faster by over three orders of magnitude with respect to the MC computation. The above comparison confirms the strength of the proposed method, that allows to generate accurate predictions of the statistical behavior of a realistic interconnect with a great efficiency improvement.

TABLE I
CPU TIME REQUIRED BY MC AND THE PROPOSED PC-BASED METHOD FOR THE SIMULATION OF THE PROPOSED EXAMPLE ON THE WHOLE FREQUENCY SWEEP.

Method	Overhead	Simulation time
MC	-	2 h 27 min
PC	3.5 sec	3.5 sec

V. CONCLUSIONS

The generation of a stochastic model for a distributed multiconducor interconnect and its corresponding loading networks is addressed in this paper. The proposed model inherently includes uncertainties that may arise from thermal and process variations and is based on the expansion of the system variables into a sum of a limited number of orthogonal basis functions, leading to an extended set of multiport equations.

The advocated method, while providing accurate results, turns out to be more efficient than the classical Monte Carlo technique in determining the system response sensitivity to parameters variability.

The strenght of the proposed technique is demonstrated by means of a coupled microstrip line structure and a frequency-domain analysis. The speed-up observed in the proposed example is around $1200\times$.

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