Oscillator noise: a nonlinear perturbative theory including orbital fluctuations and phase-orbital correlation

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Abstract

We derive a full statistical characterization of the noise spectrum of a free running oscillator perturbed by white Gaussian noise sources, including the effect of orbital fluctuations and of their correlation with phase noise, thus extending the previous theory based on the Floquet decomposition of the linearized oscillator equations [1]. This allows to derive explicit relationships for the relevant phase, amplitude and correlation spectra. The examples provide a validation of the theoretical results, and allow to assess the importance of the Floquet exponents and eigenvectors on the magnitude of the orbital noise contribution.

Index Terms

Circuit simulation, Autononmous systems, Oscillator noise, Floquet theory

I. INTRODUCTION

Noise in free running oscillators is a classical topic in circuit analysis, mainly because of its practical importance in the design and optimization of high sensitivity telecommunication systems [2]. The autonomous nature of oscillator operation makes noise analysis particularly challenging: the absence of a fixed time reference due to the lack of applied generators results in the appearance of fluctuations both into the timing properties of the circuit (expressed in terms of *phase noise* or, equivalently, *timing jitter*) and into the *amplitude* of the oscillator working point [1], [2], [3] (*amplitude* or *orbital noise*). In most practical cases, the very effects assuring the stability of the oscillator operation imply also a quenching of the amplitude noise component, thus making phase noise the dominant fluctuation effect [3]: amplitude noise, however, usually becomes important at frequencies far away from the nominal oscillation frequency and its harmonics. Notice that, in particular in presence of a strong adjacent channel, also the noise components far from the oscillation harmonics might have a significant impact on the dynamic range of the receiver [2]. Therefore, the assessment of amplitude noise, besides being important *per se* to fully characterize the circuit noise performance, also has, at least for specific applications, significant practical effects.

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In order to introduce the problem, let us consider an autonomous lumped circuit represented by an ordinary differential equation:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, and $\mathbf{f}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. Assuming that (1) admits a non-trivial periodic solution (limit cycle) $\mathbf{x}_{\mathrm{S}}(t)$ of period T, the noisy oscillator is modelled by perturbing (1) with a set of p stochastic noise sources $\mathbf{b}(t)$:

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} - \mathbf{f}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\mathbf{b}(t), \tag{2}$$

where the solution-dependent matrix $\mathbf{B}(\mathbf{z})$ (of size $n \times p$) takes into account the possible modulation of the noise generators. For the sake of simplicity, in this work we consider white Gaussian noise sources only.

The addition of noise sources makes (2) a nonlinear stochastic ordinary differential equation (S-ODE), and therefore $\mathbf{z}(t)$ is a stochastic process. Although general solution approaches of the S-ODE are available either in time-domain or through the transformation into a deterministic partial differential equation (the Fokker-Planck equation) [4], [5], from the standpoint of the circuit designer and, as a consequence, of the Electronic Design Automation (EDA) tools developed for circuit design, a different approach is usually preferred. The S-ODE solution is tackled using a perturbative approach, which in the simplest case leads to the Linear Time Varying (LTV) [1], [2], [6], [7] approach where the effect of noise is expressed as a purely additive term to $\mathbf{x}_{\mathrm{S}}(t)$. The LTV approach combines a relative simplicity of the mathematical machinery and an excellent accuracy of the results, at least not too close to the oscillator output harmonics where a divergence of the phase noise spectrum appears [1]. This issue has been discussed in [8] with reference to a specific example of oscillator, and overcome in the general case by the *nonlinear perturbative analysis* proposed in [1], [9], partly based on the seminal study discussed in [10], [11]. Similar results have been derived from a geometrical standpoint in [12].

The noisy solution is expressed as the superposition of a time-shifted version of the limit cycle and of an orbital deviation $\mathbf{y}(t)$

$$\mathbf{z}(t) = \mathbf{x}_{\mathrm{S}}(t + \alpha(t)) + \mathbf{y}(t), \tag{3}$$

where $\alpha(t)$ is a stochastic process responsible for the oscillator phase noise, while $\mathbf{y}(t)$ is responsible for the orbital fluctuations. Notice that the widely popular Impulse Sensitivity Function (ISF) methodology for phase noise estimation proposed in [2], [13] is actually equivalent to the nonlinear perturbation approach in [1] if the "numerical ISF" is used (see [14] for more details).

According to (3), the autocorrelation matrix of the noisy oscillator solution is given by:

$$\begin{aligned} \mathbf{R}_{\mathbf{z},\mathbf{z}}(t,\tau) = \mathrm{E}\left\{\mathbf{z}(t)\mathbf{z}^{\dagger}(t+\tau)\right\} &= \mathbf{R}_{\mathbf{x}_{\mathrm{S}},\mathbf{x}_{\mathrm{S}}}(t,\tau) \\ &+ \mathbf{R}_{\mathbf{x}_{\mathrm{S}},\mathbf{y}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{x}_{\mathrm{S}}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau) \end{aligned} \tag{4}$$

where $\mathrm{E}\{\cdot\}$ is the ensemble average operator, † denotes the complex conjugate and transpose operation, and

$$\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{x}_\mathrm{S}}(t,\tau) = \mathrm{E}\left\{\mathbf{x}_\mathrm{S}(t+\alpha(t))\mathbf{x}_\mathrm{S}^\dagger(t+\tau+\alpha(t+\tau))\right\}, \tag{5a}$$

$$\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,\tau) = \mathrm{E}\left\{\mathbf{x}_\mathrm{S}(t+\alpha(t))\mathbf{y}^\dagger(t+\tau)\right\}, \tag{5b}$$

$$\mathbf{R}_{\mathbf{y},\mathbf{x}_\mathrm{S}}(t,\tau) = \mathrm{E}\left\{\mathbf{y}(t)\mathbf{x}_\mathrm{S}^\dagger(t+\tau+\alpha(t+\tau))\right\}, \tag{5c}$$

$$\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau) = \mathrm{E}\left\{\mathbf{y}(t)\mathbf{y}^\dagger(t+\tau)\right\}. \tag{5d}$$

The first term $\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{x}_\mathrm{S}}(t,\tau)$ describes phase noise, and has been discussed in detail in [1], where the effect of orbital deviations is neglected altogether. The orbital contribution and its correlation with the fluctuations along the orbit have been sparsely addressed in the literature: a specific study of the 2-dimensional system in [15] and some of its variants has been performed without resorting to any perturbative approach in [16], [17], [18], while a general inclusion into nonlinear perturbative theories is proposed in [19], where however a simpler expression for $\mathbf{R}_{\mathbf{y},\mathbf{y}}(\tau)$ is derived based on the assumption of a negligible correlation between $\alpha(t)$ and $\mathbf{b}(t)$. On the other hand, a general theory deriving expressions for all the terms in (4) was proposed by Kaertner in [10] based on a decomposition of phase noise defined by projecting the perturbation along the noiseless orbit. As a consequence of this choice, the dynamic equation for phase fluctuations depends on orbital noise, thus requiring a self-consistent determination of the two quantities. This issue is overcome projecting the noisy signal along the adjoint vector associated to the orbit tangent, as proposed later in [11] and [1]: the main advantage of this projection choice is that the dynamic equation for phase noise becomes asymptotically independent from $\mathbf{y}(t)$, thus allowing for a decoupled analysis. The treatment of orbital noise in [11], however, still requires further developments since the orbital and phase-orbital correlation spectra in [11] are based on two approximations: they are estimated assuming amplitude fluctuations decading faster than phase noise, and orbital and phase orbital correlation are calculated on the basis of phase noise which, as discussed in [1], is in turn obtained by solving a linear small-time approximation to the full phase fluctuation stochastic nonlinear equation.

In this paper, we extend the analysis in [1] by deriving a consistent statistical characterization of the entire correlation matrix under the assumption that the orbital deviation is a small amplitude perturbation of the limit cycle. We shall follow the steps proposed in [1], in particular finding the proper stochastic differential equations and assessing the asymptotic statistical properties of the solution making use of the Floquet representation of the output of the linearized (with respect to the orbital deviation) oscillator system. This will allow us to derive a full characterization of the asymptotic oscillator spectrum, thus completing, at least as far as the orbital deviation remains small with respect to the steady-state solution, the analysis in [1] by deriving closed form expressions for the orbital-orbital noise and phase-orbital noise correlation spectra. We validate the approach and show an example of application in Section V.

II. THE STOCHASTIC DIFFERENTIAL SYSTEM

According to the derivation in [1] (see in particular equation (12) and the proof of Theorem 6.1), the following stochastic differential equation governs the dynamics of the phase and amplitude fluctuations:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \mathbf{F}\left[\mathbf{Y}(t), t\right]\mathbf{Y}(t) + \mathbf{G}\left[\mathbf{Y}(t), t\right]\mathbf{b}(t), \tag{6}$$

where $\mathbf{Y}^{\mathrm{T}}(t) = \left[\alpha(t), \mathbf{y}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and matrices $\mathbf{F}$ and $\mathbf{G}$, of size $(n+1) \times (n+1)$ and $(n+1) \times p$ respectively, are defined as

$$\mathbf{F}\left[\mathbf{Y}(t), t\right] = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}(t + \alpha(t)) \end{bmatrix} \tag{7}$$

$$\mathbf{G}\left[\mathbf{Y}(t), t\right] = \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}}(t + \alpha(t))\mathbf{B}(t + \alpha(t)) \\ \sum_{k=2}^{n} \mathbf{u}_k(t + \alpha(t))\mathbf{v}_k^{\mathrm{T}}(t + \alpha(t))\mathbf{B}(t + \alpha(t)) \end{bmatrix}. \tag{8}$$

The T periodic vector functions $\mathbf{u}_k(t)$ and $\mathbf{v}_k(t)$ are the Floquet eigenvectors associated to the direct and adjoint, respectively, linearized oscillator system corresponding to the Floquet exponent μ_k: see e.g. [1] for details. According to [1], we choose $\mu_1 = 0$ as the zero Floquet exponent always present in any autonomous system. Furthermore, in (7) and (8) we have

$$\mathbf{A}(t + \alpha(t)) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_{\mathrm{S}}(t + \alpha(t))} \tag{9}$$

$$\mathbf{B}(t + \alpha(t)) = \mathbf{B}\left[\mathbf{x}_{\mathrm{S}}(t + \alpha(t))\right]. \tag{10}$$

A thorough study of (6) carried out by estimating the characteristic function associated to process $\mathbf{Y}(t)$, here omitted for the sake of brevity, allows to prove that fluctuations at the same time along the limit cycle $\mathbf{x}_{\mathrm{S}}(t + \alpha(t))$ (i.e., the phase noise contribution) and the corresponding orbital deviation $\mathbf{y}(t)$ asymptotically become statistically independent [20]. This implies that we can avoid to discuss (5a) since the same results as in [1], [21] hold.

III. PROPERTIES OF THE CORRELATION FUNCTION AT DIFFERENT TIMES FOR WHITE NOISE SOURCES

We estimate the correlation functions by considering unit white, Gaussian noise sources, i.e. by assuming $\mathbf{b}(t) = \boldsymbol{\xi}(t)$.[1]

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[1]This would rigorously require to make use of the generalized stochastic processes as defined, e.g., in [22], i.e. we should consider a sequence of unit Gaussian processes $\mathbf{b}(t)$ dependent on a parameter a such that $\mathrm{E}\left\{\mathbf{b}(t_1)\mathbf{b}^{\dagger}(t_2)\right\} = \mathbf{R}_{\mathbf{b},\mathbf{b}}(t_1 - t_2; a)$ where

$$\lim_{a \to 0} \mathbf{R}_{\mathbf{b},\mathbf{b}}(t_1 - t_2; a) = \mathbf{I}\delta(t_1 - t_2).$$

For the sake of conciseness, we avoid to exploit this rigorous approach, but rather consider it implicit in the following discussion.

A. Calculation of $\mathbf{R}_{\mathbf{x}_S,\mathbf{y}}(t,\tau)$ and $\mathbf{R}_{\mathbf{y},\mathbf{x}_S}(t,\tau)$

In order to calculate the correlation function between the fluctuations along the limit cycle and the orbital ones, we need to prove a few preliminary Lemma.

Lemma 3.1: The correlation between random variables $\alpha(t_1)$ and $\boldsymbol{\xi}(t_2)$ is given by

$$\mathrm{E}\{\alpha(t_1)\xi_h(t_2)\} = \begin{cases} \mathrm{E}\left\{\alpha(t_2^+)\xi_h(t_2)\right\} & \text{if } t_2 < t_1 \\ 0 & \text{if } t_2 > t_1 \end{cases} \tag{11}$$

where ξ_h is the h-th component of $\boldsymbol{\xi}$, and t_2^+ denotes $t_2 + \Delta$ for any $0 < \Delta < t_1 - t_2$.

Proof: The proof is based on Ito's interpretation of stochastic integral, as in [1]. Let us consider first $t_1 < t_2$. According to the defining equation of $\alpha(t)$ [1, Eq. (12)]

$$\mathrm{E}\{\alpha(t_1)\xi_h(t_2)\} = \int_0^{t_1} \mathrm{E}\left\{\mathbf{v}_1^{\mathrm{T}}(t + \alpha(t))\boldsymbol{\xi}(t)\xi_h(t_2)\right\}\,\mathrm{d}t$$

where, as proved in [21, p. 67]

$$\left|\mathrm{E}\left\{\mathbf{v}_1^{\mathrm{T}}(t + \alpha(t))\boldsymbol{\xi}(t)\xi_h(t_2)\right\}\right| \le M\delta(t - t_2) \quad M > 0.$$

Therefore

$$|\mathrm{E}\{\alpha(t_1)\xi_h(t_2)\}| \le M \int_0^{t_1} \delta(t - t_2)\,\mathrm{d}t = 0 \tag{12}$$

since $t_1 < t_2$.

We consider now the case $t_1 > t_2$. Posing $t_2^+ = t_2 + \Delta$ $(0 < \Delta < t_1 - t_2)$, we find

$$\mathrm{E}\{\alpha(t_1)\xi_h(t_2)\} = \mathrm{E}\left\{\alpha(t_2^+)\xi_h(t_2)\right\}$$
$$+ \int_{t_2^+}^{t_1} \mathrm{E}\left\{\mathbf{v}_1^{\mathrm{T}}(t + \alpha(t))\boldsymbol{\xi}(t)\xi_h(t_2)\right\}\,\mathrm{d}t$$

where, according to [21, p. 67]

$$\left|\mathrm{E}\left\{\mathbf{v}_1^{\mathrm{T}}(t + \alpha(t))\boldsymbol{\xi}(t)\xi_h(t_2)\right\}\right| \le M\delta(t - t_2) \quad M > 0.$$

An argument similar to (12) proves that the integral is zero, and therefore $\mathrm{E}\{\alpha(t_1)\xi_h(t_2)\} = \mathrm{E}\left\{\alpha(t_2^+)\xi_h(t_2)\right\}$. ∎

Lemma 3.2: The random variables $\alpha(t_1)$ and $\boldsymbol{\xi}(t_2)$ are jointly Gaussian asymptotically with time (i.e., for $t_1, t_2 \to +\infty$).

Sketch of proof: We provide here a sketch of the formal proof of the lemma, based on Lemma 3.1 and on calculations similar to those performed in [1], [21].

Since two random variables are jointly Gaussian iff any linear combination of them is a Gaussian random variable, we consider the stochastic variable

$$\psi(t_1, t_2) = a_1\alpha(t_1) + a_2\xi_h(t_2) \qquad a_1, a_2 \in \mathbb{R} \tag{13}$$

and show that all its cumulants of order higher than 2 are zero. We consider explicitly only the third cumulant, since all the other cases can be treated similarly

$$\kappa_3 = \mathrm{E}\left\{(\psi - \mathrm{E}\{\psi\})^3\right\} \tag{14}$$

where, as shown in [1]

$$E\{\psi\} = a_1 E\{\alpha(t_1)\} + a_2 E\{\xi_h(t_2)\} = a_1 m. \tag{15}$$

Furthermore, both $\alpha(t_1)$ and $\xi_h(t_2)$ are, at least asymptotically for $t_1, t_2 \to +\infty$, Gaussian [1]. This means that κ_3 can be expressed as

$$\kappa_3 = 3a_1^2 a_2 E\left\{[\alpha(t_1) - m]^2 \xi_h(t_2)\right\}$$
$$+ 3a_1 a_2^2 E\left\{[\alpha(t_1) - m]\xi_h^2(t_2)\right\}. \tag{16}$$

For $t_1 < t_2$, Lemma 3.1 proves that $\alpha(t_1)$ and $\xi_h(t_2)$ are uncorrelated, therefore

$$E\left\{[\alpha(t_1) - m]^2 \xi_h(t_2)\right\} = E\left\{[\alpha(t_1) - m]^2\right\}E\{\xi_h(t_2)\} = 0$$

$$E\left\{[\alpha(t_1) - m]\xi_h^2(t_2)\right\} = E\{[\alpha(t_1) - m]\}E\left\{\xi_h^2(t_2)\right\} = 0.$$

In order to prove that $\kappa_3 = 0$ for $t_1 > t_2$ we need some further results, which can be derived performing calculations similar to those in [21, p. 67]. For the sake of conciseness, we provide here the result only: the two expected values $E\left\{\alpha(t_1)\xi_h^2(t_2)\right\}$ and $E\left\{[\alpha(t_1) - m]^2\xi_h(t_2)\right\}$ are continuous functions of t_1 and t_2. This implies that

$$E\left\{[\alpha(t_1) - m]\xi_h^2(t_2)\right\}$$
$$= \lim_{\Delta \to 0^+} E\left\{[\alpha(t_1) - \alpha(t_2 + \Delta) + \alpha(t_2 - \Delta) - m]\xi_h^2(t_2)\right\}$$
$$E\left\{[\alpha(t_1) - m]^2 \xi_h(t_2)\right\} = \lim_{\Delta \to 0^+} E\left\{[\alpha(t_1) - m]^2\right.$$
$$\times \left. [\xi_h(t_2) - \xi_h(t_1 - \Delta) + \xi_h(t_1 + \Delta)]\right\}.$$

This allows to prove that the two partial expected values are null, since from Lemma 3.1 it can be readily shown that, for $t_2 > t_1$, $\alpha(t_2) - \alpha(t_1^+)$ is uncorrelated with $\xi_h(t_1)$ and $\xi_h(t_1) - \xi_h(t_2^-)$ is uncorrelated with $\alpha(t_2)$ (where t_2^- denotes $t_2 - \Delta$ for any $0 < \Delta < t_2 - t_1$).

Similar calculations can be performed for the higher order cumulants, thus proving the Lemma. ∎

Lemma 3.3: The correlation function $\mathbf{R}_{\alpha,\boldsymbol{\xi}}(t_1, t_2) = E\left\{\alpha(t_1)\boldsymbol{\xi}^\dagger(t_2)\right\}$ is asymptotically (i.e., for $t_1, t_2 \to +\infty$) constant and given by

$$\mathbf{R}_{\alpha,\boldsymbol{\xi}}^\infty(t_1, t_2) = \begin{cases} \tilde{\mathbf{V}}_{1_0}^{\mathrm{T}} & \text{if } t_2 < t_1 \\ \mathbf{0} & \text{if } t_2 \geq t_1 \end{cases} \tag{17}$$

where $\tilde{\mathbf{V}}_{1_0}$ is the DC harmonic component of $\mathbf{v}_1(t)^{\mathrm{T}}\mathbf{B}(t)$.

Proof: See Appendix A. ∎

These preliminary results makes possible to finalize the estimation of the phase-orbit correlation.

Lemma 3.4: Asymptotically with time (i.e., for $t \to +\infty$), the correlation functions of the phase and orbital deviations due to a white Gaussian noise $\mathbf{b}(t) = \boldsymbol{\xi}(t)$ depend on τ only and are given by

$$\mathbf{R}^{\infty}_{\mathbf{x}_{\mathrm{S}},\mathbf{y}}(\tau)$$

$$= \begin{cases} \sum_{l=2}^{n} \sum_{h,j} \mathbf{D}_{lhj} \left\{ \exp\left[-\mathrm{i}\left(j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}\right)\tau + \mathrm{Re}\left\{\mu_l\right\}|\tau|\right] \right. \\ \left. \quad - \exp\left[-\mathrm{i}h\omega_0\tau\right]\right\} \exp\left[-\frac{1}{2}h^2\omega_0^2 c|\tau|\right] & \text{if } \tau \geq 0 \\ \mathbf{0} & \text{if } \tau < 0 \end{cases} \tag{18a}$$

$$\mathbf{R}^{\infty}_{\mathbf{y},\mathbf{x}_{\mathrm{S}}}(\tau)$$

$$= \begin{cases} \mathbf{0} & \text{if } \tau \geq 0 \\ \sum_{l=2}^{n} \sum_{h,j} \mathbf{D}^{\dagger}_{lhj} \left\{ \exp\left[-\mathrm{i}\left(j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}\right)\tau + \mathrm{Re}\left\{\mu_l\right\}|\tau|\right] \right. \\ \left. \quad - \exp\left[-\mathrm{i}h\omega_0\tau\right]\right\} \exp\left[-\frac{1}{2}h^2\omega_0^2 c|\tau|\right] & \text{if } \tau < 0 \end{cases} \tag{18b}$$

where

$$c = \frac{1}{T} \int_0^T \mathbf{v}_1^{\mathrm{T}} \mathbf{B}\mathbf{B}^{\mathrm{T}} \mathbf{v}_1 \, \mathrm{d}t \tag{19}$$

and

$$\mathbf{D}_{lhj} = \tilde{\mathbf{X}}_h \tilde{\mathbf{V}}_{1_0}^{\mathrm{T}} \tilde{\boldsymbol{\Lambda}}^*_{l_{h-j}} \tilde{\mathbf{U}}^{\dagger}_{l_j} \frac{\mathrm{i}h\omega_0}{-\mu_l^* - \mathrm{i}(h-j)\omega_0}. \tag{20}$$

Coefficients $\tilde{\mathbf{X}}_h$ are the harmonic components of $\mathbf{x}_{\mathrm{S}}(t)$, while $\tilde{\mathbf{U}}_{l_j}$ and $\tilde{\boldsymbol{\Lambda}}_{l_k}^{\mathrm{T}}$ are the Fourier coefficients of $\mathbf{u}_l(t)$ and $\mathbf{v}_l^{\mathrm{T}}(t)\mathbf{B}(t)$, respectively.

Proof: See Appendix B. ∎

B. Calculation of $\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau)$

The calculation of (5d) is performed according to the following Lemma:

Lemma 3.5: Asymptotically with time (i.e., for $t \to +\infty$), the correlation function of the orbital deviation due to a white Gaussian noise $\mathbf{b}(t) = \boldsymbol{\xi}(t)$ depends on τ only and is given by

$$\mathbf{R}^{\infty}_{\mathbf{y},\mathbf{y}}(\tau) = \sum_{l=2}^{n} \sum_{h,j} \mathbf{C}_{lhj} \exp\left[-\mathrm{i}\left[j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}\right]\tau\right]$$

$$\times \exp\left[\left[\mathrm{Re}\left\{\mu_l\right\} - \frac{1}{2}h^2\omega_0^2 c\right]|\tau|\right] \quad \text{if } \tau \geq 0 \tag{21a}$$

$$\mathbf{R}^{\infty}_{\mathbf{y},\mathbf{y}}(\tau) = \sum_{l=2}^{n} \sum_{h,j} \mathbf{C}^{\dagger}_{lhj} \exp\left[-\mathrm{i}\left[j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}\right]\tau\right]$$

$$\times \exp\left[\left[\mathrm{Re}\left\{\mu_l\right\} - \frac{1}{2}h^2\omega_0^2 c\right]|\tau|\right] \quad \text{if } \tau < 0 \tag{21b}$$

where c is defined in (19) and

$$\mathbf{C}_{lhj} = \sum_{l'=2}^{n} \sum_{j'} \frac{1}{\mathrm{i}(j-j')\omega_0 - \mu_{l'} - \mu_l^*} \tilde{\mathbf{U}}_{l'_{j'}} \tilde{\boldsymbol{\Lambda}}_{l'_{h-j'}}^{\mathrm{T}} \tilde{\boldsymbol{\Lambda}}_{l_{h-j}}^{*} \tilde{\mathbf{U}}_{l_j}^{\dagger}. \tag{22}$$

Proof: See Appendix C. ∎

Notice that the denominator in (22) is never null, since for an orbitally stable oscillator we have $\mathrm{Re}\{\mu_l\} < 0$ irrespective of $l \geq 2$. Finally, from (21) and (22) we have

$$\mathbf{R}_{\mathbf{y},\mathbf{y}}^{\infty}(0) = \sum_{l=2}^{n} \sum_{h,j} \mathbf{C}_{lhj} = \sum_{l=2}^{n} \sum_{h,j} \mathbf{C}_{lhj}^{\dagger}, \tag{23}$$

i.e. the sum of all $\mathbf{C}_{lhj}$ is a real and symmetric matrix (as should be, since $\mathbf{R}_{\mathbf{y},\mathbf{y}}^{\infty}(0)$ is the autocorrelation of $\mathbf{y}(t)$). This also implies that the autocorrelation matrix function $\mathbf{R}_{\mathbf{y},\mathbf{y}}^{\infty}(\tau)$ is not even with τ (apart from the diagonal elements), resulting in an impact on the shape of the orbital noise spectrum.

Equations (20) and (22) show that $\mathbf{C}_{lhj}, \mathbf{D}_{lhj} = 1/\mathrm{O}(\mu_l)$; therefore, the contribution of the orbital deviations is expected to be more significant for oscillators whose limit cycle is characterized by at least a second Floquet exponent near to zero. As discussed in [23], this is true for the entire class of high-Q oscillators. However, as we shall see in Sec. V, this is not a necessary condition for an oscillator to have a significant orbital noise contribution, in fact a major role in the $\mathbf{C}$ and $\mathbf{D}$ coefficients is also played by the Floquet eigenvectors, which could determine large orbital fluctuations contributions even when the Floquet exponents are not near to zero.

IV. TOTAL OSCILLATOR NOISE SPECTRUM

The spectrum of the oscillator noisy state variables is obtained in analogy with the result in [1].

Theorem 4.1: The spectrum of $\mathbf{z}(t) = \mathbf{x}_{\mathrm{S}}(t+\alpha(t)) + \mathbf{y}(t)$ is determined by the asymptotic behaviour of $\mathbf{R}_{\mathbf{z},\mathbf{z}}(t,\tau)$ for $t \to +\infty$. All the nontrivial cyclostationary components are zero, while the stationary part of the spectrum is given by

$$\mathbf{S}_{\mathbf{z},\mathbf{z}}(\omega) = \mathbf{S}_{\mathbf{x}_{\mathrm{S}},\mathbf{x}_{\mathrm{S}}}(\omega) + \mathbf{S}_{\mathrm{corr}}(\omega) + \mathbf{S}_{\mathbf{y},\mathbf{y}}(\omega) + \mathbf{X}_0 \mathbf{X}_0^{\dagger} \delta(\omega), \tag{24}$$

where ω is the (angular frequency) variable conjugated to τ, and the partial spectra are the Fourier transforms of $\mathbf{R}_{\mathbf{x}_{\mathrm{S}},\mathbf{x}_{\mathrm{S}}}^{\infty}(\tau)$, of $\mathbf{R}_{\mathbf{x}_{\mathrm{S}},\mathbf{y}}^{\infty}(\tau) + \mathbf{R}_{\mathbf{y},\mathbf{x}_{\mathrm{S}}}^{\infty}(\tau)$ and of $\mathbf{R}_{\mathbf{y},\mathbf{y}}^{\infty}(\tau)$, respectively

$$\mathbf{S}_{\mathbf{x}_{\mathrm{S}},\mathbf{x}_{\mathrm{S}}}(\omega) = \sum_{h} \tilde{\mathbf{X}}_h \tilde{\mathbf{X}}_h^{\dagger} \frac{h^2 \omega_0^2 c}{\Xi_h^2(\omega)} \tag{25}$$

$$\mathbf{S}_{\mathrm{corr}}(\omega) = \sum_{l=2}^{n} \sum_{h,j}$$

$$\left\{ \frac{\left(\mathbf{D}_{lhj}^{\dagger} + \mathbf{D}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c - \mathrm{Re}\{\mu_l\}\right]}{\Delta_{lhj}^2(\omega)} \right.$$

$$+ \frac{\mathrm{i}\left(\mathbf{D}_{lhj}^{\dagger} - \mathbf{D}_{lhj}\right)[\omega + j\omega_0 + \mathrm{Im}\{\mu_l\}]}{\Delta_{lhj}^2(\omega)}$$

$$-\frac{\left(\mathbf{D}_{lhj}^{\dagger} + \mathbf{D}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c\right]}{\Xi_h^2(\omega)}$$

$$-\frac{\mathrm{i}\left(\mathbf{D}_{lhj}^{\dagger} - \mathbf{D}_{lhj}\right)[\omega + h\omega_0]}{\Xi_h^2(\omega)}\Bigg\} \tag{26}$$

$$\mathbf{S}_{\mathbf{y},\mathbf{y}}(\omega) = \sum_{l=2}^{n}\sum_{h,j}$$

$$\left\{\frac{\left(\mathbf{C}_{lhj}^{\dagger} + \mathbf{C}_{lhj}\right)\left[\frac{1}{2}h^2\omega_0^2 c - \mathrm{Re}\left\{\mu_l\right\}\right]}{\Delta_{lhj}^2(\omega)}\right.$$

$$+\frac{\mathrm{i}\left(\mathbf{C}_{lhj}^{\dagger} - \mathbf{C}_{lhj}\right)(\omega + j\omega_0 + \mathrm{Im}\left\{\mu_l\right\})}{\Delta_{lhj}^2(\omega)}\Bigg\}, \tag{27}$$

where:

$$\Delta_{lhj}^2(\omega) = \left[\frac{1}{2}h^2\omega_0^2 c - \mathrm{Re}\left\{\mu_l\right\}\right]^2 + [\omega + j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}]^2 \tag{28}$$

$$\Xi_h^2(\omega) = \left[\frac{1}{2}h^2\omega_0^2 c\right]^2 + [\omega + h\omega_0]^2. \tag{29}$$

Proof: See Lemma 3.4 and 3.5, and the proof of Lemma 8.5 in [1]. ∎

The presence of complex conjugate Floquet exponents may, according to (28), give rise to resonance-like peaks in the oscillator noise spectrum, as discussed e.g. in [2].

Our derivation resulted into closed form equations (25)-(27) for the total (i.e., including phase and orbital noise, and their correlation) noise oscillator spectrum, at least within the limits of small amplitude fluctuations. Nevertheless, a fully closed form calculation of noise is hardly possible, besides some special cases (see e.g. Sec. V-A), mainly because the basic ingredients of the coefficients entering the spectrum expression are the Floquet exponents and the direct and adjoint Floquet eigenvectors of the linearized, noiseless oscillator equations. These quantities, in general, can be obtained through numerical analysis only. This makes the present approach more suitable for implementation into EDA tools, namely numerical circuit simulators such as Cadence SpectreRF [24] which already implements phase noise analysis according to the theory in [1]. These tools can be effectively used to refine and optimize the design initially carried out based on more circuit-dependent, albeit often less rigorous, approaches.

V. Validation and examples

A. Validation: a two-dimensional oscillator

In order to validate the results, we consider here the simple two-dimensional oscillator discussed in [16], which admits of an analytical calculation of the total noise spectrum. The autonomous system discussed in [16] ultimately

can be related to the oscillator proposed in [15] to discuss the decomposition of fluctuations into phase and orbital noise. The noisy oscillator equations are written, in polar coordinates, as [16]

$$\dot{\rho} = \frac{1}{2}(1 - \rho^2) + \epsilon\xi_\rho(t) \tag{30a}$$

$$\dot{\theta} = \omega_0 - \frac{1}{2}\upsilon(1 - \rho^2) + \frac{\epsilon}{\rho}\xi_\theta(t) \tag{30b}$$

where ρ and θ are, respectively, the radial and angular coordinates, ω_0, υ and ϵ are parameters, and $\xi(t)$ are unit white Gaussian noise sources.

In order to compare our results with those in [16], we consider the fluctuations of the first coordinate $x(t)$ in the cartesian representation of the nonlinear oscillator, and in particular we analyze the normalized correlation function [16, equation (20)]

$$R(\tau) = \frac{\mathrm{E}\left\{x(t)x(t+\tau)\right\}}{\mathrm{E}\left\{x^2(t)\right\}} \tag{31}$$

and the corresponding normalized spectrum $S(\omega)$.

This example is of great interest, since a semi-analytical solution $S_{\mathrm{an}}(\omega)$ for the oscillator Fokker-Planck equation derived in [16] can be obtained, thus providing a validation of the present theory.

Before starting the comparison of our approach with the theory we propose, we remark that the decomposition proposed here, as well as the theory in [1, p. 661], is based on the assumption that $|\dot{\alpha}(t)| \ll 1$. Since for this example

$$\dot{\alpha}(t) = \mathbf{v}_1(t + \alpha(t))\mathbf{B}(t + \alpha(t))\mathbf{b}(t) = \frac{\epsilon}{\omega_0}[\upsilon\xi_\rho(t) + \xi_\theta(t)], \tag{32}$$

we expect our (and Demir et al.) results to become less accurate as υ becomes large. In accordance with [16] we assume $\epsilon = \sqrt{10^{-3}}$, therefore for υ of the order of a few dozen our approach should become inaccurate.

A direct calculation allows to show that for $\upsilon = 0$ we find $S(\omega) = S_{\mathrm{an}}(\omega)$, thus providing a validation of our approach. This is confirmed by the results in Fig. 1 (above). In the lower part of the same figure, we show the results for the case $\upsilon = 4$. The agreement is still good, although some discrepancies start to arise with the exact (analytical) result, consistently with the fact that $\upsilon\epsilon \approx 0.1265$. Notice that the approximate full normalized spectrum is lower than the phase noise contribution, thus showing that the correlation between the phase and orbital deviations can decrease the total noise. This effect is not present for $\upsilon = 0$, since in this case the correlation spectrum is zero.

The comparison between our theory and the analytical normalized spectrum becomes, as expected, less favourable for $\upsilon = 10$ and $\upsilon = 30$ (in fact, $\upsilon\epsilon \approx 0.3162$ and $\upsilon\epsilon \approx 0.9487$, respectively), as shown in Fig. 2. Notice however that, for low frequency, our result still is a good approximation of the exact value.

B. Example: a Colpitts oscillator

The second example we discuss is the simple Colpitts oscillator based on the schematics in Fig. 3, where the transistor is the InGaP/GaAs HBT described by the Gummell Poon model in [25], including device nonlinear capacitances and parasitic effects. The circuit parameters are: $V_{\mathrm{CC}} = 6$ V, $R_1 = 10$ kΩ, $R_2 = 4.2$ kΩ, $R_{\mathrm{C}} = 300$ Ω, $C_1 = 5$ pF, $C_2 = 5$ pF, $C_{\mathrm{S}} = 1$ μF and $L = 10$ nH. The LC feedback network corresponds to an oscillation

frequency of 1 GHz, while the ratio C_1/C_2 suggests a required voltage gain slightly larger than 1: different ratios (keeping constant the oscillation frequency) might impact on the Floquet quantities, thus modifying the oscillator noise properties.

The circuit has been analyzed with the harmonic balance technique including 300 harmonics, while the Floquet exponents have been determined with the method in [26]. Harmonic balance was implemented according to the standard technique discussed in [27], exploiting the discrete Fourier transform algorithm to efficiently calculate time-frequency transformations (see also [26]). The large number of harmonics was required by the highly nonlinear operation of the device. The oscillation frequency is found to be $f_0 = 0.9944$ GHz. The four non zero Floquet exponents of the limit cycle are

$$\mu_2 = -1159.4 \tag{33}$$

$$\mu_3 = -2.2375 \times 10^9 \tag{34}$$

$$\mu_4 = -3.2228 \times 10^9 + 3.1241 \times 10^9 \text{i} \tag{35}$$

$$\mu_5 = -4.2187 \times 10^{12}. \tag{36}$$

Notice that μ_4, albeit complex, is not present in a complex conjugate pair since the imaginary part is equal to $\omega_0/2$, thus leading to a real Floquet multiplier.

Considering as an output variable the collector current i_C, the calculation of the c costant yields $c = 2.6523 \times 10^{-19}$ s^2 Hz, assuming for simplicity that only the transistor is noisy, and affected by white shot noise. Since the determination of the fluctuation spectrum for a scalar circuit quantity (rather than the correlation matrix of all the state variables) is more efficiently performed making use of dedicated relationships, we provide the relevant derivation in Appendix D. The collector current noise spectrum, in dB$_W$/Hz, is reported in Fig. 4 as a function of frequency, showing that, at least far from the limit cycle harmonics, the spectrum is dominated by the orbital deviation contribution (98). The correlation between phase and orbital noise (99), on the other hand, is negligible.

A better insight is obtained by considering the upper (i.e., $\omega > \omega_0$) and lower (i.e., $\omega < \omega_0$) sidebands of the fundamental frequency. The two spectra as a function of the sideband frequency (i.e., $|f - f_0|$) are shown in Fig. 5, highlighting the effect of the orbital and phase-orbital contributions which are nor symmetric with respect to the central frequency f_0, neither Lorentzian in shape. This suggests that orbital effects might be responsible for asymmetries in the noise spectrum with respect to the harmonics of the oscillation frequency, as found also in [28]. This is clearly shown in Fig. 6, where the partial contributions to the orbital noise spectrum due to the four non zero Floquet exponents are reported. The asymmetric behaviour is even more visible around the second harmonic of the oscillation frequency. Furthermore, notice that six orders of magnitude separate μ_2 and μ_3, while the corresponding contribution to orbital noise are not in the same ratio. Rather, far from the oscillator harmonics, the contribution of μ_3 is dominant with respect to μ_2: this clearly shows that also the eigenvectors may give an important contribution to the orbital noise spectrum, which might also dominate over the $1/\mu_l$ factor.

VI. Conclusions

We have presented a general theory for the nonlinear perturbative analysis of noise in free running oscillators affected by white Gaussian noise sources, consistently including the effect of phase noise, orbital fluctuations and their correlation. The theory, extending the results in [1] in which phase noise only was considered, proves that, asymptotically with time, Gaussian stationary fluctuations affect the oscillator solution. General expressions for the full oscillator spectrum have been derived, based on the Fourier components of the noiseless solution and on the Floquet exponents and eigenvectors (direct and adjoint) of the linearized oscillator equations. This makes the theory readily implementable into any EDA tool for lumped circuit analysis.

The resulting total noise spectrum may be significantly affected by the orbital deviation and by its correlation with the fluctuations along the orbit. The identification of the oscillator classes mostly impacted by this effect is not an easy task: at first glance, the high-Q oscillators [23], for which more than one Floquet exponent is near zero, are good candidates, since the amplitude of the orbital spectra is inversely proportional to the Floquet exponents. Nevertheless, the examples discussed in Sec. V show that the magnitude of the Floquet eigenvectors may play an even more important role. Furthermore, the presence of the correlation terms, in particular, allows for the reduction of the total noise spectrum with respect to the phase noise component only, while the orbital deviation power spectrum may present additive peaks with respect to the harmonics of the steady-state in presence of Floquet exponents with non null imaginary part. Finally, the superposition of the various components (each of them Lorentzian as a function of frequency) may result into a non strictly Lorentzian frequency shape.

Acknowledgements

Appendix A

Proof of Lemma 3.3

The first component of (6) yields

$$\alpha(t_1) = \int_0^{t_1} \mathbf{v}_1^{\mathrm{T}}(r + \alpha(r))\mathbf{B}(r + \alpha(r))\boldsymbol{\xi}(r) \, \mathrm{d}r. \tag{37}$$

Therefore, using the Fourier expansion of $\mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}(t)$ and denoting the corresponding j-th Fourier coefficient as $\tilde{\mathbf{V}}_{1_j}^{\mathrm{T}}$, we find for the correlation function between $\alpha(t_1)$ and the h-th component $\xi_h(t_2)$ of the (real) unit Gaussian white noise source $\boldsymbol{\xi}(t_2)$

$$R_{\alpha,\xi_h}(t_1, t_2) = \mathrm{E}\left\{\alpha(t_1)\xi_h(t_2)\right\} = \sum_{k=1}^{p} \sum_j \left(\tilde{\mathbf{V}}_{1_j}^{\mathrm{T}}\right)_k$$

$$\times \int_0^{t_1} \exp\left[\mathrm{i}j\omega_0 r\right] \mathrm{E}\left\{\exp\left[\mathrm{i}j\omega_0\alpha(r)\right]\xi_k(r)\xi_h(t_2)\right\} \, \mathrm{d}r \tag{38}$$

where $\left(\tilde{\mathbf{V}}_{1_j}^{\mathrm{T}}\right)_k$ is the k-th component of vector $\tilde{\mathbf{V}}_{1_j}^{\mathrm{T}}$. Because of Lemma 3.2, the three random variables in the expectation value in (38) are jointly Gaussian, therefore the stochastic variable

$$\eta(r, t_2) = j\alpha(r) + \frac{\omega_k}{\omega_0}\xi_k(r) + \frac{\omega_h}{\omega_0}\xi_h(t_2) \tag{39}$$

is Gaussian. Furthermore, since

$$\mathrm{E}\left\{\exp\left[\mathrm{i}j\omega_0\alpha(r)\right]\xi_k(r)\xi_h(t_2)\right\}$$

$$= -\left.\frac{\partial^2}{\partial\omega_k\partial\omega_h}\mathrm{E}\left\{\exp\left[\mathrm{i}\omega_0\eta(r, t_2)\right]\right\}\right|_{\omega_h=\omega_k=0} \tag{40}$$

to estimate (38) we need the characteristic function of $\eta(r, t_2)$, which is completely defined by the first two momenta because of the Gaussian nature of the same variable. By defining

$$f_{\xi_k}(r) = \mathrm{E}\left\{\xi_k^2(r)\right\} \tag{41}$$

$$f_{\xi_{k,h}}(r, s) = \mathrm{E}\left\{\xi_k(r)\xi_h(s)\right\} \tag{42}$$

$$f_k(r, s) = \mathrm{E}\left\{\alpha(r)\xi_k(s)\right\} \tag{43}$$

we can express, asymptotically with time

$$\mathrm{E}\left\{\eta(r, t_2)\right\} = mj \tag{44}$$

$$\mathrm{E}\left\{\eta^2(r, t_2)\right\} - \mathrm{E}\left\{\eta(r, t_2)\right\}^2 = j^2 cr + \frac{\omega_k^2}{\omega_0^2}f_{\xi_k}(r)$$

$$+ \frac{\omega_h^2}{\omega_0^2}f_{\xi_h}(t_2) + \frac{2}{\omega_0^2}\left[j\omega_0\omega_h f_h(r, t_2)\right.$$

$$\left. + j\omega_0\omega_k f_k(r, r) + \omega_h\omega_k f_{\xi_{k,h}}(r, t_2)\right] \tag{45}$$

therefore

$$\mathrm{E}\left\{\exp\left[\mathrm{i}\omega_0\eta(r, t_2)\right]\right\} = \exp\left[\mathrm{i}jm\omega_0\right]$$

$$\times \exp\left[-\omega_0^2 j^2 cr/2 - \omega_k^2 f_{\xi_k}(r)/2\right]$$

$$\times \exp\left[-\omega_h^2 f_{\xi_h}(t_2)/2 - j\omega_0\omega_h f_h(r, t_2) - j\omega_0\omega_k f_k(r, r)\right]$$

$$\times \exp\left[-\omega_h\omega_k f_{\xi_{k,h}}(r, t_2)\right]. \tag{46}$$

From (40) follows

$$\mathrm{E}\left\{\exp\left[\mathrm{i}j\omega_0\alpha(r)\right]\xi_k(r)\xi_h(t_2)\right\} = \left[f_{\xi_{k,h}}(r, t_2)\right.$$

$$\left. - j^2\omega_0^2 f_h(r, t_2)f_k(r, r)\right]\exp\left[\mathrm{i}jm\omega_0\right]\exp\left[-\omega_0^2 j^2 cr/2\right] \tag{47}$$

which, substituted into (38), yields the integral equation

$$f_h(t_1, t_2) = \int_{t_2}^{t_1} f_h(r, t_2) H(r, \{f_k(r, r)\}_{k=1}^p) \, dr$$

$$+ L_h(t_1, t_2) \tag{48}$$

where the integral limits include the condition $f_k(r, s) = 0$ if $s > r$ (since $\alpha(r)$ is uncorrelated with any "future" noise source $\xi_k(s)$), and

$$H(r, \{f_k(r, r)\}_{k=1}^p) = \sum_j \sum_{k=1}^p \left(\tilde{\mathbf{V}}_{1_j}^T \right)_k j^2 \omega_0^2 f_k(r, r)$$

$$\times \exp[ij\omega_0 r] \exp[ijm\omega_0] \exp\left[-\omega_0^2 j^2 cr/2 \right] \tag{49}$$

$$L_h(t_1, t_2) = \sum_j \sum_{k=1}^p \left(\tilde{\mathbf{V}}_{1_j}^T \right)_k$$

$$\times \int_0^{t_1} \exp\left[ij\omega_0 r + ijm\omega_0 - \omega_0^2 j^2 cr/2 \right] \delta_{k,h} \delta(r - t_2) \, dr$$

$$= \begin{cases} \sum_j \left(\tilde{\mathbf{V}}_{1_j}^T \right)_h \exp[ij\omega_0(t_2 + m)] \\ \quad \times \exp\left[-\omega_0^2 j^2 ct_2/2 \right] & \text{if } t_2 \le t_1 \\ 0 & \text{if } t_2 > t_1 \end{cases} \tag{50}$$

For $t_2 < t_1$, (48) can be derived with respect to t_1 obtaining

$$\frac{df_h(t_1, t_2)}{dt_1} = f_h(t_1, t_2) H(t_1, \{f_k(t_1, t_1)\}_{k=1}^p), \tag{51}$$

whose solution is

$$f_h(t_1, t_2) = a_h(t_2) \exp\left[\int_{t_2}^{t_1} H(s, \{f_k(s, s)\}_{k=1}^p) \, ds \right]. \tag{52}$$

Substituting into (48) we get

$$a_h(t_2) = L_h(t_1, t_2), \tag{53}$$

which is only apparently a contradiction, in fact for $t_2 < t_1$ (our initial assumption for estimating $f_h(t_1, t_2)$) $L_h(t_1, t_2)$ is actually a function of t_2 only (see (50)). Furthermore, (53) can be extended for $t_2 \ge t_1$ since in this case $L_h(t_1, t_2) = 0$, and therefore we correctly have $f_h(t_1, t_2) = 0$. Finally, since

$$f_q(r, r) = L_q(r, r)$$

$$\exp\left[\int_r^r H(s, \{f_k(s, s)\}_{k=1}^p) \, ds \right] = L_q(r, r), \tag{54}$$

we can conclude

$$f_h(t_1, t_2) = L_h(t_1, t_2) \exp\left[\int_{t_2}^{t_1} H(s, \{L_k(s, s)\}_{k=1}^p) \, ds \right]. \tag{55}$$

The integral can be evaluated explicitly obtaining

$$
\int_{t_2}^{t_1} H(s, \{L_k(s,s)\}_{k=1}^p) \, \mathrm{d}s =
$$

$$
- \sum_{j_1,j_2} \sum_{k=1}^p \left(\tilde{\mathbf{V}}_{1_{j_1}}^{\mathrm{T}}\right)_k \left(\tilde{\mathbf{V}}_{1_{j_2}}^{\mathrm{T}}\right)_k j_1^2 \omega_0^2
$$

$$
\times \left[\frac{\exp\left[\mathrm{i}(j_1+j_2)\omega_0(t_1+m)\right] \exp\left[\omega_0^2(j_1^2+j_2^2)ct_1/2\right]}{\mathrm{i}(j_1+j_2)\omega_0 - \omega_0^2(j_1^2+j_2^2)c/2} \right.
$$

$$
\left. - \frac{\exp\left[\mathrm{i}(j_1+j_2)\omega_0(t_2+m)\right] \exp\left[\omega_0^2(j_1^2+j_2^2)ct_2/2\right]}{\mathrm{i}(j_1+j_2)\omega_0 - \omega_0^2(j_1^2+j_2^2)c/2} \right] \tag{56}
$$

which tends to zero for $t_1, t_2 \to +\infty$. Therefore, (55) asymptotically yields (17).

APPENDIX B

PROOF OF LEMMA 3.4

The proof of Lemma 3.4 starts from the Fourier expansions of the orbit

$$
\mathbf{x}_{\mathrm{S}}(t) = \sum_h \tilde{\mathbf{X}}_h \exp\left[\mathrm{i}h\omega_0 t\right] \tag{57}
$$

and of the orbital deviation $\mathbf{y}(t)$. According to [1, p. 661]

$$
\mathbf{y}(t) = \sum_{l=2}^n \mathbf{u}_l(\hat{t}) \int_0^{\hat{t}} \exp\left[\mu_l(\hat{t}-s)\right] \mathbf{v}_l^{\mathrm{T}}(s)\mathbf{B}(s)\hat{\mathbf{b}}(s) \, \mathrm{d}s, \tag{58}
$$

where $\hat{t} = t + \alpha(t)$. Exploiting the Fourier series representation of the periodic functions, (58) reads

$$
\mathbf{y}(t) = \sum_{l=2}^n \sum_{j,k} \tilde{\mathbf{U}}_{l_j} \tilde{\boldsymbol{\Lambda}}_{l_k}^{\mathrm{T}} \exp\left[(\mathrm{i}j\omega_0 + \mu_l)\hat{t}\right]
$$

$$
\times \int_0^{\hat{t}} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right] \hat{\mathbf{b}}(s) \, \mathrm{d}s, \tag{59}
$$

where $\tilde{\mathbf{U}}_{l_j}$ and $\tilde{\boldsymbol{\Lambda}}_{l_k}^{\mathrm{T}}$ are the Fourier coefficients of $\mathbf{u}_l(t)$ and $\mathbf{v}_l^{\mathrm{T}}(t)\mathbf{B}(t)$, respectively. For the integral in (59) we make the ansatz

$$
\int_0^{\hat{t}} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right] \hat{\mathbf{b}}(s) \, \mathrm{d}s = \frac{\exp\left[(\mathrm{i}k\omega_0 - \mu_l)\hat{t}\right] \hat{\mathbf{f}}(\hat{t}) - \hat{\mathbf{f}}(0)}{\mathrm{i}k\omega_0 - \mu_l}, \tag{60}
$$

where $\hat{\mathbf{f}}(\hat{t}) = \hat{\mathbf{f}}(\hat{\mathbf{b}}(\hat{t}))$. Deriving both sides of (60) with respect to $\hat{t}$ and using the relations (see [1])

$$
\hat{\mathbf{b}}(\hat{t}) = \mathbf{b}(t) \tag{61}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}\hat{t}} = \frac{1}{1+\dot{\alpha}(t)} \frac{\mathrm{d}}{\mathrm{d}t} \approx \frac{\mathrm{d}}{\mathrm{d}t}, \tag{62}
$$

we derive the following differential equation for the unknown function $\hat{\mathbf{f}}(\hat{t}) = \hat{\mathbf{f}}(\hat{\mathbf{b}}(\hat{t})) = \mathbf{f}(t)$:

$$
\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} + (\mathrm{i}k\omega_0 - \mu_l)\mathbf{f}(t) = (\mathrm{i}k\omega_0 - \mu_l)\mathbf{b}(t), \tag{63}
$$

whose general solution is

$$\mathbf{f}(t) = \exp\left[(\mu_l - \mathrm{i}k\omega_0)t\right]\mathbf{f}(0) + (\mathrm{i}k\omega_0 - \mu_l)$$

$$\times \exp\left[(\mu_l - \mathrm{i}k\omega_0)t\right]\int_0^t \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right]\mathbf{b}(s)\ \mathrm{d}s. \tag{64}$$

A consistent evaluation of $\mathbf{f}(0)$ requires some further remarks. Let us assume, in accordance with [1], that $\alpha(t) = \epsilon_\alpha t$ with $0 < \epsilon_\alpha \ll 1$, and that the noise source is an harmonic function $\mathbf{b}(t) = \epsilon_b \exp(\mathrm{i}\omega t)$. This corresponds to represent $\mathbf{b}(t)$ as a superposition of stochastic amplitude sinusoidal terms. From (61)

$$\hat{\mathbf{b}}(s) = \mathbf{b}(s - \alpha(s)) = \epsilon_b \exp\left[\mathrm{i}\omega(s - \alpha(s))\right], \tag{65}$$

therefore (60) can be evaluated as

$$\int_0^{\hat{t}} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right]\hat{\mathbf{b}}(s)\ \mathrm{d}s$$

$$= \epsilon_b \int_0^{t+\epsilon_\alpha t} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right]\exp\left[\mathrm{i}\omega(1 - \epsilon_\alpha)s\right]\ \mathrm{d}s$$

$$= \epsilon_b \frac{\exp\left[(\mathrm{i}k\omega_0 - \mu_l)(1 + \epsilon_\alpha)t\right]\exp\left[\mathrm{i}\omega(1 - \epsilon_\alpha)(1 + \epsilon_\alpha)t\right] - 1}{\mathrm{i}k\omega_0 - \mu_l + \mathrm{i}\omega(1 - \epsilon_\alpha)}$$

$$\approx \epsilon_b \frac{\exp\left[(\mathrm{i}k\omega_0 - \mu_l)t\right]\exp\left[\mathrm{i}\omega t\right] - 1}{\mathrm{i}k\omega_0 - \mu_l + \mathrm{i}\omega} \tag{66}$$

since $\epsilon_\alpha \ll 1$. Similarly, substituting (64) into (60) and using the condition $\epsilon_\alpha \ll 1$ we find

$$\int_0^{\hat{t}} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right]\hat{\mathbf{b}}(s)\ \mathrm{d}s$$

$$\approx \frac{\exp\left[(\mathrm{i}k\omega_0 - \mu_l)\epsilon_\alpha t\right] - 1}{\mathrm{i}k\omega_0 - \mu_l}\mathbf{f}(0)$$

$$+ \epsilon_b \frac{\exp\left[(\mathrm{i}k\omega_0 - \mu_l)t\right]\exp\left[\mathrm{i}\omega t\right] - 1}{\mathrm{i}k\omega_0 - \mu_l + \mathrm{i}\omega}, \tag{67}$$

therefore comparing (66) and (67) we conclude that $\mathbf{f}(0) = \mathbf{0}$.

Substituting into (60) and using $\mathbf{f}(0) = \mathbf{0}$ we finally find

$$\int_0^{\hat{t}} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right]\hat{\mathbf{b}}(s)\ \mathrm{d}s$$

$$= \exp\left[(\mathrm{i}k\omega_0 - \mu_l)\alpha(t)\right]\int_0^t \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right]\mathbf{b}(s)\ \mathrm{d}s. \tag{68}$$

Using (68) into (59), we can express $\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,\tau)$ as

$$\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,\tau) = \sum_{l=2}^{n}\sum_{j,k,h}\tilde{\mathbf{X}}_h \exp\left[\mathrm{i}h\omega_0 t\right]$$

$$\times \exp\left[(-\mathrm{i}j\omega_0 + \mu_l^*)(t + \tau)\right]\int_0^{t+\tau}\exp\left[-(\mathrm{i}k\omega_0 + \mu_l^*)r\right]$$

$$\times \mathrm{E}\left\{\exp\left[\mathrm{i}h\omega_0\alpha(t) - \mathrm{i}\omega_0(j + k)\alpha(t + \tau)\right]\mathbf{b}^\mathrm{T}(r)\right\}\ \mathrm{d}r$$

$$\times \tilde{\mathbf{\Lambda}}_{l_k}^* \tilde{\mathbf{U}}_{l_j}^\dagger. \tag{69}$$

The next step consists of the determination of the average value in (69), which is performed taking into consideration that

$$
\mathrm{E}\left\{\exp\left[\mathrm{i}h\omega_0\alpha(t) - \mathrm{i}\omega_0(j+k)\alpha(t+\tau)\right]\mathbf{b}^{\mathrm{T}}(r)\right\}
$$

$$
= \nabla_{\boldsymbol{\omega}_b}^{\mathrm{T}}\mathrm{E}\left\{\exp\left[\mathrm{i}\omega_0\beta(t,\tau,r)\right]\right\}\big|_{\boldsymbol{\omega}_b=\mathbf{0}} \tag{70}
$$

where $\boldsymbol{\omega}_b$ is a p-dimensional vector, $\nabla_{\boldsymbol{\omega}_b}^{\mathrm{T}}$ is the (row) gradient operator with respect to $\boldsymbol{\omega}_b$, and

$$
\beta(t,\tau,r) = h\alpha(t) - (j+k)\alpha(t+\tau) - \mathrm{i}\frac{\boldsymbol{\omega}_b^{\mathrm{T}}}{\omega_0}\mathbf{b}(r) \tag{71}
$$

is a Gaussian random variable because of Lemma 3.2. From [1] we have asymptotically with time

$$
\mathrm{E}\left\{\beta(t,\tau,r)\right\} = (h-j-k)m, \tag{72}
$$

while because of [1, Theorem 7.1, Lemma 7.2, Corollary 7.1] we find

$$
\mathrm{E}\left\{\beta^2(t,\tau,r)\right\} - \mathrm{E}\left\{\beta(t,\tau,r)\right\}^2 = (h-j-k)^2 ct
$$

$$
+ (j+k)^2 c\tau - 2h(j+k)c\min(0,\tau)
$$

$$
- \frac{\boldsymbol{\omega}_b^{\mathrm{T}}}{\omega_0^2}\mathbf{w}_1(r)\boldsymbol{\omega}_b - 2\mathrm{i}\frac{\boldsymbol{\omega}_b^{\mathrm{T}}}{\omega_0}\mathbf{w}_2(r,t,\tau) \tag{73}
$$

where, using Lemma 3.3

$$
\mathbf{w}_1(r) = \mathrm{E}\left\{\mathbf{b}(r)\mathbf{b}^{\mathrm{T}}(r)\right\} \tag{74}
$$

$$
\mathbf{w}_2(r,t,\tau) = \mathrm{E}\left\{[h\alpha(t) - (j+k)\alpha(t+\tau)]\mathbf{b}(r)\right\}
$$

$$
= \begin{cases} (h-j-k)\tilde{\mathbf{V}}_{1_0} & \text{if } r \le t \text{ and } r \le t+\tau \\ (-j-k)\tilde{\mathbf{V}}_{1_0} & \text{if } r > t \text{ and } r \le t+\tau \\ h\tilde{\mathbf{V}}_{1_0} & \text{if } r \le t \text{ and } r > t+\tau. \end{cases} \tag{75}
$$

Since $\beta(t,\tau,r)$ is Gaussian, (72) and (73) completely define its characteristic function. Taking the limit for $t \to +\infty$

$$
\mathrm{E}\left\{\exp\left[\mathrm{i}\omega_0\beta(t,\tau,r)\right]\right\} = \delta_{h,j+k}\exp\left[-\frac{\omega_0^2}{2}h^2c|\tau|\right]
$$

$$
\times \exp\left[\frac{\omega_0^2}{2}\left(\frac{\boldsymbol{\omega}_b^{\mathrm{T}}}{\omega_0^2}\mathbf{w}_1(r)\boldsymbol{\omega}_b + 2\mathrm{i}\frac{\boldsymbol{\omega}_b^{\mathrm{T}}}{\omega_0}\mathbf{w}_2(r,t,\tau)\right)\right] \tag{76}
$$

therefore, from (70)

$$
\mathrm{E}\left\{\exp\left[\mathrm{i}h\omega_0\alpha(t) - \mathrm{i}\omega_0(j+k)\alpha(t+\tau)\right]\mathbf{b}^{\mathrm{T}}(r)\right\}
$$

$$
= \delta_{h,j+k}\mathrm{i}\omega_0\mathbf{w}_2^{\mathrm{T}}(r,t,\tau)\exp\left[-\frac{\omega_0^2}{2}h^2c|\tau|\right]. \tag{77}
$$

Substituting (75) and (77) into (69), and performing the integral yields

$$
\mathbf{R}_{\mathbf{x}_S,\mathbf{y}}^{\infty}(\tau)
$$

$$
= \begin{cases}
\displaystyle\sum_{l=2}^{n} \sum_{h,j} \tilde{\mathbf{X}}_h \tilde{\mathbf{V}}_{1_0}^{\mathrm{T}} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^{*} \tilde{\mathbf{U}}_{l_j}^{\dagger} \, ih\omega_0 \exp\left[-\dfrac{\omega_0^2}{2} h^2 c|\tau|\right] \\
\qquad \times \dfrac{\exp\left[(-ij\omega_0 + \mu_l^{*})\tau\right] - \exp\left[-ih\omega_0\tau\right]}{-\mu_l^{*} - i(h-j)\omega_0} & \text{if } \tau \geq 0 \\
0 & \text{if } \tau > 0.
\end{cases}
\tag{78}
$$

Defining the coefficients $\mathbf{D}_{lhj}$ as in (20), (78) reduces to (18a).

Repeating the calculation for $\mathbf{R}_{\mathbf{y},\mathbf{x}_S}(t,\tau)$, we find (18b).

APPENDIX C
PROOF OF LEMMA 3.5

We consider first the case $\tau \geq 0$. Using (68) into (59), we can express $\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau)$ as

$$
\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau) = \sum_{l,l'=2}^{n} \sum_{j,k,j',k'} \tilde{\mathbf{U}}_{l_j} \tilde{\mathbf{\Lambda}}_{l_k}^{\mathrm{T}} \exp\left[(ij\omega_0 + \mu_l)t\right]
$$

$$
\times \exp\left[(-ij'\omega_0 + \mu_{l'}^{*})(t+\tau)\right] \int_0^t \int_0^{t+\tau} \exp\left[(ik\omega_0 - \mu_l)s\right]
$$

$$
\times \exp\left[(-ik'\omega_0 - \mu_{l'}^{*})s'\right] \mathrm{E}\left\{\exp\left[i(j+k)\omega_0\alpha(t)\right]\right.
$$

$$
\times \exp\left[-i(j'+k')\omega_0\alpha(t+\tau)\right] \mathbf{b}(s)\mathbf{b}^{\dagger}(s')\Big\} \, \mathrm{d}s'\mathrm{d}s
$$

$$
\times \tilde{\mathbf{\Lambda}}_{l'_{k'}}^{*} \tilde{\mathbf{U}}_{l'_{j'}}^{\dagger}.
\tag{79}
$$

Expanding the expectation value in the integral according to its components, we have

$$
\mathrm{E}\left\{\exp\left[i(j+k)\omega_0\alpha(t)\right] \exp\left[-i(j'+k')\omega_0\alpha(t+\tau)\right] \xi_l(s)\right.
$$

$$
\left.\times\xi_m(s')\right\} = \frac{\partial}{\partial\omega_b} \frac{\partial}{\partial\omega_{b'}} \mathrm{E}\left\{\exp\left[i\omega_0\zeta(t,\tau,s,s')\right]\right\}\Bigg|_{\omega_b=\omega_{b'}=0}
\tag{80}
$$

where ω_b and $\omega_{b'}$ are scalars, and

$$
\zeta(t,\tau,s,s') = (j+k)\alpha(t) - (j'+k')\alpha(t+\tau)
$$

$$
- i\frac{\omega_b}{\omega_0}\xi_l(s) - i\frac{\omega_{b'}}{\omega_0}\xi_m(s')
\tag{81}
$$

is a Gaussian random variable because of Lemma 3.2. A calculation similar to that performed in Appendix B shows that asymptotically with time t

$$\int_0^t \int_0^{t+\tau} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right] \exp\left[(-\mathrm{i}k'\omega_0 - \mu_{l'}^*)s'\right]$$

$$\times \mathrm{E}\left\{\exp\left[\mathrm{i}(j+k)\omega_0\alpha(t)\right] \exp\left[-\mathrm{i}(j'+k')\omega_0\alpha(t+\tau)\right]\right.$$

$$\left. \times \xi_l(s)\xi_m(s')\right\} \, \mathrm{d}s'\mathrm{d}s$$

$$= \int_0^t \int_0^{t+\tau} \exp\left[(\mathrm{i}k\omega_0 - \mu_l)s\right] \exp\left[(-\mathrm{i}k'\omega_0 - \mu_{l'}^*)s'\right]$$

$$\times \delta_{j+k,j'+k'} \exp\left[-\frac{\omega_0^2}{2}(j+k)^2 c|\tau|\right] \delta(s-s') \, \mathrm{d}s'\mathrm{d}s. \tag{82}$$

Since $\tau \geq 0$, the double integral can be decomposed as follows:

$$\int_0^t \int_0^{t+\tau} \cdot \, \mathrm{d}s'\mathrm{d}s = \int_0^t \int_0^t \cdot \, \mathrm{d}s'\mathrm{d}s + \int_0^t \int_t^{t+\tau} \cdot \, \mathrm{d}s'\mathrm{d}s, \tag{83}$$

and a discussion similar to that in [21, pp. 66–68] shows that the second double integral at the rhs of (83) is null. The first integral can be calculated explicitly, obtaining

$$\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau) = \sum_{l,l'=2}^n \sum_{\substack{j,k,j',k' \\ j+k=j'+k'}} \tilde{\mathbf{U}}_{l_j} \tilde{\mathbf{\Lambda}}_{l_k}^\mathrm{T} \tilde{\mathbf{\Lambda}}_{l'_{k'}}^* \tilde{\mathbf{U}}_{l'_{j'}}^\dagger$$

$$\times \frac{1 - \exp\left[(\mathrm{i}j\omega_0 + \mu_l)t\right] \exp\left[(-\mathrm{i}j'\omega_0 + \mu_{l'}^*)t\right]}{\mathrm{i}(k-k')\omega_0 - \mu_l - \mu_{l'}^*}$$

$$\times \exp\left[(-\mathrm{i}j'\omega_0 + \mu_{l'}^*)\tau\right] \exp\left[-(j+k)^2\omega_0^2 c|\tau|/2\right]. \tag{84}$$

The asymptotic behaviour of $\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau)$ is obtained taking the limit for $t \to +\infty$ in (84), posing $h = j+k = j'+k'$ and using the fact that because of the assumed orbital stability $\mathrm{Re}\{\mu_l\} < 0$ $(l = 2,\ldots,n)$

$$\mathbf{R}_{\mathbf{y},\mathbf{y}}^\infty(\tau) = \sum_{l,l'=2}^n \sum_{h,j,j'} \tilde{\mathbf{U}}_{l_j} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^\mathrm{T} \tilde{\mathbf{\Lambda}}_{l'_{h-j'}}^* \tilde{\mathbf{U}}_{l'_{j'}}^\dagger$$

$$\times \frac{\exp\left[(-\mathrm{i}j'\omega_0 + \mu_{l'}^*)\tau\right]}{\mathrm{i}(j'-j)\omega_0 - \mu_l - \mu_{l'}^*} \exp\left[-h^2\omega_0^2 c|\tau|/2\right]. \tag{85}$$

A similar derivation can be carried out for $\tau < 0$, obtaining

$$\mathbf{R}_{\mathbf{y},\mathbf{y}}^\infty(\tau) = \sum_{l,l'=2}^n \sum_{h,j,j'} \tilde{\mathbf{U}}_{l_j} \tilde{\mathbf{\Lambda}}_{l_{h-j}}^\mathrm{T} \tilde{\mathbf{\Lambda}}_{l'_{h-j'}}^* \tilde{\mathbf{U}}_{l'_{j'}}^\dagger$$

$$\times \frac{\exp\left[(-\mathrm{i}j\omega_0 - \mu_l)\tau\right]}{\mathrm{i}(j'-j)\omega_0 - \mu_l - \mu_{l'}^*} \exp\left[-h^2\omega_0^2 c|\tau|/2\right]. \tag{86}$$

After defining the coefficients $\mathbf{C}_{lhj}$ as in (22), (85) and (86) yield (21).

APPENDIX D

DERIVED VARIABLE OSCILLATOR NOISE SPECTRUM

Although Theorem 4.1 fully characterizes the spectrum of the state variables noise of the oscillator, in most cases we are interested into the noise spectrum of other circuit variables. The details of this calculation are, at least for

the orbital part, not trivial. Let us consider an output scalar variable $\gamma(\mathbf{x}(t))$ derived from the circuit state variables. When noise is included, assuming a small orbital contribution we have the noisy output

$$\gamma(\mathbf{z}(t)) \approx \gamma(\mathbf{x}_S(t + \alpha(t))) + \mathbf{J}_\gamma(t + \alpha(t))\mathbf{y}(t) \tag{87}$$

where $\mathbf{J}_\gamma(t)$ is the Jacobian of $\gamma(\mathbf{x})$ evaluated in the steady-state solution $\mathbf{x}_S(t)$, and therefore is a T-periodic function of time. Clearly, we can decompose the autocorrelation function of $\gamma(t)$ as

$$R_{\gamma,\gamma}(t,\tau) = R_{\gamma,\gamma}^{(\mathbf{x}_S,\mathbf{x}_S)}(t,\tau) + R_{\gamma,\gamma}^{(\mathbf{x}_S,\mathbf{y})}(t,\tau)$$

$$+ R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{x}_S)}(t,\tau) + R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{y})}(t,\tau). \tag{88}$$

A simple calculation allows to estimate the phase noise contribution as

$$S_{\gamma,\gamma}^{(\mathbf{x}_S,\mathbf{x}_S)}(\omega) = \sum_h \left|\tilde{\Gamma}_h\right|^2 \frac{h^2\omega_0^2 c}{\Xi_h^2(\omega)} \tag{89}$$

where $\tilde{\Gamma}_h$ is the h-th harmonic component of $\gamma(\mathbf{x}_S(t))$.

The correlation between phase and orbital fluctuations is treated in the following

Lemma D.1: The asymptotic value for $t \to +\infty$ of the correlation functions $R_{\gamma,\gamma}^{(\mathbf{x}_S,\mathbf{y})}(t,\tau) = \mathrm{E}\left\{\gamma(\mathbf{x}_S(t + \alpha(t)))\mathbf{y}^T(t + \tau)\mathbf{J}_\gamma^T(t + \tau\right.$ and $R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{x}_S)}(t,\tau) = \mathrm{E}\left\{\mathbf{J}_\gamma(t + \alpha(t))\mathbf{y}(t)\gamma(\mathbf{x}_S(t + \tau + \alpha(t + \tau)))\right\}$ of an output variable of the oscillator are given by

$$R_{\gamma,\gamma}^{(\mathbf{x}_S,\mathbf{y})\infty}(\tau)$$

$$= \begin{cases} \sum\limits_{l=2}^{n}\sum\limits_{h,j} d_{lhj} \left\{\exp\left[-\mathrm{i}\left(j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}\right)\tau\right]\right. \\ \qquad \times \exp\left[\mathrm{Re}\left\{\mu_l\right\}|\tau|\right] - \exp\left[-\mathrm{i}h\omega_0\tau\right]\right\} \\ \qquad \times \exp\left[-\dfrac{1}{2}h^2\omega_0^2 c|\tau|\right] \qquad\quad \text{if } \tau \geq 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\quad \text{if } \tau < 0 \end{cases} \tag{90}$$

$$R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{x}_S)\infty}(\tau)$$

$$= \begin{cases} 0 \qquad\qquad\qquad\qquad\qquad\quad \text{if } \tau \geq 0 \\ \sum\limits_{l=2}^{n}\sum\limits_{h,j} d_{lhj}^* \left\{\exp\left[-\mathrm{i}\left(j\omega_0 + \mathrm{Im}\left\{\mu_l\right\}\right)\tau\right]\right. \\ \qquad \times \exp\left[\mathrm{Re}\left\{\mu_l\right\}|\tau|\right] - \exp\left[-\mathrm{i}h\omega_0\tau\right]\right\} \\ \qquad \times \exp\left[-\dfrac{1}{2}h^2\omega_0^2 c|\tau|\right] \qquad\quad \text{if } \tau < 0 \end{cases} \tag{91}$$

where

$$d_{lhj} = \tilde{\Gamma}_h \tilde{\mathbf{V}}_{1_0}^T \tilde{\mathbf{\Lambda}}_{l_{h-j}}^* \Psi_{l_j}^* \frac{\mathrm{i}h\omega_0}{-\mu_l^* - \mathrm{i}(h-j)\omega_0} \tag{92}$$

and $\Psi_{l_j}^*$ is the j-th harmonic amplitude of $\mathbf{J}_\gamma(t)\mathbf{u}_l(t)$.

Proof: Using (76), we find

$$R_{\gamma,\gamma}^{(\mathbf{x_S},\mathbf{y})}(t,\tau) = \sum_{l=2}^{n} \sum_{j,h,k} \tilde{\Gamma}_h \exp\left[ih\omega_0 t + (-\mathrm{i}j\omega_0 + \mu_l^*)(t+\tau)\right]$$

$$\times \int_0^{t+\tau} \exp\left[(-\mu_l^* - \mathrm{i}k\omega_0)r\right]$$

$$\times \mathrm{E}\left\{\exp\left[\mathrm{i}h\omega_0\alpha(t) - \mathrm{i}\omega_0(j+k)\alpha(t+\tau)\right]\mathbf{b}^{\mathrm{T}}(r)\right\}\,\mathrm{d}r$$

$$\times \tilde{\mathbf{\Lambda}}_{l_{h-j}}^* \Psi_{l_j}^*. \tag{93}$$

Performing calculations similar to those in Appendix B we find (90).

A similar procedure yields also (91). ∎

Concerning the orbital noise autocorrelation function, we prove the following:

Lemma D.2: The asymptotic value for $t \to +\infty$ of the correlation function $R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{y})}(t,\tau) = \mathrm{E}\left\{\mathbf{J}_\gamma(t+\alpha(t)\mathbf{y}(t)\mathbf{y}^{\mathrm{T}}(t+\tau)\mathbf{J}_\gamma^{\mathrm{T}}(t+\tau+\right.$ of an output variable of the oscillator is given by

$$R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{y})\infty}(\tau)$$

$$= \begin{cases} \displaystyle\sum_{l=2}^{n}\sum_{h,j} c_{lhj}\exp\left[-\mathrm{i}\left(j\omega_0 + \mathrm{Im}\{\mu_l\}\right)\tau\right] \\ \qquad\times \exp\left[\mathrm{Re}\{\mu_l\}|\tau|\right]\exp\left[-\dfrac{1}{2}h^2\omega_0^2 c|\tau|\right] & \text{if } \tau \geq 0 \\ \displaystyle\sum_{l=2}^{n}\sum_{h,j} c_{lhj}^*\exp\left[-\mathrm{i}\left(j\omega_0 + \mathrm{Im}\{\mu_l\}\right)\tau\right] \\ \qquad\times \exp\left[\mathrm{Re}\{\mu_l\}|\tau|\right]\exp\left[-\dfrac{1}{2}h^2\omega_0^2 c|\tau|\right] & \text{if } \tau < 0 \end{cases} \tag{94}$$

where

$$c_{lhj} = \sum_{l'=2}^{n}\sum_{j'} \frac{1}{\mathrm{i}(j-j')\omega_0 - \mu_{l'} - \mu_l^*}\Psi_{l'_{j'}}\tilde{\mathbf{\Lambda}}_{l'_{h-j'}}^{\mathrm{T}}\tilde{\mathbf{\Lambda}}_{l_{h-j}}^*\Psi_{l_j}^*. \tag{95}$$

Proof: The proof is easily carried out expressing the variables to be included into the expectation operator according to their Fourier series, and performing calculations similar to those in Appendix C. ∎

Finally, we can calculate the spectrum of the output variable according to the Theorem below.

Theorem D.1: The spectrum of $\gamma(\mathbf{z}(t)) = \gamma(\mathbf{x_S}(t+\alpha(t))) + \mathbf{J}_\gamma(t+\alpha(t))\mathbf{y}(t)$ is determined by the asymptotic behaviour of $R_{\gamma,\gamma}(t,\tau)$ for $t \to +\infty$. All the nontrivial cyclostationary components are zero, while the stationary part of the spectrum is given by

$$S_{\gamma,\gamma}(\omega) = S_{\gamma,\gamma}^{(\mathbf{x_S},\mathbf{x_S})}(\omega) + S_{\gamma,\gamma}^{(\mathrm{corr})}(\omega) + S_{\gamma,\gamma}^{(\mathbf{y},\mathbf{y})}(\omega) + \left|\tilde{\Gamma}_0\right|^2\delta(\omega), \tag{96}$$

where ω is the (angular frequency) variable conjugated to τ, and the partial spectra are, respectively, the Fourier transforms of the asymptotic correlation functions $R_{\gamma,\gamma}^{(\mathbf{x_S},\mathbf{x_S})\infty}(\tau)$, $R_{\gamma,\gamma}^{(\mathbf{x_S},\mathbf{y})\infty}(\tau) + R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{x_S})\infty}(\tau)$ and $R_{\gamma,\gamma}^{(\mathbf{y},\mathbf{y})\infty}(\tau)$

calculated in Lemma D.1 and D.2

$$S_{\gamma,\gamma}^{(\mathbf{x_s},\mathbf{x_s})}(\omega) = \sum_h \left|\tilde{\Gamma}_h\right|^2 \frac{h^2\omega_0^2 c}{\Xi_h^2(\omega)} \tag{97}$$

$$S_{\gamma,\gamma}^{(\mathbf{y},\mathbf{y})}(\omega) = \sum_{l=2}^n \sum_{h,j}$$

$$2\left\{\frac{\text{Re}\left\{c_{lhj}\right\}\left[\frac{1}{2}h^2\omega_0^2 c - \text{Re}\left\{\mu_l\right\}\right]}{\Delta_{lhj}^2(\omega)}\right.$$

$$\left. + \frac{\text{Im}\left\{c_{lhj}\right\}\left[j\omega_0 + \text{Im}\left\{\mu_l\right\} + \omega\right]}{\Delta_{lhj}^2(\omega)}\right\} \tag{98}$$

$$S_{\gamma,\gamma}^{(\text{corr})}(\omega) = \sum_{l=2}^n \sum_{h,j}$$

$$2\left\{\frac{\text{Re}\left\{d_{lhj}\right\}\left[\frac{1}{2}h^2\omega_0^2 c - \text{Re}\left\{\mu_l\right\}\right]}{\Delta_{lhj}^2(\omega)}\right.$$

$$+ \frac{\text{Im}\left\{d_{lhj}\right\}\left[j\omega_0 + \text{Im}\left\{\mu_l\right\} + \omega\right]}{\Delta_{lhj}^2(\omega)}$$

$$\left. - \frac{\text{Re}\left\{d_{lhj}\right\}\frac{1}{2}h^2\omega_0^2 c + \text{Im}\left\{d_{lhj}\right\}\left[h\omega_0 + \omega\right]}{\Xi_h^2(\omega)}\right\}. \tag{99}$$

Proof: See Lemma D.1, Lemma D.2 and the proof of Lemma 8.5 in [1]. ∎

REFERENCES

[1] A. Demir, A. Mehrotra, J. Roychowdhury, "Phase noise in oscillators: A unifying theory and numerical methods for characterization," *IEEE Trans. Circ. Syst. I: Fund. Theory and App.*, vol. 47, no. 5, pp. 655–674, May 2000.

[2] A. Hajimiri, T.H. Lee, *The design of low noise oscillators,* Dordrect: Kluwer Academic Publishers, 1999.

[3] T.H. Lee, A. Hajimiri, "Oscillator phase noise: a tutorial," *IEEE J. Solid State Circ.*, vol. 35, no. 3, pp. 326–336, Mar. 2000.

[4] A. Demir, A. Sangiovanni-Vincentelli, *Analysis and simulation of noise in nonlinear electronic circuits and systems*, Heidelberg: Springer Verlag, 1997.

[5] H. Risken, *The Fokker-Planck equation*, Second Ed., Heidelberg: Springer Verlag, 1989.

[6] A. Brambilla, G. Gruosso, M.A. Redaelli, G. Storti Gajani, D.D. Caviglia, "Improved small signal analysis for circuits working in periodic steady state," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 2, pp. 427–437, Feb. 2010.

[7] D. Murphy, J.J. Rael, A.A. Abidi, "Phase noise in *LC* oscillators: a phasor-based analysis of a general result and of loaded *Q*," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 6, pp. 1187–1203, June 2010.

[8] A. Mirzaei, A.A. Abidi, "The spectrum of a noisy free-running oscillator explained by random frequency pulling," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 3, pp. 642–653, Mar. 2010.

[9] A. Demir, "Phase noise and timing jitter in oscillators with colored-noise sources," *IEEE Trans. Circ. Syst. I: Fund. Theory and App.*, vol. 49, no. 12, pp. 1782–1791, Dec. 2000.

[10] F. X. Kaertner, "Determination of the correlation spectrum of oscillators with low noise", *IEEE Trans. Microwave Theory and Techniques*, vol. 37, no. 1, pp.90-101, Jan. 1989.

[11] F. X. Kaertner, "Analysis of white and $f^{-\alpha}$ noise in oscillators", *Int. J. Circ. Theory App.*, vol. 18, pp. 485–519, 1990.

[12] T. Djurhuus, V. Krozer, J. Vidkjaer, T.K. Johansen, "Oscillator phase noise: a geometrical approach," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 56, no. 7, pp. 1373–1382, Jul. 2009.

[13] A. Hajimiri, T.H. Lee, "A general theory of phase noise in electrical oscillators," *IEEE J. Solid State Circ.*, vol. 33, no. 2, pp. 179–194, Feb. 1998.

[14] S. Srivastava, J. Roychowdury, "Analysitcal equations for nonlinear phase errors and jitter in ring oscillators," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 54, no. 10, pp. 2321–2329, Oct. 2007.

[15] G.J. Coram, "A simple 2-D oscillator to determine the correct decomposition of perturbations into amplitude and phase noise," *IEEE Trans. Circuits and Systems I*, Vol. 48, pp. 896–898, July 2001.

[16] J.P. Gleeson, F. O'Doherty, M.P. Kennedy, "Non-Lorentzian line shape in noisy self-sustained oscillators", Boole centre for Research in Informatics Preprint No.01/2005, February 2005.

[17] F. O'Doherty, J.P. Gleeson, "Amplitude-phase coupling effects on the spectral lineshape of oscillators", Proc. of *Fluctuations and Noise – Noise in Communication Systems*, C.N. Georghiades, L.B. White Eds., pp. 31-41, Proc. of SPIE Vol. 5847, 2005.

[18] R.S. Swain, J.P. Gleeson, M.P. Kennedy, "Influence of noise intensity on the spectrum of an oscillator," *IEEE Trans. Circuits and Systems II*, Vol. 52, pp. 789–793, Nov. 2005.

[19] A. Carbone, F. Palma, "Considering noise orbital deviations on the evaluation of power density spectrum of oscillators," *IEEE Trans. Circ. Syst. II: Express Briefs*, vol. 53, no. 6, pp. 438–442, June 2006.

[20] F.L. Traversa, F. Bonani, "Asymptotic stochastic characterization of phase and amplitude noise in free-running oscillators", accepted for publication in *Fluctuations and Noise Letters*, 2011.

[21] A. Mehrotra, A. Sangiovanni-Vincentelli, *Noise analysis of radiofrequency circuits*, London: Kluwer Academic Publishers, 2004.

[22] K. Sobczyk, *Stochastic Differential Equations*, London: Kluwer Academic Publishers, 1991.

[23] A. Demir, "Floquet theory and non-linear perturbation analysis for oscillators with differential-algebraic equations," *Int. J. Circ. Theory App.*, vol. 28, pp. 163–185, 2000.

[24] H. Thibieroz, *Using Spectre RF Noise-Aware PLL Methodology to Predict PLL Behavior Accurately*, http://www.cadence.com/rl/Resources/conference_papers/ctp_cdnlivesv2007_Thibieroz.pdf.

[25] A.A. Sweet, *Designing bipolar transistor radio frequency integrated circuits,* Norwood: Artech House, 2008.

[26] F.L. Traversa, F. Bonani, S. Donati Guerrieri, "A frequency-domain approach to the analysis of stability and bifurcations in nonlinear systems described by differential-algebraic equations", *Int. J. Circuit Theory & App.*, Vol. 36, No. 4, pp. 421–439, June 2008.

[27] K. S. Kundert, A. Sangiovanni-Vincentelli, J. K. White, *Steady-state methods for simulating analog and microwave circuits,* Kluwer Academic Publisher, Boston, 1990.

[28] A. Brambilla, G. Gruosso, M.A. Redaelli, G. Storti Gajani, D.D. Caviglia, "Improved small signal analysis for circuits working in periodic steady state," *IEEE Trans. Circ. Syst. I: Regular Papers*, vol. 57, no. 2, pp. 427–437, Feb. 2010.
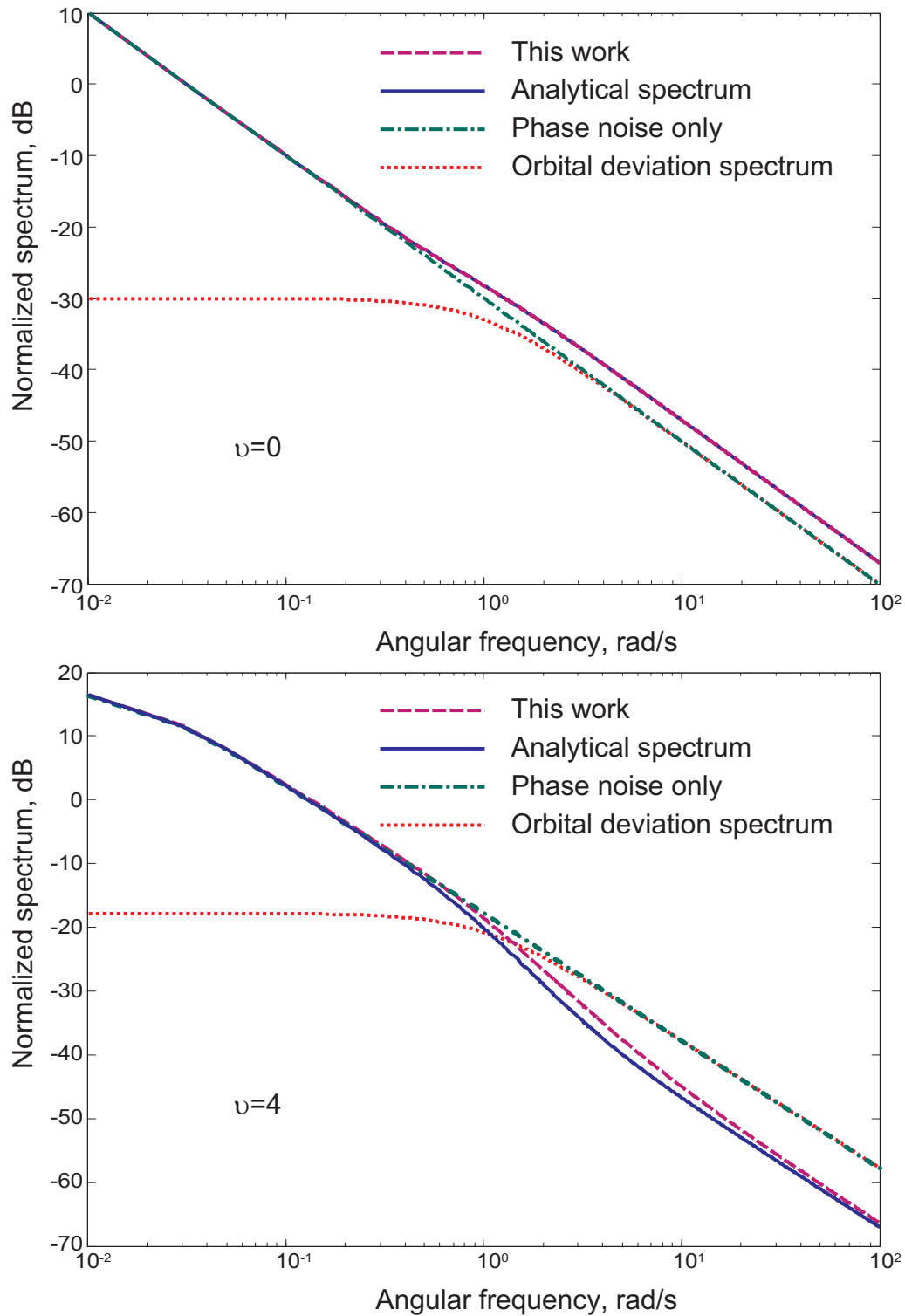
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Fig. 1. Spectrum of the $x(t)$ variable for the oscillator in [16]: $\upsilon = 0$ (above: the analytical spectrum and our result are superimposed) and $\upsilon = 4$ (below).
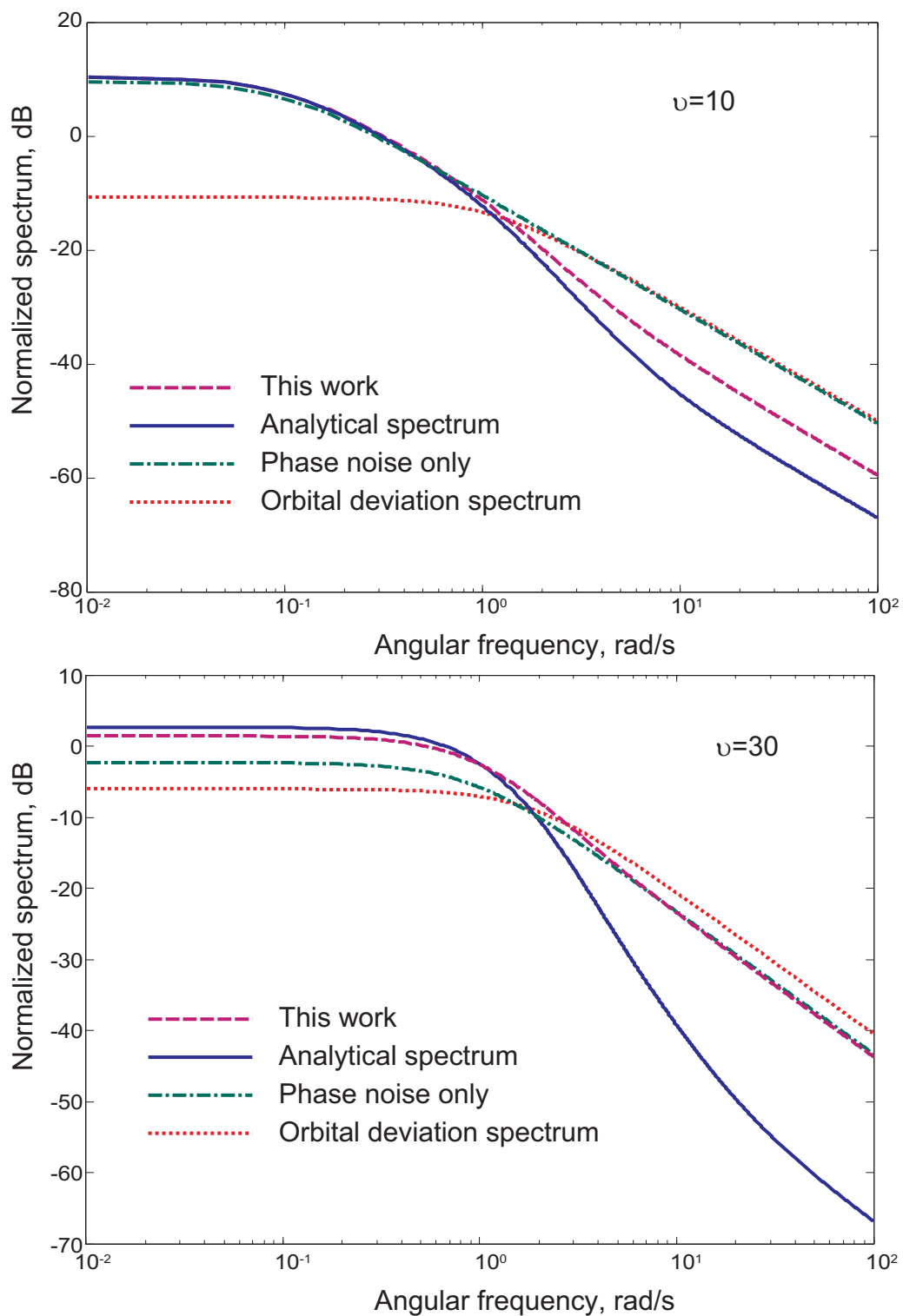
Fig. 2. Spectrum of the $x(t)$ variable for the oscillator in [16]: $\upsilon = 10$ (above) and $\upsilon = 30$ (below).
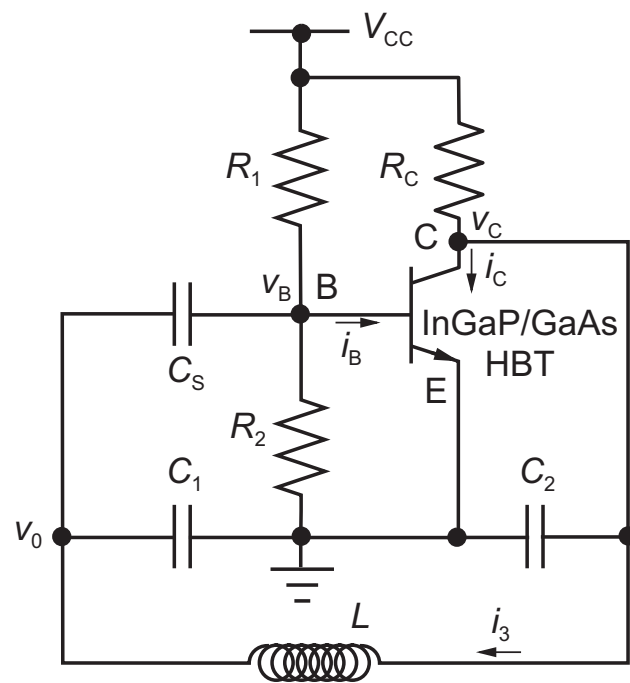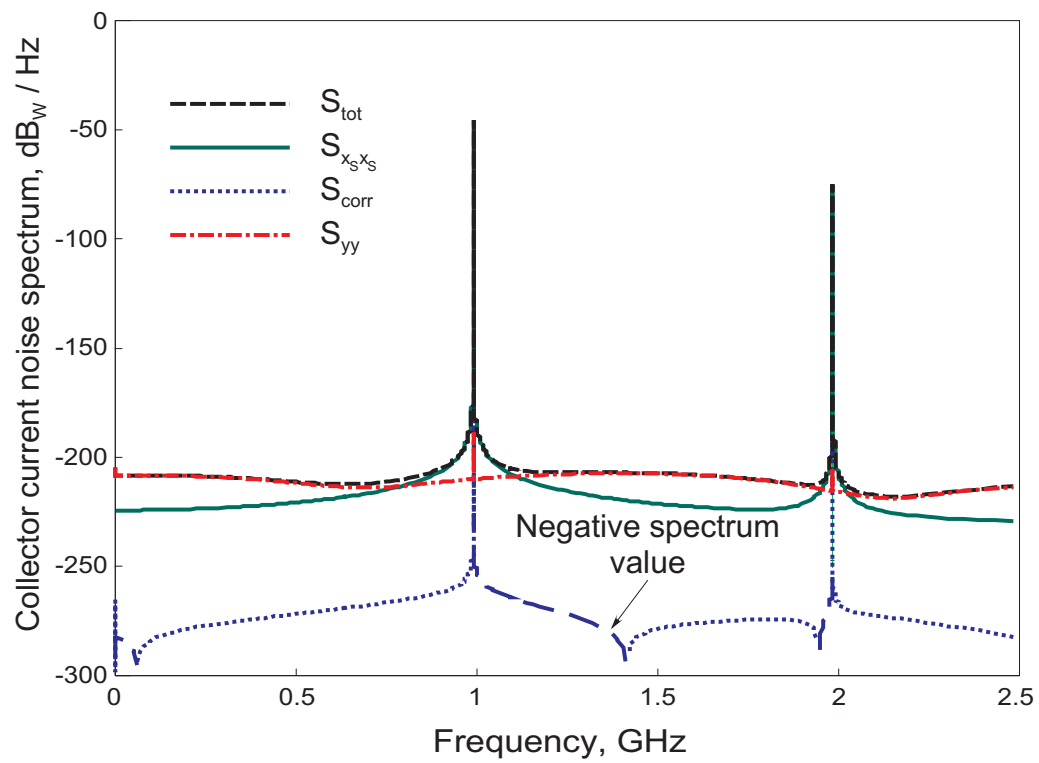
Fig. 3.   Circuit of the Colpitts oscillator.

Fig. 4.   Collector current noise spectrum of the Colpitts oscillator as a function of the absolute frequency.
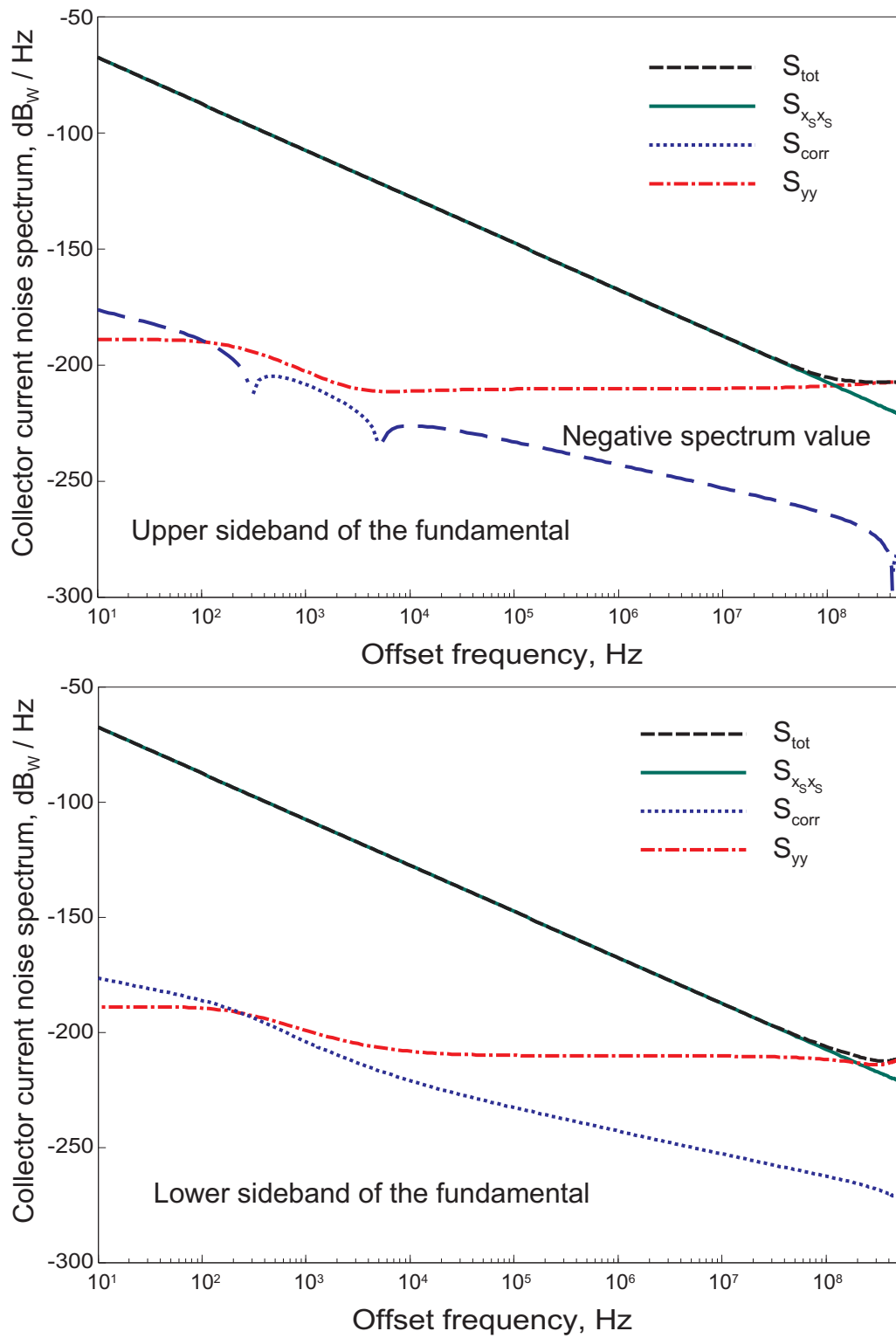
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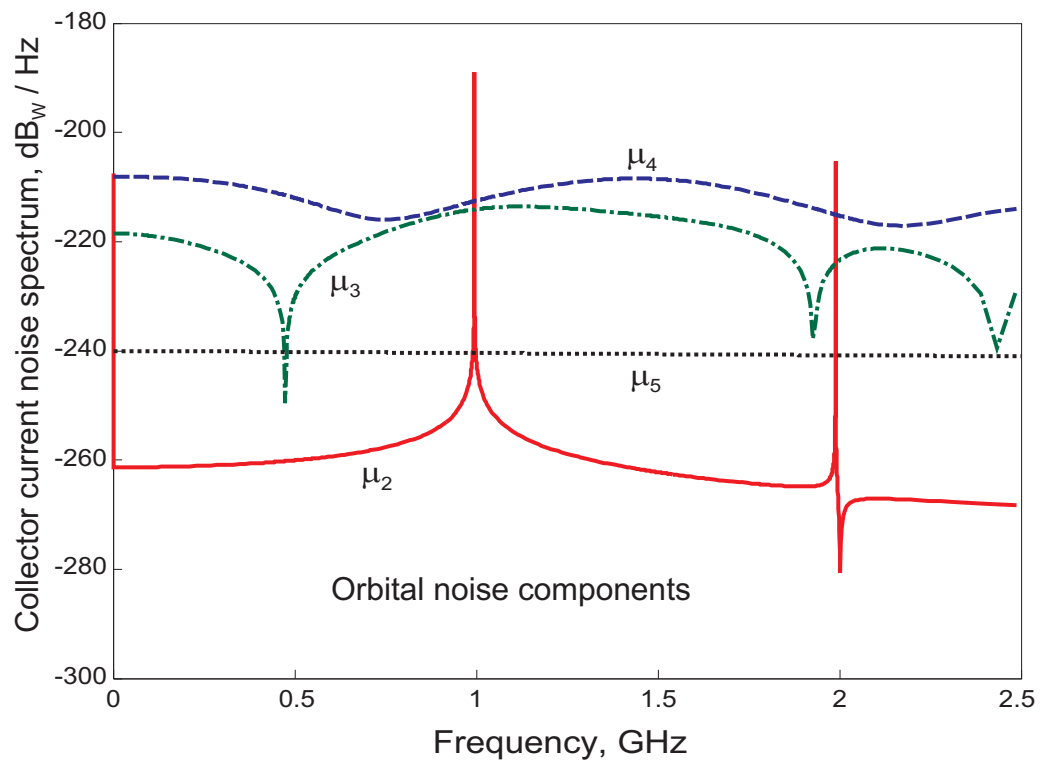
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