Cramér-Rao Bound for Hybrid GNSS-Terrestrial Cooperative Positioning

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***Abstract*—In this contribution we derive an expression of the Cramér-Rao bound for hybrid cooperative positioning, where GNSS information is combined with terrestrial range measurements through exchange of peer-to-peer messages. These results provide a theoretical characterization of achievable performance of hybrid positioning schemes, as well as allow to identify critical network configurations and devise optimized node placement strategies.**

***Index Terms*—Cooperative positioning, GNSS, GPS**

I. Introduction

Cooperative positioning methods (e.g., [1]) have been recently proposed for wireless networks operating in critical environments where GNSS is not available. However, such cooperative schemes can be also used in combination with GNSS, so as to improve localization accuracy in cases where satellite measurements are available intermittently, or from a limited number of satellites, or are strongly affected by noise or multi-path. "Hybrid cooperative positioning" schemes can thus be designed, where heterogeneous information obtained from satellites, anchors, and peers is fused together.

In this contribution, we analyze the theoretical performance limits of hybrid positioning, by expressing the Cramér-Rao lower bound (CRLB), i.e., the best possible positioning accuracy achievable by any algorithm in the aforementioned scenario. These results extend and complement previous CRLB analyses derived for cooperative-only positioning in WSNs [2] and wideband cooperative localization [3]. Compared to a purely cooperative scenario [3], the hybrid one involves one more variable – the bias – which increases the dimensionality of the problem. Compared to [2], it takes into account satellites, in addition to terrestrial devices.

The paper is organized as follows: Sec. II illustrates model and problem statement; in Sec. III expressions for the Fisher information matrix are derived for the non-cooperative and the cooperative case; numerical results and examples are presented in Sec. IV.

II. Problem Formulation

Consider a network with N nodes, of which S satellite nodes with known clock bias and known position, A anchor nodes with known position but unknown clock bias, and $M = N - S - A$ agents with unknown clock bias and unknown position.

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Let $\mathcal{M}$ be the set of agents, $\mathcal{S}$ the set of satellites, $\mathcal{A}$ the set of anchors; denote by $\mathcal{S}_m$ the set of satellites agent m can see, by $\mathcal{A}_m$ the set of anchors agent m can communicate with, and by $\mathcal{M}_m$ the set of peers agent m can communicate with. Position of satellite $s \in \mathcal{S}$, of anchor $a \in \mathcal{A}$, and of agent $m \in \mathcal{M}$, are indicated respectively by $\mathbf{x}_s$, $\mathbf{x}_a$, $\mathbf{x}_m$. The dimension of position vectors, indicated by D, may be 2 or 3. The variable b_m represents the clock bias of agent m, expressed in distance units.

Three types of measurements are available:

- $r_{a\to m}$ is the measured distance between agent m and anchor $a \in \mathcal{A}_m$:
$$r_{a\to m} = \|\mathbf{x}_a - \mathbf{x}_m\| + v_{a\to m}, \quad (1)$$
where $v_{a\to m}$ is measurement noise with variance $\sigma^2_{a\to m}$.
- $r_{n\to m}$ is a peer-to-peer distance measurement between nodes m and $n \in \mathcal{M}_m$:
$$r_{n\to m} = \|\mathbf{x}_n - \mathbf{x}_m\| + v_{n\to m}, \quad (2)$$
where $v_{n\to m}$ is measurement noise with variance $\sigma^2_{n\to m}$.
- $\rho_{s\to m}$ is a pseudorange measurement between node m and satellite $s \in \mathcal{S}_m$:
$$\rho_{s\to m} = \|\mathbf{x}_s - \mathbf{x}_m\| + b_m + v_{s\to m}, \quad (3)$$
where $v_{s\to m}$ is measurement noise with variance $\sigma^2_{s\to m}$.

We will assume that all measurement noise is zero-mean Gaussian; for peer-to-peer measurements, the link variance is symmetric: $\sigma^2_{n\to m} = \sigma^2_{m\to n}$.

Our goal is to compute the CRLB of the deterministic unknown $[\mathbf{X}, \mathbf{b}]$, where $\mathbf{X} = \{\mathbf{x}_{m\in\mathcal{M}}\}$ and $\mathbf{b} = \{b_{m\in\mathcal{M}}\}$, as a function of the (range and pseudorange) measurement noise statistics and the network geometry.

III. Fisher Information Matrix

The CRLB of any unbiased estimator of $[\mathbf{X}, \mathbf{b}]$ is obtained by inverting the corresponding Fisher information matrix (FIM). Let $\mathbf{F}$ be the FIM for our hybrid scenario. We will first compute the FIM under a non-cooperative setting, and then extend this result to the cooperative case.

A. Non-cooperative Case

We focus on a single agent, say m. The log-likelihood function of its measurements with respect to anchors and satellites is

$$\begin{aligned}&\log p\left(\{r_{a\to m}\}_{a\in\mathcal{A}_m},\{\rho_{s\to m}\}_{s\in\mathcal{S}_m}|\mathbf{x}_m,b_m\right)\\&=\sum_{a\in\mathcal{A}_m}\log p\left(r_{a\to m}|\mathbf{x}_m\right)+\sum_{s\in\mathcal{S}_m}\log p\left(\rho_{s\to m}|\mathbf{x}_m,b_m\right)\\&\doteq\Lambda_m\left(\mathbf{x}_m,b_m\right).\end{aligned}$$

Under Gaussian measurement noise:

$$\log p\left(r_{a\to m}|\mathbf{x}_m\right)=C-\frac{\left|r_{a\to m}-\|\mathbf{x}_a-\mathbf{x}_m\|\right|^2}{2\sigma^2_{a\to m}}$$

and

$$\log p\left(\rho_{s\to m}|\mathbf{x}_m,b_m\right)=C'-\frac{\left|\rho_{s\to m}-\|\mathbf{x}_s-\mathbf{x}_m\|-b_m\right|^2}{2\sigma^2_{s\to m}},$$

where C, C' are constant terms. The Fisher information matrix is given by

$$\mathbf{F}_m=-\mathbb{E}\left\{H_m\left(\Lambda_m\left(\mathbf{x}_m,b_m\right)\right)\right\},$$

where the expectation is with respect to the measurements, and $H_m(\cdot)$ is the Hessian operator containing the second-order partial derivatives with respect to each element of $[\mathbf{x}_m,b_m]$. $\mathbf{F}_m$ is a $(D+1)\times(D+1)$ matrix:

$$\mathbf{F}_m=\begin{bmatrix}\mathbf{F}_{\mathbf{x}_m} & \mathbf{f}_{\mathbf{x}_m,b_m}\\ \mathbf{f}^T_{\mathbf{x}_m,b_m} & F_{b_m}\end{bmatrix}\succeq 0, \quad (4)$$

where

$$\begin{aligned}\mathbf{F}_{\mathbf{x}_m}&=\sum_{a\in\mathcal{A}_m}\frac{1}{\sigma^2_{a\to m}}\mathbf{q}_{am}\mathbf{q}^T_{am}+\sum_{s\in\mathcal{S}_m}\frac{1}{\sigma^2_{s\to m}}\mathbf{q}_{sm}\mathbf{q}^T_{sm}\\F_{b_m}&=\sum_{s\in\mathcal{S}_m}\frac{1}{\sigma^2_{s\to m}}\\\mathbf{f}_{\mathbf{x}_m,b_m}&=\sum_{s\in\mathcal{S}_m}-\frac{1}{\sigma^2_{s\to m}}\mathbf{q}_{sm},\end{aligned}$$

in which $\mathbf{q}_{im}=\frac{\mathbf{x}_i-\mathbf{x}_m}{\|\mathbf{x}_i-\mathbf{x}_m\|}$ is a unit-length column vector between $\mathbf{x}_m$ and $\mathbf{x}_i$.

Considering all M agents, the global non-cooperative FIM is a block-diagonal matrix:

$$\mathbf{F}_{\text{non-coop}}=\begin{bmatrix}\mathbf{F}_1 & & & \\ & \mathbf{F}_2 & & \\ & & \ddots & \\ & & & \mathbf{F}_M\end{bmatrix}. \quad (5)$$

B. Cooperative Case

The log-likelihood function is now

$$\begin{aligned}&\log p\Big(\big\{\{r_{a\to m}\}_{a\in\mathcal{A}_m},\{\rho_{s\to m}\}_{s\in\mathcal{S}_m},\\&\quad\{r_{n\to m}\}_{n\in\mathcal{M}_m}\big\}_{m\in\mathcal{M}}|\mathbf{X},\mathbf{b}\Big)\\&=\sum_{m\in\mathcal{M}}\Lambda_m\left(\mathbf{x}_m,b_m\right)+\underbrace{\sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{M}_m}\log p\left(r_{n\to m}|\mathbf{x}_m\right)}_{\doteq\Lambda_{\text{coop}}(\mathbf{X})}.\end{aligned}$$

The Fisher information matrix is of the form

$$\mathbf{F}=\mathbf{F}_{\text{non-coop}}+\mathbf{F}_{\text{coop}} \quad (6)$$

and has dimension $(D+1)M\times(D+1)M$. The first term $\mathbf{F}_{\text{non-coop}}$, representing the non-cooperative contribution, is again (5). The cooperative part $\mathbf{F}_{\text{coop}}$ can be expressed as

$$\mathbf{F}_{\text{coop}}=-\mathbb{E}\left\{\begin{bmatrix}H_{11} & \dots & H_{M1}\\ \vdots & \ddots & \vdots\\ H_{M1} & \dots & H_{MM}\end{bmatrix}\Lambda_{\text{coop}}\left(\mathbf{X}\right)\right\}$$

where the cross-Hessian matrices H_{mn} are defined as (assuming $\mathbf{x}_i=[x_{1,i},\dots,x_{D,i}]$):

$$H_{mn}\doteq\begin{bmatrix}\frac{\partial^2}{\partial x_{1,m}\partial x_{1,n}} & \cdots & \frac{\partial^2}{\partial x_{1,m}\partial x_{D,n}} & \frac{\partial^2}{\partial x_{1,m}\partial b_n}\\ \vdots & \ddots & \vdots & \vdots\\ \frac{\partial^2}{\partial x_{1,m}\partial x_{D,n}} & \cdots & \frac{\partial^2}{\partial x_{D,m}\partial x_{D,n}} & \frac{\partial^2}{\partial x_{D,m}\partial b_n}\\ \frac{\partial^2}{\partial x_{1,m}\partial b_n} & \cdots & \frac{\partial^2}{\partial x_{D,m}\partial b_n} & \frac{\partial^2}{\partial b_m\partial b_n}\end{bmatrix}$$

Notice that $\Lambda_{\text{coop}}(\mathbf{X})$ does not depend on the bias. Under the hypothesis of Gaussian measurement noise in peer-to-peer communication,

$$\log p\left(r_{n\to m}|\mathbf{x}_m\right)=C''-\frac{\left|r_{n\to m}-\|\mathbf{x}_n-\mathbf{x}_m\|\right|^2}{2\sigma^2_{n\to m}},$$

leading to a block matrix of the form

$$\mathbf{F}_{\text{coop}}=\begin{bmatrix}\mathbf{F}'_1 & \mathbf{0} & \mathbf{K}_{12} & \mathbf{0} & \dots & \mathbf{K}_{1M} & \mathbf{0}\\ \mathbf{0}^T & 0 & \mathbf{0}^T & 0 & & \mathbf{0}^T & 0\\ \mathbf{K}_{21} & \mathbf{0} & \mathbf{F}'_2 & \mathbf{0} & & & \\ \mathbf{0}^T & 0 & \mathbf{0}^T & 0 & & & \\ \vdots & & & & \ddots & & \\ \mathbf{K}_{M1} & \mathbf{0} & & & & \mathbf{F}'_M & \mathbf{0}\\ \mathbf{0}^T & 0 & & & & \mathbf{0}^T & 0\end{bmatrix}\succeq 0. \quad (7)$$

where

$$\begin{aligned}\mathbf{F}'_m&=\sum_{n\in\mathcal{M}_m}\frac{1}{\sigma^2_{n\to m}}\mathbf{q}_{nm}\mathbf{q}^T_{nm}\\\mathbf{K}_{mn}&=\begin{cases}-\frac{1}{\sigma^2_{n\to m}}\mathbf{q}_{nm}\mathbf{q}^T_{nm}, & \text{if } n\in\mathcal{M}_m\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

and $\mathbf{0}$ is a $D\times 1$ zero-vector.

The above results allow to compute $\mathbf{F}$ for a given network configuration and, by inverting (6), to express the CRLB.

IV. Numerical Results

The analytical results derived in the previous section are now illustrated by a practical example. Consider the network depicted in Fig. 1, with six agents arranged in a star topology. Each agent can communicate with two neighbors, except agent 6, located in the center, that can communicate with all other agents. Agent 1 has visibility of all satellites; agent 2 can see four (the minimum number needed to estimate position and bias unambiguously); agents 3, 4, 5, and 6, on the contrary, are only connected to three, two, one, and no satellites, respectively. This configuration is representative of a network located in an indoor environment, where only agents close to windows or outer walls can receive satellite measurements.

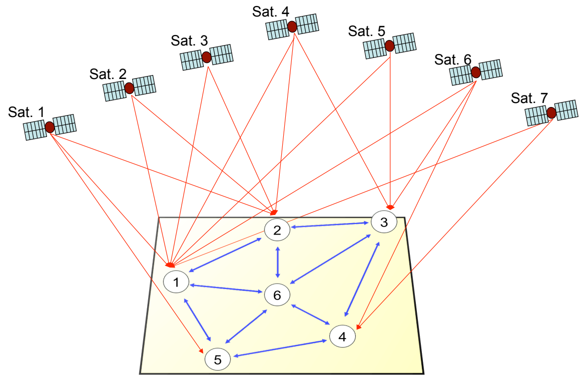

Figure 1. Example network topology.

Position of agent 6 (45.06° lat., 7.66° long., 311.96 m height) is taken as the origin of the reference system; the relative positions of the other agents, expressed in east-north-up (ENU) coordinates, are:

Agent no.	1	2	3	4	5	6
E [m]	-50	0	50	30	-30	0
N [m]	10	30	40	-20	-40	0
U [m]	0.27	0.92	-1.13	0.43	0.15	0

Satellites' positions are drawn according to real GPS data for the considered location. Their values, again expressed in ENU coordinates with respect to agent 6, are:

Sat. no.	1	2	3	4	5	6	7
E [$\cdot 10^6$m]	-16.17	-9.18	-1.71	-13.97	14.28	22.95	-12.90
N [$\cdot 10^6$m]	-4.02	-18.36	-10.50	10.83	6.46	4.86	21.68
U [$\cdot 10^6$m]	14.02	10.78	18.15	13.31	15.01	5.83	2.44

The variance of pseudorange and range measurements is set, respectively, to $\sigma_{s\to m} = 5$ m $\forall m \in \mathcal{M}, s \in \mathcal{S}_m$ and $\sigma_{n\to m} = 0.10$ m [1] $\forall m \in \mathcal{M}, n \in \mathcal{M}_m$.

Under this setting, the CRLB is computed to compare the achievable positioning accuracy in the non-cooperative and in the hybrid scenario. Let $\mathbf{J}$ be the CRLB matrix obtained by inversion of $\mathbf{F}_{\text{non-coop}}$ (5) or $\mathbf{F}$ (6), after removing rows and columns corresponding to non-estimable variables[1], and denote by $\mathbf{J}_m$ the $(D+1)\times(D+1) = 4\times 4$ block of $\mathbf{J}$ corresponding to agent m. Then, the positioning accuracy for each agent m can be decomposed into: a *horizontal* component, i.e, the trace of the x-y block of $\mathbf{J}_m$,

$$\sigma_{\text{CRLB-hor}}(m) \doteq \sqrt{\mathbf{J}_m[1,1] + \mathbf{J}_m[2,2]},$$

a *vertical* component

$$\sigma_{\text{CRLB-vert}}(m) \doteq \sqrt{\mathbf{J}_m[3,3]},$$

and a *bias* component

$$\sigma_{\text{CRLB-bias}}(m) \doteq \sqrt{\mathbf{J}_m[4,4]}.$$

The unit of all components is meters.

These performance metrics, plotted in Fig. 2, illustrate the benefits arising from cooperation. With the exception of agent 1, which has full visibility of all the available GPS satellites, the other agents obtain a significant performance improvement

[1]Non-estimable variables are: positions and biases, for agents whose total number of connections is less than $D+1$; biases, for agents connected to no satellites. These variables generate matrix singularities, hence CRLB $\to \infty$.

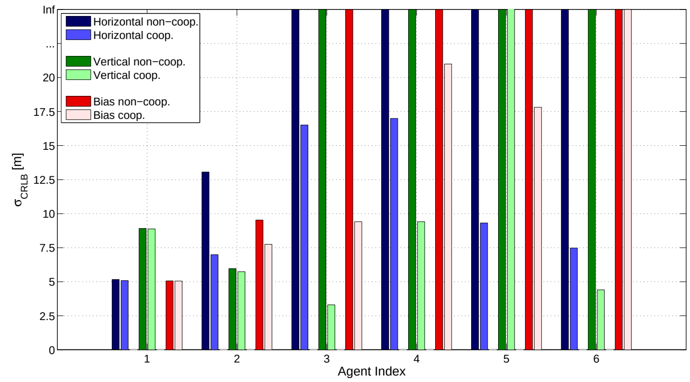

Figure 2. Comparison of position- and bias-CRLB in realistic 3D scenario: non-cooperative (GNSS only) vs. hybrid (GNSS + peer-to-peer communication) setting.

in the hybrid case. For agent 2, which sees four satellites, the CRLB reduces by one half; for agents 3, 4, and 5, the CRLB in the non-cooperative case is extremely large or infinite, while it takes relatively low values when peer-to-peer communication is introduced. Cooperation thus proves to be essential in GPS-challenged environments. Agent 6, finally, is able to estimate its position thanks to peer-to-peer information, but cannot estimate its bias in any case: at least one satellite connection is necessary, since range measurements do not carry any information about clock bias.

V. Conclusion and Further Work

The results derived in this paper give insight into the potential of implementing peer-to-peer cooperation protocols in combination with satellite-based positioning. Also, they provide a theoretical tool to evaluate the achievable positioning accuracy for a given network configuration, and can be used to detect *a priori* critical configurations or as a reference to compare the performance of practical positioning algorithms.

Related subjects of ongoing and future research are: development of practical, distributed algorithms for hybrid cooperative positioning, e.g., by extending the SPAWN algorithm [1] to combine both range and pseudorange measurements, and comparison of their performance with the CRLB.

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References

[1] H. Wymeersch, J. Lien, and M. Z. Win "Cooperative localization in wireless networks," *Proc. IEEE*, vol. 97, no. 2, Feb. 2009, pp. 427-450.

[2] E.G. Larsson, "Cramér-Rao Bound Analysis of Distributed Positioning in Sensor Networks," *IEEE Signal Processing Letters*, 2004, vol. 11, part 3, pp. 334-337

[3] Y. Shen, H. Wymeersch, and M.Z. Win, "Fundamental Limits of Wideband Cooperative Localization via Fisher Information", *IEEE Wireless Communications and Networking Conference*, pp.3951-3955, March 2007