Kähler immersions of the disc into polydiscs.

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Abstract

In this short note we give an example of a non trivial, i.e. non totally geodesic, Kähler immersion of a disc into a polydisc.

In this short note we give an example of a non trivial, i.e. non totally geodesic, Kähler immersion of a disc into a polydisc. This example is a counter-example of a conjecture posed in [CU03]. The author discover this example in 2007 [LA07]. In [M09] Mok produce similar examples to the ones contained in this note but using the half-plane model of the hyperbolic disc. A complete description of this non trivial maps can be found in [Ng09] which is the Ph.D. Thesis at Hong Kong University of Mok's student Sui Chung Ng.

Let $\Delta = (\Delta, \omega_{\text{hyp}})$ be the unit disc endowed with hyperbolic Kähler form given by the potential $N = -\log(1-|z|^2)$, i.e. $\omega_{\text{hyp}} := \frac{i}{2}\partial\overline{\partial}N$. The polidisc Δ^n is endowed with the Kähler form ω_{poly} given by the potential $\sum_{k=1}^{n} -\log(1-|z_k|^2)$.
Let $\xi \in S^1 := \{z \in \mathbb{C} : |z| = 1\}$ then the map $f_j(z) = (0, 0, \cdots, 0, \xi z_j, 0, \cdots, 0)$ is a Kähler embedding of Δ into Δ^n. Such embeddings or the composition of such an embedding with isometries of the disc or the polydisc are the so called *trivial* embeddings.
Let $z \to f(z) = (f_1(z), f_2(z))$ be a holomorphic immersion of the disc Δ into the bidisc $\Delta \times \Delta$. Then f is a Kähler map if and only if there exists $U \in U(2)$ such that:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = U \begin{pmatrix} z \\ f_1 f_2 \end{pmatrix} .$$

Let us call $\psi(z) := f_1(z)f_2(z)$. Let U be the following matrix :

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Then

$$\sqrt{2} f_1 = z + \psi \ ,$$
$$\sqrt{2} f_2 = z - \psi \ .$$

So

$$2\psi = z^2 - \psi^2$$

and then

$$\psi(z) = -1 + \sqrt{1 + z^2}\ .$$

Conversely, if ψ is as above then we can define f_1 and f_2 by the equations :

$$\sqrt{2} f_1 = z + \psi\ ,$$

$$\sqrt{2} f_2 = z - \psi\ .$$

The map $\Psi : z \hookrightarrow (f_1(z), f_2(z)) \in \mathbb{C}^2$ is well defined since there are no problems with the square root in the open disc $|z - 1| < 1$, i.e. we can take a good branch of the square root by deleting the negative axis.
The map $\Psi : z \hookrightarrow (f_1(z), f_2(z))$ is one to one since $f_1(z) + f_2(z) = \sqrt{2}z$.
To show that $\Psi(z) \in \Delta \times \Delta$ notice that for all $z \in \Delta$ we have:

$$0 < 1 - |z|^2 = (1 - |f_1(z)|^2)(1 - |f_2(z)|^2) < 1\ .$$

Observe that $f_1(0) = f_2(0) = 0$ so we get, by continuity reasons, that $(1 > |f_1(z)|^2)$ and $(1 > |f_2(z)|^2)$.
Notice that Ψ is actually an embedding since $\Psi(\Delta) = \{(z_1, z_2) : \sqrt{2}(z_1 - z_2) = 2z_1z_2\} \subset \Delta \times \Delta$.

Finally it is not hard to see that Ψ is a non trivial Kähler embedding of Δ into Δ^n.

References

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