Bivariate Aging Properties under Archimedean Dependence Structures

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Abstract

Let $\mathbf{X} = (X, Y)$ be a pair of lifetimes whose dependence structure is described by an Archimedean survival copula, and let $\mathbf{X}_t = [(X - t, Y - t)|X > t, Y > t]$ denotes the corresponding pair of residual lifetimes after time $t \geq 0$. Multivariate aging notions, defined by means of stochastic comparisons between $\mathbf{X}$ and $\mathbf{X}_t$, with $t \geq 0$, have been studied in Pellerey (2008), who considered pairs of lifetimes having the same marginal distribution. Here we present the generalizations of his results, considering both stochastic comparisons between $\mathbf{X}_t$ and $\mathbf{X}_{t+s}$ for all $t, s \geq 0$ and the case of dependent lifetimes having different distributions. Comparisons between two different pairs of residual lifetimes, at any time $t \geq 0$, are discussed as well.

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1 Introduction

Let X be a random variable, and for each real $t \in \{t : P\{X > t\} > 0\}$ let $X_t = [X - t \big| X > t]$ denotes a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is a lifetime of a device then X_t can be interpreted as the residual lifetime of the device at time t, given that the device is alive at time t. Several characterizations of aging notions of items, components or individuals by means of stochastic comparisons between the residual lifetimes X_0, X_t and X_{t+s}, with $t, t+s \in \{t : P\{X > t\} > 0\}$, have been considered and studied in literature. These characterizations serve a few purposes; they can be used when one wants to prove analytically that some random variable has an aging property, and they also throw a new light of understanding on the intrinsic meaning of the aging notions that are involved. Among others, the following well-known aging notion can be defined by comparisons among residual lifetimes: given a non–negative random lifetime X defined on $[0, +\infty)$ we say that

$$X \in \text{IFR [DFR]} \iff X_{t+s} \leq_{st} [\geq_{st}]\ X_t \text{ whenever } t, s \geq 0.$$

An exhaustive list of applications and properties of the *Increasing Failure Rate* (IFR) and *Decreasing Failure Rate* (DFR) notions may be found in Barlow and Proschan (1981). Here $\leq_{st}$ denotes the usual stochastic order (see below for definition, and Shaked and Shanthikumar, 2007, for details about this stochastic comparison).

For the same reasons as above, stochastic inequalities between the residual lifetimes of two different non-negative variables are commonly considered in reliability and survival analysis. In particular, considered two lifetimes X and Y, conditions for $X_t \leq_{st} Y_t$ for all $t \geq 0$, have been studied. See Shaked and Shanthikumar (1994) for a long list of applications of this stochastic comparison (commonly called *hazard rate order*).

Let us consider now a pair $\mathbf{X} = (X, Y)$ of non-negative random variables. Let

$$\overline{F}(x, y) = P(X > x, Y > y)$$

be the corresponding joint survival function, and let

$$\overline{G}_X(x) = \overline{F}(x, 0) = P(X > x) \quad \text{and} \quad \overline{G}_Y(x) = \overline{F}(0, x) = P(Y > x)$$

be the marginal univariate survival functions of X and Y, respectively. Assume that $\overline{F}$ is a continuous survival function which is strictly decreasing on each argument, and that $\overline{G}_X(0) = \overline{G}_Y(0) = 1$. Natural bivariate extensions of the IFR and DFR properties can be given recalling that different definitions of the usual stochastic order can be considered in the multivariate setting. In particular, the following two multivariate generalizations of the usual stochastic order are well–know (again, see Shaked and Shanthikumar, 2007, for details, properties and applications of these orders): given two bivariate random vectors $\mathbf{X}$ and $\mathbf{Y}$ we say that

(i) $\mathbf{X}$ is smaller than $\mathbf{Y}$ in usual stochastic order ($\mathbf{X} \leq_{st} \mathbf{Y}$) if, and only if, $\mathbf{E}[h(\mathbf{X})] \leq \mathbf{E}[h(\mathbf{Y})]$ for every non-decreasing function $h : \mathbb{R}^2 \to \mathbb{R}$ such that the two expectations exist;

(ii) $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the lower orthant order ($\mathbf{X} \leq_{lo} \mathbf{Y}$) if, and only if, $F_{\mathbf{X}}(x, y) \geq F_{\mathbf{Y}}(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

Note that $\mathbf{X} \leq_{st} \mathbf{Y}$ strictly implies $\mathbf{X} \leq_{lo} \mathbf{Y}$.

Let now $\mathbf{X}_t = [(X - t, Y - t)|X > t, Y > t]$ be the pair of the residual lifetimes at time $t \geq 0$, i.e., the pair of non–negative random variables having joint survival function

$$\overline{F}_t(x, y) = P(X > t + x, Y > t + y|X > t, Y > t) = \frac{\overline{F}(x + t, y + t)}{\overline{F}(t, t)}.$$

Bivariate generalizations of the IFR and DFR notions can be defined considering the stochastic inequalities

$$\mathbf{X}_{t+s} \leq_{st} [\geq_{st}] \mathbf{X}_t \ \text{ for all } \ t, s \geq 0. \tag{1.1}$$

and

$$\mathbf{X}_{t+s} \leq_{lo} [\geq_{lo}] \mathbf{X}_t \ \text{ for all } \ t, s \geq 0. \tag{1.2}$$

We will denote with $\mathcal{A}^{+}_{FR}$ $[\mathcal{A}^{-}_{FR}]$ the class of bivariate lifetimes that satisfy (1.1), and $\mathcal{A}^{w+}_{FR}$ $[\mathcal{A}^{w-}_{FR}]$ the class of bivariate lifetimes that satisfy (1.2) (here w means "weakly"). Also, one can consider the class $\mathcal{A}^0$ of bivariate lifetimes such that in (1.1) the equality $=_{st}$ (equality in law) holds for every $t, s \geq 0$. This last case is usually referred in the literature as *weak multivariate lack of memory property* (see, e.g., Ghurye and Marshall, 1984).

Conditions (1.1) and (1.2) are of course of interest in different fields of applied probability, like reliability and actuarial sciences. In reliability theory, in particular, they provide sufficient conditions for the usual stochastic comparison of two systems having the same coherent life function τ but builted using used components: in fact, for example, for every $t, s \geq 0$ one has $\tau(\mathbf{X}_{t+s}) \leq_{st} \tau(\mathbf{X}_t)$ if (1.1) holds, as follows from the fact that coherent functions are non–decreasing in their arguments (see also Theorem 6.B.16(a) in Shaked and Shanthikumar, 2007).

Even for the stochastic inequalities between residual lifetimes it is of course possible to consider multivariate generalizations. Reasoning as above one can interested, for example, in comparisons between the residual lifetimes of two vectors of lifetimes $\mathbf{X}$ and $\mathbf{Y}$ of the kind

$$\mathbf{X}_t \leq_{st} \mathbf{Y}_t \text{ for all } t \geq 0. \tag{1.3}$$

Similarly as above, inequalities as in (1.3) can be used to compare the lifetimes of systems builted using used components: for every coherent life function τ and for all $t \geq 0$ one has $\tau(\mathbf{X}_t) \leq_{st} \tau(\mathbf{Y}_t)$ if (1.3) holds. In insurance theory, they can be obviously used to compare the residual lifetimes of two pairs of ensured persons when the assumption of independence in the couples does not apply.

The aim of this paper is to describe conditions for the inequalities described above in the case that $\mathbf{X}$ and $\mathbf{Y}$ are bivariate vectors of lifetimes whose dependence structure is described by an Archimedean survival copula. In Sections 3 we will provide some conditions for $\mathbf{X}$ to satisfy (1.1) and (1.2) or to be in the no–aging class $\mathcal{A}^0$. The results presented here generalize the ones appeared in Pellerey (2008) where the same distribution for the margins X and Y is assumed, and where multivariate generalizations of the NBU and NWU aging notions are considered. In Section 4 some conditions for the stochastic comparison of two pairs of residual lifetimes at any time $t \geq 0$ will be provided as well.

2 Preliminaries

As pointed out in recent literature (see, e.g., Nelsen, 1999), the dependence structure of a bivariate vector $\mathbf{X}$ can be usefully described by its survival copula K, defined as

$$K(u, v) = \overline{F}(\overline{G}_X^{-1}(u), \overline{G}_Y^{-1}(v)),$$

where $(u, v) \in [0, 1] \times [0, 1]$. This function (which is unique under the assumption of continuity of F) together with the marginal survival functions $\overline{G}_X$ and $\overline{G}_Y$ allows for a different representation of $\overline{F}$ in terms of the triplet $(\overline{G}_X, \overline{G}_Y, K)$, useful to analyze dependence properties between X and Y. Survival copulas, instead of ordinary copulas, are in particular considered in reliability and actuarial sciences, where survival distributions instead of cumulative distributions are commonly studied.

Among survival copulas, particularly interesting is the class of Archimedean survival copulas: a survival copula is said to be *Archimedean* if it can be written as

$$K(u, v) = W(W^{-1}(u) + W^{-1}(v)) \ \forall u, v \in [0, 1] \tag{2.1}$$

for a suitable one-dimensional, continuous, strictly positive and strictly decreasing and convex survival function $W : \mathbb{R}^+ \to [0, 1]$ such that $W(0) = 1$. The inverse

W^{-1} of the function W is usually called the *generator* of the Archimedean survival copula K. As pointed out in Nelsen (1999), many standard survival copulas (such as the ones in Gumbel, Frank, Clayton and Ali-Mikhail-Haq families) are special cases of this class. Vectors of lifetimes having Archimedean survival copulas are of great interest in reliability and actuarial sciences, but also in many other applied contexts, being of this kind the dependence structure of frailty models (see Oakes, 1989). We refer the reader to Müller and Scarsini (2005) or Bassan and Spizzichino (2005a), and references therein, for details, properties and recent applications of Archimedean survival copulas.

In the following sections particular attention will be given to the case of *Clayton survival copulas*, i.e., when $W(x) = (x+1)^{-\theta}$ for $\theta \in [0, +\infty)$, or, equivalently, when

$$K(u, v) = max\{(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1)^{-\theta}, 0\}.$$

These copulas have been introduced by Clayton (1978), who applied them in epidemiology, and further considered in hydrology and credit risks problems by Cook and Johnson (1981) and Charpentier and Juri (2006), for example. Recently, properties and characterizations of Clayton copulas have been also studied in Sungur (2002) and Javid (2009).

It is important to observe that when the vector $\mathbf{X} = (X, Y)$ has an Archimedean survival copula then its joint survival function $\overline{F}$ can be written in the form

$$\overline{F}(x, y) = W(R_X(x) + R_Y(y)) \tag{2.2}$$

for two suitable continuous and strictly increasing functions $R_X, R_Y : [0, +\infty) \to [0, +\infty)$ such that $R_X(0) = R_Y(0) = 0, \lim_{x\to\infty} R_X(x) = \lim_{y\to\infty} R_Y(y) = \infty$, where W is the survival function appearing in (2.1) (see Bassan and Spizzichino, 2005b, and references therein for details). For example, in the frailty approach it is assumed that X and Y are independent conditionally on some random environmental factor Θ, having conditional survival marginals $\overline{G}_{X,\theta}(t) = \mathbb{P}[X > t|\Theta = \theta] = \overline{H}_X(t)^\theta$ for some survival function $\overline{H}_X$ (and similarly for $\overline{G}_{Y,\theta}(t)$). Thus, for this model,

$$\begin{aligned}\overline{F}(t, s) &= \mathbf{E}[\overline{H}_X(t)^\Theta \overline{H}_Y(s)^\Theta] = \mathbf{E}[\exp(\Theta(\ln \overline{H}_X(t))) \exp(\Theta(\ln \overline{H}_Y(s)))] \\ &= W(-\ln \overline{H}_X(t) - \ln \overline{H}_Y(s)) = W(R_X(t) + R_Y(s)), \quad t, s \geq 0,\end{aligned}$$

where $W(x) = \mathbf{E}[\exp(-x\Theta)]$, and $R_X(t) = -\ln \overline{H}_X(t)$ (and similarly for R_Y). In this context, the survival copula is of Clayton type when the random parameter Θ has distribution in the Gamma family.

Note that when $\overline{F}$ is defined as in (2.2) then $\overline{G}_X(x) = \overline{F}(x, 0) = W(R_X(x))$, $\overline{G}_Y(y) = \overline{F}(0, y) = W(R_Y(y))$ and $W^{-1}(x) = R_X(\overline{G}_X^{-1}(x)) = R_Y(\overline{G}_Y^{-1}(x))$.

A useful property of vectors having Archimedean survival copulas is the following. Assume that $\mathbf{X}$ has joint survival function defined as in (2.2). Then, as one can prove with straightforward calculation, the corresponding vector $\mathbf{X}_t$ of residual lifetimes at time t is defined as

$$\overline{F}_t(x,y) = W_t(R_{X_t}(x) + R_{Y_t}(y))$$

where

$$W_t(x) = \frac{W(R_X(t) + R_Y(t) + x)}{W(R_X(t) + R_Y(t))},$$

and where

$$R_{X_t}(x) = R_X(t+x) - R_X(t), \qquad R_{Y_t}(y) = R_Y(t+y) - R_Y(t)$$

for $t, x \geq 0$.

Thus, the survival copula of $\mathbf{X}_t$ is defined as

$$K_t(u,v) = W_t(W_t^{-1}(u) + W_t^{-1}(v)),$$

while its univariate marginal survival functions are given by $\overline{G}_{X_t}(x) = W_t(R_{X_t}(x))$ and $\overline{G}_{Y_t}(y) = W_t(R_{Y_t}(y))$.

It should be also observed that both K and K_t are bivariate distribution functions. In the subsequent sections we will denote with $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Y})$ and $\tilde{\mathbf{X}}_t = (\tilde{X}_t, \tilde{Y}_t)$ the two bivariate vectors having uniformly $[0,1]$ univariate marginals and joint distributions K and K_t, respectively. Obviously, it holds $\mathbf{X} =_{st} (\overline{G}_X^{-1}(\tilde{X}), \overline{G}_Y^{-1}(\tilde{Y}))$ and $\tilde{\mathbf{X}} =_{st} (\overline{G}_X(X), \overline{G}_Y(Y))$, and similarly for $\mathbf{X}_t$ and $\tilde{\mathbf{X}}_t$.

We conclude the preliminary section recalling two more multivariate stochastic orders that will be considered in the prosecution. Given two bivariate random vectors $\mathbf{X}$ and $\mathbf{Y}$, having joint survival functions $\overline{F}_{\mathbf{X}}$ and $\overline{F}_{\mathbf{Y}}$, we say that

(i) $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the *upper orthant* order ($\mathbf{X} \leq_{uo} \mathbf{Y}$) if, and only if, $\overline{F}_{\mathbf{X}}(x,y) \leq \overline{F}_{\mathbf{Y}}(x,y)$ for all $(x,y) \in \mathbb{R}^2$;

(ii) $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the *Positive Quadrant Dependence order* ($\mathbf{X} \leq_{PDQ} \mathbf{Y}$) if, and only if, they have the same marginals and both stochastic inequalities $\mathbf{X} \leq_{uo} \mathbf{Y}$ and $\mathbf{X} \geq_{lo} \mathbf{Y}$ hold.

We also recall that $\mathbf{X} \leq_{st} \mathbf{Y}$ implies both $\mathbf{X} \leq_{uo} \mathbf{Y}$ and $\mathbf{X} \leq_{lo} \mathbf{Y}$, while $\mathbf{X} \leq_{PQD} \mathbf{Y}$ holds if, and only if, they have the same marginal distributions and $K_{\mathbf{X}}(u,v) \leq K_{\mathbf{Y}}(u,v)$ for all $(u,v) \in [0,1] \times [0,1]$, where $K_{\mathbf{X}}$ and $K_{\mathbf{Y}}$ are the survival copulas of $\mathbf{X}$ and $\mathbf{Y}$, respectively.

3 Conditions for bivariate aging properties

In this section we describe conditions on the function W, R_X and R_Y such that the vector $\mathbf{X}$ satisfies the bivariate aging notions described above.

First we provide a preliminary result, that deals with comparison in PQD order of the vectors $\tilde{\mathbf{X}}_{t_1}$ and $\tilde{\mathbf{X}}_{t_2}$ having as distributions the survival copulas of the bivariate residual lifetimes $\mathbf{X}_{t_1}$ and $\mathbf{X}_{t_1}$, with $t_1 \neq t_2$.

Lemma 3.1. *Let $\boldsymbol{X}$ have joint survival function defined as in (2.2). If the function W is DFR [IFR] then $\tilde{\boldsymbol{X}}_{t_1} \geq_{PQD} [\leq_{PQD}]\ \tilde{\boldsymbol{X}}_{t_2}$ for all $0 \leq t_1 \leq t_2$.*

Proof. Fix $0 \leq t_1 \leq t_2$ and let Z be a random lifetime having distribution $F_Z(u) = 1 - W(u)$. Also, let $\tilde{t_1} = R_X(t_1) + R_Y(t_1)$ and $\tilde{t_2} = R_X(t_2) + R_Y(t_2)$. Observe that $0 \leq \tilde{t_1} \leq \tilde{t_2}$ because the functions R_X and R_Y are increasing.

Since W is DFR, by Lemma 2.1 and Theorem 2.2 in Pellerey and Shaked (1997), it follows $Z_{\tilde{t_1}} \leq_{disp} Z_{\tilde{t_2}}$, where $\leq_{disp}$ denotes the *dispersive order* (see Shaked and Shanthikumar, 2007, for definition and properties of this variability order).

Moreover, since W is DFR then it also follows $Z_{\tilde{t_1}} \leq_{st} Z_{\tilde{t_2}}$. Thus it is possible to apply Theorem 3.B.10 (b) in Shaked and Shanthikumar 2007, so obtaining that $\phi(Z_{\tilde{t_1}}) \geq_{disp} \phi(Z_{\tilde{t_2}})$ for every non-decreasing and concave function ϕ. In particular, it follows that $\log(Z_{\tilde{t_1}}) \geq_{disp} \log(Z_{\tilde{t_2}})$ which is equivalent to the monotonicity property

$$\frac{W_{t_1}^{-1}(u)}{W_{t_2}^{-1}(u)} \text{ is non-decreasing in } u. \tag{3.1}$$

Now the assertion follows observing that the condition (3.1) implies superadditivity of $W_{t_1}^{-1} \circ W_{t_2}$, which in turn implies, by Proposition 4 in Avérous and Dortet-Bernadet (2000), $K_{t_1}(u,v) \geq K_{t_2}(u,v)$ for all $u, v \in [0,1]$, i.e., $\tilde{X_{t_1}} \geq_{PQD} \tilde{X_{t_2}}$. The proof of the assertion in the brackets is similar. $\square$

It should be recalled here that equality in law between $\tilde{\mathbf{X}}_{t_1}$ and $\tilde{\mathbf{X}}_{t_2}$ is satisfied only in the case $W(x) = (x+1)^{-\theta}$ for $\theta \in [0, +\infty)$, i.e., in case the survival copula is of Clayton tipe, as shown in recent works by Charpentier (2003 and 2006) and Oakes (2005). This property of Clayton copulas will be extensively used in the prosecution.

The first main result of this section gives conditions to compare the bivariate residual lifetimes at different ages t and $t+s$ in lower othant order, i.e., for $\mathbf{X}$ to to satisfy the inequalities in (1.2).

Theorem 3.1. *Let $\boldsymbol{X}$ have joint survival function defined as in (2.2). If the function W is DFR [IFR] and R_X and R_Y are concave [convex], then $\boldsymbol{X} \in \mathcal{A}_{FR}^{w-}$ [$\boldsymbol{X} \in \mathcal{A}_{FR}^{w+}$].*

Proof. We give the proof of the assertion without the brackets, the order being similar.

Let $0 \leq t_1 \leq t_2$ and $u \geq 0$. First we show that if the functions R_X and R_Y are concave then

$$\overline{G}_{X_{t_1}}(u) \leq \overline{G}_{X_{t_2}}(u) \text{ and } \overline{G}_{Y_{t_1}}(u) \leq \overline{G}_{Y_{t_2}}(u). \tag{3.2}$$

For it, let $\tilde{t_i} = R_X(t_i) + R_Y(t_i)$ and $\delta_i = R_X(t_i + u) - R_X(t_i)$, $i = 1, 2$, and observe that $\delta_1 \geq \delta_2$, by the concavity of R_X, and $0 \leq \tilde{t_1} \leq \tilde{t_2}$, since R_X and R_Y are non-decreasing functions.

Thus,

$$\begin{aligned}
\overline{G}_{X_{t_2}}(u) &= \frac{W(R_X(t_2+u) + R_Y(t_2))}{W(R_X(t_2) + R_Y(t_2))} = \frac{W(\tilde{t_2} + \delta_2)}{W(\tilde{t_2})} \geq \frac{W(\tilde{t_1} + \delta_2)}{W(\tilde{t_1})} \\
&\geq \frac{W(\tilde{t_1} + \delta_1)}{W(\tilde{t_1})} = \frac{W(R_X(t_1+u) + R_Y(t_1))}{W(R_X(t_1) + R_Y(t_1))} = \overline{G}_{X_{t_1}}(u)
\end{aligned}$$

The first inequality follows from the DFR property of W (which is equivalent of requiring that the ratio $\frac{W(t+u)}{W(t)}$ is non-decreasing in t for every fixed $u \geq 0$), while the second one by $\delta_1 \geq \delta_2$.

Now we have:

$$\begin{aligned}
\mathbf{X}_t &=_{st} (\overline{G}^{-1}_{X_t}(\tilde{X}_t), \overline{G}^{-1}_{Y_t}(\tilde{Y}_t)) \geq_{PQD} (\overline{G}^{-1}_{X_t}(\tilde{X}_{t+s}), \overline{G}^{-1}_{Y_t}(\tilde{Y}_{t+s})) \\
&\leq_{a.s.} (\overline{G}^{-1}_{X_{t+s}}(\tilde{X}_{t+s}), \overline{G}^{-1}_{Y_{t+s}}(\tilde{Y}_{t+s})) = \mathbf{X}_{t+s},
\end{aligned}$$

for all $s, t \geq 0$. The first inequality follows from Lemma 3.1, while the second one from (3.2). Thus, there exists $\mathbf{Z}$ such that $\mathbf{X}_t \geq_{PQD} \mathbf{Z} \leq_{st} \mathbf{X}_{t+s}$ and therefore $\mathbf{X}_t \leq_{lo} \mathbf{X}_{t+s}$. $\square$

One example where Theorem 3.1 can be applied is the following. Let K be a Gumbel-Hougaard's copula, i.e., let $W(x) = e^{-x^{1/\theta}}$ with $\theta \geq 1$. It is easy to see that such survival function W is DFR. Thus, for every pair of concave functions R_X and R_Y it holds $\mathbf{X}_t \leq_{lo} \mathbf{X}_{t+s}$ for all $t, s \geq 0$.

Viceversa, let the dependence structure of $\mathbf{X}$ be described by the Gumbel-Barnett copula, i.e., $W(x) = e^{\frac{1-e^x}{\theta}}$ with $\theta \in (0, 1]$. It holds that W is IFR, thus $\mathbf{X}_t \geq_{lo} \mathbf{X}_{t+s}$ for all $t, s \geq 0$ whenever R_X and R_Y are convex.

Let us now restrict our attention to the case that K is a Clayton survival copula, i.e., let us assume that $\mathbf{X}$ is a bivariate vector having joint survival function defined as in (2.2), where $W(x) = (x + 1)^{-\theta}$ for some positive constant θ. As we will see, stronger bivariate aging notions can be proved in this case.

Actually, observing that $\tilde{\mathbf{X}}_t =_{st} \tilde{\mathbf{X}}_{t+s}$ when the underlying survival copula is of Clayton type (thus replacing the inequality $\geq_{PQD}$ with $=_{st}$ in the proof of Theorem 3.1) and observing that in this case the survival distribution W is DFR, one can immediately assert that the vector $\mathbf{X}$ satisfies the property $\mathcal{A}_{FR}^{w-}$ if $W(x) = (x+1)^{-\theta}$, with $\theta > 0$, and the functions R_X and R_Y are concave. However, similar conditions can be given for $\mathbf{X}$ to satisfy the stronger properties $\mathcal{A}^0$, $\mathcal{A}_{FR}^+$ or $\mathcal{A}_{FR}^-$, as proved in the following statement.

Theorem 3.2. *Let $\boldsymbol{X}$ have joint survival function defined as in (2.2), and let $W(x) = (x+1)^{-\theta}$ for some positive constant θ (i.e., let $\boldsymbol{X}$ have a Clayton survival copula). Then*

(i) it satisfies the weak multivariate lack of memory property $\mathcal{A}^0$ if, and only if, R_X and R_Y are defined as

$$R_X(x) = \frac{e^{b_X x} - 1}{\alpha_X} \quad \text{and} \quad R_Y(y) = \frac{e^{b_Y y} - 1}{\alpha_Y} \tag{3.3}$$

where α_X, α_Y, b_X and b_Y are strictly positive real numbers satisfying

$$\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} = 1 \text{ and } b_X = b_Y.$$

(ii) $\boldsymbol{X} \in \mathcal{A}_{FR}^+$ $[\mathcal{A}_{FR}^-]$ if and and only if the functions R_X and R_Y satisfy

$$\frac{R_X(t+u) + R_Y(t) + 1}{R_X(t) + R_Y(t) + 1} \leq [\geq] \frac{R_X(t+s+u) + R_Y(t+s) + 1}{R_X(t+s) + R_Y(t+s) + 1} \tag{3.4}$$

and

$$\frac{R_X(t) + R_Y(t+u) + 1}{R_X(t) + R_Y(t) + 1} \leq [\geq] \frac{R_X(t+s) + R_Y(t+s+u) + 1}{R_X(t+s) + R_Y(t+s) + 1}. \tag{3.5}$$

for every $t, s, u \geq 0$.

Proof. (i) Recall that in order to satisfy the condition $\mathbf{X} =_{st} \mathbf{X}_t$ the vectors $\mathbf{X}$ and $\mathbf{X}_t$ should necessarily have the same survival copula (which is unique, being F absolutely continuous). This means that the equality

$$W(W^{-1}(u) + W^{-1}(v)) = \frac{W(R_X(t) + R_Y(t) + W_t^{-1}(u) + W_t^{-1}(v))}{W(R_X(t) + R_Y(t))} \tag{3.6}$$

should be satisfied for all $u, v \in [0, 1]$ and $t \geq 0$, i.e., that K should be a Clayton survival copula (with any positive value for the parameter θ). Moreover, it should also be satisfied $\overline{G}_X(s) = \overline{G}_{X_t}(s)$ and $\overline{G}_Y(s) = \overline{G}_{Y_t}(s)$ for every $s, t \geq 0$. Letting $W(x) = (x+1)^{-\theta}$, this equalities are actually verified only when the functions

R_X and R_Y are defined as in (3.3) and $\alpha_X, \alpha_Y, b_X, b_Y$ satisfy the equality in the statement (see Aczél, 1966, for details).

Thus, under these assumptions it holds

$$\begin{aligned}\mathbf{X} =_{st} (\overline{G}_X^{-1}(\tilde{X}), \overline{G}_Y^{-1}(\tilde{Y})) &=_{st} (\overline{G}_X^{-1}(\tilde{X}_t), \overline{G}_Y^{-1}(\tilde{Y}_t)) \\ &=_{a.s.} (\overline{G}_{X_t}^{-1}(\tilde{X}_t), \overline{G}_{Y_t}^{-1}(\tilde{Y}_t)) = \mathbf{X}_t,\end{aligned}$$

i.e., $\mathbf{X} =_{st} \mathbf{X}_t$ for all $t \geq 0$, and also $\mathbf{X}_t =_{st} \mathbf{X}_{t+s}$ for all $t, s \geq 0$. The reversed implication follows observing that such equalities are satisfied only by the functions W, R_X and R_Y defined above.

(ii) It is enough to reason as in the proof of Theorem 3.1 but replacing the inequality $\geq_{PQD}$ with $=_{st}$ (because of the property of the Clayton survival copula). Moreover, when $W(x) = (x+1)^{-\theta}$ the inequalities $\overline{G}_{X_t}^{-1}(u) \leq [\geq]\ \overline{G}_{X_{t+s}}^{-1}(u)$ and $\overline{G}_{Y_t}^{-1}(u) \leq [\geq]\ \overline{G}_{Y_{t+s}}^{-1}(u)$ for all $t, s, u \geq 0$ hold if and only if assumptions (3.4) and (3.5) are satisfied, as one can easily prove with a direct calculation. $\square$

It should be observed that the resulting joint survival function and univariate marginal survival functions of the vector $\mathbf{X}$ satisfying the assertions of Theorem 3.2(i) are, respectively,

$$\overline{F}(x,y) = \left(1 + \frac{e^{bx}-1}{\alpha_X} + \frac{e^{by}-1}{\alpha_Y}\right)^{-\theta}, \tag{3.7}$$

$$\overline{G}_X(x) = \left(\frac{e^{bx} + (\alpha_X - 1)}{\alpha_X}\right)^{-\theta} \quad \text{and} \quad \overline{G}_Y(y) = \left(\frac{e^{by} + (\alpha_Y - 1)}{\alpha_Y}\right)^{-\theta}, \tag{3.8}$$

where $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} = 1$, with $\theta, b, \alpha_X, \alpha_Y \in \mathbb{R}^+$.

It is interesting also to observe that univariate survival functions defined as in (3.8) become exponential distributions when $\alpha = 1$, are the survival functions of a DFR lifetime when $\alpha > 1$, and, viceversa, of an IFR lifetime when $\alpha < 1$. Since assumption $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} = 1$ should be satisfied for a bivariate vector $\mathbf{X}$ to be in the $\mathcal{A}^0$ class, this means that a necessary condition for the weak multivariate lack of memory property is that the two marginal distributions should be DFR.

Examples of bivariate vectors $\mathbf{X}$ that are in the $\mathcal{A}_{FR}^-$ class can be provided reasoning as in the $\mathcal{A}^0$ case. In fact, we can again take the functions R_X and R_Y defined as in (3.3), always letting $b = b_X = b_Y$ to be any strictly positive real number, but this time assuming that $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} > 1$. In this case, in fact, inequalities (3.4) and (3.5) are satisfied with $\geq$, as one can verify, and the joint survival function and univariate marginal survival functions of the corresponding vector $\mathbf{X}$ are as in (3.7) and (3.8), respectively, but with $\frac{1}{\alpha_X} + \frac{1}{\alpha_Y} > 1$.

Recalling what said previously regarding the univariate survival functions in (3.8), one can observe that four possible cases can happen when $\frac{1}{\alpha_X}+\frac{1}{\alpha_Y}>1$: if α_X, $\alpha_Y<1$, then both the margins X and Y are DFR; if $\alpha_X>1$, $\alpha_Y<1$ then X is DFR and Y is IFR; if $\alpha_X<1$ and $\alpha_Y>1$ then X is DFR and Y is IFR and, unexpectedly, if both α_X and α_Y are greater than 1 then both the marginals X and Y are univariate IFR, even if $\mathbf{X}$ is in the $\mathcal{A}_{FR}^-$ class.

Viceversa, in case $\frac{1}{\alpha_X}+\frac{1}{\alpha_Y}<1$, then the bivariate vector $\mathbf{X}$ results to be in the $\mathcal{A}_{FR}^+$ class, since in this case both the inequalities $\leq$ are satisfied in (3.4) and (3.5).

This happens, for example, assuming again that R_X and R_Y are defined as in (3.3), and thus again in case that the joint survival function and univariate marginal survival functions of the vector $\mathbf{X}$ are as in (3.7) and (3.8), but this time letting $\frac{1}{\alpha_X}+\frac{1}{\alpha_Y}<1$. Under this assumption there is only one possible case for the univariate aging: $\alpha_X,\alpha_Y>1$, i.e., both margins X and Y are univariate IFR.

These considerations can be somehow generalized to the case that the functions R_X and R_Y are different than the ones in (3.3). However, in this case we should assume the same marginal distribution for X and Y (i.e., letting $R_X=R_Y=R$).

Theorem 3.3. *Let $\mathbf{X}$ have joint survival function defined as in (2.2), equal marginal distributions and Clayton survival copula. Let R be such that*

$$R(t+s)R(t+u)\geq R(t)R(t+s+u)\quad \forall t,s,u\geq 0, \tag{3.9}$$

i.e., let $\frac{R(t+s)}{R(t)}$ be decreasing in t for all $s\geq 0$. Then:

(i) If the marginals of $\mathbf{X}$ are DFR then $\mathbf{X}$ is in the $\mathcal{A}_{FR}^-$ class;

(ii) If $\mathbf{X}$ is in the $\mathcal{A}_{FR}^+$ class then its marginal distributions are IFR.

Proof. Let $W(x)=(x+1)^{-\theta}$ for some positive constant θ, and observe that, when $R_X=R_Y=R$, the marginal distributions of $\mathbf{X}$ are DFR if, and only if,

$$R(t+s+u)R(t)\leq R(t+s)+R(t+u)-R(t+s+u)-R(t)+R(t+s)R(t+u) \tag{3.10}$$

for all $t,s,u\geq 0$. Inequality (3.10) clearly implies

$$\begin{aligned} R(t+s+u)R(t) \;\leq\;& R(t+s)+R(t+u)-R(t+s+u)-R(t)+R(t+s)R(t+u)\\ &+[R(t+s)R(t+u)-R(t)R(t+s+u)] \end{aligned} \tag{3.11}$$

for all $t,s,u\geq 0$, which, in turns, is equivalent to the inequalities (3.4) and (3.5) with $\geq$. Thus, assertion (i) is proved.

Viceversa, let $\mathbf{X}$ be in the class $\mathcal{A}_{FR}^+$, i.e., let

$$\begin{aligned} R(t+s+u)R(t) \;\geq\;& R(t+s)+R(t+u)-R(t+s+u)-R(t)+R(t+s)R(t+u)\\ &+[R(t+s)R(t+u)-R(t)R(t+s+u)] \end{aligned} \tag{3.12}$$

for all $t, s, u \geq 0$ (this follows from (3.4) and (3.5)). Under assumption (3.9) this inequality implies

$$R(t+s+u)R(t) \geq R(t+s)+R(t+u)-R(t+s+u)-R(t)+R(t+s)R(t+u) \quad (3.13)$$

for all $t, s, u \geq 0$, which, in turns, implies that the marginals of $\mathbf{X}$ are IFR. $\square$

One example where Theorem 3.3 can be applied is the following. Let H be any absolutely continuous univariate cumulative distribution, with corresponding density h and survival function $\overline{H}$, and let $R(t) = [\overline{H}(t)]^{-1} - 1$, so that the marginals of $\mathbf{X}$ have the same survival distribution $\overline{G}(t) = W(R(t)) = (R(t)+1)^{-\theta} = \overline{H}(t)^{\theta}$. Referring to univariate distributions this is what in the literature is usually called proportional hazard model or Lehman's alternative. We have here a bivariate generalization where the proportionality factor is given by the parameter θ that describes the degree of dependence between $\mathbf{X}$ and $\mathbf{Y}$.

In this case the assumption $R(t+s)/R(t)$ decreasing in t for all $s \geq 0$ means that

$$\frac{H(t+s)\overline{H}(t)}{\overline{H}(t+s)H(t)}$$

is decreasing in t for all $s \geq 0$. Deriving it, with some calculations we can see that this is equivalent to:

$$\frac{h(t)}{\overline{H}(t)H(t)} = \frac{h(t)}{\overline{H}(t)}\frac{1}{H(t)} \quad \text{decreasing in } t.$$

Since $1/H(t)$ is always decreasing, the assumption on R is satisfied if $h(t)/\overline{H}(t)$ is decreasing, i.e., if H is DFR. Thus, for this model, if the underlying distribution H is DFR then it follows that the random vector $\mathbf{X}$ is in the $\mathcal{A}^-_{FR}$ class.

It is interesting to observe that the same statement does not hold for positive aging. In fact, letting $\overline{H}(x) = \frac{\alpha}{e^{bx}+\alpha-1}$, i.e., $R(x) = \frac{e^{bx}-1}{\alpha}$, we have a counterexample where H is IFR but $\mathbf{X}$ is in the $\mathcal{A}^-_{FR}$ (as shown before).

4 Conditions for comparisons of bivariate residual lifetimes

In this section we consider two different bivariate vectors of lifetimes, $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, and we provide conditions for comparisons of their residual bivariate lifetimes; the reasons of interest in these stochastic inequalities are the same than for the univariate case.

Throughout this section we will assume that the dependence structure of both the random pairs is described by Clayton survival copulas. In particular, in Theorem 4.1(i) we assume that $\mathbf{X}$ and $\mathbf{Y}$ have the same Clayton survival copula, so obtaining a comparison in usual stochastic order sense between $\mathbf{X}_t$ and $\mathbf{Y}_t$, while in Theorem 4.1(ii) we remove such assumption, so obtaining the weaker comparison in upper orthant order.

Moreover, in Theorem 4.1(i) and 4.1(ii) we assume exchangeability for the X_i and for the Y_i, i.e., $R_{X_1} = R_{X_2} = R_X$ and $R_{Y_1} = R_{Y_2} = R_Y$. In Theorem 4.1(iii) we provide conditions for $\mathbf{X}_t \leq_{st} \mathbf{Y}_t$ for all $t \geq 0$ even in the case of different margins.

Theorem 4.1. *Let $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ be two bivariate vectors having joint survival function defined as in (2.2), and let the survival copulas of $\mathbf{X}$ and $\mathbf{Y}$ be Clayton copulas parametrized by θ_X and θ_Y, respectively.*

(i) If $\theta_X = \theta_Y$, $R'_Y(0) \leq R'_X(0)$ and the ratio $R'_Y(t)/R'_X(t)$ is a decreasing function in t, then $\boldsymbol{X}_t \leq_{st} \boldsymbol{Y}_t$ for all $t \geq 0$.

(ii) If $\theta_X \geq \theta_Y$, $R'_Y(0) \leq R'_X(0)$ and the ratio $R'_Y(t)/R'_X(t)$ is a decreasing function in t, then $\boldsymbol{X}_t \leq_{uo} \boldsymbol{Y}_t$ for all $t \geq 0$.

(iii) If $\theta_X = \theta_Y$ and for every $t, s \geq 0$ it holds

(a) $[X_i - t \,|(X_1, X_2) > (t,t)] \leq_{st} [Y_i - t \,|(Y_1, Y_2) > (t,t)]$, $i = 1, 2$;

(b)

$$\begin{vmatrix} R_{X_1}(t+s) - R_{X_1}(t) & R_{Y_1}(t+s) - R_{Y_1}(t) \\ R_{X_2}(t) & R_{Y_2}(t) \end{vmatrix} \geq 0; \tag{4.1}$$

(c)

$$\begin{vmatrix} R_{X_2}(t+s) - R_{X_2}(t) & R_{Y_2}(t+s) - R_{Y_2}(t) \\ R_{X_1}(t) & R_{Y_1}(t) \end{vmatrix} \geq 0; \tag{4.2}$$

then $\boldsymbol{X}_t \leq_{st} \boldsymbol{Y}_t$ for every fixed $t \geq 0$.

Proof. (i) Let $\overline{G}_X$ and $\overline{G}_Y$ be the survival functions of X_i and Y_i, respectively, for $i = 1, 2$. It is not hard to verify that when $W(x) = (x+1)^{-\theta}$ for some positive constant θ then $\overline{G}_{X,t}(u) \leq \overline{G}_{Y,t}(u)$ for all $t, u \geq 0$ if, and only if, $A(t,s) + 2B(t,s) \geq 0$, where

$$A(t,s) = [R_X(t+s) - R_X(t)] - [R_Y(t+s) - R_Y(t)]$$

and

$$B(t,s) = R_Y(t)R_X(t+s) - R_X(t)R_Y(t+s).$$

Since $R'_Y(t)/R'_X(t)$ is a decreasing function in t, by the Basic Composition Formula (see Karlin, 1968), it follows that also $R_Y(t)/R_X(t)$ is a decreasing function in t and therefore that $B(t,s) \geq 0$. Moreover, if $R'_Y(0) \leq R'_X(0)$ and $R'_Y(t)/R'_X(t)$ is a decreasing function in t then $R'_Y(t) \leq R'_X(t)$ for any t, and therefore $A(t,s) \geq 0$. Thus $\overline{G}_{X,t}(u) \leq \overline{G}_{Y,t}(u)$ holds for all $t, u \geq 0$.

It follows

$$\begin{aligned}
\mathbf{X}_t &=_{st} (\overline{G}^{-1}_{X,t}(U_1^t), \overline{G}^{-1}_{X,t}(U_2^t)) =_{st} (\overline{G}^{-1}_{X,t}(U_1), \overline{G}^{-1}_{X,t}(U_2)) \\
&\leq_{a.s.} (\overline{G}^{-1}_{Y,t}(U_1), \overline{G}^{-1}_{Y,t}(U_2)) =_{st} (\overline{G}^{-1}_{Y,t}(U_1^t), \overline{G}^{-1}_{Y,t}(U_2^t)) = \mathbf{Y}_t,
\end{aligned}$$

where $\mathbf{U} = (U_1, U_2)$ is the vector having as distribution the Clayton survival copula, and $\mathbf{U}_t = (U_1^t, U_2^t) =_{st} \mathbf{U}$ is the vector having as distribution the common survival copula of $\mathbf{X}_t$ and $\mathbf{Y}_t$. The assertion follows.

(ii) As shown in (i), from the assumption on R_X and R_Y it follows that $A(t,s) + 2B(t,s) \geq 0$, which can be rewritten as

$$\left(\frac{1 + R_X(t) + R_X(t+s)}{1 + 2R_X(t)}\right)^{-1} \leq \left(\frac{1 + R_Y(t) + R_Y(t+s)}{1 + 2R_Y(t)}\right)^{-1}. \tag{4.3}$$

Thus

$$\begin{aligned}
\overline{G}_{X,t}(s) &= \left(\frac{1 + R_X(t) + R_X(t+s)}{1 + 2R_X(t)}\right)^{-\theta_X} \leq \left(\frac{1 + R_Y(t) + R_Y(t+s)}{1 + 2R_Y(t)}\right)^{-\theta_X} \\
&\leq \left(\frac{1 + R_Y(t) + R_Y(t+s)}{1 + 2R_Y(t)}\right)^{-\theta_Y} = \overline{G}_{Y,t}(s)
\end{aligned}$$

Moreover, from $\theta_X \geq \theta_Y$ it follows that $K_{\theta_X} \leq_{PQD} K_{\theta_Y}$, which is equivalent to $\tilde{X} \leq_{PQD} \tilde{Y}$. So we have, for any $t \geq 0$,

$$\begin{aligned}
\mathbf{X}_t &= (\overline{G}^{-1}_{X,t}(\tilde{X}_{1,t}), \overline{G}^{-1}_{X,t}(\tilde{X}_{2,t})) =_{st} (\overline{G}^{-1}_{X,t}(\tilde{X}_1), \overline{G}^{-1}_{X,t}(\tilde{X}_2)) \\
&\leq_{PQD} (\overline{G}^{-1}_{X,t}(\tilde{Y}_1), \overline{G}^{-1}_{X,t}(\tilde{Y}_2)) \leq_{a.s.} (\overline{G}^{-1}_{Y,t}(\tilde{Y}_1), \overline{G}^{-1}_{Y,t}(\tilde{Y}_2)) \\
&=_{st} (\overline{G}^{-1}_{Y,t}(\tilde{Y}_{1,t}), \overline{G}^{-1}_{Y,t}(\tilde{Y}_{2,t})) = \mathbf{Y}_t.
\end{aligned}$$

The assertion follows.

(iii) Let us consider the residual lifetimes $X_{i,t} = [X_i - t \,|(X_1, X_2) > (t,t)]$ and $Y_{i,t} = [Y_i - t \,|(Y_1, Y_2) > (t,t)]$ of the marginal random variables X_i and Y_i, and let $\overline{G}_t^{X_i} = \frac{W(R_{X_i}(t+s))}{W(R_{X_i}(t))}$ and $\overline{G}_t^{Y_i} = \frac{W(R_{Y_i}(t+s))}{W(R_{Y_i}(t))}$ be their survival functions, for $i = 1, 2$. It is easy to verify that for every fixed t and $i = 1, 2$ it holds $X_{i,t} \leq_{st} Y_{i,t}$ if, and only if,

$$R_{X_i}(t+s) - R_{X_i}(t) - (R_{Y_i}(t+s) - R_{Y_i}(t)) + R_{X_i}(t+s)R_{Y_i}(t) - R_{X_i}(t)R_{Y_i}(t+s) \geq 0 \tag{4.4}$$

for any $s \geq 0$.

On the other hand, let $\mathbf{X}_t = [(X_1 - t, X_2 - t)|X_1 > t, X_2 > t]$ be the vector of the residual lifetimes of $\mathbf{X}$ at time $t \geq 0$. Its marginal residual lifetimes $X_{t,1}$ and $X_{t,2}$ have the survival functions $\overline{G}_1^{X_t}(s) = \frac{W(R_{X_1}(t+s)+R_{X_2}(t))}{W(R_{X_1}(t)+R_{X_2}(t))}$ and $\overline{G}_2^{X_t}(s) = \frac{W(R_{X_1}(t)+R_{X_2}(t+s))}{W(R_{X_1}(t)+R_{X_2}(t))}$, respectively. Similarly, let us call $\overline{G}_1^{Y_t}$ and $\overline{G}_2^{Y_t}$ the survival functions of $Y_{t,1}$ and $Y_{t,2}$, which are the marginals of $\mathbf{Y}_t$.

For every $t, s \geq 0$, it holds $\overline{G}_1^{X_t}(s) \leq \overline{G}_1^{Y_t}(s)$ and $\overline{G}_2^{X_t}(s) \leq \overline{G}_2^{Y_t}(s)$ if, and only if,

$$\begin{aligned} & R_{X_1}(t+s) - R_{X_1}(t) - (R_{Y_1}(t+s) - R_{Y_1}(t)) + R_{X_1}(t+s)R_{Y_1}(t) - R_{X_1}(t)R_{Y_1}(t+s) \\ & + R_{X_1}(t+s)R_{Y_2}(t) - R_{X_1}(t)R_{Y_2}(t) + R_{X_2}(t)R_{Y_1}(t) - R_{X_2}(t)R_{Y_1}(t+s) \geq 0 \quad (4.5) \end{aligned}$$

and

$$\begin{aligned} & R_{X_2}(t+s) - R_{X_2}(t) - (R_{Y_2}(t+s) - R_{Y_2}(t)) + R_{X_2}(t+s)R_{Y_2}(t) - R_{X_2}(t)R_{Y_2}(t+s) + \\ & + R_{X_2}(t+s)R_{Y_1}(t) - R_{X_1}(t)R_{Y_1}(t) + R_{X_1}(t)R_{Y_2}(t) - R_{X_1}(t)R_{Y_2}(t+s) \geq 0 \quad (4.6) \end{aligned}$$

From (4.1), (4.2) and (4.4) it follows that (4.5) and (4.6) are satisfied and therefore also

$$\begin{aligned} \mathbf{X}_t \quad & =_{st} \quad ((\overline{G}_t^{X_1})^{-1}(U_1^t), (\overline{G}_t^{X_2})^{-1}(U_2^t)) =_{st} ((\overline{G}_t^{X_1})^{-1}(U_1), (\overline{G}_t^{X_2})^{-1}(U_2)) \\ & \leq_{a.s.} \quad ((\overline{G}_t^{Y_1})^{-1}(U_1), (\overline{G}_t^{Y_2})^{-1}(U_2)) =_{st} ((\overline{G}_t^{Y_1})^{-1}(U_1^t), (\overline{G}_t^{Y_2})^{-1}(U_2^t)) = \mathbf{Y}_t. \end{aligned}$$

□

An example where the assumptions of Theorem 4.1(i) are satisfied is for $R_X(t) = \log(1+b_X t)$, $R_Y(t) = \log(1+b_Y t)$ with $b_X \leq b_Y$. In the context of frailty models, for example, this means that if $\mathbf{X}$ and $\mathbf{Y}$ are such that $\overline{F}_{\mathbf{X}}(t,s) = \mathbf{E}[\overline{H}_X(t)^\Theta \overline{H}_X(s)^\Theta]$ and $\overline{F}_{\mathbf{Y}}(t,s) = \mathbf{E}[\overline{H}_Y(t)^\Theta \overline{H}_Y(s)^\Theta]$, where Θ is Gamma distributed, $\overline{H}_X(t) = 1/(1+b_X t)$ and $\overline{H}_Y(t) = 1/(1+b_Y t)$, then $\mathbf{X}_t \leq_{st} \mathbf{Y}_t$ for all $t \geq 0$ whenever $0 \leq b_X \leq b_Y$.

One case where $\mathbf{X} = (X_1, X_2)$ and $\mathbf{Y} = (Y_1, Y_2)$ are vectors of lifetimes satisfying the assumptions in Theorem 4.1(iii) is when $R_{X_i}(t) = \frac{e^{bt}-1}{\alpha_{X_i}}$, $R_{Y_i}(t) = \frac{e^{bt}-1}{\alpha_{Y_i}}$, for $i = 1, 2$, where $\alpha_{X_1} \cdot \alpha_{Y_2} = \alpha_{Y_1} \cdot \alpha_{X_2}$ and $\alpha_{X_i} \leq \alpha_{Y_i}$.

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