Connected subgroups of $SO(2,n)$ acting irreducibly on $\mathbb{R}^{2,n}$

Antonio J. Di Scala* Thomas Leistner†

Abstract

We classify all connected subgroups of $SO(2,n)$ that act irreducibly on $\mathbb{R}^{2,n}$. Apart from $SO_0(2,n)$ itself these are $U(1,n/2)$, $SU(1,n/2)$, if n even, $S^1 \cdot SO(1,n/2)$ if n even and $n \geq 2$, and $SO_0(1,2)$ for $n=3$. Our proof is based on the Karpelevich Theorem and uses the classification of totally geodesic submanifolds of complex hyperbolic space and of the Lie ball. As an application we obtain a list of possible irreducible holonomy groups of Lorentzian conformal structures, namely $SO_0(2,n)$, $SU(1,n)$, and $SO_0(1,2)$.

Keywords: Irreducible representations of Lie groups; Lie ball; complex hyperbolic space; totally geodesic submanifolds of symmetric spaces; holonomy groups, conformal holonomy.

1 Background, result, and applications

One of the results at the origins of modern differential geometry is Marcel Berger's classification of irreducible connected holonomy groups of complete semi-Riemannian manifolds, [Ber55]. The holonomy group at a point of a semi-Riemannian manifold is the group of all parallel transports (w.r.t. the Levi-Civita connection of the metric) along loops starting at this point. It is represented on the tangent space as a subgroup of $SO(r,s)$ if (r,s) is the signature of the metric. The symmetries of the curvature of the Levi-Civita connection impose algebraic constraints on this representation that were used by Berger in order to determine all possible holonomy groups which act irreducibly. The most striking feature of this *Berger list* is that it is so short. This is more surprising in some signatures and more natural in others. For example, in signature $(1,n-1)$ the only possible holonomy group of a Lorentzian manifold acting irreducibly is $SO_0(1,n-1)$, the connected component of the unit in $SO(1,n-1)$. This is due to the fact

*Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy, Fax: +39 011 564 7599, antonio.discala@polito.it
†Department Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany, Fax: +49 40 42838 5190, leistner@math.uni-hamburg.de

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that there are no proper connected subgroups of $SO_0(1,n-1)$ acting irreducibly on $\mathbb{R}^{1,n-1}$ [DSO01]. In contrary to that, for positive definite metrics, it is more surprising that only so few groups occur as holonomy groups of Riemannian manifolds, taking into account that any representation of a compact group, in particular an irreducible one, is orthogonal with respect to a positive definite scalar product. For a recent proof of Berger's theorem for Riemannian manifolds see [Olm05].

In general, for a given orthogonal group $SO(r,s)$, there is no classification of connected subgroups of $SO(r,s)$ groups that act irreducibly. In this paper we consider the case of signature $(2,n)$. The result is a classification of connected subgroups of $SO(2,n)$ that act irreducibly on $\mathbb{R}^{2,n}$:

Theorem 1. *Let $G \subset SO(2,n)$ be a connected Lie group that acts irreducibly on $\mathbb{R}^{2,n}$. Then G is conjugated to one of the following,*

1. *for arbitrary $n \geq 1$: $SO_0(2,n)$,*
2. *for $n = 2p$ even: $U(1,p)$, $SU(1,p)$, or $S^1 \cdot SO_0(1,p)$ if $p > 1$,*
3. *for $n = 3$: $SO_0(1,2) \subset SO(2,3)$.*

Our interest in signature $(2,n)$ is twofold. One aspect is the more general interest in the Berger list. Our result shows that there is only one group, namely $S^1 \cdot SO(1,n)$, that does not appear in the Berger list, i.e. that is not a holonomy group for a metric of signature $(2,n)$. For this group one can show that the algebraic constraints imposed by the curvature symmetries are not satisfied.

More important is the relation to conformal Lorentzian structures. To a Lorentzian conformal structure in dimension n, which is defined as an equivalence class of Lorentzian metrics modulo multiplication by a scaling function, one may assign a conformally invariant Cartan connection, that depends only on the conformal class but not on a representative. This conformal Cartan connectional also defines a holonomy group that is contained in $SO(2,n)$, the so-called *conformal holonomy*. For this group and its representation the algebraic restrictions are much more difficult to handle than the Berger criteria in case of metric holonomy algebras. Hence, it was natural to ask first: What are possible connected subgroups of $SO(2,n)$ that act irreducibly on $\mathbb{R}^{2,n}$? Our answer to this question gives a list of possible candidates for *special conformal Lorentzian structures*, a name which refers to — in analogy to special Riemannian structures — Lorentzian conformal structures with *irreducibly* acting conformal holonomy group. For indecomposable, non-irreducible Lorentzian conformal structures we refer the reader to [Lei07].

Now, two of the groups in Theorem 1 are known to be Lorentzian conformal holonomy groups, $SO_0(2,n)$ itself and $SU(1,n/2)$. The first is the generic Lorentzian conformal holonomy, the second is that of a Fefferman space (see for example [Bau07]). In [Leit08] it is proven that if a *connected* conformal holonomy group is contained $U(1,n/2)$ then it is already contained in $SU(1,n/2)$. Hence, $S^1 \cdot SO_0(1,n/2)$ and $U(1,n)$ cannot occur as connected conformal holonomy group of a Lorentzian conformal structure, because they are not contained in $SU(1,n/2)$. We obtain the following consequence.

Corollary 1. *Let $G \subset SO(2,n)$ be the connected conformal holonomy group of a Lorentzian conformal structure. If G acts irreducibly on $\mathbb{R}^{2,n}$, then*

$$G = SO_0(2,n),\ \text{or}\ G = SU(1,n/2)\ \text{if } n \text{ is even, or}\ G = SO_0(1,2)\ \text{if } n = 3.$$

Unfortunately, we cannot yet exclude the exceptional case of $SO_0(1,2) \subset SO(2,3)$ as a possible conformal holonomy of a 3-dimensional Lorentzian manifold. We only know that $SO(1,2)$ does not define a *conformal Cartan reduction* in the sense of [Alt08, Section 3.3]. Such a conformal Cartan reduction of $SO(p+1,q+1)$ to a group $G \subset SO(p+1,q+1)$ exists if and only if G acts transitively on the Möbius-sphere $S^{p,q} = SO(p+1,q+1)/P$, where P is the parabolic subgroup defined as the stabiliser of a light-like line in $\mathbb{R}^{p+1,q+1}$. Examples of conformal Cartan reductions are given by $SU(p+1,q+1) \subset SO(2p+2,2q+2)$, see [Bau07] or [CG06], the non-compact $G_{2(2)} \subset SO(3,4)$ in [Nur02, Nur08], and $Spin(3,4) \subset SO(4,4)$ in [Bry06], and they are linked to so-called *Fefferman constructions*. Now, the action of $SO_0(1,2)$ on $S^{1,2} = SO(2,3)/P$ is not transitive. In fact, transitivity would imply that $\mathfrak{so}(2,3) = \mathfrak{so}(1,2) + \mathfrak{p}$ which is a contradiction to $\mathfrak{so}(1,2) \cap \mathfrak{p} \neq \{0\}$ and $\dim \mathfrak{p} = 7$. Hence, $SO_0(1,2) \subset SO(2,3)$ does not define a conformal Cartan reduction, but we do not know if this already excludes it as an irreducible conformal holonomy. To clarify this question lies beyond the scope of this paper and will be subject to further studies.

Our proof of Theorem 1 is based on the Theorem of Karpelevich and Mostow.

Theorem 2. *(Karpelevich [Kar53], Mostow [Mos55], also [DSO07]) Let $M = Iso(M)/K$ be a Riemannian symmetric space of non-compact type. Then any connected and semisimple subgroup G of the full isometry group $Iso(M)$ has a totally geodesic orbit $G \cdot p \subset M$.*

We will apply this theorem to a connected subgroup G of $SO(2,n)$ that acts irreducibly on $\mathbb{R}^{2,n}$ and to the Riemannian symmetric spaces that are related to $SO(2,n)$: the complex hyperbolic space $\mathbb{C}H^n = SU(1,n)/U(n)$ and the Grassmannian of negative definite planes in $\mathbb{R}^{2,n}$ given as $SO_0(2,n)SO(2) \cdot SO(n)$ and as $SO(2,n)/SO(2) \cdot SO(n)$ if one considers *oriented* negative planes. The latter has two connected components and can be realised in $\mathbb{C}P^{n+1}$ as the submanifold of negative definite lines in $\mathbb{C}^{2,n}$. Its connected component is called *Lie ball*. In applying Karpelevich's Theorem we have to deal with two difficulties that are related to each other: First, we cannot assume that G is semisimple, and secondly, if T is a totally geodesic orbit with isometry group $H = Iso(T)$ our group G in question can be the product of H with the group $I(T)$ that is defined as

$$I(T) := \{A \in G \mid A|_T = \mathrm{Id}_T\}.$$

$I(T)$ is a normal subgroup in $Iso(T)$. We know that G is reductive but it may be that its semisimple part does not act irreducibly. On the other hand, it might happen that T is the orbit of a group H that does not act irreducibly but that $I(T) \cdot H$ acts irreducibly. Overcoming these difficulties, our proof will consist of three main steps:

1. Show that if $G \subset SO_0(2,n)$ acts irreducibly, then it is *simple* or contained in $U(1,n)$.
2. Classify connected subgroups of $U(1,n)$ acting irreducibly on $\mathbb{R}^{2,n}$ using:
   (a) G is reductive with possible centre S^1,
   (b) By Karpelevich's Theorem applied to $\mathbb{C}H^n$, the orbits T of the semisimple part are isometric to either $\mathbb{C}H^k$ or to real hyperbolic spaces $\mathbb{R}H^k$ for $k \leq n$.
   (c) $I(T)$ can be calculated.

3. If G is not in $U(1,n)$, G is simple and we apply Karpelevich's Theorem to the Lie ball $SO_0(2,n)/SO(2)\cdot SO(n)$. Then we use the classification of totally geodesic orbits in the complex quadric $SO(n+2)/SO(2)\cdot SO(n)$ by [CN77] and [Kle08], transfer it by duality to the Lie ball and obtain G as isometry group of these orbits. As G is simple, $I(T)$ can be ignored.

In the last section we will describe explicitly the inclusions of the totally geodesic submanifolds of the Lie ball.

2 Algebraic preliminaries

2.1 Irreducible representations of real Lie algebras

In this section we will review some basic results about real representations of real Lie algebras. To keep the paper self contained we will also explain and prove facts that are well known to the experts.

Let $\mathfrak{g}$ be a real Lie algebra and E a *real* representation. We say that E is of *real type* if both E and $E^{\mathbb{C}} := E \otimes \mathbb{C}$ are irreducible. If only E is irreducible we say that E is *not of real type*. In the latter case there is a splitting of $E^{\mathbb{C}}$ as $E^{\mathbb{C}} = V \oplus \overline{V}$ where V is an irreducible complex representation and $\overline{V}$ the conjugate representation w.r.t. the real form $E \subset E^{\mathbb{C}}$. Indeed, if V is a complex invariant subspace of $E^{\mathbb{C}}$, the complex subspaces $V + \overline{V}$ and $V \cap \overline{V}$ are invariant as well. On the other hand, they are equal to their complex conjugate, and thus, complexifications of real invariant subspaces. As E is irreducible, we obtain that $V \oplus \overline{V} = E^{\mathbb{C}}$. On the other hand, it is $(V_{\mathbb{R}})^{\mathbb{C}} = V \oplus \overline{V}$ and multiplication with i defines an invariant complex structure J on $V_{\mathbb{R}}$ and by complexification on $(V_{\mathbb{R}})^{\mathbb{C}} = E^{\mathbb{C}}$. Hence, $E^{\mathbb{C}}$ splits into the invariant eigen spaces of J given by V and $\overline{V}$, respectively. In particular, $E \simeq V_{\mathbb{R}} \simeq \overline{V}_{\mathbb{R}}$ as real representations inducing an invariant complex structure J on E.

In the other case, where E is of real type, $W := E^{\mathbb{C}}$ considered as a real vector space, denoted by $W_{\mathbb{R}}$, is reducible, with invariant real form E. This is equivalent to W being *self-conjugate* with a conjugation that squares to the identity. In case of a representation of real type, the invariant real form E then is given as the $+1$ eigen-space of this conjugation.

After this change of the viewpoint, it is natural to say that a *complex* irreducible representation is of *real type* if it is self-conjugate with a conjugation squaring to one. Otherwise it is called of *non-real type*. Examples of representations of real type are complexifications of the standard representations of $\mathfrak{so}(p,q)$ on $\mathbb{R}^{p,q}$. Examples of representations of non-real type are representations of $\mathfrak{u}(p,q)$ and $\mathfrak{su}(p,q)$ on $\mathbb{C}^{p,q}$ and $\mathbb{R}^{2p,2q}$ respectively.

For a complex irreducible representation V of $\mathfrak{g}$ there is a further distinction beyond being of real type or not. If V is not self-conjugate, then V is called *of complex type*. If V is self-conjugate with respect to a conjugation C, then C^2 is a $\mathbb{C}$-linear invariant automorphism of V. By the Schur lemma, it is a multiple of the identity, say $C = \lambda\cdot\mathrm{Id}$, with $\lambda \in \mathbb{R}$ because C is a conjugation. By scaling C we can assume that $\lambda^2 = \pm 1$. In one case, V was of real type, in the case where $C^2 = -\mathrm{Id}$ one says that V is *of quaternionic type*, because C defines another complex structure that anti-commutes with the multiplication with i. To summarise these standard facts, a complex irreducible representation is either of real, complex or quaternionic type. If it is not of real type, then $V_{\mathbb{R}}$ is irreducible, if it is of real type, it is the complexification of an irreducible real representation. These explanations imply the following standard fact:

Lemma 1. *$\mathfrak{g} \subset \mathfrak{so}(p,q)$ acting irreducibly is not of real type if and only if p and q are even and $\mathfrak{g} \subset \mathfrak{u}(p/2, q/2)$.*

The following lemma is also known. We prove it here for the sake of being self-contained.

Lemma 2. *Let $\mathfrak{g}$ be a real Lie algebra and V a complex irreducible representation of quaternionic type and of complex dimension $2m$.*

1. *If V is symplectic, then $\mathfrak{g} \subset \mathfrak{sp}(p,q) \subset \mathfrak{u}(2p,2q)$ with $p+q=m$.*
2. *If V is orthogonal, then $\mathfrak{g} \subset \mathfrak{so}^*(2m) \subset \mathfrak{u}(m,m)$.*

Proof. Let J be the anti-linear invariant automorphism of V with $J^2 = -1$, and let V be of complex dimension $2m$. Assume that ω is an invariant symplectic form on V. First we show that we can assume the following relation between ω and J:

$$\omega(Jx, Jy) = \overline{\omega(x,y)}. \tag{1}$$

In fact, $\hat{\omega} := \overline{\omega(J., J.)}$ gives another invariant symplectic form on V. By the Schur lemma, they are a complex multiple of each other, i.e. $\overline{\omega(J., J.)} = \lambda\omega$ for a $\lambda \in \mathbb{C}^*$, which implies that

$$\omega(J., J.) = \overline{\lambda}\ \overline{\omega(J^2., J^2.)} = \overline{\lambda}\lambda\omega(J., J.),$$

and thus $\lambda = \mathrm{e}^{i\theta} \in S^1$. Rescaling ω by $\mathrm{e}^{-i\frac{\theta}{2}}$ enables us to assume equation (1). Now Equation (1) implies that $\omega(J., .) = -\overline{\omega(., J.)}$ yielding an invariant hermitian form $\langle ., . \rangle$ on V via $\langle x, y \rangle := \omega(x, Jy)$. This is indeed hermitian,

$$\langle y, x \rangle = \omega(y, Jx) = -\overline{\omega(Jy, x)} = \overline{\omega(x, Jy)} = \overline{\langle x, y \rangle}$$

and compatible with J,

$$\langle Jx, Jy \rangle = -\omega(Jx, y) = \omega(y, Jx) = -\overline{\omega(Jy, x)} = \overline{\omega(x, Jy)} = \overline{\langle x, y \rangle}.$$

This shows that $\mathfrak{g} \subset \mathfrak{u}(2p, 2q) \cap \mathfrak{sp}(m, \mathbb{C}) = \mathfrak{sp}(p,q)$, with $p+q=m$.

Now assume that σ is an invariant symmetric bilinear form on V. By the same argument as in the symplectic case we get that

$$\sigma(Jx, Jy) = \overline{\sigma(x,y)} \quad \text{and} \quad \sigma(Jx, y) = -\overline{\sigma(x, Jy)}.$$

A hermitian form is now defined by $\langle x, y \rangle := \mathrm{i}\sigma(x, Jy)$ that is compatible with J,

$$\langle Jx, Jy \rangle = -\mathrm{i}\sigma(Jx, y) = \mathrm{i}\overline{\sigma(x, Jy)} = -\overline{\langle x, y \rangle},$$

showing that $\langle ., . \rangle$ has neutral signature (m, m) and that an orthonormal basis of σ is a light-like basis of $\langle ., . \rangle$. A calculation in a basis then shows that $\mathfrak{g} \subset \mathfrak{u}(V, \langle ., . \rangle) \cap \mathfrak{so}(2m, \mathbb{C}) = \mathfrak{so}^*(2m)$, which is defined as follows

$$\mathfrak{so}^*(2m) := \left\{ \begin{pmatrix} A & B \\ -\overline{B}^t & \overline{A}^t \end{pmatrix} \mid A \in \mathfrak{so}(m, \mathbb{C}), B = \overline{B}^t \right\}$$

(see [Hel78, p. 446]). $\square$

We conclude this section by verifying that $S^1 \cdot SO_0(1,n) \subset U(1,n) \subset SO(2,2n)$ acts irreducibly.

Proposition 1. *Let $\mathfrak{g} \subset \mathfrak{so}(p,q)$ act irreducibly on $\mathbb{R}^{p,q}$. This representation is of real type if and only if $\widetilde{\mathfrak{g}} := \imath \cdot \mathbb{R} \oplus \mathfrak{g}$ acts irreducibly on $\mathbb{C}^{p+q}$ as real representation.*

Proof. If the representation of $\mathfrak{g}$ on $\mathbb{R}^{p,q}$ is of real type then its complexification is still irreducible, and so is the representation of $\widetilde{\mathfrak{g}}$ on $\mathbb{C}^{p+q}$. But by definition, there is no conjugation that is invariant under $\widetilde{\mathfrak{g}}$, and thus the representation of $\widetilde{\mathfrak{g}}$ on $\mathbb{C}^{p+q}$ is of complex type, which means that it is still irreducible as a real representation.

On the other hand assume that $\mathbb{C}^{p+q}$ is irreducible as real and therefore as a complex representation of $\widetilde{\mathfrak{g}}$. Assume furthermore that the representation of $\mathfrak{g}$ on $\mathbb{R}^{p,q}$ is not of real type. By the remarks in the previous section, this is equivalent to the existence of a $\mathfrak{g}$-invariant complex structure J on $\mathbb{R}^{p,q}$ and to the existence of a complex $\mathfrak{g}$-invariant subspace $V \subset \mathbb{C}^{p+q}$. Extending J complex linear gives an invariant complex structure on $\mathbb{C}^{p+q}$. Note that $J \neq \imath \cdot \mathrm{Id}$, because otherwise $\widetilde{\mathfrak{g}}$ could no longer act irreducibly on $\mathbb{C}^{p,q}$. Hence, $I := \imath \cdot J$ is $\widetilde{\mathfrak{g}}$-invariant, satisfies $I^2 = \mathrm{Id}$ and is not a multiple of the identity. Hence, it has non-trivial invariant eigen spaces to the eigen values ± 1. But this again contradicts to the irreducibility of $\mathbb{C}^{p,q}$ under $\widetilde{\mathfrak{g}}$. Therefore, $\mathbb{R}^{p,q}$ must be of real type for $\mathfrak{g}$. $\square$

This gives the following conclusion in the case $p = 1$.

Corollary 2. *For $n > 1$, $S^1 \cdot SO_0(1,n)$ is an irreducible subgroup of $U(1,n) \subset SO_0(2,n)$, is not contained in $SU(1,n)$, and has no further irreducible subgroups.*

Proof. From the previous section we know that irreducible representations of non-real type are unitary, but this is not possible for $\mathfrak{g} \subset \mathfrak{so}(1,n)$. In fact, there is no proper irreducibly acting subalgebra of $\mathfrak{so}(1,n)$, see [DSO01]. But $\mathfrak{so}(1,n)$ is of real type, and the result follows from the proposition.

For the minimality assume that $\mathfrak{g} \subset \imath\mathbb{R} \oplus \mathfrak{so}(1,n)$ acts irreducibly. But then the projection of $\mathfrak{g}$ onto $\mathfrak{so}(1,n)$ acts irreducibly and thus has to be equal to $\mathfrak{so}(1,n)$. But this implies $\mathfrak{so}(1,n) = [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \subset \imath\mathbb{R} \cdot \mathfrak{so}(1,n)$. Hence, $\mathfrak{g} = \imath\mathbb{R} \cdot \mathfrak{so}(1,n)$. That $S^1 \times SO(1,n)$ is not contained in $SU(1,n)$ is obvious. $\square$

2.2 Reduction to simple Lie algebras and consequences

In this section we show that an irreducible subalgebra of $\mathfrak{so}(2,n)$ is either contained in $\mathfrak{u}(1,n/2)$ or simple. Based on the distinction of real representation into those of real and complex type and on the description of the center in [DLN05] we proved the following:

Proposition 2. *Let $G \subset SO_0(p,q)$ a connected Lie subgroup of $SO_0(p,q)$ which acts irreducibly. If G is not semisimple, then p and q are even and G is a subgroup of $U(p/2,q/2)$ with centre $U(1)$. In particular, if $G \subset SO(2,n)$, then $G \subset U(1,n/2)$ or semi-simple.*

Here we will strengthen this result for the case $G \subset SO(2,n)$ by replacing "semi-simple" by "simple". This will be based on the following general fact on complex irreducible representation of semi-simple complex Lie algebras (for a reference, see for example [Oni04, p. 11]): *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a semi-simple Lie algebra decomposing into non-trivial ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then V is a complex irreducible representation of $\mathfrak{g}$ if and only if $V = V_1 \otimes V_2$ where V_i are irreducible representations of $\mathfrak{g}_i$.*

Lemma 3. *Let $\mathfrak{g}\oplus\mathfrak{h}$ be semi-simple and $W=U\otimes V$ an irreducible complex representation. Then W is self-dual if and only if both, U and V are self-dual. The invariant isomorphisms are related by $\psi=\psi_1\otimes\psi_2$.*

Proof. The 'if'-direction is obvious, $\psi=\psi_1\otimes\psi_2$ defines the required invariant isomorphism. For the other direction we consider the identification $\tau: U\simeq U\otimes v_0$ for a fixed $v_0\in V$. τ is not only an isomorphism of vector spaces but also of representations of $\mathfrak{g}$. Let $\psi: W\simeq W^*$ be the the isomorphism yielding the self-duality of W. This implies that there are $u_0,\hat{u}_0\in U$ and $v_1\in V$ such that

$$[\psi(u_0\otimes v_0)]\,(\hat{u}_0\otimes v_1)\neq 0.$$

Otherwise, $u_0\otimes v_0$ would be in the kernel of ψ. Hence by defining

$$[\psi_1(u)]\,(\hat{u}):=[\psi(u\otimes v_0)]\,(\hat{u}\otimes v_1)$$

we obtain a $\mathfrak{g}$-invariant homomorphism $\psi_1: U\simeq U^*$ which is non trivial. By the Schur lemma, ψ_1 is an isomorphism. Obviously, for V one can proceed in the same way. The Schur-lemma also gives the uniqueness of the invariant structures and the relation between them. $\square$

Lemma 4. *Let $\mathfrak{g}\oplus\mathfrak{h}$ be semi-simple and $W=U\otimes V$ an irreducible complex self-dual representation. Then W is self-conjugate if and only if both, U and V are self-conjugate. The invariant isomorphisms are related by $\psi=\psi_1\otimes\psi_2$.*

Proof. As W is self-dual, both U and V are self dual. Hence, $\overline{U}\simeq\overline{U}^*$ and $\overline{V}\simeq\overline{V}^*$. If $\psi: W\simeq W^*$ and $C: W\simeq\overline{W}$, analogously as in the proof of the previous lemma, one defines $\phi_1: U\to\overline{U}^*$ via

$$[\phi_1(u)]\,(\hat{u}):=[\psi(u\otimes v_0)]\,(C(\hat{u}\otimes v_1)).$$

Again, by the Schur lemma, this is an isomorphism, yielding an isomorphism $\psi_1: U\simeq\overline{U}$. All invariant structures are uniquely defined. $\square$

Theorem 3. *Let $\mathfrak{g}\subset\mathfrak{so}(2,n)$ be an irreducibly acting Lie algebra. Then $\mathfrak{g}\subset\mathfrak{u}(1,n/2)$, $n=2$ and $\mathfrak{g}=\mathfrak{so}(2,2)$, or $\mathfrak{g}$ is simple.*

Proof. By Proposition 2 we can suppose that $\mathfrak{g}$ is semisimple and that the representation of $\mathfrak{g}$ on $\mathbb{R}^{2,n}$ is of real type. Assume that $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ is not simple. Then its complexification is semisimple and not simple, and thus, the complexified representation $\mathbb{C}^{n+2}$ of $\mathbb{R}^{2,n}$ is a tensor product, $\mathbb{C}^{n+2}=V_1\otimes V_2$ of irreducible representations of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. As $\mathbb{C}^{n+2}$ is of real type, the second lemma implies that V_1 and V_2 are either both of real type or both of quaternionic type. Since $\mathfrak{g}\subset\mathfrak{so}(n+2,\mathbb{C})$, by the first lemma both are self-dual, defined by either two complex linear symmetric or symplectic forms.

Assume first that both, V_1 and V_2 are of real type, i.e. $V_i=E_i^{\mathbb{C}}$ where E_i are irreducible real representations of $\mathfrak{g}_i$. If $\mathfrak{g}_i\subset\mathfrak{so}(V_i)$, also both E_i are orthogonal, i.e. $\mathfrak{g}_1\subset\mathfrak{so}(p,q)$ and $\mathfrak{g}_2\subset\mathfrak{so}(r,s)$ with $2=ps+qr$. W.l.o.g. this yields two cases: The first is $\mathfrak{g}_1=\mathfrak{so}(2)$ and $\mathfrak{g}_2\subset\mathfrak{so}(1,n/2)$ acting on $\mathbb{R}^2\otimes\mathbb{R}^{1,\frac{n}{2}}$, or $\mathfrak{g}_1=\mathfrak{g}_2=\mathfrak{so}(1,1)$. But both cases contradict to the assumption that $\mathfrak{g}$ was semisimple.

Now we consider the case where the V_i's and thus both E_i's are symplectic representations. In this case the defining scalar product on $\mathbb{R}^{2,n}$ has neutral signature, i.e. $\mathfrak{g}\subset\mathfrak{so}(2,2)$, and $\mathfrak{g}_i\subset\mathfrak{sp}(1,\mathbb{R})=\mathfrak{sl}(2,\mathbb{R})$ acting irreducible. Hence, $\mathfrak{g}_i$ either one-dimensional and therefore Abelian, two-dimensional, and thus solvable, or equal to $\mathfrak{sl}(2,\mathbb{R})$. The first two possibilities are excluded by the semisimplicity assumption. We obtain that $\mathfrak{g}$ is equal to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(2,2)$.

Now we have to deal with the case where both representations, V_1 and V_2 are of quaternionic type. As $\mathfrak{g} \subset \mathfrak{so}(n+2,\mathbb{C})$, they are either both orthogonal or both symplectic.

Using Lemma 2 we can conclude the proof of the theorem: First consider the case that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i \subset \mathfrak{sp}(p_i,q_i) \subset \mathfrak{u}(2p_i,2q_i)$. The tensor product of the hermitian forms on V_i defines a hermitian form of signature $(4(p_1q_2+p_2q_1), 4(p_1p_2+q_1q_2))$ on $V = E^{\mathbb{C}}$. Since V is an irreducible representation of $\mathfrak{g}$, the space of hermitian forms on V is one-dimensional. Hence, the defined hermitian form is a multiple of the hermitian form obtained by extending the signature $(2,n)$ scalar product on E to V. But $2 \neq 4(p_1q2+p_2q_1)$ which excludes this case.

For the case $\mathfrak{g}_i \subset \mathfrak{so}^*(2m_i) \subset \mathfrak{u}(m_i,m_i)$ we obtain that $\mathfrak{g} \subset \mathfrak{u}(2m_1m_2, 2m_1m_2)$, which implies $m_i = 1$, $V_i = \mathbb{C}^2$ and $\mathfrak{g}_i = \mathfrak{so}^*(2) = \mathfrak{so}(2)$ and $\mathfrak{g}$ is no longer semisimple. □

We can now apply Karpelevich's Theorem 2 to what we have obtained so far.

Theorem 4. *Let $G \subsetneqq SO_0(2,n)$ be a connected irreducibly acting subgroup. Then $G \subset U(1,n)$ or G is simple and equal to the effectively acting isometry group of a totally geodesic submanifold in the non-compact symmetric space $SO_0(2,n)/SO(2) \times SO(n)$.*

Proof. Let $G \subset SO_0(2,n)$ but $G \not\subset U(1,n)$. From the previous section we know that G is simple. By Karpelvich's Theorem 2 it follows that G has a totally geodesic orbit $\mathcal{T}$ in the non-compact symmetric space $\mathcal{L}^n := SO_0(2,n)/SO(2) \times SO(n)$. The subgroup

$$I(\mathcal{T}) := \{A \in G \mid Ap = p \text{ for all } p \in \mathcal{T}\}$$

is a normal subgroup in G. As G is simple, $I(T)$ is trivial and G acts effectively on $\mathcal{T}$. Hence, $\mathcal{T} = G/K \subset \mathcal{L}^n$ is a non-compact symmetric space with $K \subset G$ maximally compact. □

In the next section we will determine all irreducibly acting groups $G \subset U(1,n)$ by applying Karpelevich's theorem to the complex hyperbolic space. In the last section we will then use a classification of totally geodesic submanifolds in $Q^n = SO(n+2)/SO(2) \times SO(n)$ by [CN77] and [Kle08] and the duality of symmetric spaces in order to determine the remaining G's.

3 Irreducible subgroups of $U(1,n)$ and complex hyperbolic space

Using Karpelevich's Theorem in this section we will proof the following statement.

Theorem 5. *Let $G \subset U(1,n) \subset SO(2,2n)$ be a connected subgroup that acts irreducibly on $\mathbb{R}^{2,2n}$. Then $SU(1,n) \subset G$ or $G = S^1 \cdot SO_0(1,n)$.*

To this end we consider the complex vector space $\mathbb{C}^{n+1} =: \mathbb{C}^{1,n}$ endowed with the Hermitian form : $Q = -|z_0|^2 + |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$. $U(1,n) \subset GL(n+1,\mathbb{C})$ is the subgroup that preserves Q. Let $\mathcal{N} := \{p \in \mathbb{C}^{n,1} : Q(p) < 0\}$ be the set of negative points. Notice that $\mathcal{N}$ is a cone preserved by the $U(1,n)$-action. $\mathbb{C}H^n$ is the projectivization of $\mathcal{N}$. Thus, by taking $z_0 = 1$ we see that $\mathbb{C}H^n$ is identified with the unit ball of $\mathbb{C}^n$,

$$\mathbb{C}H^n \cong \{Z \in \mathbb{C}^n : |Z|^2 < 1\}.$$

It is standard to see that the Hermitian form Q induces on $\mathbb{C}H^n$ a $U(1,n)$-invariant Riemannian metric of constant holomorphic curvature. Indeed, we get $\mathbb{C}H^n \cong SU(1,n)/U(n)$ as symmetric space of rank one. Notice that the $U(1,n)$-action on $\mathbb{C}H^n$ is not effective since the matrices $e^{i\theta}Id \in U(1,n)$ leaves invariant any complex line. Recall also that the presentation $\mathbb{C}H^n \cong SU(1,n)/U(n)$ as symmetric quotient is unique. Namely, if $\mathbb{C}H^n \cong G/K$ where G is semisimple and $K \subset G$ maximal compact then $G = SU(1,n)$ and $K = U(n)$. The following fact about totally geodesic submanifolds of $\mathbb{C}H^n$ can be found in [Gol99, pp. 74], for example.

Proposition 3. *Let $\mathcal{T} \subset \mathbb{C}H^n$ be a complete totally geodesic submanifold. Then $\mathcal{T}$ is either a totally real submanifold or a complex submanifold. In the totally real case $\mathcal{T}$ is isometric to real hyperbolic space, otherwise $\mathcal{T}$ is biholomorphic and isometric to a lower dimensional complex hyperbolic space. In particular, there exists a real vector subspace $V \subset \mathbb{C}^{n,1}$ such that $\mathcal{T} = V \bigcap \mathbb{C}H^n$.*

Now we are ready to deduce Theorem 5 from Karpelevich's Theorem.

Proof of Theorem 5. Let $H \subset U(1,n)$ be connected and acting irreducibly on $\mathbb{R}^{2,2n}$ then H is reductive, i.e. $H = Z \cdot S$ where Z is the centre and S semisimple. From Proposition 2 we know that the centre Z is trivial or equal to S^1. Hence, the semisimple part S cannot be trivial. Now, according to Karpelevich's Theorem S has a totally geodesic orbit T of $\mathbb{C}H^n$. If T is a complex submanifold then Proposition 3 implies that S must be transitive on $\mathbb{C}H^n$ since otherwise the complex subspace V associated to T is invariant by S and $Z = S^1$. Thus H can not be irreducible. So S is transitive and we get by the uniqueness of the representation of the symmetric quotient that $SU(1,n) = S$.

Assume now that T is not a complex submanifold. Then, the classification of totally geodesic submanifolds of $\mathbb{C}H^n$ imply $T \cong \mathbb{R}H^n$. Otherwise T is contained in a proper complex totally geodesic submanifold of $\mathbb{C}H^n$ and this imply that H is not irreducible as above. Thus, T is a totally real totally geodesic submanifold. Without lost of generality we can assume that $T = \mathbb{R}H^n$ where $\mathbb{R}H^n = \{Z \in \mathbb{R}^n \subset \mathbb{C}^n : |Z|^2 < 1\}$. Notice that the Lie algebra of the group $I(T)$ is trivial. Indeed, if $u \in Lie(I(T))$ then the tangent space to T at $0 \in \mathbb{R}^n \subset \mathbb{C}^n$ is contained in the kernel of u. Since T is totally real and $u \in \mathfrak{u}(1,n)$ we get that also the normal space of T at 0 is contained in the kernel of u. Thus u vanish. Since $I(T)$ is trivial we get that $S = SO_0(1,n) \subset SU(1,n)$. Now the center must be S^1 and so $H = S^1 \cdot SO_0(1,n)$. $\square$

4 The Lie ball and its totally geodesic submanifolds

4.1 The projective model of the Lie ball.

Let $\mathbb{R}^{2,n}$ the real vector space $\mathbb{R}^{n+2}$ endowed with the quadratic form

$$q(X,Y) := \langle X, Y\rangle := -x_0y_0 - x_1y_1 + \sum_{j=2}^{n+1} x_jy_j,$$

where $X = (x_0, \cdots, x_{n+1})$ and $Y = (y_0, \cdots, y_{n+1})$. Let $\Pi \subset \mathbb{R}^{2,n}$ be a 2-dimensional subspace. The 2-plane Π is called *negative* if $q|_\pi$ is negative definite. Let us define the Lie ball $\mathcal{L}^n$ as one connected component of the set of oriented negative definite 2-planes of $\mathbb{R}^{2,n}$. For more details about this model see [Sat80, p. 285, §6] or [Wol72, p. 347]. Note that $SO(2, n)$ acts transitively on the oriented negative definite 2-planes, and that $SO_0(2, n)$ acts transitively on $\mathcal{L}^n$.

Let $\mathbb{C}^{2,n}$ be the complexification of the $\mathbb{R}^{2,n}$, i.e. q becomes

$$q(Z, W) = -z_0\overline{w_0} - z_1\overline{w_1} + \sum_{j=2}^{n+1} z_j\overline{w_j},$$

where $Z = (z_0, \cdots, z_{n+1})$ and $W = (w_0, \cdots, w_{n+1})$. Let $\Pi = span_{\mathbb{R}}\{A, B\} \subset \mathbb{R}^{2,n}$, $A, B \in \mathbb{R}^{2,n}$, be an oriented negative definite 2-plane. We can assume that $\langle A, B\rangle = 0$ and $q(A, A) = q(B, B) < 0$. Put $Z = A + iB \in \mathbb{C}^{2,n}$. Then it is immediate that

$$Z \in Q^{2,n} := \{Z = (z_0, \cdots, z_{n+1}) \in \mathbb{C}^{2,n} \mid -z_0^2 - z_1^2 + \sum_{j=2}^{n+1} z_j^2 = 0\}$$

and that $q(Z, Z) < 0$. Call $Q^{2,n}_+$ the subset of $Q^{2,n}$ of negative points, i.e.

$$Q^{2,n}_+ = \{Z = (z_0, \cdots, z_{n+1}) \in \mathbb{C}^{2,n} \mid -z_0^2 - z_1^2 + \sum_{j=2}^{n+1} z_j^2 = 0 \text{ and } q(Z, Z) < 0\}.$$

It follows that we can identify the Lie ball $\mathcal{L}^n$ with a subset of the projective space $\mathbb{C}P^{n,1}$, namely, with a connected component of the image of the canonical projection $\pi : \mathbb{C}^{n+2} \setminus 0 \to \mathbb{C}P^{n,1}$. Thus, we have homogeneous coordinates $[z_0 : z_1 : \cdots : z_{n+1}]$ to work with the Lie ball $\mathcal{L}^n$.

Let $\Pi_0 = span_{\mathbb{R}}\{e_0, e_1\}$ be the "canonical" negative definite 2-plane. From now on we will assume that the Lie ball $\mathcal{L}^n$ is the connected component of Π_0. Then Π_0 corresponds to the point $Z_0 = e_0 + \imath e_1 = (1, \imath, 0, \ldots, 0)$. Thus $\Pi_0 \cong [1 : \imath : 0 : \ldots : 0] \in \pi(Q^{2,n}_+) \cong \mathcal{L}^n$. The isotropy group at Π_0 is $SO(2) \times SO(n)$.

4.2 The complex quadric and its totally geodesic submanifolds

The complex quadric $Q^n = SO(n + 2)/SO(2) \times SO(n)$ can be viewed in to ways. First, as the Grassmannian of Z(oriented 2-planes in $\mathbb{R}^{n+2}$. Secondly, taking into account its complex nature, one can view it as a complex hypersurface in complex projective space, namely as

$$Q^n := \left\{ [z_0 : \ldots : z_{n+1}] \in \mathbb{C}P^{n+1} \mid \sum_{k=0}^{n+1} (z_k)^2 = 0 \right\}.$$

The subgroup of $SU(n + 2)$ acting on $\mathbb{C}^{n+2}$ and thus on $\mathbb{C}P^{n+1}$ that leaves invariant Q^n is $SO(n + 2)$ with isotropy group $SO(2) \times SO(n)$. The correspondence to the Grassmannian is given by

$$P = \mathrm{span}(x, y) \ \mapsto \ \pi(x + iy) \in \mathbb{C}P^{n+1}$$

where $\pi : \mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}$ is the canonical projection.

Now we will list the totally geodesic submanifolds in Q^n and their isometry groups as classified in [CN77] and [Kle08, Theorem 4.1 and Section 5]. Apart from geodesics, there are the following types:

(I1,*k*) for $1 \leq k \leq n/2$: This orbit is defined by the following totally geodesic isometric embedding

$$\mathbb{C}P^k \ni [z_0 : \ldots : z_k] \ \mapsto \ [z_0 : \ldots : z_k : \imath z_0 : \ldots : \imath z_k : 0 : \ldots : 0] \in Q^n.$$

Its image is a *maximal* totally geodesic submanifolds if $2k = n$ and $n \geq 4$. Its isometry group is $SU(k+1)$ and the totally geodesic submanifold is isometric to $SU(k+1)/U(k)$.

(I2,*k*) for $1 \leq k \leq n/2$: Here the embedding is give by the restriction of the map for type (I1,k) to real projective space $\mathbb{R}P^k$ in $\mathbb{C}P^k$. Hence, it is never maximal. Nevertheless, it will be interesting for our purposes. It is isometric to $O(k+1)/O(k)$.

(G1,*k*) for $1 \leq k \leq n-1$: This is the embedding of a lower dimensional quadric Q^k into Q^n,

$$Q^k \ni [z_0 : \ldots : z_{k+1}] \ \mapsto \ [z_0 : \ldots : z_{k+1} : 0 : \ldots : 0] \in Q^n.$$

It is maximal for $k = n-1 \geq 2$. Its isometry group is $SO(k+2)$ and it is isometric to $SO(k+2)/SO(2) \times SO(k)$.

(G2,$\boldsymbol{k_1}$,$\boldsymbol{k_2}$) for $1 \leq k_1 + k_2 \leq n$: This is a totally geodesic isometric embedding of a product of two spheres with radius $1/\sqrt{2}$ and of dimension k_1 and k_2 given by

$$((x_0, \ldots, x_{k_1}), (y_0, \ldots, y_{k_2})) \mapsto [x_0 : \ldots : x_{k_1} : \imath y_0 : \ldots : \imath y_{k_2} : 0 : \ldots : 0] \in Q^n$$

This orbit is maximal for $k_1 + k_2 = n \geq 3$. Its isometry group is given by $SO(k_1+1) \times SO(k_2+1)$.

(G3) The quadric Q^2 is isometric to $\mathbb{C}P^1 \times \mathbb{C}P^1$ i.e., $\mathbb{C}P^1 \times \mathbb{C}P^1 \equiv Q^2$. Let $C = \mathbb{R}P^1 \subset \mathbb{C}P^1$ be the trace of a closed geodesic in $\mathbb{C}P^1$. Then the map

$$\mathbb{C}P^1 \times C \to \mathbb{C}P^1 \times \mathbb{C}P^1 \equiv Q^2 \to Q^n$$

where the last embedding represents the embedding of type (G1,2) described above. So the embedding $\mathbb{C}P^1 \times C \hookrightarrow Q^m$ is maximal only for $n = 2$.

(P1,*k*) for $1 \leq k \leq n$. This is given as the embedding of type (G2, k_1, k_2) for k_1 or k_2 equal to zero. Its image is maximal for $k = n$. The isometry group is given as $SO(k+1)$.

(P2) This is the embedding of type (G1, k) for $k = 1$. It is maximal only for $n = 2$ and its isometry group is $SO(3)$.

(A) The totally geodesic submanifold is isometric to the 2-sphere of radius $\sqrt{10}/2$. It is maximal only for $n = 3$ and its isometry group is given by $SO(3)$.

4.3 Totally geodesic submanifolds of the Lie ball

We will now use Cartan's duality and Klein's classification as listed in the previous section in order to classify totally geodesic submanifolds in the Lie ball. Cartan's duality implies that there is a one-to-one correspondence between totally geodesic submanifolds G/K in the complex quadric $Q^n = SO(n+2)/SO(2) \times SO(n)$ and totally geodesic submanifolds G^*/K in the Lie ball $SO(2,n)/SO(2)\times SO(n)$. Here G^* is the non compact dual of G. For details on symmetric spaces refer to [Hel78] and to [BCO03, Chapter 9] for their totally geodesic submanifolds.

In the following, the immersions u will be equivariant. So they are useful to compute the corresponding immersion of the group into $SO(2,n)$.

Type (I1,k) Here we have $1 \le k \le n/2$. Let us consider the following map,

$$u \; : \; [z_0 : \ldots : z_k] \longrightarrow [z_0 : \imath z_0 : \ldots : z_k : \imath z_k : 0 : \ldots : 0].$$

The image of u is contained in $\pi(Q^{2,n})$. In order to see which point is taken by u to $\mathcal{L}^n$ it is enough to see that

$$-|z_0|^2 - |\imath z_0|^2 + \sum_{i=1}^{n+1}\left(|z_i|^2 + |\imath z_i|^2\right) = 2\left(-|z_0|^2 + \sum_{i=1}^{n+1}|z_i|^2\right)$$

Thus, $u[z_0 : \ldots : z_k : 0 : \ldots : 0] \in \mathcal{L}^n$ if and only if $-|z_0|^2 + \sum_{i=1}^{n+1}|z_i|^2 < 0$. Hence, u gives an holomorphic immersion from the complex hyperbolic space $\mathbb{C}H^k$ into our Lie ball $\mathcal{L}^n$. Namely, $\mathbb{C}H^k$ is regarded as the projective submanifold of $\mathbb{C}P^{k,1}$ defined by $-|z_0|^2 + \sum_{i=1}^{n+1}|z_i|^2 < 0$.

The group of isometries of $\mathbb{C}H^k$ is $SU(1,k) \subset SO(2,n)$ which acts irreducibly on $\mathbb{R}^{2,n}$ only for $k = n/2$. To see this it is enough to identify $\mathbb{R}^{2,n}$ with $\mathbb{C}^{1,n/2}$ endowed with the quadratic form $-|w_0|^2 + \sum_{i=1}^{n/2}|w_i|^2$. The action of $SU(1,n/2)$ is transitive on the set of negative 2-planes of $\mathbb{C}^{1,n/2}$ given by complex lines. For example, the complex line generated by the vector $(1,0,\ldots,0) \in \mathbb{C}^{1,n/2}$ is the negative definite 2-plane Π_0 of $\mathbb{R}^{2,n}$. Let $w = (w_0,\ldots,w_{n/2}) \in \mathbb{C}^{1,n/2}$ be a vector. Then the 2-plane generated by w, i.e. the complex line, is given by the homogeneous coordinates $[\overline{w_0} : \imath\overline{w_0} : \ldots : \overline{w}_{n/2} : \imath\overline{w}_{n/2}]$. This show that the image of our map u is the set of 2-planes coming from complex lines of $\mathbb{C}^{1,n/2}$. Thus, the image $u(\mathbb{C}H^{n/2})$ is the orbit of $SU(1,n/2)$ through Π_0.

Type (I2,*k*) Here it is $1 \le k \le n/2$. The map u is the "real form" of the above map:

$$[x_0 : \ldots : x_k] \xrightarrow{u} [x_0 : \imath x_0 : \ldots : x_k : \imath x_k : 0 : \ldots : 0]$$

Thus we get an embedding of $\mathbb{R}H^k$ (in the projective Klein model[1]) into the Lie ball. Notice that the subgroup $SO(1,k) \subset SU(1,n/2)$ acts reducibly on $\mathbb{R}^{2,n}$, even for $k = n/2$. In the light of Theorem 4 we do not get another irreducible subgroup of $SO(2,n)$. But we should point out that the group $I(u(\mathbb{R}H^k))$ i.e. the isometries that fix all points of the image of u is given as $I(u(\mathbb{R}H^k)) = SO(2)$ acting diagonally, i.e. $SO(2) \cong e^{\mathrm{i}\,\theta}Id$. For $k = n/2$ this group makes $G = I(u(\mathbb{R}H^k)) \cdot SO(1,n/2)$ act irreducibly on $\mathbb{R}^{2,n}$.

Type (G1,*k*) This is the embedding of a lower dimensional Lie ball. Its isometry group is given by $SO(2,k)$, which does not act irreducibly on $\mathbb{R}^{2,n}$.

[1] Here we refer to Felix Klein.

Type (G2,k_1, k_2) In this case $1 \leq k_1 + k_2 \leq n$ and the map u is given by:

$$([x_0 : \ldots : x_{k_1}], [y_0 : \ldots : y_{k_2}]) \stackrel{u}{\mapsto} [x_0 : \imath y_0 : x_1 : \ldots : x_{k_1} : \imath y_1 : \ldots : \imath y_{k_2} : 0 : \ldots : 0]$$

The image lies in $\mathcal{L}^n$ if and only if:

$$-x_0^2 - y_0^2 + \sum_{i=1}^{k_1} x_i^2 + \sum_{j=1}^{k_2} y_j^2 < 0 \quad \text{and} \quad -x_0^2 + y_0^2 + \sum_{i=1}^{k_1} x_i^2 - \sum_{j=1}^{k_2} y_j^2 = 0.$$

Since the map is given in homogeneous coordinates we can assume that $-x_0^2 + \sum_{i=1}^{k_1} x_i^2 = -y_0^2 + \sum_{j=1}^{k_2} y_j^2$ which shows that the image of u is in the Lie ball if and only if $[x_0 : \ldots : x_{k_1}]$ and $[y_0 : \ldots : y_{k_2}]$ lie in the real hyperbolic spaces of dimensions k_1 and k_2. Hence, u is an embedding of $\mathbb{R}H^{k_1} \times \mathbb{R}H^{k_2}$ into the Lie ball. The isometry groups is given by $SO(1,k_1) \times SO(1,k_2) \subset SO(2,k_1+k_2) \subset SO(2,n)$. Thus, the isometry group of this totally geodesic submanifold does not act irreducibly, since it fixes $\mathbb{R}^{1,k_1}$ and $\mathbb{R}^{1,k_2}$.

Type (P1,k) Here it it $1 \leq k \leq n$ and the embedding is given by the one of type (G2, k_1, k_2) for k_1 or k_2 equal to zero. Hence, we can write it as

$$[x_0 : \ldots : x_k] \mapsto [\imath : x_0 : \ldots : x_k : 0 \ldots : 0]$$

yielding an immersion from $\mathbb{R}H^k$ (as in the usual Lorentzian model) into the Lie ball $\mathcal{L}^n$. The isometry group of the totally geodesic submanifold is $SO(1,k)$ acting reducibly even for $k = n$ by fixing the first basis vector e_0.

Type (P2) This is the embedding of type (G1, k) for $k = 1$ and thus the isometry group of the totally geodesic submanifolds is given as $SO(2,1) \subset SO(2,n)$ acting reducibly by fixing $e_3, \ldots, e_{n+1}$.

Type (G3) This totally geodesic submanifold is a Riemannian product. Then its isometry group G is not simple. Thus this case reduces to $G \subset U(1,n)$.

Type (A) Here it is $n \geq 3$. This is an embedding of 3-dimensional real hyperbolic space into the Lie ball. The only irreducible acting (simple) subgroup of $SO(2,n)$ which did not appear as isometry group of a totally geodesic orbit in the Lie ball is $SO(1,2) \subset SO(2,3)$. Thus we conclude that this embedding of $SO(1,2)$ gives the isometry group of a totally geodesic orbit of type (A) for $n = 3$. For $n > 3$ it is reducible, of course.

We conclude that the only irreducibly acting simple proper subgroups of $SO(2,n)$ that appear as isometry group of a totally geodesic submanifold in the Lie ball are $SU(1,n/2)$ and $SO(1,2) \subset SO(2,3)$. Because of the reduction to simple groups in Section 2.2, together with Theorem 5, this proves our Theorem 1 in the Introduction.

References

[Alt08] J. Alt. *Fefferman Constructions in Conformal Holonomy.* PhD thesis, Humboldt-University Berlin, 2008.

[BCO03] J. Berndt, S. Console, and C. Olmos. Submanifolds and holonomy, 2003. Chapman & Hall/CRC , Research Notes in Mathematics 434.

[Bau07] H. Baum. The conformal analog of Calabi-Yau manifolds. In *Handbook of Pseudo-Riemannian Geometry*, IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2007. In press.

[Ber55] Marcel M. Berger. Sur les groupes d'holonomie homogène des variétés a connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[Bry06] R. L. Bryant. Conformal geometry and 3-plane fields on 6-manifolds. In *Developments of Cartan Geometry and Related Mathematical Problems*, volume 1502 of *RIMS Symposium Proceedings*, pages 1–15, 2006.

[CG06] A. Cap and A. R. Gover. A holonomy characterisation of Fefferman spaces, 2006. http://arXiv.org:math/0611939.

[CN77] B.-Y. Chen and T. Nagano. Totally geodesic submanifolds of symmetric spaces. I. *Duke Math. J.*, 44(4):745–755, 1977.

[DLN05] A. J. Di Scala, T. Leistner, and T. Neukirchner. Geometric applications of irreducible representations of Lie groups, 2005. In *Handbook of Pseudo-Riemannian Geometry*, IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2007. In press, arXiv:math.DG/0507047.

[DSO01] A. J. Di Scala and C. Olmos. The geometry of homogeneous submanifolds of hyperbolic space. *Math. Z.*, 237(1):199–209, 2001.

[DSO07] A. J. Di Scala and C. Olmos. A geometric proof of the Karpelevich-Mostow's theorem, 2007. Preprint, arXiv:math/0702201.

[Gol99] W. M. Goldman. *Complex hyperbolic geometry.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1999. Oxford Science Publications.

[Hel78] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, 1978.

[Kar53] F. I. Karpelevič. Surfaces of transitivity of a semisimple subgroup of the group of motions of a symmetric space. *Doklady Akad. Nauk SSSR (N.S.)*, 93:401–404, 1953.

[Kle08] S. Klein. Totally geodesic submanifolds of the complex quadric. *Differential Geom. Appl.*, 26:79, 2008.

[Lei07] T. Leistner. Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds. *Differential Geom. Appl.*, 24(5):458–478, 2006.

[Leit08] F. Leitner. A remark on unitary conformal holonomy. In *Symmetries and overdetermined systems of partial differential equations*, volume 144 of *IMA Vol. Math. Appl.*, pages 445–460. Springer, New York, 2008.

[Mos55] G. D. Mostow. Some new decomposition theorems for semi-simple groups. *Mem. Amer. Math. Soc.*, 1955(14):31–54, 1955.

[Nur02] P. Nurowski. Differential equations and conformal structures. *J. Geom. Phys.*, 43(4):327–340, 2002.

[Nur08] P. Nurowski. Conformal structures with explicit ambient metrics and conformal G_2 holonomy. In *Symmetries and overdetermined systems of partial differential equations*, volume 144 of *IMA Vol. Math. Appl.*, pages 515–526. Springer, New York, 2008.

[Olm05] C. Olmos. A geometric proof of the Berger holonomy theorem. *Ann. of Math.* (2) 161 (2005), no. 1, 579–588.

[Oni04] A. L. Onishchik. *Lectures on real semisimple Lie algebras and their representations.* ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2004.

[Sat80] I. Satake. *Algebraic structures of symmetric domains*, volume 4 of *Kanô Memorial Lectures*. Iwanami Shoten, Tokyo, 1980.

[Wol72] J. A. Wolf. Fine structure of Hermitian symmetric spaces. In *Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970)*, pages 271–357. Pure and App. Math., Vol. 8. Dekker, New York, 1972.