Weak helix submanifolds of euclidean spaces

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Abstract

Let $M \subset \mathbb{R}^n$ be a submanifold of a euclidean space. A vector $d \in \mathbb{R}^n$ is called a helix direction of M if the angle between d and any tangent space T_pM is constant. Let $\mathcal{H}(M)$ be the set of helix directions of M. If the set $\mathcal{H}(M)$ contains r linearly independent vectors we say that M is a weak r-helix. We say that M is a strong r-helix if $\mathcal{H}(M)$ is a r-dimensional linear subspace of $\mathbb{R}^n$. For curves and hypersurfaces both definitions agree. The object of this article is to show that these definitions are not equivalent. Namely, we construct (non strong) weak 2-helix surfaces of $\mathbb{R}^4$.

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1 Introduction

Recently, M. Ghomi solved in [Gh] the shadow problem formulated by H. Wente. He used the concept of shadow boundary (or horizon) in his work. In [RH, pag. 2] Ruiz-Hernández observed that shadow boundaries are naturally related to helix submanifolds i.e. submanifolds whose tangent space makes constant angle with a fixed direction d. Helix surfaces has also been studied in non flat ambient spaces (see for example [DM, DFVV]). An interesting motivation for the study of helix hypersurfaces comes also from the physics of interfaces of liquid cristals (see [CD] for details). The concept of (strong) r-helix submanifold of $\mathbb{R}^n$ was introduced in [DRH]. Let $M \subset \mathbb{R}^n$ be a submanifold of a euclidean space. A vector $d \in \mathbb{R}^n$ is called a helix direction

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of M if the angle between d and any tangent space T_pM is constant. Let $\mathcal{H}(M)$ be the set of helix directions of M. If the set $\mathcal{H}(M)$ contains r linearly independent vectors we say that M is a weak r-helix. We say that M is a strong r-helix if $\mathcal{H}(M)$ is a r-dimensional linear subspace of $\mathbb{R}^n$. For curves and hypersurfaces both definitions agree.

The object of this article is to show that these definitions are not equivalent. Namely, we prove the following theorem.

Theorem 1.1 *There exist non strong weak 2-helix surfaces of $\mathbb{R}^4$.*

In order to prove the above theorem we give in Theorem 3.1 the classification of strong 2-helix surfaces of $\mathbb{R}^4$. Then we study a quasi-linear PDE with analytic coefficients to prove our main theorem.

In the last section we explain the relation between strong/weak helix submanifolds and the helix-property introduced by F. Dillen and S. Nölker in [DN].

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2 Definitions and basic facts

Let $M \subset \mathbb{R}^n$ be a submanifold. A vector $d \in \mathbb{R}^n$ is called a helix direction of M if the angle θ between d and the tangent space T_pM is constant for all $p \in M$. In such case we call θ the helix angle of d. As a convention we let the zero vector $\overrightarrow{0}$ to be a helix direction for every submanifold. Let $d \neq 0$ be a helix direction of M. The helix angle $\theta \in [0, \pi/2]$ is given by the decomposition $\frac{d}{\|d\|} = \cos(\theta)T(p) + \sin(\theta)\xi(p)$, where $T(p) \in T_pM$ and $\xi(p) \in \nu_p(M)$ are unitary vectors. Let $\pi_p : \mathbb{R}^n \to T_pM$ be the orthogonal projection.

Proposition 2.1 *Let $M \subset \mathbb{R}^n$ be a submanifold and let $d \in \mathbb{R}^n$ a vector. The following conditions are equivalent:*

- *d is a helix direction of M.*
- *$\|\pi_p(d)\|^2 = c$ is a constant (i.e. does not depend upon $p \in M$). In such case we have*

$$\|d \wedge e_1(p) \wedge e_2(p) \wedge \cdots \wedge e_m(p)\|^2 = c\|e_1(p) \wedge e_2(p) \wedge \cdots \wedge e_m(p)\|^2 ,$$

where $(e_1(p), e_2(p), \cdots, e_m(p))$ is any basis of T_pM.

- *$\langle \pi_p(d), d\rangle = c$ is a constant.*

Let $M \subset \mathbb{R}^n$ be a submanifold and let $\mathcal{H}(M) := \{d : d \text{ is a helix direction of } M\}$ be the set of helix directions of M.

Here is the strong definition of an r-helix.

Definition 2.2 *A submanifold $M \subset \mathbb{R}^n$ is a strong r-helix if the set $\mathcal{H}(M)$ is a linear subspace of $\mathbb{R}^n$ of dimension greater or equal to r.*

Here is the weak definition of an r-helix.

Definition 2.3 *A submanifold $M \subset \mathbb{R}^n$ is a weak r-helix if the set $\mathcal{H}(M)$ contains r linearly independent vectors $d_i \in \mathbb{R}^n$.*

Notice that for curves and hypersurfaces of $\mathbb{R}^n$ both definitions agree.

Given a weak r-helix and r independent helix directions $d_i \in \mathcal{H}(M)$ ($1 \le i \le r$) we can split them as normal and tangent components. Namely,

$$\frac{d_i}{\|d_i\|} = \cos(\theta_i)T_i + \sin(\theta_i)\xi_i \ .$$

Proposition 2.4 *A weak r-helix M is strong if and only if the inner products $\langle T_i, T_j\rangle$ (resp. $\langle \xi_i, \xi_j\rangle$) are constant functions on M.*

Proof. Let $x_1 d_1 + x_2 d_2 + \cdots + x_r d_r$ be a linear combination of the d_i's with constant coefficients. Then

$$\sum_i x_i d_i = \sum_i x_i \cos(\theta_i) T_i + \sum_i x_i \sin(\theta_i)\xi_i.$$

So

$$\|\sum_i x_i \cos(\theta_i)T_i\|^2 = \sum_{i\,j} \langle x_i \cos(\theta_i)T_i, x_j \cos(\theta_j)T_j\rangle =$$

$$= \sum_{i\,j} x_i \cos(\theta_i) x_j \cos(\theta_j)\langle T_i, T_j\rangle = X^t.G.X$$

where $X^t := (x_1\cos(\theta_1), \cdots, x_r\cos(\theta_r))$ and $G = (\langle T_i, T_j\rangle)$. Thus if G is a constant matrix then Proposition 2.1 implies that $\mathcal{H}(M)$ is a linear subspace of $\mathbb{R}^n$. Reciprocally, if $X^t.G.X = f(X)$ then by taking derivatives with respect to any tangent vector $v_p \in T_pM$ we get:

$$X^t.\frac{\partial G}{\partial v_p}.X = 0 \ .$$

Thus, the quadratic form of the symmetric matrix $\frac{\partial G}{\partial v_p}$ vanishes identically. So $\frac{\partial G}{\partial v_p}$ vanishes identically and we are done. □

3 Strong 2-helix in $\mathbb{R}^4$

Here is the local classification of strong 2-helix surfaces of $\mathbb{R}^4$.

Theorem 3.1 *A strong 2-helix $M^2 \subset \mathbb{R}^4$ is flat, i.e. the Gauss curvature of M is zero. Moreover such 2-helix comes (locally) from:*

(i) a 1-helix $H^2 \subset \mathbb{R}^3$, i.e. locally $H^2 = M^2 \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$,

(ii) a 1 helix $H^1 \subset \mathbb{R}^3$, i.e. locally $M = \mathbb{R} \times H^1 \subset \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$.

Moreover if there are two orthogonal helix directions with the same angle $\frac{\pi}{4}$ then there exist a helix direction d of angle $\theta = 0$, i.e. M is a product as in (ii).

Proof. Let $M^2 \subset \mathbb{R}^4$ be a strong 2-helix. It is not difficult to see that if $dim(\mathcal{H}(M)) \geq 3$ then M^2 is totally geodesic. Indeed, if $dim(\mathcal{H}(M)) \geq 3$ then $dim(\mathcal{H}(M)) \cap \nu_p(M)) \geq 1$. So we can split orthogonally $\mathbb{R}^4 = \mathbb{R}^3 \oplus \mathbb{R}d$, where $d \in \mathcal{H}(M) \cap \nu_p(M))$, i.e. M is contained in an affine hyperplane. Let $V = (\mathbb{R}d)^\perp \subset \mathcal{H}(M)$ be the orthogonal complement of $\mathbb{R}d$ in $\mathcal{H}(M)$. Then $dim(V \cap \mathbb{R}^3) \geq 2$ so M^2 is a 2-helix of $\mathbb{R}^3$. The same argument as above shows that a 2-helix of $\mathbb{R}^3$ is totally geodesic.
So we can assume that $dim(\mathcal{H}(M)) = 2$. It is not difficult to see that if there exists a helix direction $d \in \mathcal{H}(M)$ of angle $\theta \in \{0, \frac{\pi}{2}\}$ we have that M^2 comes from (i) or (ii). So assume also that there are no helix directions of angle $\theta \in \{0, \frac{\pi}{2}\}$. We are going to show that this is not possible. Let $d_1, d_2 \in \mathbb{R}^4$ be two helix directions of M^2. Decompose the vectors d_1, d_2 as

$$\frac{d_1}{\|d_1\|} = \cos(\theta_1)T_1 + \sin(\theta_1)\xi_1 \ ,$$

$$\frac{d_2}{\|d_2\|} = \cos(\theta_2)T_1 + \sin(\theta_2)\xi_2 \ .$$

Then T_1, T_2 (resp. ξ_1, ξ_2) are linearly independent. Indeed, if T_1, T_2 (resp. ξ_1, ξ_2) are dependent then we can find a helix direction d of M of angle $\frac{\pi}{2}$ (resp. of angle 0).

For any tangent vector $X \in TM^2$ we have the following equations for $j = 1, 2$:

$$0 = \cos(\theta_j)\nabla_X \mathrm{T}_j(p) - \sin(\theta_j)A^{\xi_j}(X) \text{ and} \quad (1)$$

$$0 = \cos(\theta_j)\alpha(X, \mathrm{T}_j(p)) + \sin(\theta_j)\nabla^\perp_X \xi_j. \quad (2)$$

where $\nabla_X Y := (D_X Y)^\top$ is the Levi-Civita connection of the surface $M^2 \subset \mathbb{R}^4$ (i.e. the tangent component of the derivative D of $\mathbb{R}^n$ to the submanifold M^2), α is its second fundamental form, $A^\xi(X) := -(D_X\xi)^\perp$ is its shape operator and $\nabla^\perp_X \xi := (D_X \xi)^\perp$ is the normal connection (see [BCO] for details). Let us compute the covariant derivatives of the tangent fields T_1, T_2. Namely,

$$\nabla_{T_1} T_1 = 0, \tag{3}$$
$$\nabla_{T_2} T_1 = \frac{\langle T_1, T_2\rangle_2}{1-\langle T_1, T_2\rangle^2}(-\langle T_1, T_2\rangle T_1 + T_2), \tag{4}$$
$$\nabla_{T_1} T_2 = \frac{\langle T_1, T_2\rangle_1}{1-\langle T_1, T_2\rangle^2}(T_1 - \langle T_1, T_2\rangle T_2), \tag{5}$$
$$\nabla_{T_2} T_2 = 0\,, \tag{6}$$
$$\tag{7}$$

where $\langle T_1, T_2\rangle_i = T_i\langle T_1, T_2\rangle = \frac{\partial\langle T_1, T_2\rangle}{\partial T_i}$ for $i = 1, 2$.

Notice that the above equations implies that the vector fields T_1, T_2 are ∇-parallel if their angle is constant. Thus, from Proposition 2.4 we get that T_1, T_2 are ∇-parallel.

For the shape operator of M^2 we get:

$$A^{\xi_1}(T_1) = 0, \tag{8}$$
$$A^{\xi_1}(T_2) = \cot(\theta_1)\nabla_{T_2} T_1, \tag{9}$$
$$A^{\xi_2}(T_1) = \cot(\theta_2)\nabla_{T_1} T_2, \tag{10}$$
$$A^{\xi_2}(T_2) = 0 \tag{11}$$

Since T_1, T_2 are ∇-parallel we get that M^2 is totally geodesic (i.e. its shape operator A^ξ is zero) which is a contradiction since we have assumed above M^2 to be non totally geodesic. So a strong 2-helix of $\mathbb{R}^4$ is given locally as in (i) or (ii). Notice that in both cases the Gauss curvature is identically zero.

To prove the last part it is enough to assume that the strong 2-helix M^2 comes from (i). That is to say $M^2 \subset \mathbb{R}^3$ and $M^2 \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$.

Then $\mathcal{H}(M^2) = span_{\mathbb{R}}\{d, e_4\}$, where $d \in \mathbb{R}^3$ is a helix direction of angle $\theta \in [0, \frac{\pi}{2}]$. If $b_1, b_2 \in \mathcal{H}(M^2)$ are orthogonal then there exists α such that:

$$b_1 = \cos(\alpha)d + \sin(\alpha)e_4,$$
$$\pm b_2 = -\sin(\alpha)d + \cos(\alpha)e_4.$$

Since the helix angles of b_1, b_2 are both equal to $\frac{\pi}{4}$ we get the following two possibilities:

$$\frac{1}{\sqrt{2}} = \cos(\alpha)\cos(\theta) = -\sin(\alpha)\cos(\theta)$$

or

$$\frac{1}{\sqrt{2}} = \cos(\alpha)\cos(\theta) = \sin(\alpha)\cos(\theta).$$

In both cases we get $\cos(\alpha) = \pm\frac{1}{\sqrt{2}}$ which imply $\theta = 0$. Thus in this case M also comes from (ii). □

4 Constructing a weak 2-helix of $\mathbb{R}^4$

The goal of this section is to show the existence of a weak (non strong) 2-helix of $\mathbb{R}^4$. Namely, to prove Theorem 1.1.

The idea is to look for immersions $F : (x, y) \to (x, y, u(x, y), v(x, y))$ where we impose the condition of being a weak 2-helix w.r. to e_3 and e_4 with the same angle $\frac{\pi}{4}$. Notice that the last part of Proposition 3.1 imply that such immersion F is a strong 2-helix if and only if the functions u, v are linear. Indeed, there can not exist a helix direction $d \in span\{e_3, e_4\}$ of helix angle $\theta = 0$. Thus, we have to show that such immersion F does exist.

Proposition 4.1 *Let $F : \Omega \to \mathbb{R}^4$, where $\Omega \subset \mathbb{R}^2$ is open and $F : (x, y) \to (x, y, u(x, y), v(x, y))$. Then F is a weak 2-helix w.r. to e_3 and e_4 with the same angle $\frac{\pi}{4}$ if and only if the following system is satisfied on Ω:*

$$(H) = \begin{cases} \|\nabla v\| = \|\nabla u\| , \\ Det(\nabla v, \nabla u) = \pm 1 \end{cases}$$

Proof. The conditions to be helix w.r. to e_3 and e_4 with angle $\frac{\pi}{4}$ on Ω are (see Proposition 2.1):

$$(*) = \begin{cases} \|e_3 \wedge F_x \wedge F_y\|^2 = \frac{1}{2}\|F_x \wedge F_y\|^2 , \\ \|e_4 \wedge F_x \wedge F_y\|^2 = \frac{1}{2}\|F_x \wedge F_y\|^2 \end{cases}$$

From

$$\begin{aligned} F_x \wedge F_y &= (e_1 + u_x e_3 + v_x e_4) \wedge (e_2 + u_y e_3 + v_y e_4) = \\ &= e_1 \wedge e_2 + u_y e_1 \wedge e_3 + v_y e_1 \wedge e_4 + \\ &+ u_x e_3 \wedge e_2 + u_x v_y e_3 \wedge e_4 + \\ &+ v_x e_4 \wedge e_2 + v_x u_y e_4 \wedge e_3 \end{aligned}$$

$$e_3 \wedge F_x \wedge F_y = e_3 \wedge e_1 \wedge e_2 + v_y e_3 \wedge e_1 \wedge e_4 + v_x e_3 \wedge e_4 \wedge e_2 ,$$

$$e_4 \wedge F_x \wedge F_y = e_4 \wedge e_1 \wedge e_2 + u_y e_4 \wedge e_1 \wedge e_3 + u_x e_4 \wedge e_3 \wedge e_2 ,$$

we get

$$\|F_x \wedge F_y\|^2 = 1 + \|\nabla u\|^2 + \|\nabla v\|^2 + Det(\nabla u, \nabla v)^2$$

$$\|e_3 \wedge F_x \wedge F_y\|^2 = 1 + \|\nabla v\|^2$$
$$\|e_4 \wedge F_x \wedge F_y\|^2 = 1 + \|\nabla u\|^2$$

Now $(*)$ holds if and only if (**) holds where

$$(**) = \begin{cases} \|e_3 \wedge F_x \wedge F_y\|^2 + \|e_4 \wedge F_x \wedge F_y\|^2 = \|F_x \wedge F_y\|^2 , \\ \|e_3 \wedge F_x \wedge F_y\|^2 = \|e_4 \wedge F_x \wedge F_y\|^2 \end{cases}$$

and this is equivalent to system (H). $\square$

4.1 The non linear operator L

Let $L : \mathbb{R}^2 \setminus D \to \mathbb{R}^2$, where D is the unit disc, be given by:

$$L(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} A(x,y) \\ B(x,y) \end{bmatrix} = \begin{bmatrix} \frac{-y + x\sqrt{(x^2+y^2)^2-1}}{x^2+y^2} \\ \frac{x + y\sqrt{(x^2+y^2)^2-1}}{x^2+y^2} \end{bmatrix}$$

The operator L has the following properties:

$$\|L(\overrightarrow{v})\| = \|\overrightarrow{v}\| ,$$
$$Det(\overrightarrow{v} , L(\overrightarrow{v})) = 1$$

Proposition 4.2 *Let u, v be smooth functions on Ω such that $L(\nabla u) = \nabla v$. If ∇u is not constant on Ω then $F(x,y) = (x, y, u(x,y), v(x,y))$ is a weak non strong 2-helix.*

Proof. Since ∇u is not constant F can not be totally geodesic. Proposition 4.1 implies that F is a weak 2-helix. The last part of Theorem 3.1 implies that F can not be a strong 2-helix. Indeed, if F is strong then there exists a helix direction $d \in span\{e_3, e_4\}$ of angle $\theta = 0$ which is impossible. $\square$

4.2 The hyperbolic quasi-linear PDE associated to the operator L

It is standard to get a quasi-linear PDE from L. Namely, given ∇u we impose the condition on $L(\nabla u)$ to be a gradient, i.e. equality of mixed derivatives. Thus, such PDE is :

$$(A(u_x, u_y))_y = (B(u_x, u_y))_x ,$$

equivalently,

$$A_1 u_{xy} + A_2 u_{yy} = B_1 u_{xx} + B_2 u_{yx} .$$

Thus we get the following quasi-linear PDE

$$(QL) \quad 0 = B_1 u_{xx} + (-A_1 + B_2) u_{xy} + (-A_2) u_{yy} \ .$$

A long but straightforward computation shows that the above equation is hyperbolic, i.e.

$$-B_1 A_2 - \frac{(-A_1 + B_2)^2}{4} = \frac{-1}{((u_x)^2 + (u_y)^2)^2 - 1} < 0 \ .$$

4.3 The existence of a non strong weak 2-helix in $\mathbb{R}^4$

Notice that A_1, A_2, B_1, B_2 are real analytic functions. So we can apply Cauchy-Kowalevski theorem to solve the equation (QL) as soon as we can find non characteristic real analytic initial data (see [J, p. 56] for details).

So in order to show the existence of a real analytic function u solving (QL) equation and such that ∇u is not constant we have to find adequated analytic inital data for the Cauchy problem, i.e. a non characteristic hypersurface S with the normal derivatives of u along S.

Let us call, as is standard, $p = u_x, q = u_y$ and let $I_0 = (q_0, p_0)$ be a point such that $p_0^2 + q_0^2 > 1$. Let g be the following bilinear form defined near the point I_0:

$$g = B_1 dq^2 + (-A_1 + B_2) dpdq + (-A_2) dp^2 \ .$$

Since the PDE (QL) is hyperbolic g gives a Lorentz metric around I_0. So we can find an analytic vector field V such that

$$g(V, V) \neq 0$$

around I_0. We can also regard (p_0, q_0) as point in the (x, y) plane. So $V(x, y)$ is also a vector field around (p_0, q_0) in the plane (x, y). It is not difficult to see that there exists an analytic curve $\gamma(t)$ such that $\gamma(0) = (p_0, q_0)$ and $\langle \gamma'(t), V(\gamma(t)) \rangle = 0$. That is to say $V(t)$ is normal to $\gamma(t)$. Consider the following initial conditions on $\gamma(t)$ for the Cauchy problem for the quasi-linear PDE (QL):

$$u(t) = \frac{\|\gamma(t)\|}{2}$$

$$\frac{\partial u(t)}{\partial V} = \langle \gamma(t), V(t) \rangle$$

Then this initial condition is analytic and non characteristic. Indeed, the condition $g(V, V) \neq 0$ holds for the initial data $(u(t), \frac{\partial u(t)}{\partial V})$ and this is exactly the condition on the initial data to be non characteristic.
Thus we can apply Cauchy-Kowalevski theorem to get a solution u around (p_0, q_0). Observe that $(\nabla u)(t) = \gamma(t)$. Thus ∇u is not constant. We have proved the following theorem.

Theorem 4.3 *There exists a non linear function* u *such that* $L(\nabla u)$ *is a gradient.*

So Theorem 1.1 follows from the above theorem by using Proposition 4.2.

5 The helix-property of Dillen-Nölker

In [DN] the authors introduced the concept of *helix-property* for submanifolds of a pseudo-euclidean space.

Definition 5.1 *[DN, Definition 3.1, p.48] An isometric immersion* $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ *satisfies the helix-property if there is a fixed vector subspace* $\mathbb{R}^l$ *of* $\mathbb{R}^m$ *and a fixed linear map* $C : \mathbb{R}^l \to \mathbb{R}^n$ *such that for all* $p \in U$ *,* $v \in T_pU$ *and* $b \in \mathbb{R}^l$

$$\langle f_* v, Cb \rangle = \langle v, b \rangle \,.$$

It is possible to give a characterization of submanifolds who satisfies the helix-property in terms of its second fundamental form α. Recall that an isometric immersion $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ (resp. a submanifold $M \subset \mathbb{R}^n$) is called *full* if the image $f(U)$ (resp. M) is not contained in a proper afine subspace of $\mathbb{R}^n$.

Proposition 5.2 *[DN, Proposition 3.4, p.49] A full isometric immersion* $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ *satisfies the helix-property if and only if there is a fixed linear subspace* $\mathbb{R}^l$ *of* $\mathbb{R}^n$ *such that* $\langle \alpha(X, Y), V \rangle = 0$ *for all* $V \in \mathbb{R}^l$ *and for all tangent vectors* X *and* Y *.*

The above proposition imply that the helix-property is related with the extrinsic geometry of the geodesics of $f(U) = M \subset \mathbb{R}^n$. Namely, any geodesic of M is a helix, in the classical sense, w.r. to any direction of the subspace $\mathbb{R}^l$.

The following two propositions explain the relation between the helix-property of Dillen-Nölker and the concept of weak/strong helix submanifold introduced in this paper.

Proposition 5.3 *Assume that the full submanifold* $M \subset \mathbb{R}^n$ *satisfies the helix-property w.r. to the subspace* $\mathbb{R}^l$ *as in Proposition 5.2. Then the subspace* $\mathbb{R}^l$ *is contained in* $\mathcal{H}(M)$ *, i.e. any* $V \in \mathbb{R}^l$ *is a helix direction of* M *. In particular,* $\mathcal{H}(M)$ *contains* l *linearly independent helix directions and so* M *is a weak* l *-helix submanifold.*

Before giving the proof, let us explain why we can not conclude that the submanifold M as in the above proposition is a strong l-helix. This is so since the set $\mathcal{H}(M)$ of helix directions of M can be bigger than the subspace $\mathbb{R}^l$ of helix directions coming from the helix-property. For example, let $F : U \subset \mathbb{R}^2 \to \mathbb{R}^4$ be a weak non strong 2-helix given by Theorem 1.1.

Define $f : U \times \mathbb{R} \to \mathbb{R}^5$ by $f(x, y, z) = (F(x, y), z)$. Then f is an isometric immersion which satisfies the helix-property (see [DN, Example 3.2, p.49]) but f is not a strong helix since $\mathcal{H}(M = f(U \times \mathbb{R}))$ is not a linear subspace of $\mathbb{R}^5$.

Proof of Proposition 5.3. According to Proposition 2.1 it is enough to show that the length of the projection $\pi_p(V)$ for a fixed $V \in \mathbb{R}^l$ does not depends upon $p \in M$. Let us call V^M the vector field on M given by the projection of V, i.e., $V^M(p) = \pi_p(V)$. Let h_V be the height function $h_V : \mathbb{R}^n \to \mathbb{R}$, $h_V(p) := \langle p, V \rangle$ associated to the fixed $V \in \mathbb{R}^l$. Notice that the vector field V^M on M is the gradient of the restriction of h_V to M. Thus, V^M satisfies

$$\langle \nabla_X V^M, Y \rangle = \langle \nabla_Y V^M, X \rangle \ ,$$

for all tangent vectors X and Y of M, where ∇ is the Levi-Civita connection of M. Let now $\gamma(t)$ be an arbitrary geodesic of M. Then

$$\frac{d\langle \gamma(t), V \rangle}{dt} = \langle \gamma'(t), V \rangle$$

and

$$\frac{d^2\langle \gamma(t), V \rangle}{dt^2} = \langle \gamma''(t), V \rangle = \langle \alpha(\gamma'(t), \gamma'(t)), V \rangle = 0,$$

where the last equality is due to Proposition 5.2. So we get

$$\langle \gamma'(t), V \rangle = \langle \gamma'(t), V^M \rangle = cte \ .$$

By using the Levi-Civita connection of M we get:

$$0 = \frac{d\langle \gamma'(t), V^M \rangle}{dt} = \langle \gamma'(t), \nabla_{\gamma'(t)} V^M \rangle \ .$$

Since the geodesic $\gamma(t)$ is arbitrary we get that V^M is a Killing vector field of M, i.e., $\langle X, \nabla_X V^M \rangle = 0$ for all tangent vectors X of M. Thus, we get that V^M is a parallel vector field of M since $\langle X, \nabla_Y V^M \rangle$ is also symmetric in X, Y being V^M a gradient. Now the length of V^M is clearly constant on M and this shows that V is a helix direction of M. □

Proposition 5.4 *Let $M \subset \mathbb{R}^n$ be a full strong helix submanifold of the euclidean space. Then M satisfies the helix-property w.r. to the subspace $\mathcal{H}(M)$ if and only if the projection V^M is a parallel vector field of M for all $V \in \mathcal{H}(M)$, where $V^M(p) := \pi_p(V)$.*

Proof. Notice that the proof of the *only if* part is identical to the proof of Proposition 5.3. Let now $V \in \mathcal{H}(M)$ be a helix direction. Let $V = \cos(\theta)\,\mathrm{T} + \sin(\theta)\xi$ be the decomposition of V into tangent and normal components. By taking derivatives w.r. to $X \in TM$ we get

$$0 = \cos(\theta)\nabla_X \mathrm{T}(p) - \sin(\theta)A^{\xi}(X) \text{ and} \tag{12}$$
$$0 = \cos(\theta)\alpha(X, \mathrm{T}(p)) + \sin(\theta)\nabla^{\perp}_X \xi. \tag{13}$$

Assume now that the projection $V^M = \cos(\theta)\,\mathrm{T}$ of V onto M is parallel. Then Equation 12 imply that $A^{\xi} \equiv 0$ and so $\xi \perp span\{\alpha(X,Y) : X, Y \in TM\}$. Now Proposition 5.2 imply that M satisfies the helix-property w.r. to $\mathcal{H}(M)$. $\square$

An example of a strong 1-helix that does not satisfies the helix-property is provided by the standard cone $\mathcal{C} := \{(x,y,z) : x^2 + y^2 = z^2, z > 0\}$. The normal space of the cone $\mathcal{C}$ makes a constant angle with the z-axis, so $\mathcal{C}$ is a strong 1-helix of $\mathbb{R}^3$. Notice that the linear span of the second fundamental form of the cone $\mathcal{C}$ is $\mathbb{R}^3$. Thus, Proposition 5.2 imply that the cone $\mathcal{C}$ does not satisfies the helix-property. Actually it is not difficult to see that cylinders over plane curves (i.e., $\mathbb{R} \times \gamma \subset \mathbb{R} \times \mathbb{R}^2$, where γ is a curve in $\mathbb{R}^2$) are the only surfaces of $\mathbb{R}^3$ satisfying the helix-property.

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