Analysis of Diffraction by three semi-infinite planes using Wiener-Hopf method

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Abstract

The diffraction of an incident plane wave on three equally spaced half-planes is formulated in terms of Wiener-Hopf (W-H) equations using an equivalent circuit model. The order of matrix factorization is reduced due to the symmetry of the problem. A new solution of this classic Wiener-Hopf problem is proposed and the formulation of the problem is presented for the general case of skew incident plane wave.

1. Formulation

The well-known in literature diffraction problem of three equally spaced semi-infinite PEC planes [1] is formulated in terms of Wiener-Hopf (W-H) equations. We consider the general case of an arbitrary plane wave incident (1) on the structure:

$$\begin{cases} E_z^i = E_o e^{j\tau_o \rho\cos(\varphi-\varphi_o)} e^{-j\alpha_o z} \\ H_z^i = H_o e^{j\tau_o \rho\cos(\varphi-\varphi_o)} e^{-j\alpha_o z} \end{cases} \tag{1}$$

where k is the free-space wave number, $\tau_o = k\sin\beta$, $\alpha_o = k\cos\beta$, and β and φ_o are respectively the zenithal and azimuthal angles of the incident wave. Figure 1 illustrates the geometry of the problem for an E-polarized normal incident plane wave ($\beta = \pi/2$).

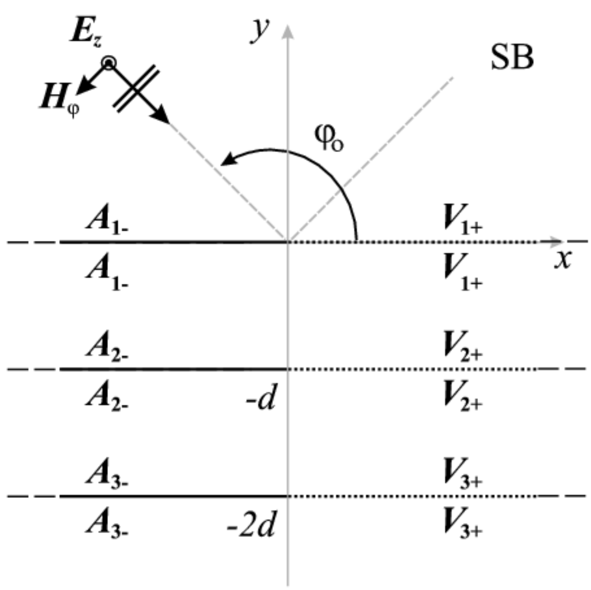

Figure 1: Diffraction on three equally spaced semi-infinite PEC planes: E-polarization.
The figure reports the Wiener-Hopf unknowns.

In order to formulate the problem in the spectral domain we apply the Fourier transform to the transverse components of the electromagnetic fields (2):

$$\mathbf{V}(\eta, y) = e^{j\alpha_o z}\int_{-\infty}^{\infty} \hat{y}\times\mathbf{E}_t(x,y,z)e^{j\eta x}dx$$
$$\mathbf{I}(\eta, y) = e^{j\alpha_o z}\int_{-\infty}^{\infty} \mathbf{H}_t(x,y,z)e^{j\eta x}dx \qquad (2)$$

We define an equivalent circuit problem, see Figure 2, where the characteristic admittance of the free space is a matrix [2]:

$$\mathbf{Y_c} = \frac{Y_o}{k\sqrt{\tau_O^2-\eta^2}}\begin{vmatrix} \tau_O^2 & -\eta\,\alpha_O \\ -\eta\,\alpha_O & k^2-\eta^2 \end{vmatrix} \qquad (3)$$

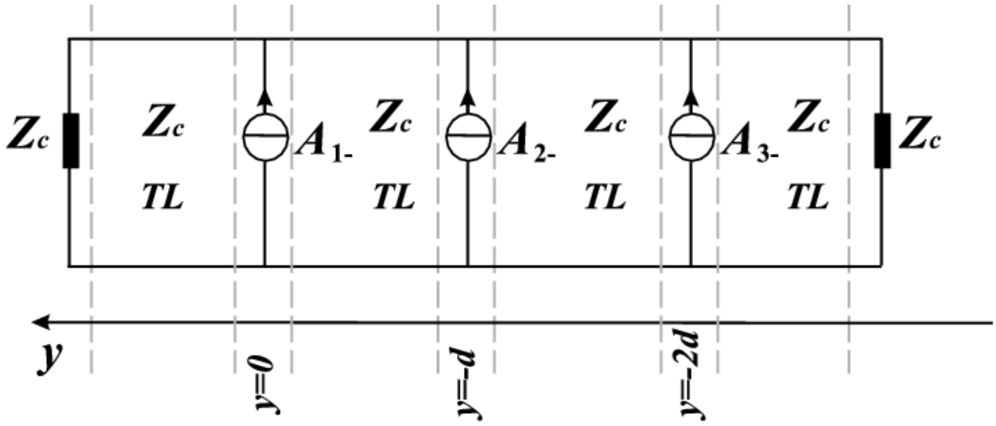

Figure 2: Equivalent Circuit model of the problem under study.

The equivalent circuit models the propagation inside the region defined by the half-planes (region 1, 2 and 3) using a Π model of matrix admittance:

$$\mathbf{Y}_1 = \mathbf{Y}_2 = j\mathbf{Y}_c \tan\frac{\tau d}{2},\ \mathbf{Y}_3 = -j\mathbf{Y}_c\frac{1}{\sin\tau d} \qquad (4)$$

where $\tau = \sqrt{\tau_o^2-\eta^2}$.

The circuit nodal analysis provide the three following equations

$$\text{node } 1 \Rightarrow \ (\boldsymbol{Y}_c+\boldsymbol{Y}_1+\boldsymbol{Y}_3)\boldsymbol{V}_{1+} - \boldsymbol{Y}_3\boldsymbol{V}_{2+} = \boldsymbol{A}_{1-}$$
$$\text{node } 3 \Rightarrow \ (\boldsymbol{Y}_c+\boldsymbol{Y}_1+\boldsymbol{Y}_3)\boldsymbol{V}_{3+} - \boldsymbol{Y}_3\boldsymbol{V}_{2+} = \boldsymbol{A}_{3-} \qquad (5)$$
$$\text{node } 2 \Rightarrow \ 2(\boldsymbol{Y}_1+\boldsymbol{Y}_3)\boldsymbol{V}_{2+} - \boldsymbol{Y}_3\boldsymbol{V}_{1+} - \boldsymbol{Y}_3\boldsymbol{V}_{3+} = \boldsymbol{A}_{2-}$$

where $\boldsymbol{V}_{1+}, \boldsymbol{V}_{2+}$ and $\boldsymbol{V}_{3+}$ and $\hat{\boldsymbol{y}}\times\boldsymbol{A}_{1-}, \hat{\boldsymbol{y}}\times\boldsymbol{A}_{2-}$ and $\hat{\boldsymbol{y}}\times\boldsymbol{A}_{3-}$ are respectively the Fourier transforms of the transverse electric field on the three apertures $y = 0,-d,-2d$ and the total currents induced on the three half- planes. The previous equations constitute a W-H system of third order where the unknowns and the admittance are respectively vector and matrix of size 2, therefore the W-H problem is a scalar matrix problem of size 6.

The symmetry of the geometry allows to reduce the order of the vector W-H problem. In fact by summing the first two equations we get the following system of equations:

$$\begin{cases} \left(\boldsymbol{Y_c}+\boldsymbol{Y_1}+\boldsymbol{Y_3}\right)(\boldsymbol{V_{1+}}+\boldsymbol{V_{3+}}) - 2\boldsymbol{Y_3}\boldsymbol{V_{2+}} = \boldsymbol{A_{1-}}+\boldsymbol{A_{3-}} \\ -\boldsymbol{Y_3}(\boldsymbol{V_{1+}}+\boldsymbol{V_{3+}}) + 2\left(\boldsymbol{Y_1}+\boldsymbol{Y_3}\right)\boldsymbol{V_{2+}} = \boldsymbol{A_{2-}} \end{cases} \tag{6}$$

Equation (6) is a homogeneous vector W-H problem of order 2 ($\boldsymbol{G}\cdot\boldsymbol{V}_+ = \boldsymbol{A}_-$), where the plus unknowns are $(\boldsymbol{V_{1+}}+\boldsymbol{V_{3+}})$ and $2\boldsymbol{V_{2+}}$, the minus unknowns are $(\boldsymbol{A_{1-}}+\boldsymbol{A_{3-}})$ and $\boldsymbol{A_{2-}}$ and the kernel takes the following form:

$$\boldsymbol{G} = \begin{vmatrix} \left(\boldsymbol{Y_c}+\boldsymbol{Y_1}+\boldsymbol{Y_3}\right) & -\boldsymbol{Y_3} \\ -\boldsymbol{Y_3} & \left(\boldsymbol{Y_1}+\boldsymbol{Y_3}\right) \end{vmatrix} = \boldsymbol{Y_c}\otimes\boldsymbol{H} \tag{7}$$

where the operator $\otimes$ is the Kronecker matrix product and where

$$\boldsymbol{H} = \begin{vmatrix} 1-j\,Cot(\tau d) & j\,Csc(\tau d) \\ j\,Csc(\tau d) & -j\,Cot(\tau d) \end{vmatrix} = \frac{2}{1-e^{-2j\tau d}} \begin{vmatrix} 1 & -e^{-j\tau d} \\ -e^{-j\tau d} & \left(\dfrac{1+e^{-j\tau d}}{2}\right) \end{vmatrix} \tag{8}$$

2. Solution

In order to obtain the solution of the Wiener-Hopf problem we need to factorize (7).
By applying the mixed-product property of the Kronecker product (9) to the plus and minus factorized matrices of $\boldsymbol{Y_c}$ and $\boldsymbol{H}$ we obtain the factorization of $\boldsymbol{G}$, see (10).

$$A\cdot C\otimes B\cdot D = A\otimes B\cdot C\otimes D \tag{9}$$

$$\boldsymbol{G} = \boldsymbol{G}_-\cdot\boldsymbol{G}_+ = \boldsymbol{Y}_{c-}\cdot\boldsymbol{Y}_{c+}\otimes\boldsymbol{H}_-\cdot\boldsymbol{H}_+ = \boldsymbol{Y}_{c-}\otimes\boldsymbol{H}_-\cdot\boldsymbol{Y}_{c+}\otimes\boldsymbol{H}_+ \tag{10}$$

From the homogenous Wiener-Hopf equation (6) we obtain the normal form ($G(\alpha)F_+(\alpha) = F_-(\alpha) + F_{o+}(\alpha)$) by extracting the source term related to the incident plane wave:

$$\boldsymbol{G}\cdot\boldsymbol{V}_+ = \boldsymbol{A}_-^{d} + \frac{\boldsymbol{R_o}}{\eta-\eta_o} \tag{11}$$

$\boldsymbol{R_o}$ is the source obtained as GO component from the incident field.
We remind the spectral quantities are functions of the spectral variable η. The solution takes the following form:

$$\begin{cases} \boldsymbol{V}_+(\eta) = \boldsymbol{G}_+^{-1}(\eta)\cdot\boldsymbol{G}_+^{-1}(\eta_o)\cdot\dfrac{\boldsymbol{R_o}}{\eta-\eta_o} \\ \boldsymbol{A}_-(\eta) = \boldsymbol{G}\ (\eta)\cdot\boldsymbol{V}_+(\eta) = \boldsymbol{G}_-(\eta)\cdot\boldsymbol{G}_+^{-1}(\eta_o)\cdot\dfrac{\boldsymbol{R_o}}{\eta-\eta_o} \end{cases} \tag{11}$$

We note that the matrix $\boldsymbol{Y_c}$ can be factorized in closed form. However no exact factorization has been obtained for matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ up to now, therefore we can apply the approximate procedure presented in [3, 4] based on integral equations.
Numerical results will be presented at the Conference.

3. Acknowledgments

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4. References

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