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A geometric proof of the Karpelevich-Mostow's Theorem

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Abstract

In this paper we give a geometric proof of the Karpelevich's theorem that asserts that a semisimple Lie subgroup of isometries, of a symmetric space of non compact type, has a totally geodesic orbit. In fact, this equivalent to a well-known result of Mostow about exixtence of compatible Cartan decompositions.

1. *Introduction.*

In this paper we address the problem of giving a geometric proof of the following theorem of Karpelevich.

Theorem 1.1. *(Karpelevich [7]) Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected and semisimple subgroup $G \subset Iso(M)$ has a totally geodesic orbit $G.p \subset M$.*

It is well-known that Karpelevich's theorem is equivalent to the following algebraic theorem.

Theorem 1.2. *(Mostow [8, Theorem 6]) Let $\mathfrak{g}'$ be a real semisimple Lie algebra of non compact type and let $\mathfrak{g} \subset \mathfrak{g}'$ be a semisimple Lie subalgebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$. Then there exists a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ for $\mathfrak{g}'$ such that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$.*

The proof of the above theorems is very algebraic in nature and uses delicate arguments related to automorphisms of semisimple Lie algebras.

For the real hyperbolic spaces, i.e. when $\mathfrak{g}' = \mathfrak{so}(n,1)$, there are two geometric proofs of Karpelevich's theorem [4], [2]. The proof in [4] is based on the study of minimal orbits of isometries subgroups, i.e. orbits with zero mean curvature. The approach in [2] is based on hyperbolic dynamics. It is interesting to note that both proofs are strongly based on the fact that the boundary at infinity of real hyperbolic spaces has a simple structure.

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The only non-trivial algebraic tool that we will use is the existence of a Cartan decomposition of a non compact semisimple Lie algebra. But this can also be proved geometrically as was explained by S.K. Donaldson in [**5**].

Here is a brief explanation of our proof of Theorem 1.1. We first show that a simple subgroup $G \subset \mathrm{Iso}(M)$ has a minimal orbit $G.p \subset M$. Then, by using a standard totally geodesic embedding $M \hookrightarrow \mathcal{P}$, where $\mathcal{P} = SL(n, \mathbb{R})/SO(n)$, we will show that $G.p$ is, actually, a totally geodesic submanifold of M.

2. *Preliminaries.*

The results in this section are well known and are included to orient the non-specialist reader.

The equivalence between Theorems 1.1 and 1.2 is a consequence of the following Elie Cartan's famous and remarkable theorem.

Theorem 2.1. *(Elie Cartan) Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then the Lie group Iso(M) is semisimple of non compact type. Conversely, if $\mathfrak{g}$ is a semisimple Lie algebra of non compact type then there exist a Riemannian symmetric space M of non positive curvature without flat factor such that $\mathfrak{g}$ is the Lie algebra of Iso(M).*

The difficult part of the proof of the above theorem is the second part. Namely, the construction of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is maximal compact subalgebra of $\mathfrak{g}$ and the Killing form B of $\mathfrak{g}$ is positive definite on $\mathfrak{p}$. The standard and well-known proof of the existence of a Cartan decomposition is long and via the classification theory of complex semisimple Lie algebras, i.e. the existence of a real compact form (see e.g. [**6**]). There is also a direct and geometric proof of the existence of a Cartan decomposition [**5**].

On the other hand, when $\mathfrak{g} = \mathrm{Lie}(\mathrm{Iso}(M))$, where M is a Riemannian symmetric space of non positive curvature without flat factor, a Cartan decomposition of $\mathfrak{g}$ can be constructed geometrically. Namely, $\mathfrak{g} = \mathrm{Lie}(\mathrm{Iso}(M)) = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the isotropy group $K_p \subset \mathrm{Iso}(M)$ and $\mathfrak{p} := \{X \in \mathrm{Lie}(\mathrm{Iso}(M)) : (\nabla X)_p = 0\}$.

It is well-known that the Riemannian symmetric spaces $\mathcal{P} = SL(n, \mathbb{R})/SO(n)$ are the *universal* Riemannian symmetric space of non positive curvature. Namely, any Riemannian symmetric space of non compact type $M = G/K$ can be totally geodesically embedded in some $\mathcal{P}$ (up to rescaling the metric in the irreducible De Rham factors). A proof of this fact follows from the following well-known result (c.f. Theorem 1 in [**5**]).

Proposition 2.2. *Let $\mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{R})$ be a semisimple Lie subalgebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then there exists a Cartan decomposition $\mathfrak{sl}(n, \mathbb{R}) = \mathcal{A} \oplus \mathcal{S}$ such that $\mathfrak{k} \subset \mathcal{A}$ and $\mathfrak{p} \subset \mathcal{S}$. Thus, if $G \subset SL(n)$ is semisimple, G has a totally geodesic orbit in $\mathcal{P} = SL(n)/SO(n)$. Indeed, any Riemannian symmetric space of non positive curvature M, without flat factor, can be totally geodesically embedded in some $\mathcal{P} = SL(n)/SO(n)$.*

Proof. Notice that any Cartan decomposition of $\mathfrak{sl}(n, \mathbb{R})$ is given by the anti-symmetric $\mathcal{A}$ and symmetric matrices $\mathcal{S}$ w.r.t. a positive definite inner product on $\mathbb{R}^n$. Since $g^* := \mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$, there exists a positive definite Hermitian form $(\ |\ )$ of $\mathbb{C}^n$

invariant by g^*. By defining $\langle\, ,\, \rangle := Real(\mid)$ it follows that $\mathfrak{k} \subset \mathcal{A}$ and $\mathfrak{p} \subset \mathcal{S}$. □

Let $S_\infty(M)$ be the *sphere or boundary at infinity* of M, i.e. $S_\infty(M)$ is the set of equivalence classes of asymptotic geodesics rays (see [**5**] or [**3**, Chapter II.8] for details).

Here is another corollary of the existence of the totally geodesic embedding $M \hookrightarrow \mathcal{P}$.

Corollary 2.3. *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then a connected and semisimple Lie subgroup $G \subset$ Iso(M) of non compact type has no fixed points in $S_\infty(M)$.*
This corollary is false see Corrigendum in http://arxiv.org/abs/1104.0892

We include the following proposition.

Proposition 2.4. *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Let $S = \mathbb{R}^N \times M$ be a symmetric space of non positive curvature with flat factor $\mathbb{R}^N$. If $G \subset$ Iso(S) is a connected non compact simple Lie group then $G \subset$ Iso(M).*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. Then the projection $\pi : \mathfrak{g} \mapsto Lie(Iso(\mathbb{R}^N))$ is injective or trivial i.e. $\pi \equiv 0$. If π is injective then a further composition with the projection to $so(N)$ gives that $\mathfrak{g}$ must carry a bi-invariant metric. So, $\mathfrak{g}$ can not be simple and non compact. □

Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. A subspace $T \subset \mathfrak{p}$ is called a *Lie triple system* if $[T, [T, T]] \subset T$. It is well-known that there is a 1-1 correspondence between Lie triple systems T of $\mathfrak{p}$ and totally geodesic submanifolds through the base point $[K] \in G/K$ (see [**6**]).

3. *Minimal and totally geodesic orbits.*

We will need the following proposition (see Lemma 3.1. in [**4**] or Proposition 5.5. in [**1**]).

Proposition 3.1. *Let M be a Riemannian symmetric space of non positive curvature without flat factor and let $G \subset$ Iso(M) be a connected group of isometries. Assume that G has a totally geodesic orbit $G.p$. Then any other minimal orbit $G.q$ is also a totally geodesic submanifold of M. Moreover, if G is semisimple then the union of totally geodesic G-orbits T_G is a totally geodesic submanifold of M which is a Riemannian product $T_G = (G.p) \times A$ where A is a totally geodesic submanifold of M.*

Proof. Let $G.p$ be the totally geodesic orbit and let $G.q \neq \{q\}$ be another orbit. Let γ be a geodesic in M that minimizes the distance between q and $G.p$ (such geodesic do exists since totally geodesic submanifolds of M are closed and embedded). Eventually by changing the base point p by another in the orbit we may assume that $\gamma(0) = p$ and $\gamma(1) = q$. A simple computation using the Killing equation shows that $\dot{\gamma}(t)$ is perpendicular to $T_{\gamma(t)}(G.\gamma(t))$, for all t.

Let X be a Killing field in the Lie algebra of G such that $X.q \neq 0$ and let ϕ^X_s be the one-parameter group of isometries generated by X. Define $h : I \times \mathbb{R} \to M$ by $h_s(t) := \phi^X_s.\gamma(t)$. Note that $X.h_s(t) = \frac{\partial h}{\partial s}$ and that, for a fixed s, $h_s(t)$ is a geodesic.

Let $A_{\dot\gamma(t)}$ be the shape operator, in the direction of $\dot\gamma(t)$ of the orbit $G.\gamma(t)$. Define $f(t) := -\langle A_{\dot\gamma(t)}(X.\gamma(t)), X.\gamma(t)\rangle = \langle \frac{D}{\partial s}\frac{\partial h}{\partial t}, X.h_s(t)\rangle\mid_{s=0}$. Now a computation as in Lemma 3.1. in [**4**] or Proposition 5.5. in [**1**] implies that $\frac{d}{dt}f(t) \geq 0$. Since $f(0) = 0$, due to the fact that $G.p$ is totally geodesic, we obtain that $f(1) = -\langle A_{\dot\gamma(1)}(X.q), X.q\rangle \geq 0$. Hence $A_{\dot\gamma(1)}$ is negative semidefinite. Since $G.q$ is minimal, trace$(A_{\dot\gamma(1)}) = 0$, we get that $f(t) \equiv 0$. Thus, $\langle R(\dot\gamma(t), X.\gamma(t))\dot\gamma(t), X.\gamma(t)\rangle \equiv 0$ and $\nabla_{\dot\gamma(t)}(X.\gamma(t)) \equiv 0$. Notice that the tangent spaces $T_{\gamma(t)}G\cdot\gamma(t)$ are parallel along $\gamma(t)$ in M. So the normal spaces $\nu_{\gamma(t)}G\cdot\gamma(t)$ are also parallel along $\gamma(t)$ in M. Since M is a symmetric space of non positive curvature the condition $\langle R(\dot\gamma(t), X.\gamma(t))\dot\gamma(t), X.\gamma(t)\rangle \equiv 0$ implies $R(\dot\gamma(t), X.\gamma(t))(\cdot) \equiv 0$. Let $\eta(t) \in \nu_{\gamma(t)}G.\gamma(t)$ be a parallel vector along $\gamma(t)$ and let X, Y two Killing vector fields in the Lie algebra of G. Then $\frac{d}{dt}\langle \nabla_X Y, \eta(t)\rangle = \langle \nabla_{\dot\gamma(t)}\nabla_X Y, \eta(t)\rangle = \langle \nabla_X\nabla_{\dot\gamma(t)}Y, \eta(t)\rangle + \langle R(\dot\gamma(t), X.\gamma(t))(Y), \eta(t)\rangle \equiv 0$. Since $\langle \nabla_X Y, \eta(0)\rangle = 0$ we get that the G-orbits $G.\gamma(t)$ are totally geodesic submanifolds of M. This show the first part.

For the second part let $K' := \mathrm{Iso}(M)_p$ be the isotropy subgroup at $p \in M$ and let $\mathfrak{k}'$ its Lie algebra. Let $\mathfrak{p}' \subset \mathrm{Lie}(\mathrm{Iso}(M))$ be such that $X \in \mathfrak{p}'$ iff $(\nabla X)_p = 0$. Thus, $\mathrm{Lie}(\mathrm{Iso}(M)) = \mathfrak{k}' \oplus \mathfrak{p}'$ is a Cartan decomposition of $\mathrm{Lie}(\mathrm{Iso}(M))$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Since $G.p$ is totally geodesic in M we get that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$. Let $\alpha := \{Y \in \mathfrak{p}' : Y \perp \mathfrak{p} \text{ and } [Y, \mathfrak{p}] = 0\}$ which is a Lie triple system of $\mathfrak{p}'$. Moreover, $\mathfrak{n} := \mathfrak{p} \oplus \alpha$ is also a Lie triple system of $\mathfrak{p}'$. So, $N := exp_p(\mathfrak{n}) = exp_p(\mathfrak{p}) \times exp_p(\alpha)$ is a G-invariant totally geodesic submanifold of M. Notice that (by construction) $N \subset T_G$.

Let $G.q$ any other totally geodesic G-orbit. From the computation in the first part we get $R(\dot\gamma(t), X.\gamma(t))(\cdot) \equiv 0$ which implies $\gamma'(0) \in \alpha$. This shows $T_G \subset N$. Then $N = T_G = (G.p) \times A$ where $A := exp_p(\alpha)$ is a totally geodesic submanifold of M associated to the Lie triple system α. □

4. *Karpelevich's Theorem for G a simple Lie group.*

Here is the first step to prove Theorem 1.1.

Theorem 4.1. *Let M be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected, simple and non compact Lie subgroup $G \subset Iso(M)$ has a minimal orbit $G.p \subset M$.*

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g} := Lie(G)$ and let $K \subset G$ be the maximal compact subgroup associated to $\mathfrak{k}$. Let Σ be the set of fixed points of K. Notice that $\Sigma \neq \emptyset$ by Cartan's fixed point theorem. Since G is simple all G-orbits $G.x$ through points in $x \in \Sigma$ are homothetic i.e. the Riemannian metric induced on $G.x$ and $G.y$ differ from a constant multiple for $x, y \in \Sigma$. Let $x_0 \in \Sigma$ be a point in Σ and let g_0 be the Riemannian metric on $G.x_0 = G/K$ induced by the Riemannian metric $g = \langle,\rangle$ of M. So if $y \in \Sigma$ the Riemannian metric g_y on $G.y$ is given by $g = \lambda(y)\cdot g_0$. Notice that if $X \in \mathfrak{p}$ is unitary at x_0 (i.e. $g_0(X(x_0), X(x_0)) = 1$) then $\lambda(y) = g(X(y), X(y)) = \|X(y)\|^2$. We claim that $\lambda(y)$ has a minimum in Σ. Indeed, if $y_n \to \infty \in S_\infty(\Sigma) \subset S_\infty(M)$ ($y_n \in \Sigma$) and $\lambda(y_n) \leq const$ then the monoparametric Lie group $\psi_t^X \subset G$ associated to any unitary $X \in \mathfrak{p}$ at $x_0 \in \Sigma$ must fix $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$. Thus, since $X \in \mathfrak{p}$ is arbitrary and $\mathfrak{p}$ generate $\mathfrak{g}$ we get that $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$ is a fixed point of G. This contradicts Corollary 2.3. So there exist $y_0 \in \Sigma$ such that λ has a minimum. Notice that the volume element Vol_y of an orbit $G.y$ is given by $\lambda^{\frac{n}{2}}Vol_{x_0}$, where $n = dim(G/K)$. Now a simple computation shows that the mean curvature vector of $G.y_0$ vanish and we are done. □

Now we are ready to prove Karpelevich's Theorem 1.1 for G a simple non compact Lie subgroup of $Iso(M)$.

THEOREM 4.2. *Let M be a Riemannian symmetric space of non positive curvature. Then any connected, simple and non compact Lie subgroup $G \subset Iso(M)$ has a totally geodesic orbit $G.p \subset M$.*

Proof. According to Proposition 2.4 we can assume that M has no flat factor. Let $i : (M, g) \hookrightarrow (\mathcal{P}, h)$ be a totally geodesic embedding as in Proposition 2.2. Notice that the pull-back metric i^*h can eventually differ (up to constant factors) from g on each irreducible De Rham factor of M. Anyway, totally geodesic submanifolds of (M, g) and (M, i^*h) are the same since totally geodesic submanifolds are defined in terms of the same Levi-Civita connection $\nabla^g = \nabla^{i^*h}$. Notice that G also acts by isometries on (M, i^*h). Indeed, G can be also regarded as a subgroup of $Iso(\mathcal{P})$. Now Proposition 2.2 implies that G has a totally geodesic orbit $G.p$ in $\mathcal{P}$. The above proposition shows that G has a minimal orbit $G.y_0$ in (M, i^*h). Since the embedding $M \hookrightarrow \mathcal{P}$ is totally geodesic we get that the G-orbit $G.y_0$ is also a minimal submanifold of $\mathcal{P}$. Then Proposition 3.1 implies that $G.y_0$ is a totally geodesic submanifold of $\mathcal{P}$. Thus, $G.y_0$ is a totally geodesic submanifold of (M, i^*h) and so $G.y_0$ is also a totally geodesic submanifold of (M, g). □

5. *Karpelevich's Theorem.*

Let $G \subset Iso(M)$ be a semisimple, connected Lie group. Then the Lie algebra $\mathfrak{g} = Lie(G) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a sum of a simple Lie algebra $\mathfrak{g}_1$ and a semisimple Lie algebra $\mathfrak{g}_2$. Due to Cartan's fixed point theorem we can assume that each simple factor of $\mathfrak{g}$ is non compact. We are going to make induction on the number of simple factors of the semisimple Lie algebra $\mathfrak{g}$. Let G_1 (resp. G_2) be the simple Lie group associated to $\mathfrak{g}_1$ (resp. the semisimple Lie subgroup associated to $\mathfrak{g}_2$). Let $T_{G_1} \subset M$ be the union of the totally geodesic orbits of the simple subgroup G_1 acting on M. Notice that Theorem 4.2 implies that $T_{G_1} \neq \emptyset$ and Proposition 3.1 implies that $T_{G_1} = (G_1 \cdot p) \times A$ is a totally geodesic submanifold of M, where $G_1 \cdot p$ is a totally geodesic G_1-orbit. Notice that G_2 acts on $T_{G_1} = (G_1 \cdot p) \times A$. Then $\mathfrak{g}_2$ (or eventually a quotient $\mathfrak{g}_2/\sim$ of it) acts on A. Since A is symmetric space of non positive curvature we get (by induction) that the semisimple subgroup G_2 (or eventually a quotient $G_2/\sim$ of it) has a totally geodesic orbit $S \subset A$. Then $(G_1 \cdot p) \times S$ is a totally geodesic orbit of G and this finish our proof of Karpelevich's Theorem 1.1.

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