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# CONVEX COMPARISONS FOR RANDOM SUMS IN RANDOM ENVIRONMENTS AND APPLICATIONS

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Recently, Belzunce, Ortega, Pellerey, and Ruiz [3] have obtained stochastic comparisons in increasing componentwise convex order sense for vectors of random sums when the summands and number of summands depend on a common random environment, which prove how the dependence among the random environmental parameters influences the variability of vectors of random sums. The main results presented here generalize the results in Belzunce et al. [3] by considering vectors of parameters instead of a couple of parameters and the increasing directionally convex order. Results on stochastic directional convexity of families of random sums under appropriate conditions on the families of summands and number of summands are obtained, which lead to the convex comparisons between random sums mentioned earlier. Different applications in actuarial science, reliability, and population growth are also provided to illustrate the main results.

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## 1. INTRODUCTION

Much research has been devoted to study conditions for the increasing convex order (also known as variability order, second stochastic dominance, or stop-loss order) of random sums (see Shaked and Shanthikumar [39], Pellerey [28] and [29], Denuit, Genest, and Marceau [7] or Kulik [17], among others). These results have found a wide field of applications in actuarial science, reliability, epidemics, economics, or queuing, where the random sums have been used to describe total claim amounts over a fixed time, accumulated wear of systems during time in cumulative damage shock models, number of individuals in a population that grows by means of a branching process, number of infected individuals in epidemic models, and so forth.

Dependencies between summands and number of summands are common in applicative problems and several models for such dependence have been studied in the last few years. In real problems, the random variables in the sum usually depend on some economical, physical, or geographical random environment. Recently, the impact of dependencies among the random environments on variability comparisons of multivariate vectors of random sums has been studied in Belzunce, Ortega, Pellerey and Ruiz [3] and Frostig and Denuit [12]. In addition, stochastic comparisons of random sums involving Bernoulli random variables have become of growing in interest and have been applied in insurance, engineering, and medicine (see Lefèvre and Utev [18], Hu and Wu [14], Frostig [11], or Hu and Ruan [13]).

In the literature, there are different multivariate extensions of the convex order from several extensions of convexity: in particular, the multivariate convex order, the componentwise convex order, and the directionally convex order (see the monograph by Shaked and Shanthikumar [39]). The directional convexity takes into account the order structure on the space, which the usual notion of convexity does not. The directionally convex order was introduced by Shaked and Shanthikumar [38] and has been proved to be useful in problems involving dependence in several contexts of applied probability (see, e.g., Meester and Shanthikumar [23,24]), Bäuerle and Rolski [2], Li and Xu [19], or Rüschendorf [35]). This order is strictly weaker than the supermodular order, which compares only dependence structure of vectors with fixed equal marginals. The directionally convex order tells about the dependence and variability of the marginals, which are not necessarily equal.

Belzunce et al. [3] have studied variability comparisons by means of the increasing componentwise convex order for two vectors of random sums. In that work, the summands and the number of summands are dependent by means of a couple of random parameters, which represent some environmental conditions. They have considered random sums defined by

$$Z_i(\theta_1,\theta_2) = \sum_{k=1}^{N_i(\theta_1)} X_{k,i}(\theta_2) \tag{1.1}$$

for $i = 1, 2, \ldots, m$, where $(\theta_1, \theta_2) \in \mathcal{T} \subseteq \mathbb{R}^2$ and $\boldsymbol{X}_i(\theta_2) = \{X_{k,i}(\theta_2), k \in \mathbb{N}\}$, $i = 1, \ldots, m$, is a sequence of nonnegative random variables, $(N_1(\theta_1), \ldots, N_m(\theta_1))$

is a vector of integer-valued random variables, and $\boldsymbol{X}_1(\theta_2), \ldots, \boldsymbol{X}_m(\theta_2)$ and $N_1(\theta_1), \ldots, N_m(\theta_1)$ are mutually independent.

In this article, we extend the above setting by considering dependence by means of a multivariate random vector of parameters. A main motivation for introducing multivariate random environments is clear from a practical point of view. For example, severity and number of claims in insurance for nature catastrophes such as hurricanes or earthquakes depend on geography as well as some other physical factors; in motor third-party liability insurance, there are several factors influencing the driving abilities (see Denuit, Dhaene, Goovaerts, and Kaas [6] for other examples).

Formally, let $\mathcal{T} \subseteq \mathbb{R}^{n_1}$ and $\mathcal{L} \subseteq \mathbb{R}^{n_2}$ be two sublattices in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, and let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{n_1}) \in \mathcal{T}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n_2}) \in \mathcal{L}$. Consider the sums defined by

$$Z_i(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \sum_{k=1}^{N_i(\boldsymbol{\theta})} X_{k,i}(\boldsymbol{\lambda}) \qquad \textbf{(1.2)}$$

for $i = 1, 2, \ldots, m$, where $X_{1,1}(\boldsymbol{\lambda})$, $X_{2,1}(\boldsymbol{\lambda}), \ldots, X_{1,m}(\boldsymbol{\lambda})$, $X_{2,m}(\boldsymbol{\lambda}), \ldots$ and $N_1(\boldsymbol{\theta}), \ldots, N_m(\boldsymbol{\theta})$ are mutually independent.

Now, let $(\boldsymbol{\Theta}, \boldsymbol{\Lambda}) = (\Theta_1, \ldots, \Theta_{n_1}, \Lambda_1, \ldots, \Lambda_{n_2})$ be a random vector taking on values in $\mathcal{T} \times \mathcal{L}$. We are interested in stochastic comparisons of vectors of random sums given by

$$\mathbf{Z}(\boldsymbol{\Theta}, \boldsymbol{\Lambda}) = (Z_1(\boldsymbol{\Theta}, \boldsymbol{\Lambda}), \ldots, Z_m(\boldsymbol{\Theta}, \boldsymbol{\Lambda})). \qquad \textbf{(1.3)}$$

Here, the random sum

$$Z_i(\boldsymbol{\Theta}, \boldsymbol{\Lambda}) = \sum_{k=1}^{N_i(\boldsymbol{\Theta})} X_{k,i}(\boldsymbol{\Lambda}) \qquad \textbf{(1.4)}$$

can be considered as a mixture of $\{Z_i(\boldsymbol{\theta}, \boldsymbol{\lambda}) | (\boldsymbol{\theta}, \boldsymbol{\lambda}) \in \mathcal{T} \times \mathcal{L}\}$, with respect to a vector $(\boldsymbol{\Theta}, \boldsymbol{\Lambda})$ of random parameters describing the environmental conditions.

Another generalization that we will consider in the article gives rise when some of the parameters of the random sum appear both in the summands and the number of summands. The presence of duplicates of parameters is useful in some applicative contexts (see, e.g., Section 4.3). Formally, let $\mathcal{D} \subseteq \mathbb{R}^n$ be a sublattice in $\mathbb{R}^n$ and let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n) \in \mathcal{D}$. Consider the sums defined by

$$Z_i(\boldsymbol{\delta}) = \sum_{j=1}^{N_i(\boldsymbol{\delta})} X_{j,i}(\boldsymbol{\delta}) \qquad \textbf{(1.5)}$$

for $i = 1, 2, \ldots, m$, where $X_{j,i}(\boldsymbol{\delta}) \geq 0$ a.s. and $X_{1,1}(\boldsymbol{\delta}), X_{2,1}(\boldsymbol{\delta}), \ldots, X_{1,m}(\boldsymbol{\delta}), X_{2,m}(\boldsymbol{\delta}), \ldots$ and $N_1(\boldsymbol{\delta}), \ldots, N_m(\boldsymbol{\delta})$ are mutually independent. Note that (1.5) includes, as a particular case, the case when the $X_{j,i}(\boldsymbol{\delta})$ or the $N_i(\boldsymbol{\delta})$ are actually parametrized only by a subset of the parameters $\delta_1, \ldots, \delta_n$.

Assuming that

$$\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_n)$$

is a random vector taking on values in $\mathcal{D}$, it is interesting to study the stochastic properties of the vector of random sums

$$\mathbf{Z}(\mathbf{\Delta}) = (Z_1(\mathbf{\Delta}), \ldots, Z_m(\mathbf{\Delta})), \quad \textbf{(1.6)}$$

where $Z_i(\mathbf{\Delta})$ is a mixture of $\{Z_i(\boldsymbol{\delta}) | \boldsymbol{\delta} \in \mathcal{D}\}$ with respect to the vector $\mathbf{\Delta}$ of random parameters.

In this article we obtain results on stochastic directional convexity (see Shaked and Shanthikumar [38]) of families of random sums, under appropriate conditions on the families of summands and number of summands. From these results, we study how the dependence among multivariate random environments influences the variability of random sums and the dependence and variability of vectors of random sums by means of the increasing directionally convex order, which are the main purposes of this article; that is, we provide sufficient conditions to model, to compare, and to bound the variability as well as the strength of dependence between two vectors of random sums parameterized on multivariate random environments. In this way, this article completes the study started in Belzunce et al. [3].

The article proceeds as follows. In Section 2 we provide notation and tools on stochastic comparisons and multivariate stochastic convexity that will be used in the article. In Section 3 we state and prove the main results mentioned earlier concerning stochastic comparisons and stochastic directional convexity of families of random sums. Finally, applications for some models in insurance, reliability, and populations growth, defined by means of random sums, are dealt with in Section 4.

## 2. UTILITY NOTIONS AND PRELIMINARIES

In this section we focus on providing notation and mathematical tools for the results in the article. In particular, we will recall the definitions of some stochastic orders as well as multivariate notions of stochastic convexity for a family of parameterized random variables. For that, we will consider different notions of convexity in the multivariate setting.

Some conventions and notations that are used throughout the article were given previously. Let $\leq$ denote the coordinatewise ordering (i.e., for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for $i = 1, 2, \ldots, n$) and $[\boldsymbol{x}, \boldsymbol{y}] \leq \boldsymbol{z}$ as shorthand for $\boldsymbol{x} \leq \boldsymbol{z}$ and $\boldsymbol{y} \leq \boldsymbol{z}$. The operators $+$, $\vee$, and $\wedge$ denote respectively the componentwise sum, maximum, and minimum. The notation $=_{\text{st}}$ stands for equality in law and a.s. is shorthand for almost surely. For any family of parameterized random variables $\{X_\theta | \theta \in \mathcal{T}\}$, with $\mathcal{T} \subseteq \mathbb{R}$, such that every $\theta$ is a value from a random variable $\Theta$, whose distribution is concentrated on $\mathcal{T}$, we denote by $X(\Theta)$ the mixture of the family $\{X_\theta | \theta \in \mathcal{T}\}$ with mixing distribution $\Theta$. For any random variable (or vector) $X$ and an event $A$,

$[X|A]$ denotes a random variable whose distribution is the conditional distribution of $X$ given $A$. Also, according to most of the reliability literature, throughout this article we write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing."

## 2.1. Univariate Stochastic Orderings

Some of the main results in this article deal with the increasing convex order of random sums. Let us recall the definition of this ordering, also known as variability order, second stochastic dominance or stop-loss order, jointly with the stochastic order. For a comprehensive discussion on these stochastic orders, we refer to Shaked and Shanthikumar [39] and Müller and Stoyan [26].

DEFINITION 2.1: *Let $X$ and $Y$ be two nonnegative random variables, with survival functions $\overline{F}_X$ and $\overline{F}_Y$, respectively, then $X$ is said to be smaller than $Y$ in the stochastic (increasing convex) order (denoted by $X \leq_{\text{st}(icx)} Y$) if*

$$\boldsymbol{E}[\phi(X)] \leq \boldsymbol{E}[\phi(Y)]$$

*for all increasing (increasing convex) functions $\phi$ for which the expectations exist. Equivalently, $X \leq_{\text{st}} Y$ if for all $t \geq 0$ it holds that $\overline{F}_X(t) \leq \overline{F}_Y(t)$.*

A characterization of the stochastic ordering that will play a crucial role in this article is recalled now (see Theorem 1.A.1 in Shaked and Shanthikumar [39]). Given two random variables $X$ and $Y$, $X \leq_{\text{st}} Y$ if and only if there exist two random variables $\widehat{X}$ and $\widehat{Y}$, defined on the same probability space, such that $X =_{\text{st}} \widehat{X}$, $Y =_{\text{st}} \widehat{Y}$, and $\widehat{X} \leq \widehat{Y}$, a.s.

The increasing convex order has been applied in several contexts, such as reliability and actuarial science. It allows one to compare the stop-loss transforms of two insurance policies for a kind of reinsurance contract (see Müller and Stoyan [26] for applications in risk theory).

## 2.2. Multivariate Notions of Convexity

Next, we recall the concepts of convex, directionally convex, and supermodular functions. For a complete discussion on convex functions, we refer to the monograph by Rockafellar [31]. For a definition and properties of directionally convex functions, see Shaked and Shanthikumar [38] or Meester and Shanthikumar [23]. For a discussion and background on supermodular functions (that are also called superadditive functions in the literature) we refer to Marshall and Olkin [22].

DEFINITION 2.2: *A real-valued function $\phi$ defined on $\mathbb{R}^n$ is said to be the following:*

*(i) Convex (concave) (denoted by $\phi \in cx(cv)$) if*

$$\phi(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq (\geq)\, \alpha\phi(\boldsymbol{x}) + (1-\alpha)\phi(\boldsymbol{y})$$

*for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\alpha \in [0,1]$. If in addition, $\phi$ is increasing (decreasing), [i.e., for all $\boldsymbol{x} \leq \boldsymbol{y}$, then $\phi(\boldsymbol{x}) \leq (\geq)\, \phi(\boldsymbol{y})$], then we say that $\phi$ is increasing (decreasing) and convex (denoted by $\phi \in icx(icv)$).*

*(ii) Increasing componentwise convex (denoted by $\phi \in iccx$) if it is increasing and it is convex in each argument when the others are held fixed.*

*(iii) Supermodular (denoted by $\phi \in sm$) if*

$$\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$$

*for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.*

*(iv) Directionally convex (concave) (denoted by $\phi \in dcx(dcv)$) if for any $\boldsymbol{x}_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, such that $\boldsymbol{x}_1 \leq [\boldsymbol{x}_2, \boldsymbol{x}_3] \leq \boldsymbol{x}_4$ and $\boldsymbol{x}_1 + \boldsymbol{x}_4 = \boldsymbol{x}_2 + \boldsymbol{x}_3$, then*

$$\phi(\boldsymbol{x}_1) + \phi(\boldsymbol{x}_4) \geq (\leq)\, \phi(\boldsymbol{x}_2) + \phi(\boldsymbol{x}_3).$$

*If, in addition, $\phi$ is increasing (decreasing), then we say that $\phi$ is increasing (decreasing) and directionally convex (denoted by $\phi \in idcx(idcv)$).*

*A function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $\phi(\boldsymbol{x}) = (\phi_1(\boldsymbol{x}), \ldots, \phi_m(\boldsymbol{x}))$ is directionally convex (concave) if each of the coordinate functions $\phi_i$, $i = 1, 2, \ldots, m$, is directionally convex (concave).*

Directional convexity neither implies nor is implied by usual convexity (see Shaked and Shanthikumar [38]). The composition of functions preserves increasing directional convexity (see Lemma 2.4 in Meester and Shanthikumar [23]). In particular, the composition of an icx function with an idcx function is an idcx function (see Corollary 2.5 in Meester and Shanthikumar [23]). A useful characterization of dcx functions is given now (see Proposition 2.1 in Shaked and Shanthikumar [38]). Given $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\phi \in$ dcx if and only if $\phi$ is supermodular and coordinatewise convex.

*Remark 2.1:* We note that $\phi$ is a supermodular function if and only if $\phi$ is supermodular in any couple of arguments when the others are held fixed (see Marshall and Olkin [22]). From this property and the previous characterization, observe that a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is increasing and directionally convex in $(\theta_1, \ldots, \theta_n)$ if and only if $\phi$ is increasing, supermodular in any couple $(\theta_i, \theta_l)$, whenever all other arguments are held fixed, and convex in any $\theta_i$, whenever all other arguments are held fixed.

LEMMA 2.1: *Let $\mathcal{T} \subseteq \mathbb{R}^n$ and let $g : \mathcal{T} \longrightarrow \mathbb{N}$ be an increasing and directionally convex function. If $\{x_j,\ j \in \mathbb{N}\}$ is any increasing sequence of real numbers, then the function $\psi(\boldsymbol{\theta}) := \sum_{j=1}^{g(\boldsymbol{\theta})} x_j$ is increasing and directionally convex.*

PROOF: First, let us write the function $\psi$ as $\psi(\boldsymbol{\theta}) = S_{g(\boldsymbol{\theta})}$, where $S_n = \sum_{j=1}^{n} x_j$. Note that $S_n$ is increasing and convex when $\{x_j,\ j \in \mathbb{N}\}$ is an increasing sequence of real numbers.

Thus, the composition $\psi = S \circ g$ is increasing and directionally convex by Corollary 2.5 in Meester and Shanthikumar [23] and the assertion follows. ■

## 2.3. Multivariate Notions of the Increasing Convex Order

The increasing convex order can be extended to the multivariate case in several ways. Here, we consider three of them. For a survey on these stochastic orderings, we refer to Shaked and Shanthikumar [39].

DEFINITION 2.3: *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be two n-dimensional random vectors; then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the increasing convex (increasing componentwise convex, increasing directionally convex) order (denoted by $\boldsymbol{X} \leq_{\text{icx(iccx,idcx)}} \boldsymbol{Y}$) if*

$$\boldsymbol{E}[\phi(\boldsymbol{X})] \leq \boldsymbol{E}[\phi(\boldsymbol{Y})]$$

*for all increasing convex [increasing componentwise convex, increasing directionally convex] real-valued functions $\phi$ defined on $\mathbb{R}^n$ for which the expectations exist.*

Increasing (componentwise, directionally) concave orders are defined analogously. Clearly, the iccx order is stronger than the icx order; that is, if $\boldsymbol{X} \leq_{\text{iccx}} \boldsymbol{Y}$, then $\boldsymbol{X} \leq_{\text{icx}} \boldsymbol{Y}$. Also, if $\boldsymbol{X} \leq_{\text{iccx}} \boldsymbol{Y}$, then $X_i \leq_{\text{icx}} Y_i$.

Stochastic orders defined above by means of functionals take into account variability. The following dependence order is defined in terms of supermodular functions. The supermodular order strictly implies the increasing directionally convex order, although the supermodular order compares only dependence structure of vectors with fixed equal marginals and the increasing directionally convex order additionally compares the variability of the marginals, which might be different. For a further discussion on supermodular order of random vectors, see Marshall and Olkin [22], Shaked and Shanthikumar [40] and Müller and Stoyan [26].

DEFINITION 2.4: *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ be two n-dimensional random vectors, with equal marginal distributions; then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the supermodular order (denoted by $\boldsymbol{X} \leq_{\text{sm}} \boldsymbol{Y}$) if*

$$\boldsymbol{E}[\phi(\boldsymbol{X})] \leq \boldsymbol{E}[\phi(\boldsymbol{Y})],$$

*for every supermodular real-valued function $\phi$ defined on $\mathbb{R}^n$ for which the expectations exist.*

For $n = 2$, the supermodular order is equivalent to the well-known positive quadrant dependence order (for short, PQD) (see Joe [15]). The supermodular order has been recently used in several applied contexts (see Shaked and Shanthikumar [40], Müller [25], Bäuerle and Müller [1], Denuit et al. [7], Lillo, Pellerey, Semeraro [20], Frostig [11], Rüschendorf [35], Lillo and Semeraro [21], or Belzunce et al. [3], among others).

## 2.4. Multivariate Stochastic Convexity

At this point, we recall some notions of multivariate stochastic convexity for a family of parameterized random variables. Shaked and Shanthikumar [36,37] introduced the notion of stochastic convexity. Multivariate stochastic directional convexity was introduced in Shaked and Shanthikumar [38] and it was also studied in Chang, Chao, Pinedo, and Shanthikumar [4] and Meester and Shanthikumar [23].

Stochastic directional convexity was generalized to a general space in Meester and Shanthikumar [24]. Below, we will consider a family of multivariate random variables $\boldsymbol{X}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \mathcal{T}$, where $\mathcal{T}$ is a sublattice of either $\mathbb{R}^n$ or $\mathbb{N}^n$.

Definition 2.5: *A family $\{\boldsymbol{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\}$ of multivariate random variables is said to be the following:*

*(i) Stochastically increasing (denoted by $\{\boldsymbol{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\} \in SI$) if for any $\boldsymbol{\theta}_i \in \mathcal{T}$, $i = 1, 2$, $\boldsymbol{\theta}_1 \leq \boldsymbol{\theta}_2$, then $\boldsymbol{X}(\boldsymbol{\theta}_1) \leq_{\text{st}} \boldsymbol{X}(\boldsymbol{\theta}_2)$.*

*(ii) Stochastically increasing and directionally convex (denoted by $\{\boldsymbol{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\} \in SI - DCX$) if $\{\boldsymbol{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\} \in SI$ and $\boldsymbol{E}[\phi(\boldsymbol{X}(\boldsymbol{\theta}))]$ is increasing and directionally convex in $\boldsymbol{\theta}$ for any $\phi \in idcx$.*

*(iii) Stochastically increasing and directionally convex in the sample path sense (denoted by $\{\boldsymbol{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\} \in SI - DCX(sp)$) if for any four $\boldsymbol{\theta}_i \in \mathcal{T}$, $i = 1, \ldots, 4$, such that $\boldsymbol{\theta}_1 \leq [\boldsymbol{\theta}_2, \boldsymbol{\theta}_3] \leq \boldsymbol{\theta}_4$ and $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_4 = \boldsymbol{\theta}_2 + \boldsymbol{\theta}_3$, there exist four random variables $\boldsymbol{X}_i$, $i = 1, \ldots, 4$, defined on a common probability space, such that $\boldsymbol{X}_i =_{st} \boldsymbol{X}(\boldsymbol{\theta}_i)$, $i = 1, \ldots, 4$ and*

$$[\boldsymbol{X}_2, \boldsymbol{X}_3] \leq \boldsymbol{X}_4, \quad a.s. \tag{2.1}$$

*and*

$$\boldsymbol{X}_1 + \boldsymbol{X}_4 \geq \boldsymbol{X}_2 + \boldsymbol{X}_3, \quad a.s. \tag{2.2}$$

*(iv) Stochastically increasing and directionally linear in the sample path sense (denoted by $\{\boldsymbol{X}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\} \in SI - DL(sp)$) if in (iii) the inequality (2.2) is replaced by*

$$\boldsymbol{X}_1 + \boldsymbol{X}_4 = \boldsymbol{X}_2 + \boldsymbol{X}_3, \quad a.s. \tag{2.3}$$

In the case that both the parameter and the random variables are univariate, then we will use the notation $\mathrm{SI} - \mathrm{CX(sp)}$ instead of $\mathrm{SI} - \mathrm{DCX(sp)}$.

Note that stochastic directional convexity in the sample path sense strictly implies stochastic directional convexity (see Counterexample 3.1 in Shaked and Shanthikumar [38]).

Stochastic increasing directional convexity and stochastic increasing directional convexity in sample path sense are closed by composition with idcx functions (see, e.g., Lemma 2.15 in Meester and Shanthikumar [23]). Also, both notions of stochastic convexity are closed by conjunction of independent random variables (see Lemma 2.16 in Meester and Shanthikumar [23] or Theorem 3.3 and Theorem 4.4 in Meester and Shanthikumar [24]).

Some examples of stochastic directional convexity of parameterized families of random variables can be found in the literature: See Shaked and Shanthikumar [38], Chang, Shanthikumar and Yao [5] or Meester and Shanthikumar [24]. For example, the Bernoulli distribution and the Poisson distribution are $\mathrm{SI}-\mathrm{DL(sp)}$, the multinomial distribution and the gamma distribution are $\mathrm{SI}-\mathrm{DCX(sp)}$ and the multivariate geometric distribution is $\mathrm{SD}-\mathrm{DCX(sp)}$. Other examples can be obtained by using above the preservation properties. Also, under appropriate conditions, some applied stochastic models have stochastic directional convexity properties (see above references).

## 3. MAIN RESULTS

In this section we provide results on stochastic directional convexity and stochastic directional convexity in the sample path sense for a family of parameterized random sums, under appropriate conditions on the parameterized families of nonnegative summands and number of summands. From them, we provide results for comparing two random sums in the increasing convex order and two vectors of random sums in the increasing directionally convex order sense when the summands and the number of summands are dependent by means of a multivariate random environment.

THEOREM 3.1: *Consider the family of random sums* $\{Z(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\}$ *defined by*

$$Z(\boldsymbol{\delta}) = \sum_{j=1}^{N(\boldsymbol{\delta})} X_j(\boldsymbol{\delta}),$$

*where* $\mathcal{D}$ *is a sublattice in* $\mathbb{R}^n$. *If*

- *(i)* *all of the families* $\{X_j(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\}$, $j \in \mathbb{N}$, *and* $\{N(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\}$ *are independent,*
- *(ii)* $\{X_j(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\} \in SI - DCX(sp)$ *for every fixed* $j \in \mathbb{N}$,
- *(iii)* $\{N(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\} \in SI - DCX(sp)$,
- *(iv)* $\{X_j(\boldsymbol{\delta}), j \in \mathbb{N}\} \in SI$ *for every fixed* $\boldsymbol{\delta} \in \mathcal{D}$, *then* $\{Z(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\} \in SI - DCX(sp)$.

PROOF: Let $\boldsymbol{\delta}_i$, with $i = 1, \ldots, 4$, be such that $\boldsymbol{\delta}_1 \leq [\boldsymbol{\delta}_2, \boldsymbol{\delta}_3] \leq \boldsymbol{\delta}_4$ and $\boldsymbol{\delta}_1 + \boldsymbol{\delta}_4 = \boldsymbol{\delta}_2 + \boldsymbol{\delta}_3$. By assumptions (i), (ii), and (iii), we can build on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the random variables $\widehat{X}_{j,i} =_{\text{st}} X_j(\boldsymbol{\delta}_i)$, $j \in \mathbb{N}$, and $\widehat{N}_i =_{\text{st}} N(\boldsymbol{\delta}_i)$, for $i = 1, \ldots, 4$, such that, almost surely,

$$\widehat{X}_{j,1} + \widehat{X}_{j,4} \geq \widehat{X}_{j,2} + \widehat{X}_{j,3} \quad \text{and} \quad \widehat{X}_{j,4} \geq [\widehat{X}_{j,2},\ \widehat{X}_{j,3}]$$

and

$$\widehat{N}_1 + \widehat{N}_4 \geq \widehat{N}_2 + \widehat{N}_3 \quad \text{and} \quad \widehat{N}_4 \geq [\widehat{N}_2, \widehat{N}_3].$$

Note that by construction and assumption (i), the random vectors $(\widehat{X}_{j,1}, \widehat{X}_{j,2}, \widehat{X}_{j,3}, \widehat{X}_{j,4})$, $j \in \mathbb{N}$, and $(\widehat{N}_1, \widehat{N}_2, \widehat{N}_3, \widehat{N}_4)$ can be assumed independent.

Let now

$$\widehat{N}_2^* =_{\text{a.s.}} \min\{\widehat{N}_4, \widehat{N}_1 + \widehat{N}_4 - \widehat{N}_3\} \quad \text{and} \quad \widehat{N}_1^* =_{\text{a.s.}} \widehat{N}_2^* + \widehat{N}_3 - \widehat{N}_4 = \min\{\widehat{N}_1, \widehat{N}_3\}.$$

Observe that

$$\widehat{N}_2 \leq_{\text{a.s.}} \widehat{N}_2^*, \qquad \widehat{N}_1 \geq_{\text{a.s.}} \widehat{N}_1^*$$

and

$$\widehat{N}_1^* + \widehat{N}_4 =_{\text{a.s.}} \widehat{N}_2^* + \widehat{N}_3, \qquad \widehat{N}_1^* \leq_{\text{a.s.}} [\widehat{N}_2^*, \widehat{N}_3] \leq_{\text{a.s.}} \widehat{N}_4.$$

Similarly, for all $j \in \mathbb{N}$, let

$$\widehat{X}_{j,2}^* =_{\text{a.s.}} \min\{\widehat{X}_{j,4}, \widehat{X}_{j,1} + \widehat{X}_{j,4} - \widehat{X}_{j,3}\} \quad \text{and}$$
$$\widehat{X}_{j,1}^* =_{\text{a.s.}} \widehat{X}_{j,2}^* + \widehat{X}_{j,3} - \widehat{X}_{j,4} = \min\{\widehat{X}_{j,1}, \widehat{X}_{j,3}\}.$$

As above, it holds that

$$\widehat{X}_{j,2} \leq_{\text{a.s.}} \widehat{X}_{j,2}^*, \qquad \widehat{X}_{j,1} \geq_{\text{a.s.}} \widehat{X}_{j,1}^*$$

and

$$\widehat{X}_{j,1}^* + \widehat{X}_{j,4} =_{\text{a.s.}} \widehat{X}_{j,2}^* + \widehat{X}_{j,3}, \qquad \widehat{X}_{j,1}^* \leq_{\text{a.s.}} [\widehat{X}_{j,2}^*, \widehat{X}_{j,3}] \leq_{\text{a.s.}} \widehat{X}_{j,4}.$$

Also, again by construction and assumption (i), we can assume independence among all of the random vectors $(\widehat{X}_{j,1}^*, \widehat{X}_{j,2}^*, \widehat{X}_{j,3}, \widehat{X}_{j,4})$, $j \in \mathbb{N}$, and $(\widehat{N}_1^*, \widehat{N}_2^*, \widehat{N}_3, \widehat{N}_4)$.

Now, let

$$\widehat{Z}_i = \sum_{j=1}^{\widehat{N}_i} \widehat{X}_{j,i}, \quad i = 1, \dots, 4, \tag{3.1}$$

and observe that $\widehat{Z}_i =_{\mathrm{st}} Z(\boldsymbol{\delta}_i)$. Also, let

$$\widehat{Z}_i^* = \sum_{j=1}^{\widehat{N}_i^*} \widehat{X}_{j,i}^*, \quad i = 1, 2.$$

For almost all $\omega \in \Omega$, we have

$$\begin{aligned}
\widehat{Z}_1 + \widehat{Z}_4 &\geq \widehat{Z}_1^* + \widehat{Z}_4 \\
&= \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,1}^* + \sum_{j=1}^{\widehat{N}_4} \widehat{X}_{j,4} \\
&= \sum_{j=1}^{\widehat{N}_1^*} (\widehat{X}_{j,1}^* + \widehat{X}_{j,4}) + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_4} \widehat{X}_{j,4} \\
&\geq \sum_{j=1}^{\widehat{N}_1^*} (\widehat{X}_{j,2}^* + \widehat{X}_{j,3}) + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_2^*} \widehat{X}_{j,2}^* + \sum_{j=\widehat{N}_2^*+1}^{\widehat{N}_4} \widehat{X}_{j,3} \\
&\geq \sum_{j=1}^{\widehat{N}_2^*} \widehat{X}_{j,2}^* + \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_3} \widehat{X}_{j+\widehat{N}_2^*-\widehat{N}_1^*,3}.
\end{aligned}$$

Now, let $\widehat{X}'_{j,3}$ be sampled from the distribution of $\widehat{X}_{j,3}$ but using the uniform random variable $F_{j+\widehat{N}_2^*-\widehat{N}_1^*,3}(\widehat{X}_{j+\widehat{N}_2^*-\widehat{N}_1^*,3})$; that is, let $\widehat{X}'_{j,3} = F_{j,3}^{-1}(F_{j+\widehat{N}_2^*-\widehat{N}_1^*,3}(\widehat{X}_{j+\widehat{N}_2^*-\widehat{N}_1^*,3}))$, where $F_{j,i}$ is the cumulative distribution function of $\widehat{X}_{j,i}$ and $F_{j,i}^{-1}$ is its right continuous inverse. It obviously holds that $\widehat{X}'_{j,3} =_{\mathrm{st}} \widehat{X}_{j,3}$ and, by assumption (iv), $\widehat{X}'_{j,3} \leq_{\mathrm{a.s.}} \widehat{X}_{j+\widehat{N}_2^*-\widehat{N}_1^*,3}$ for all $j = \widehat{N}_1^* + 1, \dots, \widehat{N}_3$. Moreover, the variables $\widehat{X}'_{j,3}$, with $j = \widehat{N}_1^* + 1, \dots, \widehat{N}_3$, are independent from the variables $\widehat{X}_{j,3}$, with $j = 1, \dots, \widehat{N}_1^*$

Prosecuting with the above inequalities, with probability 1 we have **Q1**

$$\begin{aligned}
\widehat{Z}_1 + \widehat{Z}_4 &\geq \sum_{j=1}^{\widehat{N}_2^*} \widehat{X}_{j,2}^* + \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_3} \widehat{X}'_{j,3} \\
&= \widehat{Z}_2^* + \widehat{Z}'_3,
\end{aligned}$$

where

$$\widehat{Z}_3' = \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_3} \widehat{X}_{j,3}'. \tag{3.2}$$

Finally, observing that $\widehat{Z}_2^* \geq_{\text{a.s.}} \widehat{Z}_2$, we get

$$\widehat{Z}_1 + \widehat{Z}_4 \geq_{\text{a.s.}} \widehat{Z}_2 + \widehat{Z}_3', \tag{3.3}$$

where the $\widehat{Z}_i, i = 1, 2, 4$, are defined as in (3.1) and $\widehat{Z}_3'$ is defined as in (3.2).

It is not hard to verify that $\widehat{Z}_3' =_{\text{st}} Z(\boldsymbol{\delta}_3)$. Moreover, it is easy to verify that with probability 1, it holds that

$$\widehat{Z}_4 \geq [\widehat{Z}_2, \widehat{Z}_3']. \tag{3.4}$$

In fact, for example, we have

$$\begin{aligned}\widehat{Z}_3' &= \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_3} \widehat{X}_{j,3}' \leq_{\text{a.s.}} \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_3} \widehat{X}_{j+\widehat{N}_2^*-\widehat{N}_1^*,3} \\ &\leq_{\text{a.s.}} \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_1^*+1}^{\widehat{N}_4-\widehat{N}_2^*+\widehat{N}_1^*} \widehat{X}_{j+\widehat{N}_2^*-\widehat{N}_1^*,3} \\ &= \sum_{j=1}^{\widehat{N}_1^*} \widehat{X}_{j,3} + \sum_{j=\widehat{N}_2^*+1}^{\widehat{N}_4} \widehat{X}_{j,3} \\ &\leq_{\text{a.s.}} \sum_{j=1}^{\widehat{N}_4} \widehat{X}_{j,3} \leq_{\text{a.s.}} \sum_{j=1}^{\widehat{N}_4} \widehat{X}_{j,4} = \widehat{Z}_4.\end{aligned}$$

Thus, by inequalities (3.3) and (3.4), recalling that $\widehat{Z}_i =_{\text{st}} Z(\boldsymbol{\delta}_i)$ when $i = 1, 2, 4$ and $\widehat{Z}_3' =_{\text{st}} Z(\boldsymbol{\delta}_3)$, one gets the assertion. ■

The following results deal with comparisons of two random sums in terms of the dependence between the multivariate random environments. For that, consider a multivariate random vector of parameters $\boldsymbol{\Delta}$ taking on values in $\mathcal{D}$ and consider the family of random sums $Z(\boldsymbol{\Delta})$ defined as a mixture of $\{Z(\boldsymbol{\delta}) | \boldsymbol{\delta} \in \mathcal{D}\}$ (defined by (1.5)), with respect to the random vector $\boldsymbol{\Delta}$.

COROLLARY 3.1: *Let $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}'$ be two random vectors taking on values in $\mathcal{D}$. If the assumptions of Theorem* 3.1 *hold, then*

$$\boldsymbol{\Delta} \leq_{\text{idcx}} \boldsymbol{\Delta}'$$

*implies*

$$Z(\boldsymbol{\Delta}) \leq_{\text{icx}} Z(\boldsymbol{\Delta}')$$

PROOF: Let $u$ be any increasing and convex univariate function.

Since any univariate increasing and convex function $u$ is also increasing and directionally convex and since SI − DCX(sp) implies SI − DCX, then it follows that the function $h(\boldsymbol{\delta}) = \boldsymbol{E}[u(Z(\boldsymbol{\delta}))]$ is increasing and directionally convex.

Now, the assertion follows from Corollary 2.12 in Meester and Shanthikumar [23]. ■

Note that Corollary 3.1 does not improve Theorem 3.1 in Belzunce et al. [3] since in that result, the assumptions on the sequences $\{X_j(\lambda), \lambda \in \mathcal{L}\}, j \in \mathbb{N}$, and $\{N(\theta), \theta \in \mathcal{T}\}$ are weaker. However, in Corollary 3.1 we get the icx comparison of the random sums under the weaker idcx comparison among the random parameters.

The following result is a generalization of the previous one to the case of vectors of random sums.

COROLLARY 3.2: *Consider $m \in \mathbb{N}$ random sums defined by*

$$Z_i(\boldsymbol{\delta}) = \sum_{j=1}^{N_i(\boldsymbol{\delta})} X_{j,i}(\boldsymbol{\delta}), \qquad i = 1, \ldots, m$$

*that are independent for any fixed value of $(\boldsymbol{\delta}) \in \mathcal{D}$ and let*

$$\mathbf{Z}(\boldsymbol{\delta}) = (Z_1(\boldsymbol{\delta}), \ldots, Z_m(\boldsymbol{\delta})).$$

*If*

*(i) all of the families $\{X_{j,i}(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\}$, $j \in \mathbb{N}$, and $\{N_i(\boldsymbol{\delta}), (\boldsymbol{\delta}) \in \mathcal{D}\}$, $i = 1, \ldots, m$, are independent,*

*(ii) $\{X_{j,i}(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\} \in SI - DCX(sp)$ for every fixed $j \in \mathbb{N}$ and $i = 1, \ldots, m$,*

*(iii) $\{N_i(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\} \in SI - DCX(sp)$ for any $i = 1, \ldots, m$,*

*(iv) $\{X_{j,i}(\boldsymbol{\delta}), j \in \mathbb{N}\} \in SI$ for every fixed $\boldsymbol{\delta} \in \mathcal{D}$ and $i = 1, \ldots, m$,*

*then*

$$\boldsymbol{\Delta} \leq_{\text{idcx}} \boldsymbol{\Delta}'$$

*implies*

$$\mathbf{Z}(\boldsymbol{\Delta}) \leq_{\text{idcx}} \mathbf{Z}(\boldsymbol{\Delta}').$$

PROOF: By Theorem 3.1, we have that $\{Z_i(\boldsymbol{\delta}), \boldsymbol{\delta} \in \mathcal{D}\}$ is SI − DCX(sp) for all $i = 1, \ldots, m$. Then, by applying Theorem 4.4 in Meester and Shanthikumar [24], we have that

$$\{(Z_1(\boldsymbol{\delta}), \ldots, Z_n(\boldsymbol{\delta}))|\boldsymbol{\delta} \in \mathcal{D}\} \in \text{SI} - \text{DCX(sp)}$$

and, therefore, it is also SI − DCX.

Let $u$ be any idcx function. Since $\{(Z_1(\boldsymbol{\delta}), \ldots, Z_m(\boldsymbol{\delta})) | \boldsymbol{\delta} \in \mathcal{D}\}$ is SI-DCX, then also the function $h$ defined by

$$h(\boldsymbol{\delta}) = \mathbf{E}[u(\mathbf{Z}(\boldsymbol{\delta}))] = \mathbf{E}[u((Z_1(\boldsymbol{\delta}), \ldots, Z_m(\boldsymbol{\delta})))]$$

is increasing and directionally convex. The assertion now follows by Lemma 2.11 in Meester and Shanthikumar [23]. ■

In the two results presented above, sample path stochastic convexity properties are assumed for the families of nonnegative summands and random number of summands. In the following two results, the weaker regular stochastic convexity is assumed and proved.

In the first one of them, we make use of a different notation for the parameters, since here different parameters for the summands and the number of summands should be assumed. However, in the subsequent result some common parameters are allowed.

THEOREM 3.2: *Consider the family of random sums* $\{Z(\boldsymbol{\theta}, \boldsymbol{\lambda}), (\boldsymbol{\theta}, \boldsymbol{\lambda}) \in \mathcal{T} \times \mathcal{L}\}$ *defined by*

$$Z(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \sum_{j=1}^{N(\boldsymbol{\theta})} X_j(\boldsymbol{\lambda}).$$

*If*

*(i) all of the families* $\{X_j(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathcal{L}\}$, $j \in \mathbb{N}$, *and* $\{N(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\}$ *are independent,*

*(ii)* $\{X_j(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathcal{L}\} \in SI - DCX$ *for every fixed* $j \in \mathbb{N}$,

*(iii)* $\{N(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\} \in SI - DCX$,

*(iv)* $\{X_j(\boldsymbol{\lambda}), j \in \mathbb{N}\} \in SI$ *for every fixed* $\boldsymbol{\lambda} \in \mathcal{L}$,

*then* $\{Z(\boldsymbol{\theta}, \boldsymbol{\lambda}), (\boldsymbol{\theta}, \boldsymbol{\lambda}) \in \mathcal{T} \times \mathcal{L}\} \in SI - DCX$.

PROOF: First, observe that since the families $\{X_j(\boldsymbol{\lambda}), \boldsymbol{\lambda} \in \mathcal{L}\}$ and $\{N(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathcal{T}\}$ are SI by assumptions (ii) and (iii), respectively, then the family $\{Z(\boldsymbol{\theta}, \boldsymbol{\lambda}), (\boldsymbol{\theta}, \boldsymbol{\lambda}) \in \mathcal{T} \times \mathcal{L}\}$ is clearly SI. Thus, in order to prove the result, it is enough to prove that the function

$$h(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \boldsymbol{E}[u(Z(\boldsymbol{\theta}, \boldsymbol{\lambda}))]$$

is increasing and directionally convex whenever $u$ is any increasing and convex real function. For that, by Remark 2.1 we will prove that $h(\boldsymbol{\theta}, \boldsymbol{\lambda})$ is increasing and supermodular in any couple of arguments whenever all other arguments are held fixed, and convex in any argument whenever all other arguments are held fixed.

Let us see now that $h_{\boldsymbol{\lambda}}(\boldsymbol{\theta}) = h(\boldsymbol{\theta}, \boldsymbol{\lambda})$ is increasing and directionally convex in $\boldsymbol{\theta}$ for every fixed value $\boldsymbol{\lambda} \in \mathcal{L}$. To prove this, fix $\boldsymbol{\lambda} \in \mathcal{L}$ and consider the sum $S_n = \sum_{j=1}^{N(\boldsymbol{\theta})} X_j(\boldsymbol{\lambda})$. By Example 5.3.11 in Chang et al. [5], the family $\{S_n, n \in \mathbb{N}\}$ is SI-DCX(sp), thus also SI-CX. Now, by Theorem 8.E.1 in Shaked and Shanthikumar

[39] and by assumption (iii), it follows that $\{S_{N(\boldsymbol{\theta})}, \boldsymbol{\theta} \in \mathcal{T}\}$ is SI-DCX. Thus, by the definition of SI-DCX, the function $h_{\boldsymbol{\lambda}}(\boldsymbol{\theta}) = \boldsymbol{E}[u(S_{N(\boldsymbol{\theta})})]$ is increasing and directionally convex for every function $u$ (and, in particular, if $u$ is univariate *icx*). Thus, from Remark 2.1, $h_{\boldsymbol{\lambda}}(\boldsymbol{\theta})$ is increasing and supermodular in any couple $(\theta_i, \theta_l)$ whenever all other arguments are held fixed, and convex in any $\theta_i$ whenever all other arguments are held fixed.

Next, let us see that $h_{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = h(\boldsymbol{\theta}, \boldsymbol{\lambda})$ is increasing and directionally convex in $\boldsymbol{\lambda}$ for every fixed value of $\boldsymbol{\theta}$. For that, fix a value $\boldsymbol{\theta}$ and consider

$$\begin{aligned} h_{\boldsymbol{\theta}}(\boldsymbol{\lambda}) &= h(\boldsymbol{\theta}, \boldsymbol{\lambda}) \\ &= \boldsymbol{E}\left[ u\left( \sum_{j=1}^{N(\boldsymbol{\theta})} X_j(\boldsymbol{\lambda}) \right) \right] \\ &= \boldsymbol{E}\left[ \boldsymbol{E}\left[ u\left( \sum_{j=1}^{N(\boldsymbol{\theta})} X_j(\boldsymbol{\lambda}) \right) | N(\boldsymbol{\theta}) \right] \right] \\ &= \sum_{n=0}^{\infty} \phi_n(\boldsymbol{\lambda}) P\left[N(\boldsymbol{\theta}) = n\right], \end{aligned}$$

where $\phi_n(\boldsymbol{\lambda}) = \mathbf{E}\left[\widetilde{\psi}_n(\boldsymbol{X}_n(\boldsymbol{\lambda}))\right]$, with $\boldsymbol{X}_n(\boldsymbol{\lambda}) = (X_1(\boldsymbol{\lambda}), \ldots, X_n(\boldsymbol{\lambda}))$ and $\widetilde{\psi}_n(\boldsymbol{x}) = u(\sum_{i=1}^n x_i)$ (here $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is any nonnegative real vector).

It is easy to see that $\widetilde{\psi}_n(\boldsymbol{x})$ is increasing and directionally convex in $\boldsymbol{x}$ for every $n \in \mathbb{N}$.

Thus, since $\{\boldsymbol{X}_n(\boldsymbol{\lambda}) = (X_1(\boldsymbol{\lambda}), \ldots, X_n(\boldsymbol{\lambda})), \boldsymbol{\lambda} \in \mathcal{L}\} \in SI - DCX$ for every $n \in \mathbb{N}$ (by Theorem 3.3 in Meester and Shanthikumar [23] and assumptions (i) and (ii)), we get that $\phi_n(\boldsymbol{\lambda})$ is increasing and directionally convex in $\boldsymbol{\lambda}$ for every $n \in \mathbb{N}$. Thus, also $h_{\boldsymbol{\theta}}(\boldsymbol{\lambda}) = \boldsymbol{E}[\phi_{N(\boldsymbol{\theta})}(\boldsymbol{\lambda})]$ is increasing and directionally convex in $\boldsymbol{\lambda}$.

As above, from Remark 2.1 it follows that for any fixed $\boldsymbol{\theta}$, $h_{\boldsymbol{\theta}}(\boldsymbol{\lambda})$ is increasing and supermodular in any couple $(\lambda_i, \lambda_l)$ whenever all other arguments are held fixed, and convex in any $\lambda_i$ whenever all other arguments are held fixed.

Note also that $h$ is supermodular in any couple of arguments $(\theta_i, \lambda_l)$ whenever all other parameters are held fixed. In fact, this assertion can be proved by the same arguments as in the proof of Theorem 2.1 in Belzunce et al. [3] by taking into in account by assumption (iii) that the family $N(\boldsymbol{\theta})$ is stochastically increasing in $\theta_i$ and, analogously, from assumption (ii) that the families $X_j(\boldsymbol{\lambda}), j \in \mathbb{N}$, are stochastically increasing in $\lambda_l$.

Thus, the function $h(\boldsymbol{\theta}, \boldsymbol{\lambda})$ is supermodular and convex in any argument whenever all other arguments are held fixed. Moreover, the function $h(\boldsymbol{\theta}, \boldsymbol{\lambda})$ is clearly increasing. Hence, from Proposition 2.1 in Shaked and Shanthikumar [38], it is increasing and directionally convex and the assertion follows. ■

As immediate consequence of Theorem 3.2, we can easily get the following conditions for the icx comparison of random sums in random environments.

COROLLARY 3.3: *Consider the family of random sums* $\{Z(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}), (\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{T}\times\mathcal{L}\times\mathcal{D}\}$ *defined by*

$$Z(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}) = \sum_{j=1}^{N(\boldsymbol{\theta},\boldsymbol{\delta})} X_j(\boldsymbol{\lambda},\boldsymbol{\delta}).$$

*If*

*(i) all of the families* $\{X_j(\boldsymbol{\lambda},\boldsymbol{\delta}), (\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{L}\times\mathcal{D}\}$, $j \in \mathbb{N}$, *and* $\{N(\boldsymbol{\theta},\boldsymbol{\delta}), (\boldsymbol{\theta},\boldsymbol{\delta}) \in \mathcal{T}\times\mathcal{D}\}$ *are independent,*

*(ii)* $\{X_j(\boldsymbol{\lambda},\boldsymbol{\delta}), (\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{L}\times\mathcal{D}\} \in SI - DCX$ *for every* $j \in \mathbb{N}$,

*(iii)* $\{N(\boldsymbol{\theta},\boldsymbol{\delta}), (\boldsymbol{\theta},\boldsymbol{\delta}) \in \mathcal{T}\times\mathcal{D}\} \in SI - DCX$,

*(iv)* $\{X_j(\boldsymbol{\lambda},\boldsymbol{\delta}), j \in \mathbb{N}\} \in SI$ *for every fixed* $(\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{L}\times\mathcal{D}$,

*then*

$$(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta}) \le_{\text{idcx}} (\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}')$$

*implies*

$$Z(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta}) \le_{\text{icx}} Z(\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}')$$

PROOF: First, we will prove that for any two random vectors $(\boldsymbol{\Theta}_1,\boldsymbol{\Lambda}_1,\boldsymbol{\Delta}_1)$ and $(\boldsymbol{\Theta}_2,\boldsymbol{\Lambda}_2,\boldsymbol{\Delta}_2)$,

$$(\boldsymbol{\Theta}_1,\boldsymbol{\Lambda}_1,\boldsymbol{\Delta}_1) \le_{\text{idcx}} (\boldsymbol{\Theta}_2,\boldsymbol{\Lambda}_2,\boldsymbol{\Delta}_2) \Rightarrow ((\boldsymbol{\Theta}_1,\boldsymbol{\Delta}_1),(\boldsymbol{\Lambda}_1,\boldsymbol{\Delta}_1)) \le_{\text{idcx}} ((\boldsymbol{\Theta}_2,\boldsymbol{\Delta}_2),(\boldsymbol{\Lambda}_2,\boldsymbol{\Delta}_2)). \tag{3.5}$$

For it, note that if $g((\boldsymbol{\theta},\boldsymbol{\delta}_1),(\boldsymbol{\lambda},\boldsymbol{\delta}_2))$ is idcx, then also the function $\phi(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}) = g((\boldsymbol{\theta},\boldsymbol{\delta}),(\boldsymbol{\lambda},\boldsymbol{\delta}))$ is idcx. Therefore, if $(\boldsymbol{\Theta}_1,\boldsymbol{\Lambda}_1,\boldsymbol{\Delta}_1) \le_{\text{idcx}} (\boldsymbol{\Theta}_2,\boldsymbol{\Lambda}_2,\boldsymbol{\Delta}_2)$, then for any idcx function $g$ we have that

$$\begin{aligned} \boldsymbol{E}[g((\boldsymbol{\Theta}_1,\boldsymbol{\Delta}_1),(\boldsymbol{\Lambda}_1,\boldsymbol{\Delta}_1))] &= \boldsymbol{E}[\phi(\boldsymbol{\Theta}_1,\boldsymbol{\Lambda}_1,\boldsymbol{\Delta}_1)] \\ &\le \boldsymbol{E}[\phi(\boldsymbol{\Theta}_2,\boldsymbol{\Lambda}_2,\boldsymbol{\Delta}_2)] \\ &= \boldsymbol{E}[g((\boldsymbol{\Theta}_2,\boldsymbol{\Delta}_2),(\boldsymbol{\Lambda}_2,\boldsymbol{\Delta}_2))], \end{aligned}$$

and this proves (3.5).

We will denote $\tilde{Z}(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}_1,\boldsymbol{\delta}_2) = \sum_{j=1}^{N(\boldsymbol{\theta},\boldsymbol{\delta}_1)} X_j(\boldsymbol{\lambda},\boldsymbol{\delta}_2)$ and observe that $Z(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta}) =_{\text{st}} \tilde{Z}(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta},\boldsymbol{\Delta})$.

Now, let $u$ be any increasing and convex function and let $h$ be defined as in Theorem 3.2. Then, by Theorem 3.2 and inequality (3.5) we get

$$\begin{aligned}
\mathbf{E}[u(Z(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta}))] &= \mathbf{E}[u(\tilde{Z}(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta},\boldsymbol{\Delta}))]\\
&= \mathbf{E}[\mathbf{E}[u(\tilde{Z}(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta},\boldsymbol{\Delta}))|(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta},\boldsymbol{\Delta})]]\\
&= \mathbf{E}[h((\boldsymbol{\Theta},\boldsymbol{\Delta}),(\boldsymbol{\Lambda},\boldsymbol{\Delta}))]\\
&\le \mathbf{E}[h((\boldsymbol{\Theta}',\boldsymbol{\Delta}'),(\boldsymbol{\Lambda}',\boldsymbol{\Delta}'))]\\
&= \mathbf{E}[\mathbf{E}[u(\tilde{Z}(\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}',\boldsymbol{\Delta}'))|(\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}',\boldsymbol{\Delta}')]]\\
&= \mathbf{E}[u(\tilde{Z}(\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}',\boldsymbol{\Delta}'))]\\
&= \mathbf{E}[u(Z(\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}'))]
\end{aligned}$$

(i.e., the assertion). ■

Note that the above result can be generalized to a vector of random sum like for Corollary 3.2. In fact, the proof of the following corollary is similar to the proof of Corollary 3.2, but here we use Theorem 3.3 in Meester and Shanthikumar [24] instead of Theorem 4.4 in Meester and Shanthikumar [24].

COROLLARY 3.4: *Consider $m \in \mathbb{N}$ random sums defined by*

$$Z_i(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}) = \sum_{j=1}^{N_i(\boldsymbol{\theta},\boldsymbol{\delta})} X_{j,i}(\boldsymbol{\lambda},\boldsymbol{\delta}), \qquad i = 1,\ldots,m,$$

*that are independent for any fixed value of $(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{T}\times\mathcal{L}\times\mathcal{D}$ and let*

$$\mathbf{Z}(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}) = (Z_1(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta}),\ldots,Z_m(\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\delta})).$$

*If*

*(i) all of the families $\{X_{j,i}(\boldsymbol{\lambda},\boldsymbol{\delta}),(\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{L}\times\mathcal{D}\}$, $j \in \mathbb{N}$, and $\{N_i(\boldsymbol{\theta},\boldsymbol{\delta}),(\boldsymbol{\theta},\boldsymbol{\delta}) \in \mathcal{T}\times\mathcal{D}\}$, $i = 1,\ldots,m$, are independent,*

*(ii) $\{X_{j,i}(\boldsymbol{\lambda},\boldsymbol{\delta}),(\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{L}\times\mathcal{D}\} \in SI - DCX$ for every $j \in \mathbb{N}$ and $i = 1,\ldots,m$,*

*(iii) $\{N_i(\boldsymbol{\theta},\boldsymbol{\delta}),(\boldsymbol{\theta},\boldsymbol{\delta}) \in \mathcal{T}\times\mathcal{D}\} \in SI - DCX$ for any $i = 1,\ldots,m$,*

*(iv) the sequence $\{X_{j,i}(\boldsymbol{\lambda},\boldsymbol{\delta}), j \in \mathbb{N}\} \in SI$ for every fixed $(\boldsymbol{\lambda},\boldsymbol{\delta}) \in \mathcal{L}\times\mathcal{D}$ and $i = 1,\ldots,m$,*

*then*

$$(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta}) \le_{\text{idcx}} (\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}')$$

*implies*

$$\mathbf{Z}(\boldsymbol{\Theta},\boldsymbol{\Lambda},\boldsymbol{\Delta}) \le_{\text{idcx}} \mathbf{Z}(\boldsymbol{\Theta}',\boldsymbol{\Lambda}',\boldsymbol{\Delta}')$$

## 4. APPLICATIONS

In this section we provide some examples to illustrate how the main results can be applied.

### 4.1. Collective Risk Models in Actuarial Sciences

Consider an homogeneous portfolio of $n$ risks over a single period of time and assume that during that period, each policyholder $i$ can have a nonnegative claim $X_i$ with probability $\theta_i \in [0,1] \subseteq \mathbb{R}$. Then the total claim amount $S(\theta_1,\ldots,\theta_n)$ during that time can be represented as

$$S(\theta_1,\ldots,\theta_n) = \sum_{i=1}^{n} I_i(\theta_i)X_i,$$

where $I_i(\theta_i)$ denotes a Bernoulli random variable with parameter $\theta_i$.

As it is pointed out, for example, in Frostig [10], assumption of independence among the Bernoulli random variables $I_i(\theta_i), i = 1,\ldots,n$, is not suitable to describe real contexts, since their distributions might actually depend on some common random environment. Thus, one can replace the vector of real parameters $(\theta_1,\ldots,\theta_n)$ by a random vector $\boldsymbol{\Theta} = (\Theta_1,\ldots,\Theta_n)$, with values in $[0,1]^n \subseteq \mathbb{R}^n$ and describing both the random environment for occurrences of claims and the dependence among them. Some known results in the literature deal with stochastic comparisons of random sums involving Bernoulli random variables (see Lefèvre and Utev [18], Hu and Wu [14], Frostig [10], or Hu and Ruan [13]).

Here, we state conditions for the stochastic comparison, in the increasing convex sense, of two total claim amounts defined as above.

Proposition 4.1: *Let $\mathbf{I}(\boldsymbol{\theta}) = (I_1(\theta_1),\ldots,I_n(\theta_n))$, where the $I_i(\theta_i)$ are independent Bernoulli random variables with parameters $\theta_i$, $i = 1,\ldots,n$. Consider $N(\theta_1,\ldots,\theta_n) = \sum_{i=1}^n I_i(\theta_i)$. Then $\{N(\theta_1,\ldots,\theta_n),(\theta_1,\ldots,\theta_n) \in [0,1]^n \subseteq \mathbb{R}^n\} \in SI - DCX(sp)$.*

Proof: First, note that $\{N(\theta_1,\ldots,\theta_n),(\theta_1,\ldots,\theta_n) \in [0,1]^n \subseteq \mathbb{R}^n\}$ is clearly stochastically increasing.

Now, consider a family of Bernoulli random variables $\{I_\theta : \theta \in [0,1]\}$. It is easy to see that this family is SI-DL(sp) (see, e.g., Example 5.3.8 in Chang et al. [5]). Therefore, for any fixed $\theta_{i,k} (k = 1,\ldots,4, i = 1,\ldots,n)$ such that $\theta_{i,1} \le [\theta_{i,2},\theta_{i,3}] \le \theta_{i,4}$ and $\theta_{i,1} + \theta_{i,4} = \theta_{i,2} + \theta_{i,3}$, we can build, on the same probability space, random variables $\widehat{I}_i(\theta_k) =_{\text{st}} I_i(\theta_k)$ for $k = 1,\ldots,4$ and $i = 1,\ldots,n$, such that

$$\left[\widehat{I}_i(\theta_{i,2}),\widehat{I}_i(\theta_{i,3})\right] \le \widehat{I}_i(\theta_{i,4}), \quad \text{a.s.}$$

and

$$\widehat{I}_i(\theta_{i,1}) + \widehat{I}_i(\theta_{i,4}) = \widehat{I}_i(\theta_{i,2}) + \widehat{I}_i(\theta_{i,3}), \quad \text{a.s.}$$

Note that, by independence, we can build all of the variables $\widehat{I}_i(\theta_{i,k})$, for all $i = 1, \ldots, n$, on the same probability space.

Now, consider the random variables $\widehat{N}_k = \sum_{i=1}^n \widehat{I}_i(\theta_{i,k})$. We observe that

$$[\widehat{N}_2, \widehat{N}_3] \leq \widehat{N}_4, \quad \text{a.s.}$$

and

$$\widehat{N}_1 + \widehat{N}_4 = \widehat{N}_2 + \widehat{N}_3, \quad \text{a.s.}$$

Then $\{N(\theta_1, \ldots, \theta_n), (\theta_1, \ldots, \theta_n) \in [0,1]^n \subseteq \mathbb{R}^n\} \in \mathrm{SI} - \mathrm{DL(sp)}$, since

$$(\theta_{1,1}, \ldots, \theta_{n,1}) \leq [(\theta_{1,2}, \ldots, \theta_{n,2}), (\theta_{1,3}, \ldots, \theta_{n,3})] \leq (\theta_{1,4}, \ldots, \theta_{n,4}), \quad \text{a.s.}$$
$$(\theta_{1,1}, \ldots, \theta_{n,1}) + (\theta_{1,4}, \ldots, \theta_{n,4}) = (\theta_{1,2}, \ldots, \theta_{n,2}) + (\theta_{1,3}, \ldots, \theta_{n,3}), \quad \text{a.s.}$$

and $\widehat{N}_k =_{\mathrm{st}} N(\theta_{1,k}, \ldots, \theta_{n,k})$, for $k = 1, \ldots, 4$. The assertion follows observing that $\mathrm{SI} - \mathrm{DL(sp)}$ implies $\mathrm{SI} - \mathrm{DCX(sp)}$. ∎

As immediate consequence, we get the following result.

COROLLARY 4.1: *Let $X_1, \ldots, X_n$ be independent and identically distributed nonnegative random variables and let $I_1(\theta_1), \ldots, I_n(\theta_n)$ be independent Bernoulli random variables with parameters $\theta_1, \ldots, \theta_n$, respectively, and independent of $X_i$, $i = 1, \ldots, n$. Consider the total claim amounts $S(\theta_1, \ldots, \theta_n) = \sum_{i=1}^n I_i(\theta_i)X_i$. Then*

$$(\Theta_1, \ldots, \Theta_n) \leq_{\mathrm{idcx}} (\Theta_1', \ldots, \Theta_n')$$

*implies*

$$S(\Theta_1, \ldots, \Theta_n) \leq_{\mathrm{icx}} S(\Theta_1', \ldots, \Theta_n').$$

PROOF: Observe that since the claims $X_i$ are assumed to be independent, then

$$S(\theta_1, \ldots, \theta_n) =_{\mathrm{st}} \sum_{i=1}^{N(\theta_1, \ldots, \theta_n)} X_i.$$

The assertion now follows from Proposition 4.1 and Corollary 3.1. ∎

### 4.2. Population Growth Models

Branching processes have been considered an appropriate mathematical model for the description of populations' growth, where individuals produce offsprings according to some stochastic laws. Several applications involve medicine, molecular and cellular biology, human evolution, physics or actuarial science (see Rolski, Schmidli, Schmidt, and Teugeis [32], Ross [33], or Kimmel and Axelrod [16]). In this subsection, we provide a result dealing with stochastic comparisons between two branching processes defined on random environments, which is closely related to Theorem 2.2 in Pellerey [30].

The branching processes on random environments that we consider here are defined as follows. Let $\boldsymbol{\theta} = \{\theta_0, \theta_1, \ldots, \}$ be a sequence of values in $\mathcal{T}$ describing the evolutions of the environment, and define, recursively, the stochastic process $\boldsymbol{Z}(\boldsymbol{\theta}) = \{Z_n(\theta_0, \ldots, \theta_n), n \in \mathbb{N}\}$ by

$$Z_0(\theta_0) = X_{1,0}(\theta_0)$$

and

$$Z_n(\theta_0, \ldots, \theta_n) = \sum_{j=1}^{Z_{n-1}(\theta_0,\ldots,\theta_{n-1})} X_{j,n}(\theta_n), \qquad n \geq 1. \tag{4.1}$$

In order to deal with random evolutions of the environment, we consider a sequence $\boldsymbol{\Theta} = (\Theta_0, \Theta_1, \ldots)$ of random variables taking on values in $\mathcal{T}$ and we consider the stochastic process $\mathbf{Z}(\boldsymbol{\Theta}) = \{Z_n(\Theta_0, \ldots, \Theta_n), n \in \mathbb{N}\}$ defined by

$$Z_0(\Theta_0) = X_{1,0}(\Theta_0)$$

and

$$Z_n(\Theta_0, \ldots, \Theta_n) = \sum_{j=1}^{Z_{n-1}(\Theta_0,\ldots,\Theta_{n-1})} X_{j,n}(\Theta_n), \qquad n \geq 1, \tag{4.2}$$

where, for every $j, k \in \mathbb{N}$, $X_{j,k}(\Theta_k)$ is a nonnegative random variable such that $[X_{j,k}(\Theta_k) | \Theta_k = \theta] =_{\text{st}} X_{j,k}(\theta)$.

First, we prove the SI-DCX(sp) property of such parameterized families of branching processes.

PROPOSITION 4.2: *Let $\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots)$ be a sequence of values in $\mathcal{T} \subseteq \mathbb{R}$ and consider the stochastic process defined by (4.1). If*

*(i) the variables $\{X_{j,k}(\theta_k)\}$, $j \in \mathbb{N}$ and $k \in \mathbb{N}$ are all mutually independent,*

*(ii) $\{X_{j,k}(\theta_k), \theta_k \in \mathcal{T}\} \in SI - CX(sp)$ for every fixed $j \in \mathbb{N}$ and $k \in \mathbb{N}$,*

*(iii) $\{X_{j,k}(\theta_k), j \in \mathbb{N}\} \in SI$ for every fixed $\theta_k \in \mathcal{T}$ and $k \in \mathbb{N}$,*

*then $\{Z_n(\theta_0, \ldots, \theta_n), (\theta_0, \ldots, \theta_n) \in \mathcal{T}^{n+1}\} \in SI - DCX(sp)$ for every $n \in \mathbb{N}$.*

PROOF: We will proceed by induction. First, observe that, trivially we have that $\{Z_1(\theta_0), (\theta_0) \in \mathcal{T}\}$ is $\mathrm{SI} - \mathrm{CX(sp)}$ and, thus, $\mathrm{SI} - \mathrm{DCX(sp)}$. Now, assume that assertion is true for $n-1$; that is, assume that $\{Z_{n-1}(\theta_0, \ldots, \theta_{n-1}), (\theta_0, \ldots, \theta_{n-1}) \in \mathcal{T}^n\}$ is $\mathrm{SI} - \mathrm{DCX(sp)}$.

Then, by Theorem 3.1 and the inductive hypothesis, it follows that

$$Z_n(\theta_0, \ldots, \theta_n) = \sum_{j=1}^{Z_{n-1}(\theta_0,\ldots,\theta_{n-1})} X_{j,n}(\theta_n) \tag{4.3}$$

is $\mathrm{SI} - \mathrm{DCX(sp)}$ in $(\theta_0, \ldots, \theta_n) \in \mathcal{T}^{n+1}$ and, thus, the assertion follows. ■

From the previous result, we can easily get the following comparison result for two branching processes defined on two different random environments (see Pellerey [30] for further details)).

COROLLARY 4.2: *Consider the stochastic processes* $\mathbf{Z}(\boldsymbol{\theta}) = \{Z_n(\theta_0, \ldots, \theta_n), n \in \mathbb{N}\}$ *and* $\mathbf{Z}(\boldsymbol{\Theta}) = \{Z_n(\Theta_0, \ldots, \Theta_n), n \in \mathbb{N}\}$ *defined by (4.1) and (4.2), respectively. If the assumptions of Proposition 4.2 hold, then*

$$(\Theta_1, \ldots, \Theta_n) \leq_{\text{idcx}} (\Theta_1', \ldots, \Theta_n')$$

*implies*

$$Z_n(\Theta_1, \ldots, \Theta_n) \leq_{\text{icx}} Z_n(\Theta_1', \ldots, \Theta_n').$$

### 4.3. Cumulative Damage Shock Models

Shock models are of great interest in the context of reliability theory since they are commonly used to describe the lifetime or the reliability of systems or items subjected to shocks. In this context, compound Poisson processes are used to describe the wear accumulated by systems during time. Assume that a system is subjected to shocks arriving according to a Poisson process $N_\theta$ having rate $\theta > 0$ and that the $i$th shock causes a nonnegative damage $X_i$, where the damages accumulate additively. Then the total wear accumulated up to time $t \geq 0$ by the system is given by (see Esary, Marshall, and Proschan [9])

$$W_\theta(t) = \sum_{i=1}^{N_\theta(t)} X_i, \tag{4.4}$$

with $W_\theta(t) = 0$ in the case $N_\theta(t) = 0$.

If the system fails when the accumulated wear exceeds a fixed threshold, then some properties of the distribution of the system lifetime can be obtained from stochastic properties of the process $\mathbf{W}_\theta = \{W_\theta(t), t \in \mathbb{R}\}$.

In literature there are many articles dealing with stochastic comparisons among accumulated wear processes defined as in (4.4). However, almost all of them assume independence among all damages $X_i$ and also independence between the damages and the counting process $N_\theta$ (see, e.g., Esary et al. [9], Ross and Schechner [34], or Pellerey [27]). Here, we provide a generalization of these results under conditional independence among damages and the shock arrival process.

For it, assume that the system is subjected to shocks arriving according to a Poisson process $N_\theta$. Let $X_j(\theta, \lambda)$ denote the damage caused by the $j$th shock, parameterized by the same parameter $\theta$ of the process $N_\theta$ and a generic environmental parameter $\lambda$ that is common for all damages. Then the total wear accumulated up to time $t \geq 0$ by the system is given by

$$W_{\theta,\lambda}(t) = \sum_{j=1}^{N_\theta(t)} X_j(\theta,\lambda) \tag{4.5}$$

(where $\sum_{j=1}^{N_\theta(t)} X_j(\theta,\lambda) = 0$ in the case $N_\theta(t) = 0$).

Now, assume that the parameters are given by random environmental factors (i.e., by a random vector $(\Theta, \Lambda)$), and consider the wear process

$$W_{\Theta,\Lambda}(t) = \sum_{j=1}^{N_\Theta(t)} X_j(\Theta,\Lambda), \tag{4.6}$$

defined as a mixture of the families $W_{\theta,\lambda}$ with respect to the vector $(\Theta, \Lambda)$. Then by Corollary 3.1 and since Poisson random variables are $\mathrm{SI} - \mathrm{DL(sp)}$, we obtain the following comparison criterion.

COROLLARY 4.3: *Consider the stochastic processes $W_{\theta,\lambda}$ and $W_{\Theta,\Lambda}$ defined by (4.5) and (4.6), respectively. If*

*(i) $X_j(\theta,\lambda)$ are independent for all $j \in \mathbb{N}$ for any fixed values of $(\theta,\lambda)$,*
*(ii) $\{X_j(\theta,\lambda), (\theta,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+\} \in SI - DCX(sp)$ for any $j \in \mathbb{N}$,*
*(iii) the families $\{X_j(\theta,\lambda), (\theta,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+\}$ and $\{N_\theta, \theta \in \mathbb{R}^+\}$ are independent,*
*(iv) $\{X_j(\theta,\lambda), j \in \mathbb{N}\} \in SI$ for any $(\theta,\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+$,*

*then*

$$(\Theta, \Lambda) \le_{\mathrm{idcx}} (\Theta', \Lambda')$$

*implies*

$$W_{\Theta,\Lambda}(t) \le_{\mathrm{icx}} W_{\Theta',\Lambda'}(t) \quad \forall t \ge 0.$$

Similar results can be stated in case the damages do not accumulate additively. For example, assume that the damage caused by the $i$th shock is given by a function of the previously accumulated damage and the intensity $X_i$ of the $i$th shock. For that, consider a cumulative damage discrete-time process $\boldsymbol{W}(\boldsymbol{\lambda}) = \{W_n(\lambda_1,\ldots,\lambda_n), n \in \mathbb{N}, \lambda_i \in \mathbb{R}^+, i = 1,\ldots,n\}$ defined recursively as

$$W_1(\lambda_1) = X_1(\lambda_1)$$

and

$$W_n(\lambda_1,\ldots,\lambda_n) = W_{n-1}(\lambda_1,\ldots,\lambda_{n-1}) + g(W_{n-1}(\lambda_1,\ldots,\lambda_{n-1}), X_n(\lambda_n)), \quad n > 1.$$

Now, consider two processes defined as above but with parameters given by realizations of two vectors $(\Lambda_1,\ldots,\Lambda_n)$ and $(\Lambda'_1,\ldots,\Lambda'_n)$ describing different environmental conditions. Proceeding by induction and using arguments similar to those in the previous proof and Lemma 2.4 in Meester and Shanthikumar [23], one can easily prove the following result.

COROLLARY 4.4: *Consider* $W_n(\lambda_1, \ldots, \lambda_n), n \in \mathbb{N}, \lambda_i \in \mathbb{R}^+, i = 1, \ldots, n$ *defined as above. If*

*(i)* *the families* $\{X_i(\lambda_i), \lambda_i \in \mathbb{R}^+\}$, *with* $i = 1, \ldots, n$, *are independent,*

*(ii)* $\{X_i(\lambda_i), \lambda_i \in \mathbb{R}^+\} \in SI - CX(sp)$, *for every fixed value* $i = 1, \ldots, n$,

*(iii)* $\{X_i(\lambda_i), i = 1, \ldots, n\} \in SI$, *for every fixed value* $\lambda_i \in \mathbb{R}^+$,

*then*

$$(\Lambda_1, \ldots, \Lambda_n) \leq_{\text{idcx}} (\Lambda'_1, \ldots, \Lambda'_n) \quad \forall n \in \mathbb{N}$$

*implies*

$$W_n(\Lambda_1, \ldots, \Lambda_n) \leq_{\text{icx}} W_n(\Lambda'_1, \ldots, \Lambda'_n) \quad \forall n \in \mathbb{N}$$

*whenever the function* $g(w, x)$ *is increasing and directionally convex.*


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