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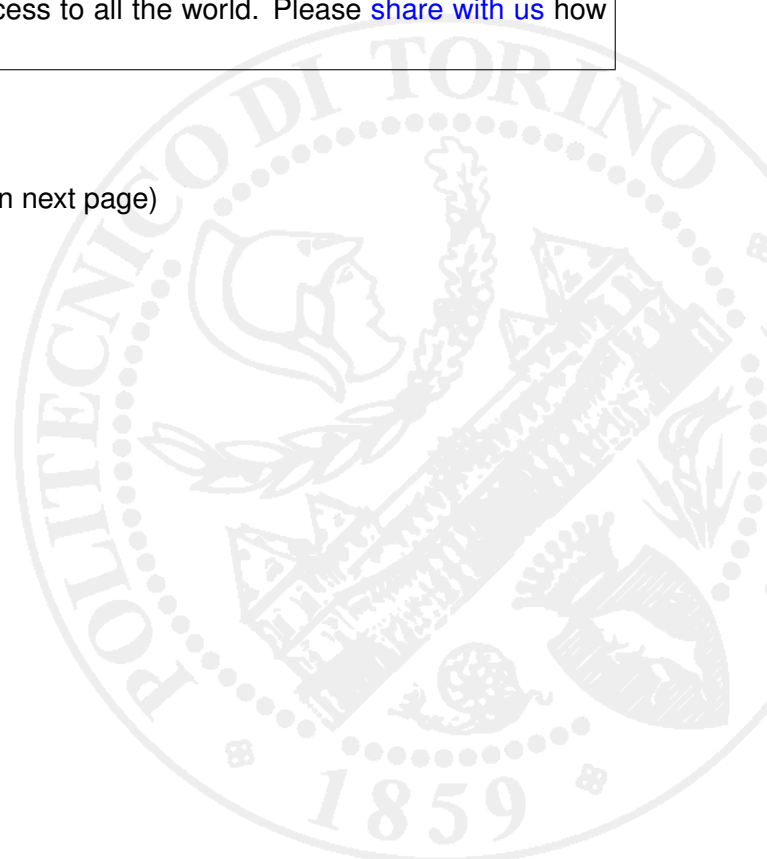
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Fractional calculus in continuum mechanics: local vs. non-local approach

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Abstract: Two fractional calculus approaches in the framework of continuum mechanics are revisited and compared. The former is a local approach, which has been proposed to investigate the behaviour of fractal media. The latter is a non-local approach, according to which long-range interactions between material particles are opportunely modelled in the equilibrium equations. Analogies and differences between the two models are outlined.

Keywords: Fractal media, local fractional calculus, long-range interactions, Marchaud derivative.

1. INTRODUCTION

Aim of the paper is to revisit and compare two applications of fractional calculus to continuum mechanics. For the sake of simplicity, only one-dimensional problems will be dealt with.

The former approach is based on the local fractional calculus introduced by Kolwnakar (Kolwankar and Gangal (1996), Kolwankar and Gangal (1998)) to address the problem of fractal media, i.e. solids where the deformation is localized on a fractal subset. By assuming that deformation takes place on a fractal subset, the displacement field is represented by devil staircase-like functions (Carpinteri et al. (2001), Carpinteri and Cornetti (2002)). As well known, these functions have zero first derivative, except in an infinite number of points where they are not differentiable. On the other hand, they admit fractional derivatives of order less than the fractal dimension of the set where strain localizes. It is hence possible to express the fractal strain as the local fractional derivative of the displacement field.

The latter approach was introduced to model non-local continua (Di Paola and Zingales (2008)), i.e. solids characterized by non-local interactions (Aifantis (1994), Eringen and Edelen (1972)). The novelty is that internal forces are described by fractional integrals and derivatives (Di Paola and Zingales (2008)). One of the most remarkable achievements of this approach is that, by the Marchaud definition of fractional derivative (Samko et al. (1993)), the fractional operators have a clear mechanical interpretation, i.e. springs connecting non-adjacent points of the body. The related stiffness decays along with the distance among the material points. However, it is seen that, in order to have a consistent mechanical model, only the

integral part of Marchaud derivative has to be retained. The two approaches are finally discussed and compared. For the sake of clarity, it is opportune to introduce in this section the definitions of the left and right Riemann-Liouville derivatives, $D_{a+}^{\alpha} f(x)$ and $D_{b-}^{\alpha} f(x)$, respectively. In particular, given a Lebesgue function $f(x)$ on the closed interval $[a, b]$, their expressions are given by ($0 < \alpha < 1$, Samko et al. (1993)):

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(\xi)}{(x-\xi)^{\alpha}} d\xi, \quad (1)$$

$$D_{b-}^{\alpha} f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(\xi)}{(\xi-x)^{\alpha}} d\xi. \quad (2)$$

2. LOCAL APPROACH

The mechanics of solids deformable over fractal subsets has been widely investigated (Carpinteri (1994a,b), Carpinteri et al. (2001), Carpinteri et al. (2004a)). When dealing with fractal media, mechanical quantities with anomalous physical dimensions, related to the fractal dimensions of the domain upon which they are defined, must be taken into account (Carpinteri (1994a,b)). Then, by means of the local fractional operators, the static and kinematic equations (as well as the principle of virtual work) may be derived (Carpinteri et al. (2001)). Before entering into details, let us remind that we are referring to a generic body whose stress flux and deformation pattern have fractal characteristics, whereas the material itself does not need to present a fractal microstructure.

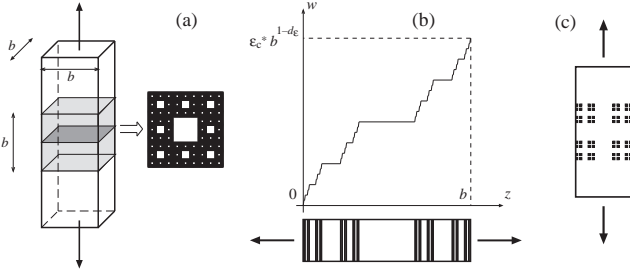


Fig. 1. Fractal localization of: stress (a), strain (b), energy dissipation (c).

2.1 Fractal mechanics quantities

The singular stress flux through fractal media can be modelled by means of lacunar fractal sets A^* of dimension $\Delta_\sigma = 2 - d_\sigma \leq 2$ (Fig. 1a), representing damaged cross sections. An original definition of the fractal stress σ^* was put forward in (Carpinteri (1994a,b)) by applying the renormalization group procedure to the nominal stress tensor $[\sigma]$. The fractal stress σ^* , whose dimensions are $[F][L]^{-(2-d_\sigma)}$, is a scale-invariant quantity.

The kinematical conjugate of the fractal stress σ^* is the fractal strain ε^* . The basic assumption is that displacement discontinuities can be localized on an infinite number of cross-sections, spreading throughout the body. This hypothesis has been suggested by several experimental investigations, for instance in metals (Kleiser and Bockel (1986)) and in highly stressed rock masses (Poliakov et al. (1995)). Considering the simplest uniaxial model, a slender bar subjected to tension, it can be argued that the projection (over the horizontal axis z) of the cross sections where deformation localizes, is a lacunar fractal set, with dimension $\Delta_\varepsilon = 1 - d_\varepsilon$ comprised between zero and one. If the Cantor set ($\Delta_\varepsilon \cong 0.631$) is assumed as the archetype of damage distributions, we may speak of the fractal Cantor bar (Fig. 1b). The dilatation strain tends to localize into singular stretched regions, while the rest of the body is considered as undeformed. The displacement function can be represented by a devil's staircase graph, that is, by a singular fractal function which is constant everywhere except at the points corresponding to a lacunar fractal set of zero Lebesgue measure (Fig. 1b). By applying the renormalization group procedure (Carpinteri et al. (2001)), the fractal strain ε^* , whose physical dimension $[L]^{d_\varepsilon}$ lies between that of pure strain $[L]^0$ and that of a displacement $[L]^1$, reveals to be the scale-independent parameter describing the kinematics of the fractal bar.

Eventually, let us remind that the lacunar fractal domain Ω^* , with dimension $\Delta_\omega = 3 - d_\omega$, where the strain energy Φ is stored during a generic loading process, must be equal to the Cartesian product of the lacunar cross-section with dimension $2 - d_\sigma$ times the cantorian projection set, with dimension $1 - d_\varepsilon$ (Fig. 1c). Thus, a fundamental relationship among the exponents is achieved (Carpinteri et al. (2001)):

$$d_\omega = d_\sigma + d_\varepsilon. \quad (3)$$

2.2 Local fractional calculus

Local fractional derivatives (LFDs) were introduced with the motivation of studying the local properties of fractal structures and processes (Kolwankar and Gangal (1998), Kolwankar and Gangal (1999)), since no fractal function can be the solution of a classical differential equation (Carpinteri and Mainardi (1997)). The LFD definition is obtained from Eq. (1) introducing two “corrections” in order to avoid some physically undesirable features of the classical definition. In fact, if one wishes to analyze the local behaviour of a function, both the dependence on the lower limit a and the fact that adding a constant to a function yields a different fractional derivative should be avoided. This can be obtained subtracting from the function the value of the function at the point where we want to study the local scaling property and choosing as the lower limit that point itself. Therefore, restricting our discussion to an order q comprised between 0 and 1, the LFD is defined as the following limit (if it exists and is finite):

$$D^q f(x) = \lim_{t \rightarrow x} D_{x+}^q [f(t) - f(x)], \quad 0 < q \leq 1. \quad (4)$$

In (Kolwankar and Gangal (1996)) the Weierstrass function was shown to be locally fractionally differentiable up to a *critical order* α between 0 and 1. More precisely, the LFD is zero if the order is lower than α , does not exist if greater, while exists and is finite only if equal to α . Thus, the LFD shows a behaviour analogous to that of the Hausdorff measure of a fractal set. Furthermore, the critical order is strictly linked to the fractal properties of the function itself. In fact, the critical order α coincides with the local Hölder exponent s (which depends, as is well-known, on the fractal dimension), as it was demonstrated by proving the following local fractional Taylor expansion of function $f(x)$ of order $q < 1$ for $x \rightarrow x_0$ (Kolwankar and Gangal (1996)):

$$f(x) = f(x_0) + \frac{D^q f(x_0)}{\Gamma(q+1)}(x - x_0)^q + R_q(x - x_0), \quad (5)$$

where $R_q(x - x_0)$ is a remainder, negligible if compared with the other terms. Let us observe that the terms in the right hand side of Eq. (5) are nontrivial and finite only if q is equal to the critical order α . Moreover, for $q = \alpha$, the fractional Taylor expansion (5) gives us the geometrical interpretation of the LFD. When q is set equal to unity, one obtains from (5) the equation of a tangent. All the curves passing through the same point x_0 with the same first derivative have the same tangent. Analogously, all the curves with the same critical order α and the same D^α form an equivalence class modelled by x^α . This is how it is possible to generalize the geometric interpretation of derivatives in terms of “tangents”.

The solution of the simple differential equation $df/dx = 1_{[0,x]}$ gives the length of the interval $[0, x]$. The solution is nothing but the integral of the unit function. Wishing to extend this idea to the evaluation of the measure of fractal sets, it can be seen immediately that the classical fractional integral does not work, as it fails to be additive because of its nontrivial kernel. On the other hand, a fractional measure of a fractal set can be obtained through

the inverse of the LFD defined as (Kolwankar and Gangal (1999)):

$$I_{[a,b]}^\alpha f = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(x_i^*) I_{x_i, x_{i+1}}^\alpha 1_{dx_i}(x_{i+1}), \quad (6)$$

where $[x_i, x_{i+1}]$, $i = 0, \dots, N-1$, $x_0 = a$ and $x_N = b$, provide a partition of the interval $[a, b]$ and x_i^* is some suitable point chosen in the subinterval $[x_i, x_{i+1}]$, while 1_{dx_i} is the unit function defined on the same subinterval. Kolwankar called $I_{[a,b]}^\alpha f$ the *fractal integral* of order α of $f(x)$ over the interval $[a, b]$ (Kolwankar and Gangal (1999)). The simple local fractional differential equation $D^\alpha f(x) = g(x)$ has not a finite solution when $g(x)$ is constant and $0 < \alpha < 1$. Interestingly, the solution exists if $g(x)$ has a fractal support whose Hausdorff dimension d is equal to the fractional order of derivation α . Consider, for instance, the triadic Cantor set C , built on the interval $[0, 1]$, whose dimension is $d = \ln 2 / \ln 3$. Let $1_C(x)$ be the function whose value is one in the points belonging to the Cantor set upon $[0, 1]$, zero elsewhere. Therefore, the solution of $D^\alpha f(x) = 1_C(x)$ when $\alpha = d$ is $f(x) = I_{[0,x]}^\alpha 1_C(t)$. Applying Eq. (6) with $x_0 = 0$ and $x_N = x$ and choosing x_i^* to be such that $1_C(x_i^*)$ is maximum in the interval $[x_i, x_{i+1}]$, one gets (Kolwankar and Gangal (1999)):

$$\begin{aligned} f(x) &= I_{[0,x]}^\alpha 1_C = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(1 + \alpha)} \\ &= \frac{S_C(x)}{\Gamma(1 + \alpha)}, \end{aligned} \quad (7)$$

where F_C^i is a flag function that takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0 otherwise. $S_C(x)$ is the Cantor (devil's) staircase (Fig. 1b), i.e. a function almost everywhere flat except on an infinite number of singular points corresponding to the underlying Cantor set where it grows from 0 to 1. Moreover, Eq. (7) introduces the *fractional measure* of a fractal set: for the Cantor set C it is defined as $\mathcal{F}^\alpha(C) = I_{[0,1]}^\alpha 1_C(x)$. In fact $\mathcal{F}^\alpha(C)$ is infinite if $\alpha < d$, and 0 if $\alpha > d$. For $\alpha = d$, we find $\mathcal{F}^\alpha(C) = \frac{1}{\Gamma(1 + \alpha)}$, since $S_C(1) = 1$. This measure definition yields the same value as that predicted by the Hausdorff measure, the difference being represented only by a different value of the normalization constant. Eventually, from Eq. (7), it follows that the fractional measure of a generalized Cantor set $C_\alpha^{[a,b]}$ of dimension α built over the interval $[a, b]$ of the x -axis is:

$$\mathcal{F}^\alpha(C_\alpha^{[a,b]}) = I_{[a,b]}^\alpha 1_{C_\alpha^{[a,b]}} = \frac{(b-a)^\alpha}{\Gamma(1 + \alpha)}, \quad (8)$$

where $1_{C_\alpha^{[a,b]}}$ is the function equal to 1 if $x \in C_\alpha^{[a,b]}$, to 0 elsewhere.

2.3 The fractal bar

In the present section, we intend to solve a simple case using the mathematical tools presented in the previous section. Our aim is to show that experimental diagrams (see, for instance, Kleiser and Bocek (1986)) such as the

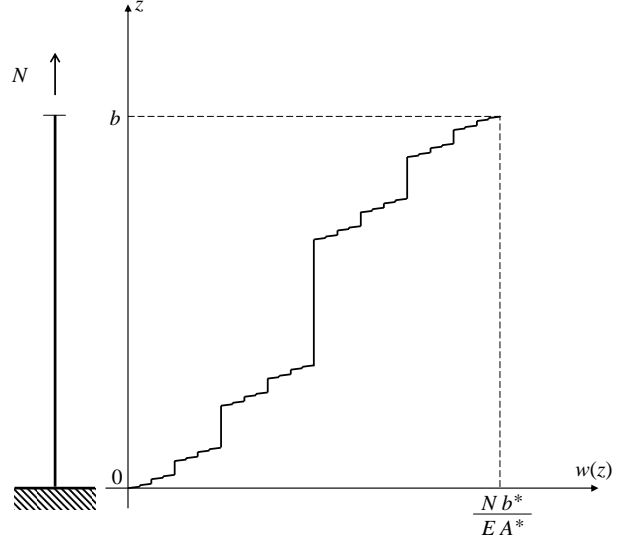


Fig. 2. The fractal bar subjected to an axial load: the displacement field.

one of Fig. 1b can be obtained also analytically. More details can be found in Carpinteri and Cornetti (2002).

Thus, let us consider a uniaxial model (Carpinteri and Cornetti (2002)), hereafter called *fractal Cantor bar*, according to Feder's terminology (Feder (1998)), i.e. a bar of length b deformable on a fractal subset of dimension $(1 - d_\varepsilon)$. The longitudinal axis is z . The bar is clamped in $z = 0$, whereas a tensile load N is applied at its end $z = b$ (Fig. 2). A strain field will arise that is zero almost everywhere except in an infinite number of points (corresponding to the deformable subset) where it is singular. The displacement singularities can be characterized by the LFD of order equal to the fractal dimension $\alpha = 1 - d_\varepsilon$ of the domain of the singularities, the unique value for which the LFD is finite and different from zero (the critical value). Therefore, we can define analytically the fractal strain ε^* as the LFD of order α of the displacement:

$$\varepsilon^*(z) = D^\alpha w(z). \quad (9)$$

Let us observe that, in Eq. (9), the noninteger physical dimensions $[L]^{d_\varepsilon}$ of ε^* are introduced by the LFD, whereas in Section 2.1 they are a geometrical consequence of the fractal dimension of the localization domain.

Without losing generality, let us assume the deformable subset to be the triadic Cantor set $C_\alpha^{[0,b]}$ built on $[0, b]$, $\alpha = \ln 2 / \ln 3$. In order to compute the displacement function $w(z)$, we need the proper constitutive law. Here, for the sake of simplicity, we use a linear elastic relation and assume $d_\sigma = d_\varepsilon$: in this case the coefficient of proportionality between fractal stress and fractal strain coincides with the one between the nominal quantities, i.e. it is the Young's modulus E . In symbols: $\sigma^* = E\varepsilon^*$.

For equilibrium reasons, the internal axial force is constant and equal to N throughout the bar. Hence, we get a fractal strain ε^* equal to N/EA^* over the deformable subset, 0 elsewhere. The kinematic equation (9) becomes:

$$D^\alpha w(z) = \frac{N}{EA^*} 1_{C_\alpha^{[0,b]}}(z). \quad (10)$$

Introducing the dimensionless quantities $\tilde{w} = w/b$, $\tilde{z} = z/b$ ($\tilde{z} \in [0, 1]$), we can apply the scaling property expressed by (for $a = 0$, Samko et al. (1993)):

$$\frac{d^q f(bx)}{[dx]^q} = b^q \frac{d^q f(bx)}{[d(bx)]^q}, \quad (11)$$

which is valid also for the LFD, to get $D^\alpha w(z) = b^{1-\alpha} D^\alpha \tilde{w}(\tilde{z})$. Eq. (10) can therefore be expressed in dimensionless form as follows:

$$D^\alpha \tilde{w}(\tilde{z}) = \frac{N}{EA^* b^{1-\alpha}} 1_C(\tilde{z}), \quad (12)$$

where C is the triadic Cantor set built on $[0, 1]$ as indicated in Section 2.1. In this form, the solution of the differential equation (12) can be obtained directly from Eq. (7):

$$\tilde{w}(\tilde{z}) = \frac{N}{EA^* b^{1-\alpha}} \frac{S_C(\tilde{z})}{\Gamma(1+\alpha)}, \quad (13)$$

where $S_C(x)$ is the Cantor staircase built on the interval $[0, 1]$ and rising from 0 to 1. Recovering the physical quantities yields:

$$w(z) = \frac{Nb^*}{EA^*} S_C\left(\frac{z}{b}\right), \quad (14)$$

where $b^* = \frac{b^\alpha}{\Gamma(1+\alpha)}$ is the fractal measure of the deformable subset. Equation (14) is plotted in Fig. 2. Let us emphasize that the Cantor staircase, introduced geometrically in Section 2.2, is now obtained analytically. Furthermore, notice that Eq. (14) provides important information about the size effect affecting the global deformation. In fact we find that the free end displacement $w(b)$ is equal to $\frac{Nb^*}{EA^*}$, i.e. $w(b) \sim b^\alpha$. This means that the displacement increases less than linearly with the bar length, as occurs with classical elastic bodies. From the point of view of the overall deformation $\varepsilon = w(b)/b$, we get $\varepsilon \sim b^{-(1-\alpha)}$: it decreases with size as a consequence of the strain localization on a lacunar fractal subset.

3. NON-LOCAL APPROACH

Different approaches have been proposed to take into account the effects of material microstructure onto the classical continuum equations (Eringen and Edelen (1972), Aifantis (1994)). A new model based on fractional calculus has recently been suggested (Di Paola and Zingales (2008)), and it will be described in the next sections.

3.1 Elastic bar with long-range interactions

Let us now consider an elastic bar of length b and subjected to external forces F . Let us discretize the bar in m equal elements, each with volume $V_j = A\Delta z$ ($j = 1, \dots, m$), where A is the cross-section area and $\Delta z = b/m$. The generic volume element V_j is located at the abscissa $z_j = j\Delta z$. Its equilibrium equation can be written taking into account external loads, contact forces provided by adjacent volumes, V_{j-1} and V_{j+1} , which are denoted by N_j and N_{j+1} , respectively, and long range interactions Q_j applied on V_j by the surrounding non-adjacent elements of the bar (Fig. 3). In formulae:

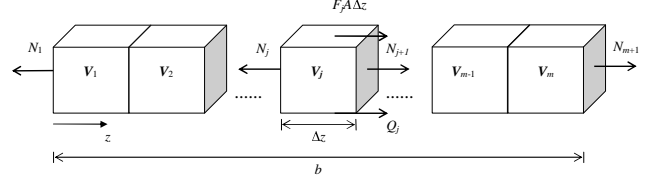


Fig. 3. Discretized elastic bar: equilibrium of the generic volume element V_j .

$$\begin{aligned} \Delta N_j + Q_j &= \Delta N_j + \sum_{h=1}^{j-1} Q^{(h,j)} - \sum_{h=j+1}^m Q^{(h,j)} = \\ &= -F_j A \Delta z, \end{aligned} \quad (15)$$

where $F_j = F(z_j)$, $\Delta N_j = N_{j+1} - N_j$ and $Q^{(h,j)}$ ($h \neq j$) are the long-range forces that surrounding volume elements apply on element V_j . These forces can be modelled as:

$$Q^{(h,j)} = \text{sgn}(z_h - z_j)[w(z_h) - w(z_j)]g(z_h, z_j)V_j V_h, \quad (16)$$

being $g(z_h, z_j)$ a real-valued monotonically decreasing function expressed as:

$$g(z_h, z_j) = \frac{Ec_\alpha \alpha}{A\Gamma(1-\alpha)|z_h - z_j|^{1+\alpha}}, \quad (0 \leq \alpha \leq 1) \quad (17)$$

and $\text{sgn}(z)$ the classical signum function:

$$\text{sgn}(z) = \begin{cases} -1 & \text{if } z < 0 \\ 0 & \text{if } z = 0 \\ +1 & \text{if } z > 0. \end{cases} \quad (18)$$

Note that the constant c_α in Eq. (17) has the anomalous dimensions $[L]^{\alpha-2}$. The classical continuum mechanics is consistently recovered for $\alpha \rightarrow 0$. Substituting Eq. (16) into (Eq. (15)), yields:

$$\begin{aligned} \Delta N_j - \frac{Ec_\alpha \alpha A}{\Gamma(1-\alpha)} \Delta z \left[\sum_{h=1}^{j-1} \frac{w(z_h) - w(z_j)}{(z_j - z_h)^{1+\alpha}} \Delta z + \right. \\ \left. - \sum_{h=j+1}^m \frac{w(z_j) - w(z_h)}{(z_h - z_j)^{1+\alpha}} \Delta z \right] = -F_j A \Delta z. \end{aligned} \quad (19)$$

Dividing Eq. (19) by $EA\Delta z$ and assuming a linear elastic material i.e., $\sigma(z) = E\varepsilon(z) = Edw/dz$, the following differential equation is obtained for $\Delta z \rightarrow 0$ (Di Paola and Zingales (2008)):

$$\frac{d^2 w}{dz^2} - c_\alpha \left(\widehat{\mathbf{D}}_{0+}^\alpha w + \widehat{\mathbf{D}}_{b-}^\alpha w \right) = -\frac{F(z)}{E}. \quad (20)$$

$\widehat{\mathbf{D}}_{0+}^\alpha w$ and $\widehat{\mathbf{D}}_{b-}^\alpha w$ in Eq. (20) represent the integrals terms in the Marchaud fractional derivatives on a finite interval (note that $a = 0$), which have been proved to coincide (Samko et al. (1993)), for a certain class of functions, with the Riemann-Liouville fractional derivatives (Eqs. (1) and (2)):

$$\begin{aligned} \mathbf{D}_{a+}^\alpha f(x) &= \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \\ &+ \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}} d\xi, \end{aligned} \quad (21)$$

$$\begin{aligned} \mathbf{D}_{b-}^{\alpha} f(x) &= \frac{f(x)}{\Gamma(1-\alpha)(b-x)^{\alpha}} + \\ &+ \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{f(x) - f(\xi)}{(\xi-x)^{1+\alpha}} d\xi, \end{aligned} \quad (22)$$

Equation (20) can be rewritten, equivalently, in terms of the Marchaud fractional derivatives ((Eqs. (21) and (22))) as:

$$\begin{aligned} \frac{d^2 w}{dz^2} - c_{\alpha} [(\mathbf{D}_{0+}^{\alpha} w + \mathbf{D}_{b-}^{\alpha} w) + \\ -w (\mathbf{D}_{0+}^{\alpha} 1 + \mathbf{D}_{b-}^{\alpha} 1)] &= -\frac{F(z)}{E}, \end{aligned} \quad (23)$$

which represents a nonhomogeneous ordinary fractional differential equation, with non-constant coefficients, since the derivatives of the unit function are neither zero nor constant.

It is worth observing that the fractional formula of integration by parts (Samko et al. (1993)):

$$\int_a^b [(\mathbf{D}_{a+}^{\alpha} f) \cdot g] dx = \int_a^b [(\mathbf{D}_{b-}^{\alpha} g) \cdot f] dx \quad (24)$$

can be exploited to verify that the resultant of the non-local forces, expressed by the second term in the left-hand side of Eq. (23), is zero. In fact, by choosing either f or g equal to 1, it is easy to check that:

$$\int_0^b [(\mathbf{D}_{0+}^{\alpha} w + \mathbf{D}_{b-}^{\alpha} w) + \\ -w (\mathbf{D}_{0+}^{\alpha} 1 + \mathbf{D}_{b-}^{\alpha} 1)] dz = 0. \quad (25)$$

Unfortunately, analytical solutions of Eq. (23) (i.e., Eq. (20)) are not available in the literature at the authors best knowledge.

3.2 Equivalent mechanical model

The physical validity of Eq. (15) can be explained by considering a simple discrete spring-point model of the bar (Di Paola and Zingales (2008)), as reported in Fig. 4 only for four points. Local forces between adjacent particles are taken into account by springs with elastic stiffness $K^l = EA/\Delta z$. On the other hand, long-distance interactions between particles are modelled by linear springs with distance-decay stiffness as $K_{jh}^{nl} = g(z_j, z_h)$, where function g is provided by Eq. (17).

Thus, the equilibrium equation for the generic node located at the abscissa z_j may be written as:

$$\begin{aligned} j = 1: \\ K^l (w_2 - w_1) + \\ + (A\Delta z)^2 \sum_{h=2}^m g(z_h, z_1)(w_h - w_1) &= -F_1, \end{aligned} \quad (26)$$

$$\begin{aligned} j = 2, \dots, m-1: \\ K^l (w_{j-1} - 2w_j + w_{j+1}) + \\ + (A\Delta z)^2 \sum_{h=1, h \neq j}^m g(z_h, z_j)(w_h - w_j) &= -F_j, \end{aligned} \quad (27)$$

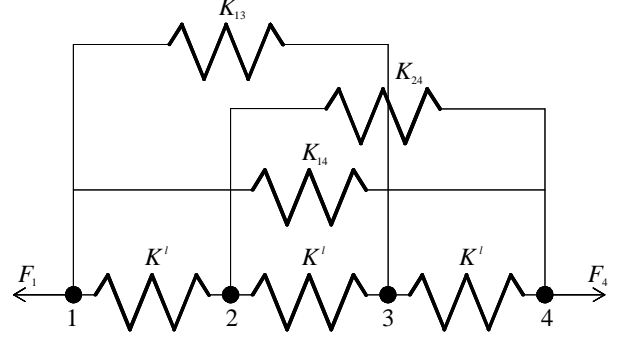


Fig. 4. Discretized elastic bar: spring-point model.

$j = m$:

$$\begin{aligned} K^l (w_{m-1} - w_m) + \\ + (A\Delta z)^2 \sum_{h=1}^{m-1} g(z_m, z_h)(w_h - w_m) &= -F_m, \end{aligned} \quad (28)$$

where the right-hand sides of Eqs. (26-28) represent the body forces applied to material particles.

Equilibrium equations (26-28) can be synthesized in a compact form by introducing the non-local coefficient matrix $\mathbf{K} = \mathbf{K}^l + \mathbf{K}^{nl}$ as:

$$\mathbf{K} \mathbf{w} = \mathbf{F}, \quad (29)$$

where \mathbf{w} and \mathbf{F} are the (m) -dimensional displacement and force vectors, respectively; \mathbf{K}^l is the tri-diagonal matrix related to contact contributions due to adjacent elements:

$$\mathbf{K}^l = \begin{bmatrix} K^l & -K^l & \dots & \dots & 0 \\ -K^l & 2K^l & -K^l & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & -K^l & 2K^l & -K^l \\ 0 & \dots & \dots & -K^l & K^l \end{bmatrix}, \quad (30)$$

while \mathbf{K}^{nl} is the fully populated, non-local stiffness matrix:

$$\begin{aligned} \mathbf{K}^{nl} &= \\ &= (A\Delta z)^2 \begin{bmatrix} K_{11}^{nl} & -g(z_2, z_1) & \dots & -g(z_m, z_1) \\ -g(z_2, z_1) & K_{22}^{nl} & \dots & -g(z_m, z_2) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -g(z_m, z_1) & \dots & \dots & K_{mm}^{nl} \end{bmatrix}, \end{aligned} \quad (31)$$

with

$$K_{jj}^{nl} = \sum_{h=1, h \neq j}^m g(z_j, z_h). \quad (32)$$

3.3 Numerical Applications

The non-local stiffness matrix obtained by the point-spring model (Eq. (31)) is found to coincide, for $\Delta z \rightarrow 0$, with that obtained by discretizing Eq. (20) with fractional finite differences (Di Paola and Zingales (2008)). Homogeneous boundary conditions are imposed to prevent divergent behaviour (Kilbas et al. (2006)). Let us consider a slender bar under tension. The bar dimensions are selected as follows: $L = 200 \text{ mm}$ and $A = 100 \text{ mm}^2$. The following material characteristics are considered: $E = 72 \cdot \text{GPa}$, $\alpha = 0.3$ and $c_{\alpha} = 0.03 \text{ mm}^{-1.5}$. The axial strain related

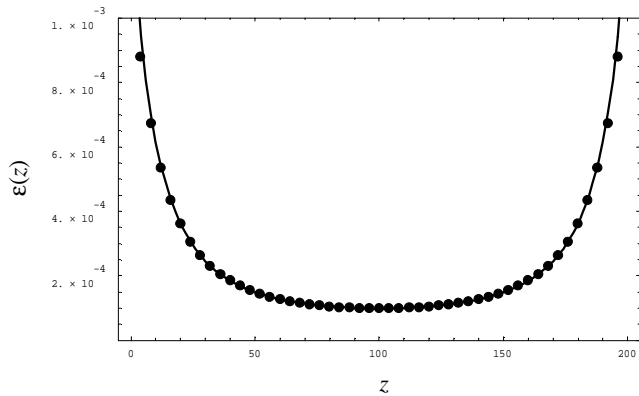


Fig. 5. Axial strain in a free-free bar: continuum fractional approach (continuous line) vs. discrete spring-point model (dots).

to such a structure is displayed in Fig. (5), showing a perfect agreement between the continuum fractional approach (continuous line) and the discrete spring-point model presented in the previous section (dots). Note that the axial strains are uniform along the bar core and increase at the boundary, as from experimental evidences. Similar results are predicted also by different non-local models (see, for instance, Aifantis (1994)).

4. CONCLUSIONS

Classical continuum theory works properly at the macro-scale, where the effect of material microstructure can be neglected. Recent technological progress (e.g. nano-devices) as well as the possibility to have a deeper insight into material behaviour forced the scientific community to analyze phenomena taking place at the meso/micro level. One way to address such problems is the use of enriched continuum mechanics models. Among these models, in the present paper we revisited two recently proposed approaches whose common feature is represented by the use of fractional calculus, which turns out to be a powerful tool to handle non-standard continua.

The former approach deals with fractal media, i.e. with solids whose microstructure is such that strain localizes onto fractal subsets. The latter approach deals with solids characterized by non-local long-range interactions. Although they are rather different, it is interesting to highlight the following analogies/differences between them.

- (1) Both the approaches yield non-standard fractional derivatives, the former one using the local fractional derivative and the latter one the integral part of Marchaud fractional derivative.
- (2) The local approach is based on kinematic arguments, whilst the non-local one stems from a static analysis.
- (3) Both the approaches provide a non-uniform strain field under a uniform stress field.
- (4) Both the models introduce a displacement derivative of order lower than the classical one, differently from what proposed by gradient theory (Aifantis (1994)), which provides derivatives of the displacement of higher orders. This is a drawback of gradient theory, since higher order differential equations need extra boundary conditions to be solved, whose physical meaning is still unclear.

- (5) In the local model, the LFD replaces the classical derivative (Eq. (9)), whereas in the non-local model the fractional derivative represents a correction to be added to the classical equilibrium equation (Eq. (19)) in order to take internal forces into account.
- (6) In the local approach, the kinematic equation appears as the straightforward generalization to fractal media of the definition of strain in continuum mechanics. This provides the key to generalize the principle of virtual work as well as to highlight the static-kinematic duality for fractal media (Carpinteri et al. (2001)). On the other hand this aspect remains hidden in the non-local approach.

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