Prime numbers in intervals starting at a fixed power of the integers

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PRIME NUMBERS IN INTERVALS STARTING AT A FIXED POWER OF THE INTEGERS

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ABSTRACT. The best known results about the distribution of prime numbers in short intervals imply that all intervals $[n, n+H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{7/12}$, and almost all intervals $[n, n+H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{1/6}$. As a natural generalization, this paper is concerned with the distribution of prime numbers in intervals of type $[n^\alpha, n^\alpha + H]$ with $\alpha > 1$.

1. INTRODUCTION

Let $\psi(x) = \sum_{n \leq x} \Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function. We consider the asymptotic formula

$$\psi(x+H) - \psi(x) \sim H \qquad x \to \infty, \tag{1}$$

which is related to the number of primes in the interval $(x, x+H]$. The Prime Number Theorem implies that (1) holds with $H \gg x$. An interval $(x, x+H]$ with $H = o(x)$ is called a short interval. The best known unconditional result about the distribution of primes in short intervals is due to M. N. Huxley [8] and asserts that (1) holds for all $H \geq x^{7/12+\varepsilon}$. This was slightly improved by D. R. Heath-Brown in [7] to $H \geq x^{7/12-o(1)}$. Under the assumption of the Riemann Hypothesis, A. Selberg [11] proved that (1) holds for all $H \geq x^{1/2} f(x) \log x$ with $f(x) \to \infty$ arbitrarily slowly. These results imply that all intervals $[n, n+H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{7/12}$ and, assuming the Riemann Hypothesis, for all $H \geq N^{1/2} f(N) \log N$ with $f(N) \to \infty$ arbitrarily slowly.

We can relax our conditions and investigate if (1) holds for "almost all" x. By this, we mean that the measure of $x \in [X, 2X]$ for which (1) does not hold is $o(X)$. Huxley's zero density estimate [8], in conjunction with the method of Selberg [11], show that (1) holds for almost all x with $H \geq x^{1/6+\varepsilon}$, slightly improved by A. Zaccagnini in [14] to $H \geq x^{1/6-o(1)}$. Under the assumption of the Riemann Hypothesis, Selberg [11] proved that (1) holds for almost all x with $H \geq f(x) \log^2 x$, where $f(x) \to \infty$ arbitrarily slowly. These results imply that almost all intervals $[n, n+H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{1/6}$ and, assuming the Riemann Hypothesis, for all $H \geq f(N) \log^2 N$ with $f(N) \to \infty$ arbitrarily slowly.

As a natural generalization of the above results, this paper is concerned with the distribution of prime numbers in intervals $[n^\alpha, n^\alpha + H]$, with fixed $\alpha > 1$. Our main unconditional result is the following.

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Theorem 1 *Let $\varepsilon > 0$ and $\alpha > 1$. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)+\varepsilon}$, where*

$$c(\alpha) = \begin{cases} \dfrac{1}{6} & \text{if} \quad 1 < \alpha \leq \dfrac{6}{5} \\[2ex] \dfrac{11\alpha - 10}{16\alpha} & \text{if} \quad \dfrac{6}{5} < \alpha \leq \dfrac{6}{5} + \Delta \\[2ex] 1 - \sup\limits_{(k,l)} \dfrac{5(1+\alpha-l+k)}{\alpha(5k+12)} & \text{if} \quad \alpha \geq 4 \end{cases}$$

with Δ suitable positive constant and (k, l) running over the exponent pairs.

For the sake of simplicity, we will explicitly work out the value of the function $c(\alpha)$ only for the extreme and more interesting values of α. However, it will be clear from the proof that the same method enables one to obtain the explicit values of the function $c(\alpha)$ in the whole range $\alpha > 1$. As one might expect, we get an increasing function $c(\alpha)$ such that $c(1) = 1/6$, $c(\alpha) < 7/12$ for every α and

$$\lim_{\alpha \to +\infty} c(\alpha) = \frac{7}{12}.$$

To bound some sums which arise in our argument we employ the counting functions $N(\sigma, T)$ and $N^*(\sigma, T)$. The former is defined as the number of zeros $\rho = \beta + i\gamma$ of Riemann zeta function which satisfy $\sigma \leq \beta \leq 1$ and $|\gamma| \leq T$, while $N^*(\sigma, T)$ is defined as the number of ordered sets of zeros $\rho_j = \beta_j + i\gamma_j$ $(1 \leq j \leq 4)$, each counted by $N(\sigma, T)$, for which $|\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4| \leq 1$. If we make the heuristic assumption that

$$N^*(\sigma, T) \ll \frac{N(\sigma, T)^4}{T}, \tag{2}$$

as in D. Bazzanella and A. Perelli [2], then we can simplify and improve Theorem 1 for large values of α as follows.

Theorem 2 *Assume (2), let $\varepsilon > 0$ and $\alpha \geq 4$. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)+\varepsilon}$ and*

$$c(\alpha) = \frac{7}{12} - \frac{5}{12\alpha}.$$

We conclude by presenting our results under the assumption of more standard hypotheses.

Theorem 3 *Let $\alpha > 1$, $\varepsilon > 0$ and assume the Lindelöf Hypothesis. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)+\varepsilon}$ and*

$$c(\alpha) = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right).$$

Theorem 4 *Let $\alpha > 1$ and assume the Riemann Hypothesis. Then almost all intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$ contain the expected number of primes for all $H \geq N^{c(\alpha)} f(N) \log^2 N$ with $f(N) \to \infty$ arbitrarily slowly and*

$$c(\alpha) = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right).$$

As one might expect, under the assumption of the Lindelöf Hypothesis or the Riemann Hypothesis, we get an increasing function $c(\alpha)$ such that $c(1) = 0$, $c(\alpha) < 1/2$ for every α and

$$\lim_{\alpha \to +\infty} c(\alpha) = \frac{1}{2}.$$

The main tools of the proofs are the Kusmin–Landau estimate for an exponential sum together with the van der Corput's method of exponent pairs, see [4], and a result about the structure of the exceptional set for the distribution of primes in short intervals due to Bazzanella and Perelli, see [2] and [1].

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2. Definitions and basic lemmas

Our starting point is the definition of the exceptional set for the number of primes in short intervals. Let $|\ \ |$ denote the modulus of a complex number or the Lebesgue measure of an infinite set of real numbers or the cardinality of a finite set. Let X be a large positive number, $\delta > 0$ and define

$$E_\delta(X, H) = \{X \leq x \leq 2X : |\psi(x + H(x)) - \psi(x) - H(x)| \geq \delta H(x)\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, H) = \emptyset$ for all $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞, the set $E_\delta(X, H)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. We will consider increasing functions $H(x)$ of the form $H(x) = x^{\theta + \varepsilon(x)}$, with some $0 < \theta < 1$ and a differentiable function $\varepsilon(x)$ such that $|\varepsilon(x)|$ is decreasing, $\varepsilon(x) = o(1)$ and

$$\varepsilon(x + y) = \varepsilon(x) + O\left(\frac{|y|}{x \log x}\right). \tag{3}$$

A function satisfying these requirements will be called of type θ. It is easy to see that functions like $x^\theta \log^c x$, with c real constant, and similar functions, are of type θ, and that for every functions $H(x)$ of type θ we have $H(2x) \ll H(x)$.

Remark In a preceding paper, the author and Perelli [2] defined in a slightly different way the set of functions $H(x)$ of type θ, and set

$$\varepsilon(x + y) = \varepsilon(x) + O\left(\frac{|y|}{x}\right)$$

instead of (3). We remark that with this weaker condition we do not have $H(2x) \ll H(x)$ as claimed.

Our first lemma is concerned with the structure of the exceptional set above.

Lemma 1 *Let $0 < \theta < 1$, let $H(x)$ be of type θ, let X be sufficiently large depending on the function $H(x)$ and let $0 < \delta' < \delta$ with $\delta - \delta' \geq exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X, H)$ then $E_{\delta'}(X, H)$ contains the interval $[x_0 - cH(X), x_0 + cH(X)] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$.*

Proof. *We will always assume that x and X are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence.*

We first observe from the definition of a function of type θ that if $y = O(x^{\alpha+\varepsilon})$ with some $0 < \alpha < 1$, then

$$H(x + y) = H(x) + O(x^{\theta+\alpha-1+\varepsilon}) \tag{4}$$

for every $\varepsilon > 0$.

From the Brun–Titchmarsh theorem (see H. L. Montgomery and R .C. Vaughan [10]*), we have that*

$$\psi(x + y) - \psi(x) \leq \frac{21}{10} y \frac{\log x}{\log y} \tag{5}$$

for all $10 \leq y \leq x$. From (5) we easily obtain that

$$\psi(x + y) - \psi(x) \leq \frac{9}{4\alpha} cY \tag{6}$$

for all $X \leq x \leq 3X$ and $0 \leq y \leq cY$, where $0 < \alpha < 1$, $X^{\alpha-\varepsilon} \leq Y \leq X$ and

$$\frac{\alpha}{5} \exp(-\sqrt{\log X}) \leq c \leq 1.$$

Let $H(x)$ be of type θ, $x_0 \in E_\delta(X, H)$,

$$x \in [x_0 - cH(X), x_0 + cH(X)] \cap [X, 2X],$$

where c satisfies the above restrictions, and

$$\Delta(x, H) = \psi(x + H(x)) - \psi(x) - H(x).$$

We have

$$|\Delta(x, H)| = |\Delta(x_0, H) + \Delta(x, H) - \Delta(x_0, H)| \geq$$

$$|\Delta(x_0, H)| - |\psi(x + H(x)) - \psi(x_0 + H(x_0))| - |\psi(x) - \psi(x_0)| - |H(x) - H(x_0)|.$$

But from (4) with $\alpha = \theta$ we get

$$H(x_0) = H(x) + O(X^{2\theta-1+\varepsilon}),$$

hence from (6) with $\alpha = \theta$ we obtain

$$|\Delta(x, H)| \geq \delta H(x) - \frac{9}{2\theta} cH(X) + O(X^{2\theta-1+\varepsilon}) \geq \delta H(x) - \frac{5}{\theta} cH(X) \geq \delta' H(x)$$

by choosing $c = (\delta - \delta')\theta/5$, since $H(x)$ is increasing. Hence $x \in E_{\delta'}(X, H)$ and the lemma follows. □

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [2], and essentially says that if we have a single exception in $E_\delta(X, H)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta'}(X, H)$, with δ' a little smaller than δ.

We now present the necessary results about the conditional and unconditional bounds for the exceptional set for the number of primes in short intervals. With this in mind, we consider $H(x)$ of type θ and define the functions

$$\mu_\delta(\theta) = \inf\{\xi \geq 0 : |E_\delta(X,H)| \ll_\delta X^\xi\}$$

and

$$\mu(\theta) = \sup_{\delta>0} \mu_\delta(\theta). \tag{7}$$

Our results are as follows.

Lemma 2 *There exists a constant $\eta > 0$ such that*

$$\mu(\theta) \leq \frac{(11-6\theta)}{10} \quad \text{if} \quad \frac{1}{6} < \theta \leq \frac{1}{6} + \eta.$$

Proof. *In order to prove Lemma 2 we use the classical explicit formula (see H. Davenport* [3, chapter 17]*) to write*

$$\psi(x+H(x)) - \psi(x) - H(x) = -\sum_{|\gamma|\leq T} x^\rho c_\rho(x) + O\left(\frac{X\log^2 X}{T}\right), \tag{8}$$

uniformly for all $X \leq x \leq 2X$, where $10 \leq T \leq X$, $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$,

$$c_\rho(x) = \frac{(1+H(x)/x)^\rho - 1}{\rho} \qquad \text{and} \qquad c_\rho(x) \ll \min\left(\frac{H(X)}{X}, \frac{1}{|\gamma|}\right). \tag{9}$$

Let $H(x)$ be of type θ. Choose

$$T = \frac{X}{H(X)} \log^3 X \tag{10}$$

and use the theorem of Montgomery (see Theorem 11.3 of A. Ivić [9]*) which asserts that*

$$N(\sigma,T) \ll T^{1600(1-\sigma)^{3/2}} \log^{15} T \tag{11}$$

for every $152/155 \leq \sigma \leq 1$. From (9) – (11) and Vinogradov's zero-free region (see E. C. Titchmarsh [12, chapter 6]*) we deduce by a standard argument that there exists a constant $d > 0$ such that*

$$\sum_{\substack{|\gamma|\leq T \\ \beta\notin I}} x^\rho c_\rho(x) \ll \frac{H(X)}{X} \log X \max_{\sigma\notin I} X^\sigma N(\sigma,T) \ll \frac{H(X)}{\log X}, \tag{12}$$

where $I = [1/2, 1-d]$, uniformly for all $X \leq x \leq 2X$.

Again by a standard argument, from (9), (10) and the Ingham–Huxley density estimates which assert that for every $\varepsilon > 0$ we have

$$N(\sigma,T) \ll \begin{cases} T^{3(1-\sigma)/(2-\sigma)+\varepsilon} & \frac{1}{2} \leq \sigma \leq \frac{3}{4} \\ T^{3(1-\sigma)/(3\sigma-1)+\varepsilon} & \frac{3}{4} \leq \sigma \leq 1 \end{cases}, \tag{13}$$

we obtain

$$\int_X^{2X} \Big| \sum_{\substack{|\gamma|\le T\\ \beta\in I}} x^\rho c_\rho(x)\Big|^2 \mathrm{d}x \ll X^{2\theta-1+\varepsilon} \max_{\sigma\in I} X^{2\sigma} N(\sigma,T) \ll X^{(11+14\theta)/10+\varepsilon},$$

for sufficiently small $\eta > 0$ *and* $1/6 < \theta \le 1/6+\eta$. *Hence for every* $\varepsilon > 0$ *and* $\delta > 0$ *we have*

$$|E_\delta(X,H)| \ll X^{(11-6\theta)/10+\varepsilon},$$

and so the lemma is proved. □

We observe that we can take $d = 2.5\cdot 10^{-7}$ and then $\eta = 3.125\cdot 10^{-7}$. The value of η could be somewhat increased by using an optimized version of density estimate (11).

Lemma 3 *Assume (2). Then we have*

$$\mu(\theta) \le \frac{7}{5}(1-\theta) \quad \text{if} \quad \frac{23}{48} < \theta < \frac{7}{12}.$$

Proof. *Let* $H(x)$ *be of type* θ *and*

$$T = \frac{X}{H(X)} \log^3 X.$$

Following the method of Heath-Brown [5], *we can write*

$$\int_X^{2X} |\psi(x+H(x)) - \psi(x) - H(x) + \Sigma|^4 \,\mathrm{d}x \ll X^{4\theta-3+\varepsilon} \max_{1/2\le\sigma\le 1} X^{4\sigma} N^*(\sigma,T),$$

with $\Sigma = o(H(X))$. *Assuming (2) and using the Ingham–Huxley zero density estimates, the above estimate implies that*

$$|E_\delta(X,H)| \ll X^{-3+\varepsilon} \max_{1/2\le\sigma\le 1} X^{4\sigma} N^*(\sigma,T) \ll X^{-3+\varepsilon} \max_{1/2\le\sigma\le 1} X^{4\sigma} \frac{N(\sigma,T)^4}{T}$$

$$\ll X^{\theta-4+\varepsilon}\left(\max_{1/2\le\sigma\le 3/4} X^{4\sigma} T^{12(1-\sigma)/(2-\sigma)} + \max_{3/4\le\sigma\le 1} X^{4\sigma} T^{12(1-\sigma)/(3\sigma-1)}\right),$$

for every $\delta > 0$ *and* $\varepsilon > 0$. *With* $23/48 < \theta < 7/12$ *the maximum is attained at* $\sigma = 3/4$, *so we have*

$$|E_\delta(X,H)| \ll X^{\frac{7}{5}(1-\theta)+\varepsilon},$$

for every $\delta > 0$ *and* $\varepsilon > 0$. *This completes the proof of the lemma.* □

Lemma 4 *Assume the Lindelöf Hypothesis, let* $\varepsilon > 0$ *and* $\delta > 0$. *For every* $H \ge 1$ *we have*

$$|E_\delta(X,H)| \ll \frac{X^{1+\varepsilon}}{H(X)}.$$

Lemma 4 may be proved along the same lines as G. Yu [13, Lemma B].

To deal with the problem of estimating the exceptional set for the distribution of primes in intervals $[n^\alpha, n^\alpha + H] \subset [N, 2N]$, suppose that $H(x)$ is of type θ, let

$$\Delta(n,H,\alpha) = \psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha),$$

and define the set

$$A_\delta(N,H,\alpha) = \{N^{1/\alpha} \le n \le (2N)^{1/\alpha} : |\Delta(n,H,\alpha)| \ge \delta H(n^\alpha)\},$$

that contains the exceptions, if any, to the expected asymptotic formula for the number of primes in intervals of type $[n^\alpha, n^\alpha + H(n^\alpha)] \subset [N, 2N]$. Our last lemmas allow us to link $|A_\delta(N,H,\alpha)|$ to the exceptional set for the distribution of primes in short intervals.

Lemma 5 *Let $H(x)$ be of type θ, with $1/6 < \theta < 7/12$. Then for every $\delta > 0$ we have*

(i) $|A_\delta(N,H,\alpha)| = o(N^{1/\alpha})$ *if* $1 < \alpha \le \frac{6}{5}$

and

(ii) $|A_\delta(N,H,\alpha)| \ll \dfrac{|E_{\delta/2}(N,H)| f(N) \log^2 N}{H(N)} + o(N^{1/\alpha})$ *if* $\alpha > \frac{6}{5}$,

with $f(N) \to \infty$ arbitrarily slowly.

Proof. *Recalling the explicit formula for $\psi(x)$ and putting*

$$T = \frac{N}{H(N)} f(N) \log^2 N,$$

where $f(N) \to \infty$ arbitrarily slowly, we have

$$\begin{aligned}\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha) &= - \sum_{|\gamma|<T} n^{\alpha\rho}\ c_\rho(n) + o(H(N)) \\ &= - \sum_{\substack{|\gamma|<T \\ \beta\in I}} n^{\alpha\rho}\ c_\rho(n) + o(H(N)),\end{aligned}$$

where d and $I = [1/2, 1-d]$ are defined as in the proof of Lemma 2,

$$c_\rho(n) = \frac{1-(1+H(n^\alpha)n^{-\alpha})^\rho}{\rho} \qquad \text{and} \qquad c_\rho(n) \ll \min\left(\frac{H(N)}{N}, \frac{1}{|\gamma|}\right).$$

Further we divide the interval I into $O(\log N)$ subintervals I_j of the form

$$I_j = \left[\frac{j-1}{\log N}, \frac{j}{\log N}\right] \cap I.$$

On applying Cauchy's inequality we find

$$|\sum_{\substack{|\gamma|<T \\ \beta\in I}} n^{\alpha\rho}\ c_\rho(n)|^2 \ll \log N \sum_j |\sum_{\substack{|\gamma|<T \\ \beta\in I_j}} n^{\alpha\rho}\ c_\rho(n)|^2,$$

and so we get

$$\begin{aligned}H(N)^2 |A_\delta(N,H,\alpha)| &\ll \sum_{n\in A_\delta(N,H,\alpha)} |\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha) + o(H(N))|^2 \\ &\le \sum_{N^{1/\alpha}\le n\le (2N)^{1/\alpha}} |\sum_{\substack{|\gamma|<T \\ \beta\in I}} n^{\alpha\rho}\ c_\rho(n)|^2\end{aligned}$$

$$\ll \log N \sum_{N^{1/\alpha}\le n\le (2N)^{1/\alpha}} \sum_j \Big| \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} n^{\alpha\rho}\ c_\rho(n)\Big|^2.$$

Squaring and using partial summation we have then

$$|A_\delta(N,H,\alpha)| \ll \frac{\log N}{H(N)^2} \sum_{N^{1/\alpha}\le n\le (2N)^{1/\alpha}} \sum_j \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j}} n^{\alpha(\rho+\overline{\rho'})}\ c_\rho(n)\ \overline{c_{\rho'}(n)}$$

$$\ll \frac{\log N}{N^2} \sum_j N^{2j/\log N} \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j}} |S|$$

where

$$S = \sum_{N^{1/\alpha}\le n\le (N_1)^{1/\alpha}} n^{\alpha i(\gamma-\gamma')} = \sum_{N^{1/\alpha}\le n\le (N_1)^{1/\alpha}} \mathrm{e}(g(n)),$$

$$e(x) = \mathrm{e}^{2\pi i x}, \quad g(x) = \frac{\alpha(\gamma-\gamma')}{2\pi}\log x$$

and $N \le N_1 \le 2N$.

Let

$$H(N) \ge \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N, \tag{14}$$

with $f(N)\to\infty$ *arbitrarily slowly. Using the theorem of Kusmin–Landau (see S. W. Graham and G. Kolesnik* [4, theorem 2.1]*) and the trivial bound, one finds that*

$$|S| \ll \frac{N^{1/\alpha}}{|\gamma-\gamma'|} \quad \textit{and} \quad |S| \ll N^{1/\alpha},$$

and hence

$$|A_\delta(N,H,\alpha)| \ll \frac{\log N}{N^2} \sum_j N^{2j/\log N} \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j,\ |\gamma-\gamma'|\le 1}} N^{1/\alpha}$$

$$+ \frac{\log N}{N^2} \sum_j N^{2j/\log N} \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j,\ |\gamma-\gamma'|> 1}} \frac{N^{1/\alpha}}{|\gamma-\gamma'|},$$

which implies

$$|A_\delta(N,H,\alpha)| \ll \frac{N^{1/\alpha}}{N^2}\log^3 N \left(\sum_j \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} N^{2j/\log N} \right). \tag{15}$$

For every $1<\alpha\le 6/5$ *and* $H(x)$ *of type* θ *with* $1/6<\theta<7/12$*, and for every* $\alpha>6/5$ *and* $H(x)$ *satisfying (14), it follows by a standard argument and the Ingham–Huxley zero density estimates that*

$$\sum_j \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} N^{2j/\log N} \ll \max_{\sigma\in I} N^{2\sigma} N(\sigma,T) \ll \frac{N^2}{\log^A N}, \tag{16}$$

for every $A>0$*. From (15) and (16), it follows that*

$$|A_\delta(N,H,\alpha)| = o(N^{1/\alpha})$$

for every $1 < \alpha \leq 6/5$ *and for every* $\alpha > 6/5$ *with*

$$H(N) \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N.$$

Finally, let $\alpha > 6/5$ *and*

$$H(N) < \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N.$$

To deal with this small H *we observe that if* $n \in A_\delta(N, H, \alpha)$ *then* $N \leq n^\alpha \leq 2N$ *and*

$$|\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha)| \geq \delta H(n^\alpha).$$

Thus $n^\alpha \in E_\delta(N, H)$*. By Lemma 1 we find a constant* $c > 0$ *such that*

$$[n^\alpha - cH(N), n^\alpha + cH(N)] \cap [N, 2N] \subset E_{\delta/2}(N, H).$$

We now consider $m \in A_\delta(N, H, \alpha)$*, with* $|m - n| \geq \frac{2}{\pi} f(N) \log^2 N$ *and similarly we get* $m^\alpha \in E_\delta(N, H)$ *and then*

$$[m^\alpha - cH(N), m^\alpha + cH(N)] \cap [N, 2N] \subset E_{\delta/2}(N, H),$$

again by Lemma 1. Since

$$|m^\alpha - n^\alpha| \geq |m - n| \alpha N^{1-1/\alpha} \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N > H(N)$$

we may deduce that

$$[m^\alpha - cH(N), m^\alpha + cH(N)] \cap [n^\alpha - cH(N), n^\alpha + cH(N)] = \emptyset,$$

for c *suitable small. This leads to the bound*

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)| f(N) \log^2 N}{H(N)},$$

for every $\delta > 0$*, which proves the lemma.* □

Lemma 6 *Assume the Lindelöf Hypothesis. Let* $H(x)$ *be of type* θ*, with* $0 < \theta < 1/2$*. Then for every* $\delta > 0$ and $\alpha > 1$ *we have*

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)| f(N) \log^2 N}{H(N)} + o(N^{1/\alpha})$$

with $f(N) \to \infty$ *arbitrarily slowly.*

Proof. *We follow the proof of the Lemma 5 until the equation (15). Under the assumption of the Lindelöf Hypothesis, which states that the Riemann zeta-function satisfies*

$$\zeta(\sigma + it) \ll t^\eta \quad (\sigma \geq \frac{1}{2}, t \geq 2),$$

for any $\eta > 0$*, we have*

$$N(\sigma, T) \ll \begin{cases} T^{(2+4\eta)(1-\sigma)} (\log T)^M & 0 \leq \sigma \leq 1 \\ T^{3\eta(1-\sigma)/(\sigma-3/4)} (\log T)^M & \frac{3}{4} < \sigma \leq 1 \end{cases}, \tag{17}$$

with $T \geq 2$ and M suitable absolute constant (see Lemma 3 of Yu [13]*). From (17) it follows that the bound (16) hold for every*

$$H(N) \geq \frac{2\alpha}{\pi} N^{1-1/\alpha} f(N) \log^2 N$$

and $\alpha > 1$. We can conclude the proof by dealing with smaller values of H in the same way as in the proof of Lemma 5. □

3. Proof of the Theorem 1

By the case (i) of the Lemma 5, we can take

$$c(\alpha) = \frac{1}{6} \quad \text{if} \quad 1 < \alpha \leq \frac{6}{5}.$$

For all $\alpha > 6/5$, by (ii) of the Lemma 5, we have

$$|A_\delta(N, H, \alpha)| \ll \frac{|E_{\delta/2}(N, H)| f(N) \log^2 N}{H(N)} + o(N^{1/\alpha}),$$

for every $H(x)$ of type θ, with $1/6 < \theta < 7/12$. Futhermore, by Lemma 2 there exists $\eta > 0$ such that we have here

$$|E_{\delta/2}(N, H)| \ll N^{(11-6\theta)/10+\varepsilon},$$

for every

$$\frac{1}{6} < \theta \leq \frac{1}{6} + \eta,$$

and every $H(x)$ of type θ. These estimates together yield

$$|A_\delta(N, H, \alpha)| \ll N^{(11-16\theta)/10+\varepsilon} + o(N^{1/\alpha}),$$

and

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

for every

$$\theta > \frac{11\alpha - 10}{16\alpha}$$

and sufficiently small $\alpha > 6/5$. It follows that

$$c(\alpha) = \frac{11\alpha - 10}{16\alpha} \quad \text{if} \quad \frac{6}{5} < \alpha \leq \frac{6}{5} + \Delta,$$

for suitable positive constant Δ. From the explicit value for η available from the Lemma 2, we can state that an admissible value is $\Delta = 7.2 \cdot 10^{-7}$.

To estimate $c(\alpha)$ for large values of α we need to follow a quite different method. In a similar way as in the proof of Lemma 5, we let

$$T = \frac{N}{H(N)} \log^3 N$$

and write

$$\psi(n^\alpha + H(n^\alpha)) - \psi(n^\alpha) - H(n^\alpha) = -\sum_{\substack{|\gamma| < T \\ \beta \in I}} n^{\alpha\rho}\, c_\rho(n) + o(H(N)),$$

where $I = [1/2, 1-d]$, for a suitable positive constant d. Next we divide the interval I into $O(\log N)$ subintervals I_j of the form

$$I_j = \left[\frac{j-1}{\log N}, \frac{j}{\log N}\right] \cap I.$$

Using Hölder's inequality, we get

$$\left| \sum_{\substack{|\gamma|<T\\ \beta\in I}} n^{\alpha\rho}\ c_\rho(n) \right|^4 \ll \log^3 N\ \sum_j \left| \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} n^{\alpha\rho}\ c_\rho(n) \right|^4$$

and then we can deduce

$$|A_\delta(N,H,\alpha)| \ll \frac{\log^3 N}{H(N)^4} \sum_{N^{1/\alpha}\le n\le (2N)^{1/\alpha}} \sum_j \left| \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} n^{\alpha\rho}\ c_\rho(n) \right|^4 \ll$$

$$\frac{\log^3 N}{H(N)^4} \sum_{N^{1/\alpha}\le n\le (2N)^{1/\alpha}} \sum_j \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j}} \sum_{\substack{|\gamma''|<T\\ \beta''\in I_j}} \sum_{\substack{|\gamma'''|<T\\ \beta'''\in I_j}} n^{\alpha(\rho+\rho'+\overline{\rho''}+\overline{\rho'''})}\ C_n$$

$$\ll \frac{\log^3 N}{N^4} \sum_j N^{4j/\log N} \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j}} \sum_{\substack{|\gamma''|<T\\ \beta''\in I_j}} \sum_{\substack{|\gamma'''|<T\\ \beta'''\in I_j}} |S| = V_1 + V_2,$$

where

$$C_n = c_\rho(n)\ c_{\rho'}(n)\ \overline{c_{\rho''}(n)}\ \ \overline{c_{\rho'''}(n)}$$

$$S = \sum_{N^{1/\alpha}\le n\le (N_1)^{1/\alpha}} \mathrm{e}(g(n)), \quad g(x) = \frac{\alpha(\gamma+\gamma'-\gamma''-\gamma''')}{2\pi}\log x,$$

$$V_1 = \frac{\log^3 N}{N^4} \sum_j N^{4j/\log N} \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j}} \sum_{\substack{|\gamma''|<T\\ \beta''\in I_j}} \sum_{\substack{|\gamma'''|<T\\ \beta'''\in I_j\\ |\gamma+\gamma'-\gamma''-\gamma'''|\le(\pi/\alpha)N^{1/\alpha}}} |S|$$

and

$$V_2 = \frac{\log^3 N}{N^4} \sum_j N^{4j/\log N} \sum_{\substack{|\gamma|<T\\ \beta\in I_j}} \sum_{\substack{|\gamma'|<T\\ \beta'\in I_j}} \sum_{\substack{|\gamma''|<T\\ \beta''\in I_j}} \sum_{\substack{|\gamma'''|<T\\ \beta'''\in I_j\\ |\gamma+\gamma'-\gamma''-\gamma'''|>(\pi/\alpha)N^{1/\alpha}}} |S|$$

We first proceed to estimate V_1. For the terms in the inner sum with

$$|\gamma+\gamma'-\gamma''-\gamma'''| < 1$$

we can estimate $|S|$ using the trivial bound. For the terms with

$$1 \le |\gamma+\gamma'-\gamma''-\gamma'''| \le \frac{\pi}{\alpha} N^{1/\alpha}$$

we can use the Kusmin–Landau theorem. Hence we obtain the estimate

$$S \ll \frac{N^{1/\alpha}}{1+|\gamma+\gamma'-\gamma''-\gamma'''|},$$

which, by Heath-Brown's method [5], implies

$$V_1 \ll \frac{N^{1/\alpha}\log^5 N}{N^4}\ \max_{\sigma\in I} N^{4\sigma} N^*(\sigma,T).$$

For $H(x)$ of type θ, with $0.342 < \theta < 7/12$, Heath-Brown's zero-density estimates

$$N^*(\sigma,T) \ll \begin{cases} T^{(10-11\sigma)/(2-\sigma)+\varepsilon} & \frac{1}{2} \le \sigma \le \frac{2}{3} \\ T^{(18-19\sigma)/(4-2\sigma)+\varepsilon} & \frac{2}{3} \le \sigma \le \frac{3}{4} \\ T^{12(1-\sigma)/(4\sigma-1)+\varepsilon} & \frac{3}{4} \le \sigma \le 1, \end{cases} \tag{18}$$

(Theorem 2 of [6]) give upper bounds for $N^{4\sigma}N^*(\sigma,T)$ that attain their maximum at $\sigma = 1-d$. A short calculation then shows that

$$\max_{\sigma\in I} N^{4\sigma}N^*(\sigma,T) \ll \frac{N^4}{(\log N)^A},$$

for every $A > 0$. Hence we conclude

$$V_1 = o(N^{1/\alpha}),$$

for every $0.342 < \theta < 7/12$.

Now we turn to estimating V_2. Let (k,l) an exponent pair then

$$S \ll \left(\frac{|\gamma+\gamma'-\gamma''-\gamma'''|}{N^{1/\alpha}}\right)^k \left(N^{1/\alpha}\right)^l \ll \left(\frac{T}{N^{1/\alpha}}\right)^k N^{l/\alpha} \ll N^{(k\alpha(1-\theta)-k+l)/\alpha+\varepsilon},$$

for every $\varepsilon > 0$ and $H(x)$ of type θ. This yields

$$\begin{aligned} V_2 &\ll N^{(k\alpha(1-\theta)-k+l)/\alpha-4+\varepsilon} \sum_j \left(\sum_{\substack{|\gamma|<T \\ \beta\in I_j}} N^{j/\log N}\right)^4 \\ &\ll N^{(k\alpha(1-\theta)-k+l)/\alpha-4+\varepsilon} \left(\max_\sigma N^\sigma N(\sigma,T)\right)^4 \end{aligned}$$

For H of type θ, with

$$\frac{23}{48} < \theta < \frac{7}{12}, \tag{19}$$

the density estimates of Ingham–Huxley give upper bounds for $N^\sigma N(\sigma,T)$ that attain their maximum at $\sigma = 3/4$. So we may deduce

$$V_2 \ll N^{(k\alpha(1-\theta)-k+l)/\alpha-4+\varepsilon}\ N^{3+12(1-\theta)/5}.$$

The above bound is $o(N^{1/\alpha})$ for every

$$\theta > 1 - \frac{5(1+\alpha-l+k)}{\alpha(5k+12)}, \tag{20}$$

if (k,l) is an exponent pair, H of type θ and α sufficiently large. Thus we can select

$$c(\alpha) = 1 - \sup_{(k,l)} \frac{5(1+\alpha-l+k)}{\alpha(5k+12)},$$

where (k,l) runs over the exponent pairs. Since all exponent pairs (k,l) have $0 \le k \le 1/2 \le l$, we obtain

$$1 - \frac{5}{12}\frac{1+\alpha}{\alpha} < 1 - \sup_{(k,l)} \frac{5(1+\alpha-l+k)}{\alpha(5k+12)} = c(\alpha),$$

which implies (19) if $\alpha \geq 4$. On the other hand from the exponent pairs

$$A^{i-1}B(0,1) = \left(\frac{1}{2(2^i-1)}, 1 - \frac{i}{2(2^i-1)}\right),$$

where

$$i = \left[\frac{5\alpha}{12}\right],$$

we get

$$1 - \sup_{(k,l)} \frac{5(1+\alpha-l+k)}{\alpha(5k+12)} = c(\alpha) < \frac{7}{12}.$$

and then, as one might expect, we conclude

$$\lim_{\alpha\to+\infty} c(\alpha) = \frac{7}{12}.$$

This completes the proof of Theorem 1.

Note We are able to obtain the function $c(\alpha)$, in a suitable interval of α, from every estimate of the counting function $N(\sigma, T)$ in a fixed interval of σ. As an example, if we recall that

$$N(\sigma,T) \ll T^{9(1-\sigma)/(7\sigma-1)} \log^C T, \tag{21}$$

with $41/53 \leq \sigma \leq 1$ and C suitable constant (see Theorem 11.4 of Ivić [9]), we can choose H of type θ, $d = (9\theta-3)/7 - \xi$ with $\xi > 0$, in (12) of Lemma 2, using the Ingham–Huxley density estimates and (21) we can obtain an estimate of $|E_\delta(X,H)|$. Hence, from Lemma 5, we can obtain

$$c(\alpha) = \begin{cases} \dfrac{5}{8} - \dfrac{7}{16\alpha} & \text{if} \quad \dfrac{3}{2} < \alpha \leq \dfrac{3339}{1138} \\[2ex] \dfrac{1969}{2809} - \dfrac{35}{53\alpha} & \text{if} \quad \dfrac{3339}{1138} \leq \alpha \leq 3.447 \end{cases},$$

that cover a great part of the gap between $6/5+\Delta$ and 4. Along the same lines we can obtain a large number of possible function $c(\alpha)$, for every $\alpha > 1$.

4. Proof of the Theorem 2, 3 and 4

In order to prove Theorem 2 we assume (2) and use Lemma 5 to see that

$$|A_\delta(N,H,\alpha)| \ll \frac{|E_{\delta/2}(N,H)| f(N) \log^2 N}{H(N)} + o(N^{1/\alpha}),$$

for every $H(x)$ of type θ, with $1/6 < \theta < 7/12$. So by Lemma 3 we have

$$|E_{\delta/2}(N,H)| \ll N^{\frac{7}{5}(1-\theta)+\varepsilon},$$

with $H(x)$ of type θ and $23/48 < \theta < 7/12$. The last two estimates together yield

$$|A_\delta(N,H,\alpha)| \ll N^{\frac{7}{5}-\frac{12}{5}\theta+\varepsilon} + o(N^{1/\alpha}),$$

so that

$$|A_\delta(N,H,\alpha)| = o(N^{1/\alpha}),$$

for every

$$\theta > \frac{7}{12} - \frac{5}{12\alpha} \quad \text{and} \quad \frac{23}{48} < \theta < \frac{7}{12}.$$

Then we can define

$$c(\alpha) = \frac{7}{12} - \frac{5}{12\alpha} \quad \text{if} \quad \alpha \geq 4.$$

Similarly we can prove Theorem 3, using Lemma 6 and Lemma 4 instead of Lemma 5 and Lemma 3, so obtaining

$$|A_\delta(N, H, \alpha)| \ll N^{1-2\theta+\varepsilon} + o(N^{1/\alpha}),$$

so that

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

for every

$$\theta > \frac{1}{2}\left(1 - \frac{1}{\alpha}\right).$$

Then we can choose

$$c(\alpha) = \frac{1}{2}\left(1 - \frac{1}{\alpha}\right) \quad \text{if} \quad \alpha > 1.$$

To prove Theorem 4 we recall that Selberg [11] proved, under the assumption of the Riemann Hypothesis, that

$$\int_X^{2X} |\psi(x+H) - \psi(x) - H|^2 \, \mathrm{d}x \ll HX \log^2 X,$$

for all $H \geq 10$, which implies

$$|E_\delta(N, H)| \ll \frac{N}{H(N)} \log^2 N,$$

for every $\delta > 0$. In conjunction with Lemma 6, this gives

$$|A_\delta(N, H, \alpha)| \ll \frac{N \log^4 N}{H(N)^2} f(N) + o(N^{1/\alpha}),$$

with $f(N) \to \infty$ arbitrarily slowly, so that

$$|A_\delta(N, H, \alpha)| = o(N^{1/\alpha}),$$

with

$$H(N) > N^{\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} f(N) \log^2 N,$$

for every $\alpha > 1$ and $\delta > 0$. This completes the proof of Theorem 4.

References

[1] D. Bazzanella, *Primes between consecutive square*, Arch. Math. **75** (2000), 29-34.
[2] D. Bazzanella and A. Perelli, *The exceptional set for the number of primes in short intervals*, J. Number Theory **80** (2000), 109-124 .
[3] H. Davenport, *Multiplicative Number Theory*, volume GTM 74. Springer - Verlag, 1980. second edition.
[4] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, Cambridge University Press, 1991.
[5] D. R. Heath-Brown, *The difference between consecutive primes II.* J. London Math. Soc. (2), 19 (1979), 207–220.
[6] D. R. Heath-Brown, *Zero density estimates for the Riemann zeta-function and Dirichlet L-function.* J. London Math. Soc. (2), 19 (1979), 221–232.
[7] D. R. Heath-Brown, *The number of primes in a short interval.* J. Reine Angew. Math., 389 (1988), 22–63.
[8] M. N. Huxley, *On the difference between consecutive primes.* Invent. Math., 15 (1972), 164–170.
[9] A. Ivić, *The Riemann Zeta-Function* , John Wiley & Sons, New York, 1985.
[10] H. L. Montgomery and R .C. Vaughan, *The large sieve*, Mathematika **20** (1973), 119–134.
[11] A. Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arc. Math. Naturvid. **47** (1943), no. 6, 87–105.
[12] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, second ed., Oxford U.P., 1986.
[13] G. Yu, *The differences between consecutive primes*, Bull. London Math. Soc. **28** (1996), no. 3, 242–248.
[14] A. Zaccagnini, *Primes in almost all short intervals.* Acta Arith., 84 (1998), 225–244.

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