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## **Robust control in presence of parametric uncertainties: observer-based feedback controller design**

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## **Abstract**

This paper proposes a complete procedure for the design of a robust controller for a nonlinear process, taking into account the various issues arising in the design and using the main theoretical results from the Literature about this topic. An extended model is set-up, linking performance and robustness to the control law: the  $H_\infty$  norm of the extended system in closed loop measures the achievement of the objectives. The result is a state feedback control law which guarantees robust performance. The problem of the design of an observer to estimate the state of the system is also addressed, as the complete knowledge of the state is required to calculate the control action; moreover, the implications of the use of the observer in the design of the controller are pointed out. The methodology is illustrated via simulation of a regulation problem in a Continuous Stirred Tank Reactor (CSTR). The application of this methodology to more complex systems will be discussed.

## **Keywords**

- $H_\infty$  control
- Nonlinear control
- Robust control
- Robust performance
- Observer design
- Uncertain process

## Introduction

In the last decades there was a relevant effort to face with the design of controllers for nonlinear multivariable processes, particularly in the chemical engineering field, where the dynamics of the systems can be a function of many parameters (such as physical properties, heat and mass transfer coefficients, kinetic constants) which can be nonlinear functions of the state variables (temperature, pressure, composition). Thus, a controller designed around a nominal operating point, as in the classical approach, cannot guarantee satisfactory performance (disturbance rejection and/or tracking control): the dynamic behaviour can be qualitatively different from one operating regime to another and instability can arise.

Simple gain-scheduling schemes were proposed even in the recent past (Shinskey, 1996): linear process models valid within a "small" region around the linearisation point are derived and a local design is performed, thus resulting in a look-up table that interpolates controller gains as the process traverses the operating region. This procedure is time consuming and expensive, but is well accepted for many applications. Galan et al. (2000) discussed the construction of a multi-linear model approximation for a Single Input Single Output (SISO) plant, but the question of how many models are required remains largely unanswered.

Input/Output (I/O) geometric linearisation (Singh and Rugh, 1972; Isidori and Ruberti, 1984; Isidori, 1989) can reduce a nonlinear system to a linear one using coordinate transformation, thus avoiding the standard Jacobian linearisation. Using the I/O linearisation approach, Kravaris and Chung (1987) proposed the Globally Linearising Controller, using a state feedback to make the I/O relationship linear and then using an external linear controller around the I/O linear system. Similarly, Sampath et al. (1998) proposed a multi-loop feedback configuration: the inner loop is a state feedback law, based on a differential geometric method, meant for Input/Output (I/O) linearisation, while the outer loop is designed for robust stability and nominal performance on the basis of the robust control theory for linear systems

(Doyle, 1982).

In principle, once the nonlinearities are cancelled, the outer-loop can be designed to impose any desired stable dynamics on the closed loop. The usual approach is to impose linear dynamics with poles in the left half plane, thus resulting in stable dynamics. However, the issue of where the poles should be placed in the left half plane was not addressed in the literature. This is of utmost importance when there is uncertainty in the model and/or the nonlinearities are not cancelled exactly. Uncertainty can have various origins, beside linearisation of nonlinear dynamics:

- the process structure is known but some parameters are not, or they may change in time or they are known only in a range of approximation;
- the process is known but some of its dynamics are willingly ignored for simplicity;
- some dynamics of the process, especially at high frequencies, are unknown.

Moreover, even differential geometric techniques in presence of uncertainties do not give perfectly linear models and these nonlinearities require, for example, standard Jacobian linearization around their steady state values. This is different from the Jacobian linearization of the original nonlinear system: only the perturbations due to uncertainties are linearised, but not the whole model (Kolavennu et al., 2000). As a consequence, the controller in the outer loop must be designed not only for nominal stability and performance, but also for robustness in face of uncertainties.

An alternative approach, based on the explicit construction of a Lyapunov function, was proposed by Chen and Leitmann (1987): although their method opens an interesting way to the control of nonlinear systems in presence of uncertainties, there exists no general procedure to find an explicit Lyapunov function for building a robust controller.

Recent papers gave extensive results on the control of nonlinear systems that handle uncertainty and constraints. El-Farra & Christofides (2003) focuses their attention on control of multi-input multi-output nonlinear processes with uncertain dynamics and actuator constraints and proposed a Lyapunov-based nonlinear

controller design approach that accounts explicitly and simultaneously for process nonlinearities, plant-model mismatch, and input constraints. Mhaskar et al. (2005) investigated a robust hybrid Model Predictive Control design that handle nonlinearity and uncertainty: the proposed method provides a safety net for the implementation of any available MPC formulation, designed with or without taking uncertainty into account, and allows for an explicit characterization of the set of initial conditions starting from where the closed-loop system is guaranteed to be stable. Also in this case the key idea is to use a Lyapunov-based robust controller, for which an explicit characterization of the closed-loop stability region can be obtained, to provide a stability region within which MPC can be implemented. A set of switching laws are designed that exploit the performance of MPC whenever possible, while using the bounded controller to provide the stability guarantees. Finally Mhaskar (2006) investigated how it is possible to incorporate appropriate stability constraints in the optimization problem, thus guaranteeing feasibility and closed-loop stability in presence of constraints and uncertainty.

In the present paper a different approach will be shown, modelling the nonlinearities as uncertainties in a linearised model, thus allowing the use of the results of robust control theory for linear systems: most important results from the Literature (Doyle, 1982; Zhou et al., 1995; Colaneri et al., 1997) will be rationalised and formalised in a full procedure for the design of a robust controller. The focus of the paper is on the design of the "extended model" of the process which includes performance requirements and parametric uncertainties. Moreover, it will be shown that when a feasible controller cannot be designed, i.e. the objectives cannot be satisfied, the methods allows to discriminate if the responsible of this is a too severe performance requirement or a too large range of uncertainty for a certain parameter, thus driving the design process to achieve the best results. Moreover, the problem of the design of a robust observer for the estimation of the state of the system in presence of uncertainties using the available measures will be addressed. This issue, which has been quite often neglected in the literature, is of great importance as the

calculation of the controller action requires the knowledge of the state of the system, which can be not completely available from the measures. Moreover, the use of the observer strongly affects the design of the controller and the robust performance of the system. Few papers in the past dealt with this issue; in particular El-Farra et al. (2005) synthesized a family of output feedback controllers using a combination of bounded state feedback controllers, high-gain observers and appropriate saturation filters to enforce asymptotic stability for the individual closed-loop modes and provided an explicit characterization of the corresponding output feedback stability regions in terms of the input constraints and the observer gain. In this paper a different approach is presented, based on the set-up of an appropriate extended model which includes not only the control requirements but also the observer specifications.

## **Robust control design**

The aim of this paragraph is to summarise the fundamentals of the theory of robust control for linear systems and to give a systematic procedure for the design of this kind of controllers.

### *Controller design*

Let us consider a generic linear system described by the following set of equations:

$$\begin{cases} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w \end{cases} \quad (1)$$

where:

- $x$  is the state of the system;
- $u$  is the manipulated input of the system, i.e. the control action;
- $w$  represents the external (bounded) disturbances, i.e. unknown signals affecting

the dynamics of the system;

- $y$  represents the measured variables (which can be affected by external disturbances) available for the control;
- $z$  represents the objectives of the process, i.e. the signals carrying requirements to be satisfied by the control.

Within this framework, the system (1) corresponds to a dynamical model with four ports constituted by two groups of inputs ( $w$  and  $u$ ), and two groups of outputs ( $z$  and  $y$ ) signals; no limitations about the dimension of  $x$ ,  $u$ ,  $w$  and  $y$  are required and the approach is truly multivariable.

The control problem can be stated very preliminary as the selection of a control signal  $u$  to apply to the process to maintain (regulate) some output signals to a set point, or to force them to follow (tracking) reference signals, in spite of the disturbances acting on the plant. Thus, in a regulation problem the goal is to get insensitivity of the objectives  $z$  with respect to the disturbances  $w$ . Also a tracking problem can be defined in terms of insensitivity: if  $r$  is the reference trajectory for the objective  $z$ , it is possible to introduce the tracking error  $e_r = z-r$  and try to achieve insensitivity of  $e_r$  with respect to  $r$ . A further performance requirement is on control activity due to measurement noises: excessive amplitude of the control action in all frequency range should be avoided, i.e. a low sensitivity of the control action with respect to measurement noises.

To quantify the desired performance it is necessary to assign a measure to input and output signals. Various types of measures can be defined, e.g. the energy of a signal, its power, its maximum value in frequency. These measures are called norms and generalize the classical concept of distance in the Euclidean spaces. The most important (and useful for our purposes) norm for an array (or a matrix) of signals  $U$  is the norm  $H_\infty$ :

$$\|U\|_\infty = \sup_{\omega} \bar{\sigma}[U(j\omega)] \quad (2)$$

where  $U(j\omega)$  is the representation in the frequency ( $\omega$ ) domain of the signals (which

can be obtained, for example, from the Laplace transform of the function, followed by the substitution of the Laplace variable  $s$  with  $j\omega$  and  $\bar{\sigma}(\omega)$  is the maximum singular value of the argument matrix for every value of  $\omega$  (the maximum singular value of a transfer function matrix has intuitively the same interpretation of the gain of a scalar system).

The definition of a norm on the spaces of the input and output signals induces a norm on the dynamic operator which establishes a mapping between the input and the output signal spaces. Given two norms, one for the input signal  $u$  and one for the output signal  $y$ , a BIBO operator is characterised by:

$$\|y\| = \|Gu\| \leq M \|u\| \quad \forall u \quad (3)$$

The norm of the operator  $G$  is defined as the lowest value of  $M$  that verifies the previous inequality, thus:

$$\|G\| = \sup_u \frac{\|y\|}{\|u\|} \quad (4)$$

This induced norm can be considered a generalization of the gain as it is intended in electronic amplifiers.

Let us consider a SISO control problem where the aim is to calculate the control  $u$  to make the objective  $z \in \mathbb{R}$  insensitive to the disturbance  $w \in \mathbb{R}$ . Let  $G_{w,z}$  be the closed loop transfer function between  $w$  and  $z$ :

$$z(s) = G_{w,z} w(s) \quad (5)$$

and let  $W_{w,z}$  be a generic filter applied to the objective  $z$ :

$$z^* = W_{w,z} z = W_{w,z} G_{w,z} w \quad (6)$$

If it is possible to find a control  $u$  such that:

$$\|W_{w,z} G_{w,z}\|_{\infty} < 1 \quad (7)$$

this means that:

$$|G_{w,z}| < |W_{w,z}|^{-1}, \quad \forall \omega \quad (8)$$

The filter  $W_{w,z}$  can thus be used to specify the performance requirements for the controlled system. An example can be useful to understand this key-point: let us



assume that the goal of the controller is to reduce the disturbance effects of 1/100 for all the frequencies till a crossover frequency of 10 rad/s, with a maximum of 6 dB in the sensitivity at high frequencies. As a first guess it is possible to assume (see Figure 1):

$$W_{w,z} = \frac{\alpha(s+\beta)}{s+\delta} \quad (9)$$

Equation (8) states that in the controlled system the module of the closed loop sensitivity function  $w \rightarrow z$  ( $G_{w,z}$ ) will be lower than  $|W_{w,z}|^{-1}$ , thus, at low frequencies ( $\omega \rightarrow 0$ , thus  $s \rightarrow 0$ ),  $|W_{w,z}|^{-1} = 10^{-2}$ , i.e.

$$|W_{w,z}(\omega \rightarrow 0)| = \frac{\alpha\beta}{\delta} = 10^2 = 40 \text{ dB} \quad (10)$$

while, at high frequencies ( $\omega \rightarrow \infty$ , thus  $s \rightarrow \infty$ ),  $|W_{w,z}|^{-1} = 6 \text{ dB}$ , i.e.

$$|W_{w,z}(\omega \rightarrow \infty)| = \alpha = \frac{1}{6 \text{ dB}} = -6 \text{ dB} \quad (11)$$

The value of the pole  $\delta$  of the filter can be assumed equal to the crossover frequency (10 rad/s), so that, given the performance requirements, the filter  $W_{w,z}$  is fully defined.

It is well documented in any text about optimum control that with perfect knowledge of all states and with performance measured by a square norm of the closed loop operator the best control is an algebraic feedback from the states:

$$u = -Hx \quad (12)$$

thus, in the design procedure, the goal is to find a controller gain  $H$  that satisfies eq. (7). The available algorithms for the calculation of the controller gain that minimise the norm of an operator looks for the minimum of  $\|W_{w,z}G_{w,z}\|_{\infty}$ : if this minimum is lower than 1, then the controller gain satisfies eq. (8) and thus the performance requirements; otherwise, if the minimum of the norm is larger than 1, the controller does not satisfy the requirements. This may occur if the crossover frequency or the disturbance reduction at low frequencies or the maximum in the sensitivity at high frequencies are too severe: the method drives the design procedure to the best feasible result.

With the control law given by eq. (12), the following equations describe the dynamics of the controlled system:

$$\begin{cases} \dot{x} = (A - B_2H)x + B_1w \\ z = (C_1 - D_{12}H)x + D_{11}w \\ y = C_2x + D_{21}w \end{cases} \quad (13)$$

Thus, beside the performance requirements, the controlled system has to be stable, i.e. the eigenvalues of the matrix  $(A - B_2H)$ , which depend on the gain  $H$ , must have negative real part. Obviously the system (13) has to be controllable, i.e. it the poles of  $(A - B_2H)$  should be placed in arbitrary positions varying  $H$  (this requirement can be relaxed in presence of poles that cannot be moved arbitrarily, but are stable; in this case the system is said to be stabilisable).

In case of multiple objectives specification (on multiple disturbance rejection, tracking error and control activity) the same procedure can be followed, defining a filter  $W_{z_i, w_i}$  to describe the performance requirements for the closed loop sensitivity between the  $i$ -th objective and the  $i$ -th disturbance ( $w_i \rightarrow z_i$ ) and calculating a controller gain  $H$  that satisfies:

$$\|W_{w_i, z_i} G_{w_i, z_i}\|_{\infty} < 1, \forall (w_i, z_i) \quad (14)$$

being  $G_{w_i, z_i}$  the closed loop transfer function between  $w_i$  and  $z_i$ . Again, from the results of the minimisation of the norm, it is possible to verify if any of the performance requirements must be made less severe to satisfy eq. (14). The various filters  $W_{z_i, w_i}$  can be designed as stated previously; for the control activity, as the goal to guarantee that the control has not excessive amplitude in all the frequency range, in particular at high frequency, it is necessary that the weight of the control is a proper dynamical operator or a pure gain.

Let us consider now the presence of uncertainties in the model, due, for example, to a parameter  $p$  that can assume values between  $p_{\min}$  and  $p_{\max}$ . In this case it is possible to define:

$$W_p = \frac{p_{\max} - p_{\min}}{p_{\max} + p_{\min}} \quad (15)$$

and then pose:

$$p = p_{\text{mean}} (1 + \Delta W_p) \quad (16)$$

where  $p_{\text{mean}}$  is the mean value of the parameter  $p$  between  $p_{\min}$  and  $p_{\max}$  and  $\Delta$  is a parameter whose value can vary in the range  $(-1;1)$ :

$$\|\Delta\|_{\infty} < 1 \quad (17)$$

Nonlinearities can be treated in the same way. Let us consider, for example, a kinetic constant  $k$  varying with the temperature according to an Arrhenius-type expression: if it is possible to assume that the temperature of the system varies in a certain interval, thus also the kinetic constant will vary between  $k_{\min}$  and  $k_{\max}$  and thus:

$$k = k_{\text{mean}} (1 + \Delta W_k) \quad (18)$$

where:

$$W_k = \frac{k_{\max} - k_{\min}}{k_{\max} + k_{\min}} \quad (19)$$

and  $k_{\text{mean}}$  is the mean value of the parameter  $k$  between  $k_{\min}$  and  $k_{\max}$ . Eq. (16) and eq. (18) are two particular cases of the more general expression:

$$G_p = G(1 + \Delta W_G) \quad (20)$$

where  $G_p$  is the true transfer function of the process,  $G$  is the approximate transfer function,  $W_G$  is the relative variation and  $\Delta$  is such that  $\|\Delta\|_{\infty} < 1$ .  $\Delta$  can be an arbitrary linear, proper, invariant, BIBO dynamic operator, with the unique constraint to have limited norm  $H_{\infty}$ : this operator parameterizes all the elements in the region of the system uncertainty. Figure 2 shows the block diagram corresponding to eq. (20) for a SISO system; the operator  $\Delta$  can be eliminated from the block diagram (and thus from the model of the system), thus giving origin to a couple of new signals:  $w_G$  is a new input to the process, playing the role of a bounded disturbance (in fact  $w_G$  is the output of the operator  $\Delta$ , which is unknown, even if bounded), while  $z_G$  plays the role of a new objective: the insensitivity between the various  $w_i$  and  $z_i$  (specifying the

performance towards external disturbances) has now to be guaranteed also for all the couples  $w_G-z_G$ , i.e. for all the values of the parameters  $\Delta$  (robust performance).

This model is known as "extended system" and it embeds nominal model, process uncertainty, disturbances and requirements. Let us indicate with  $G_{eq}$  the transfer matrix between  $w$  and  $z$  resulting from closing the loop of the extended system in nominal conditions: performances and stability must be preserved for any value of the uncertainty in the admissible region. Robust stability is the property of the control that guarantees stability not just for the nominal model, but for any model obtained from the uncertainty operator in the specified region. The property is stated by the **Small Gain Theorem** (Zames, 1966a; Zames, 1966b):

Let  $G_{eq}$  be an asymptotically stable transfer matrix, and  $\Delta$  an arbitrary asymptotically stable operator with norm  $H_\infty < 1$ , then the feedback loop is stable if and only if the loop gain satisfies  $\|\Delta G_{eq}\|_\infty < 1$ . This implies that stability is guaranteed by  $\|G_{eq}\|_\infty < 1$ .

Finally, the design of a robust controller reduces to the calculation of the gain  $H$  of the controller which satisfies eq. (14) for all the couples  $(w_i, z_i)$  representing performance specifications (through the filters  $W_{z_i, w_i}$ ) and for all those representing the uncertainties (through  $W_G$ ).

The algorithms that are used to calculate the controller gain looks for a value of  $H$  that minimise the  $H_\infty$  norm of all the dynamic system (and thus of the multivariable operator  $GW$ ) and not of all the input-output couples;  $\|GW\|_\infty < 1$  ensures robust performance, but this is not a necessary condition to guarantee robust performance. Without going into details, the problem can be modified introducing a matrix  $D$  and the algorithms, beside the gain  $H$ , calculate the values of  $D$  and of  $\|D^{-1}(GW)D\|_\infty$ ; if it is possible to find a gain  $H$  and a matrix  $D$  that verify:

$$\|D^{-1}(GW)D\|_\infty < 1 \tag{21}$$

then robust performance is guaranteed.

### Observer design

The use of the control law (12) requires the knowledge of all the state  $x$ ; some components of this array can be accessible through measurements, but some others not. Moreover, the measure of all the states  $x$  can be expensive or time consuming (thus introducing a delay in the control loop) or even not feasible. This justifies the use of an observer to get a quick and reliable estimation of the state array  $x$ , indicated as  $\hat{x}$  in the following. Figure 3 depicts the block diagram of the observer: given the value of the input  $u$  and of the measurements  $y$  (and also of the objectives  $z$ ), which are obviously available, the observer is a dynamic system returning the estimates of the states and also of the objectives ( $\hat{z}$ ) and of the measured variables ( $\hat{y}$ ): the difference between the values of the measured variables  $y$  and their estimates  $\hat{y}$  quantifies the adequacy of the observer.

Given a linear system such as that of eq. (1), the following set of equations constitutes the observer:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + K(y - \hat{y}) + B_2u \\ \hat{z} = C_1\hat{x} + D_{12}u \\ \hat{y} = C_2\hat{x} \end{cases} \quad (22)$$

If eq. (22) is subtracted from eq. (1) the equations describing the dynamics of the errors on the estimation of the states and of the objectives are obtained:

$$\begin{cases} \dot{e}_x = (A - KC_2)e_x + (B_1 - KD_{21})w \\ \dot{e}_z = C_1e_x + D_{11}w \end{cases} \quad (23)$$

Obviously, the observer has to be stable, i.e. the eigenvalues of the matrix  $(A - KC_2)$ , which are a function of the observer gain  $K$ , must have negative real part; moreover, the couple of matrices  $(A, C_2)$  must be observable, i.e. it should be possible to place the poles of  $(A - KC_2)$  in arbitrary positions varying  $K$  (this requirement can be relaxed in presence of poles that cannot be moved arbitrarily, but are stable; in this case the system is said to be detectable).

The goal is now to calculate the observer gain  $K$  so that the estimation error on the objectives  $e_z$  is insensitive with respect to the process disturbances  $w$ , i.e. to verify:

$$\|G_{w,e_z}\|_\infty < 1 \quad (24)$$

where  $G_{w,e_z}$  is the transfer function between the disturbances  $w$  and the estimation errors  $e_z$ . The same algorithms previously described for the calculation of the controller gain can be used to calculate  $K$ : the only difference is in the extended system that is used.

### *Observer based state feedback*

The control design is affected by the presence of the observer, as now:

$$u = -H\hat{x} \quad (25)$$

The separation principle states that for a linear system controlled with (25) the poles of the closed loop system are given by the poles of the controller and those of the observer, so that the two can be designed independently and then put together. Even if robust performance is guaranteed for the controller and for the observer following the previously described procedures, robust performance is not guaranteed for the closed loop system if the observer and the controller are designed independently.

Two issues should be taken into account for the design of the robust controller:

1. the first is that if the best performance that can be obtained by estimating the objectives with an observer is measured by  $\|G_{w,e_z}\|_\infty \leq \gamma_o$ , when a full-state feedback control is used the controller performance (measured by  $\|G_{e_y,\hat{z}}\|_\infty$ ), cannot be better than the estimations obtained;
2. the second intuition is that the observer model adopted for designing state feedback, cannot be any, but it must result from the best observer in the presence of disturbances in the worst conditions (Zhou et al., 1995; Colaneri et al., 1997), i.e.

$$w = \Omega x \quad (26)$$

These two intuitions integrate each other in the following result (Zhou et al., 1995; Colaneri et al., 1997):

If the best objective estimator, designed in the worst disturbance conditions for a given bound  $\|G_{w,e_z}\|_\infty \leq \gamma_o$  exists and a state feedback on the model obtained from this estimator with control performance  $\|G_{e_y,\hat{z}}\|_\infty \leq \gamma_c$  exists, then the output feedback control obtained integrating estimator and state feedback guarantees internal stability and performances given by  $\|G_{w,z}\|_\infty \leq \gamma_o$  if and only if  $\gamma_c < \gamma_o$ .

In conclusion, the observer model offers guaranteed performances for the original process only if it is derived from a specific class of observers and the model in closed loop has a norm that outperforms the error of estimating the objectives.

From previous considerations the steps constituting the design process are:

1. for a given  $\gamma_o < 1$ , the feedback  $K$  of an optimum observer in the worst disturbance condition is determined. If a solution exists,  $\gamma_o$  represents a guaranteed upper bound of the norm of the operator disturbances-objective estimation errors. For this observer the matrix  $\Omega$  corresponds to the worst disturbances. With  $K$  and  $\Omega$  the model for feedback design is built (see Figure 4, block diagram A);
2.  $H$ , the feedback from the estimated states, is computed in the second step solving a state feedback problem, assuming as new disturbances the measure reconstruction errors and, as new objectives, the objective estimates. The feedback design results in a closed loop operator  $e_y \rightarrow \hat{z}$ , with a norm bounded by  $\gamma_c$  (see Figure 4, block diagram B);
3. if  $\gamma_c < \gamma_o$ , the control law with guaranteed bound  $\gamma_o$  has been found (see Figure 4, block diagram C), otherwise a greater value to  $\gamma_o$  has to be assigned and the observer - feedback design repeated. Again, if  $\gamma_o$  becomes larger than 1 some of the requirements (the range of uncertainties of the parameters and/or the performance requirements) have to be made less severe.

## Illustrative example

The purpose of this section is to apply the proposed design procedure and to evaluate its performance through an example in the chemical engineering field. Let us consider a CSTR in which an isothermal, liquid phase, multi-component chemical reaction is carried out:



The objective is to keep the concentration of  $B$  at a desired set-point by manipulating the molar feedrate of species  $C$  (indicated as  $u$ ). The (nonlinear) balance equations are (Kravaris and Palanki, 1988):

$$\begin{cases} \dot{c}_A = -k_1 c_A + \frac{F}{V} (c_{A,feed} - c_A) + k_2 c_C^2 \\ \dot{c}_B = -\frac{F}{V} c_B + k_3 c_C^2 \\ \dot{c}_C = k_1 c_A - \frac{F}{V} c_C - (k_2 + k_3) c_C^2 + u \\ y = c_A \end{cases} \quad (28)$$

The state variables are the concentrations  $c_A$ ,  $c_B$  and  $c_C$ , while the measured concentration is  $c_A$ . The system parameters are:  $k_1 = 1 \text{ s}^{-1}$ ,  $k_3 = 5 \text{ m}^3 \text{ mol}^{-1} \text{ s}^{-1}$ ,  $F = 3 \text{ m}^3 \text{ s}^{-1}$ ,  $V = 3 \text{ m}^3$ ,  $c_{A,feed} = 2 \text{ mol m}^{-3}$ ; at the desired steady-state  $c_{A,S} = 2.18 \text{ mol m}^{-3}$ ,  $c_{B,S} = 3.93 \text{ mol m}^{-3}$  and  $c_{C,S} = 0.87 \text{ mol m}^{-3}$  and  $u_S = 5 \text{ mol s}^{-1}$ .

There is uncertainty in the parameter  $k_2$ , which can vary between 0.1 and  $5.9 \text{ m}^3 \text{ mol}^{-1} \text{ s}^{-1}$ , thus, using the representation given by eq. (16):

$$k_2 = k_{2,mean} (1 + \Delta W_{k_2}) \quad (29)$$

with  $k_{2,mean} = 3 \text{ m}^3 \text{ mol}^{-1} \text{ s}^{-1}$  and  $W_{k_2} = 0.97$ . Moreover, the system is nonlinear in the state variable  $c_C$ ; standard Jacobi linearisation around the steady-state gives:

$$c_C^2 = (c_{C,S})^2 + 2c_{C,S} (c_C - c_{C,S}) \quad (30)$$

Thus, the system of equations (28) takes the form:



$$\begin{cases} \dot{c}_A = -k_1 c_A + \frac{F}{V} (c_{A,feed} - c_A) + k_2 [(c_{C,S})^2 + 2c_{C,S} (c_C - c_{C,S})] \\ \dot{c}_B = -\frac{F}{V} c_B + k_3 [(c_{C,S})^2 + 2c_{C,S} (c_C - c_{C,S})] \\ \dot{c}_C = k_1 c_A - \frac{F}{V} c_C - (k_2 + k_3) [(c_{C,S})^2 + 2c_{C,S} (c_C - c_{C,S})] + u \\ y = c_A \end{cases} \quad (31)$$

At steady state equations (31) yield:

$$\begin{cases} 0 = -k_1 c_{A,S} + \frac{F}{V} (c_{A,feed,S} - c_{A,S}) + k_2 (c_{C,S})^2 \\ 0 = -\frac{F}{V} c_{B,S} + k_3 (c_{C,S})^2 \\ 0 = k_1 c_{A,S} - \frac{F}{V} c_{C,S} - (k_2 + k_3) (c_{C,S})^2 + u_S \\ y = c_{A,S} \end{cases} \quad (32)$$

Subtracting eq. (32) from eq. (31) the dynamics of the system is described in terms of deviations from the steady-state values:

$$\begin{cases} \dot{\bar{c}}_A = -k_1 \bar{c}_A + \frac{F}{V} (\bar{c}_{A,feed} - \bar{c}_A) + 2k_2 c_{C,S} \bar{c}_C \\ \dot{\bar{c}}_B = -\frac{F}{V} \bar{c}_B + 2k_3 c_{C,S} \bar{c}_C \\ \dot{\bar{c}}_C = k_1 \bar{c}_A - \frac{F}{V} \bar{c}_C - 2(k_2 + k_3) c_{C,S} \bar{c}_C + \bar{u} \\ y = \bar{c}_A \end{cases} \quad (33)$$

where:  $\bar{c}_A = c_A - c_{A,S}$ ,  $\bar{c}_B = c_B - c_{B,S}$ ,  $\bar{c}_C = c_C - c_{C,S}$ ,  $\bar{c}_{A,feed} = c_{A,feed} - c_{A,feed,S}$ ,  $\bar{u} = u - u_S$ . In nominal conditions  $c_{C,S} = 0.887 \text{ mol m}^{-3}$  and this value can be used for the calculation of the controller with classical techniques (PID, LQR,...); here  $c_{C,S}$  can be assumed to vary between 0 and  $1.774 \text{ mol m}^{-3}$ , i.e. a variation of  $\pm 100\%$  around the steady-state value, thus:

$$c_C = 0.887 (1 + \Delta W_{c_c}) \text{ mol m}^{-3} \quad (34)$$

with  $W_{c_c} = 1$ .

As a first guess, a reduction of the disturbance effects of 1/100 till a crossover frequency of 10 rad/s is required, with a maximum of 6 dB in the sensitivity at high frequencies, thus resulting in the following performance filter:

$$W_{w,z} = \frac{0.5(s+2000)}{s+10} \quad (35)$$

The state representation of the filter (35) is:

$$\begin{aligned} \dot{z} &= -10z + u \\ y &= 995z + 0.5u \end{aligned} \quad (36)$$

The representation (36) is required because when the filter (35) is used, a further state ( $z$ ) is introduced in the system. With respect to the control activity a constant filter is generally used and, as no requirements are given about this performance, a value of 0.1 is assumed. Figure 5 shows the block diagram of the extended system corresponding to this control problem: each block identifies an operator that acts over the input and gives, as output, the input multiplied by the function of the block (which can be either a constant, or an operator); the block that performs the integration of the input signal is indicated as '1/s'; the others are indicated with the notation used in eq. (33)-(35). The summation nodes are indicated as circles and the signals arriving to each node are summed, apart from those that have a minus sign, that are subtracted from the summation. Various couples of input-output signals emerge:

- 1)  $c_{A,feed} - z$  represents the channel disturbance-objective ( $W$  is the performance filter corresponding to eq. (36));
- 2)  $n - z_n$  represents the channel measurement noise-control activity (the performance filter is a constant equal to 0.1);
- 3)  $w_{k_2} - z_{k_2}$  represents the uncertainty channel on the kinetic constant  $k_2$  (0.97 is the value of  $W_{k_2}$ );
- 4)  $w_{c_c} - z_{c_c}$  represents the uncertainty channel due to  $c_{C,S}$ ;
- 5)  $y - u$  represents the channel measure-control action.

A simplified block diagram of the extended system is given in Figure 6, where only the input (disturbances and control actions) and output (objectives and measures) signals are evidenced; the control loop, as well as the loop due to the uncertainties, are also evidenced. The transfer functions between the various input and output of

the system are given explicitly in Table 1.

Let us assume, for the moment, that all the states of the system (i.e. the values of  $c_A$ ,  $c_B$ ,  $c_C$  and  $z$ ) are available for the control. Various software, e.g. the Robust Control Toolbox of MatLab, can be used to minimise the  $H_\infty$  norm of the input-output operator according to eq. (21) giving a value of 0.416 for the  $\|D^{-1}(GW)D\|_\infty$ : this means that the required robust performance can be achieved and the controller gain is:

$$H = [1.03737 \quad 6199.42 \quad 303641 \quad 3206.43] \quad (37)$$

Figure 7 shows the variations of  $c_B$  when  $c_{A,feed}$  changes from 2 to 7 mol m<sup>-3</sup>: the time evolution (from the steady-state value) without any control and with the previously calculated controller gain are compared, so that its effectiveness is pointed out. In order to verify the robust performance, the same test has been repeated for three values of the parameter  $k_2$ : the results are shown in Figure 8 and demonstrate that the performance requirements are fulfilled for all the values of  $k_2$  considered.

The designed controller requires the knowledge of the states of the system; actually, only  $c_A$  is measured, thus an observer is required to get the estimation of the other states. The extended system used for the design of the observer is the same depicted in Figures 5 and 6, i.e. the same uncertainties due to  $k_2$  and to the nonlinearity and the same performance specification are considered. When the calculations are performed, eq. (24) cannot be satisfied: the norm of the operator is bounded by a value of 1.52 which does not guarantee anything. The norm of the single channels is thus analysed and it is found that the responsible for the high value of the norm of the operator is the uncertainty channel due to the linearisation; thus, a lower value of  $W_c$  is assumed, passing from 1 to 0.8. The calculations are repeated and the norm of the operator can now be lowered to 0.861, giving an observer gain equal to:

$$K = [29823.4 \quad 2.73 \quad -5.32 \quad -0.19] \quad (38)$$

As an example of the various tests made to verify the effectiveness of the proposed

observer, Figure 9 shows the results obtained in presence of a change from 2 to 4 mol m<sup>-3</sup> of  $c_{A,feed}$  and from 5 to 3 of  $u$ : the evolution of molar concentration of species  $A$ ,  $B$  and  $C$  from the steady-state has been calculated by means of simulation of the system of nonlinear equations (28) and the results obtained with an observer, using the previously calculated gain and using the value of  $c_A$  as measured input, are compared, evidencing how the observer is able to follow the dynamics of the system and to give a precise estimation of the states.

This result can be useful if the aim of the observer is to monitor the process, i.e. to get an estimation of some variables that cannot be measured, but if the goal is to use the observer in a state feedback loop, the optimum observer in the worst disturbance condition has to be calculated. The calculations evidenced that the condition  $\gamma_c < \gamma_o$  can be fulfilled, but the minimum value of  $\gamma_o$  that can be obtained is about 4. Again, the norm of the single channels is analysed and it is found that the responsible for the high value of the norm of the operator is the uncertainty channel due to the linearisation; thus, a the value of  $W_{c_c}$  is lowered to 0.5 (now the observer has to estimate the state variables in a controlled system, so it is reasonable that  $c_{c,s}$  varies in a narrow range). The best results are obtained with  $\gamma_c < \gamma_o = 0.784$ . Figure 10 (upper graph) shows the variations of  $c_B$  when  $c_{A,feed}$  changes from 2 to 7 mol m<sup>-3</sup> (the time evolution from the steady-state value without any control was given in Figure 7).

## Conclusions and further remarks

The construction of an extended model for the design of a robust controller has been discussed in the paper: this model allows for the specification of the regulation and tracking performance requirements as well as of the control activity requirements; also parametric uncertainties can be included in the extended model. Performance

and robustness are linked to the control law, which consists of a simple state feedback.

Moreover, this procedure can guide the designer towards the best feasible controller: consider each requirement or source of uncertainty, e.g. disturbance rejection or steady state gain changes, and the restriction of the closed loop operator referring to the corresponding pair disturbance-objective; if the  $H_\infty$  norm of this restriction is less than 1, the performance is satisfied in nominal conditions (or robust stability is guaranteed with respect to one source of uncertainty), while if the value is greater than 1 it means that the specified performance cannot be satisfied by any control gain (or that the range of uncertainty is too large to allow for robust performance).

Nonlinear system can be easily handled in this framework by describing it as deviations of parameters from a nominal value, and then utilizing linear system based control methods. This approach obviously implies that the range of the uncertainty could be very large and, thus, yield very conservative results, in particular when uncertainty is also considered. As a result, the robust controller could be unfeasible only because of the nonlinearity has been managed in a too conservative way. Anyway, in case the controller is unfeasible, i.e. it is not possible to get a value of the norm  $H_\infty$  of the extended model lower than one, the proposed procedure allows to point out if the responsible of this is a too severe performance requirement or a too large range of uncertainty for a certain parameter (or for a certain non-linear variable). This is the main limitation of the proposed approach, which is in turn compensated by its simplicity (according to Luyben's philosophy of keeping a control system as simple as possible).

If the state of the system is not fully measured, then an observer is required: also in this case an extended model has to be set-up and again the  $H_\infty$  norm of this extended system ensures the performance of the observer. The observer model offers guaranteed performances for the original process only if it is derived from a specific class of observers and the model in closed loop has a norm that outperforms the error

of estimating the objectives.

### **Acknowledgement**

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## Notation

$c$	molar concentration, mol m <sup>-3</sup>
$A, B, C$	chemical species
$D$	matrix to be used in eq. (21)
$e$	estimation error
$e_r$	tracking error
$F$	feed flow rate, m <sup>3</sup> s <sup>-1</sup>
$G_{eq}$	equivalent transfer function
$G_p$	true transfer function of the process
$G$	approximate transfer function
$G_{w,e_z}$	transfer function between the disturbances $w$ and the estimation errors $e_z$ .
$G_{w,z}$	transfer function between $w$ and $z$
$GW$	matrix transfer function of the extended system
$H$	controller gain
$k, p$	parameter
$K$	observer gain
$j$	imaginary unit ( $j^2 = -1$ )
$r$	reference trajectory
$s$	Laplace variable
$u$	control action
$U$	array (or matrix) of signals
$V$	reactor volume, m <sup>3</sup>
$x$	state of the system
$y$	measured variables
$w$	disturbances
$W_G$	relative variation of a parameter

$W_{w,z}$	filter used to specify the requirements on the sensitivity of the output $z$ with respect to the input $w$
$z$	objectives of the controlled systems
$z^*$	filtered objectives of the controlled systems

### *Greeks*

$\alpha, \beta, \delta$	parameter of a generic filter $W_{w,z}$
$\gamma$	threshold value of the norm
$\bar{\sigma}$	maximum singular value
$\omega$	frequency, rad/s
$\Omega$	array used to calculate the worst disturbance

### *Subscripts and Superscripts*

$\wedge$	estimated value
$-$	deviation from the steady-state value
1, 2, 3	reaction identifier
$c$	controller
<i>feed</i>	feeding value
<i>max</i>	maximum value
<i>mean</i>	mean value
<i>min</i>	minimum value
$o$	observer
$s$	steady-state value

### *Abbreviations*

BIBO	Bounded Input Bounded Output
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CSTR	Continuous Stirred Tank Reactor
I/O	Input/Output
LQR	Linear Quadratic Regulator
MPC	Model Predictive Control
PID	Proportional Integral Derivative
SISO	Single Input Single Output

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- Figure 8* Time evolution (from the steady-state value) of the variations of  $c_B$  when  $c_{A,feed}$  changes from 2 to 7 mol m<sup>-3</sup> for three values of  $k_2$ . (o : 0.1 m<sup>3</sup> mol<sup>-1</sup> s<sup>-1</sup>,  $\Delta$  : 3 m<sup>3</sup> mol<sup>-1</sup> s<sup>-1</sup>,  $\square$  : 5.9 m<sup>3</sup> mol<sup>-1</sup> s<sup>-1</sup>).

*Figure 9* Comparison between the evolution of molar concentration of species  $B$  and  $C$  from the steady-state calculated by means of simulation of the system of nonlinear equations (solid lines) and by the observer (O:  $c_A$ ,  $\Delta$ :  $c_B$ ,  $\diamond$ :  $c_C$ ).

*Figure 10* Time evolution (from the steady-state value) of the variations of  $c_B$  when  $c_{A,feed}$  changes from 2 to 7 mol m<sup>-3</sup> in presence of observer+full state feedback controller (the symbols indicate the value of the control action).

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<i>Table 1</i>	Transfer functions between the various input and output signals of the extended system corresponding to the CSTR of the example.
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Figure 1

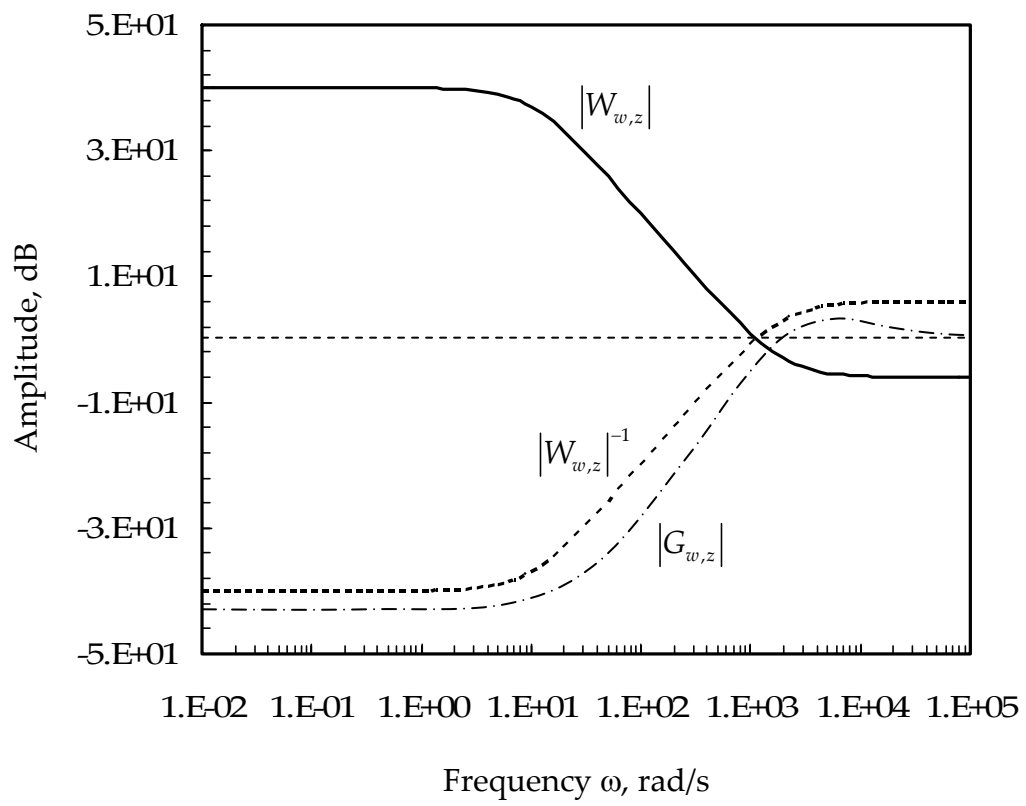


Figure 2

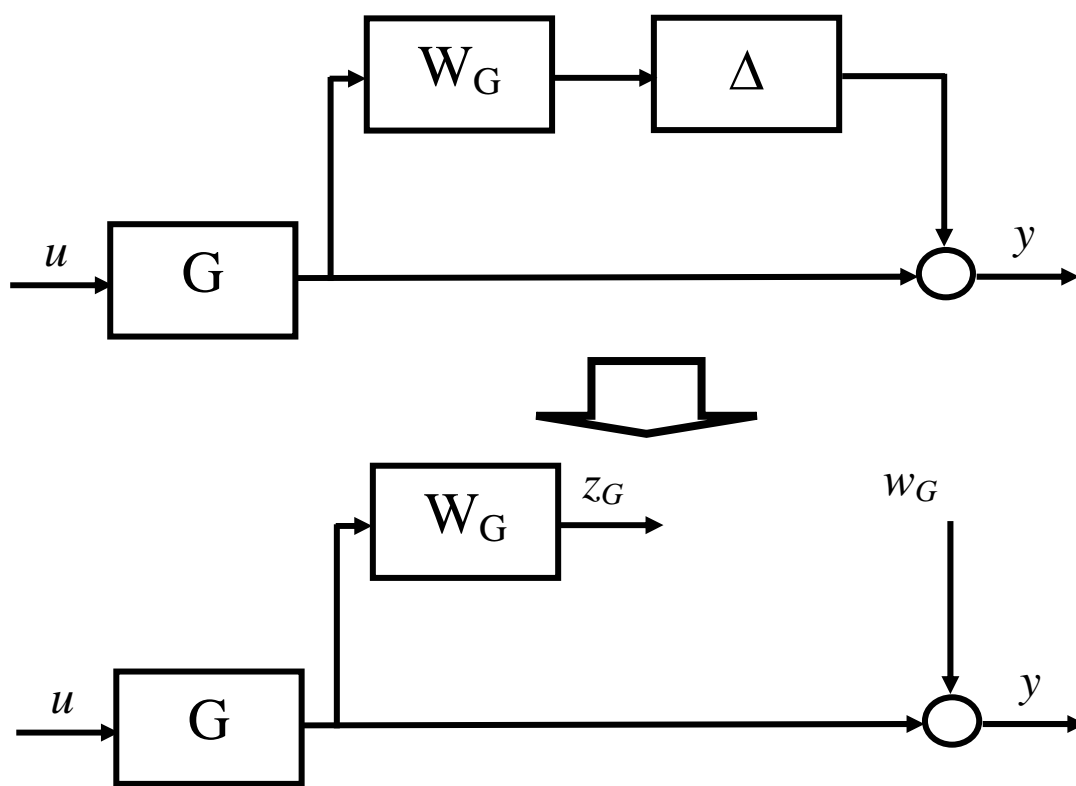




Figure 3

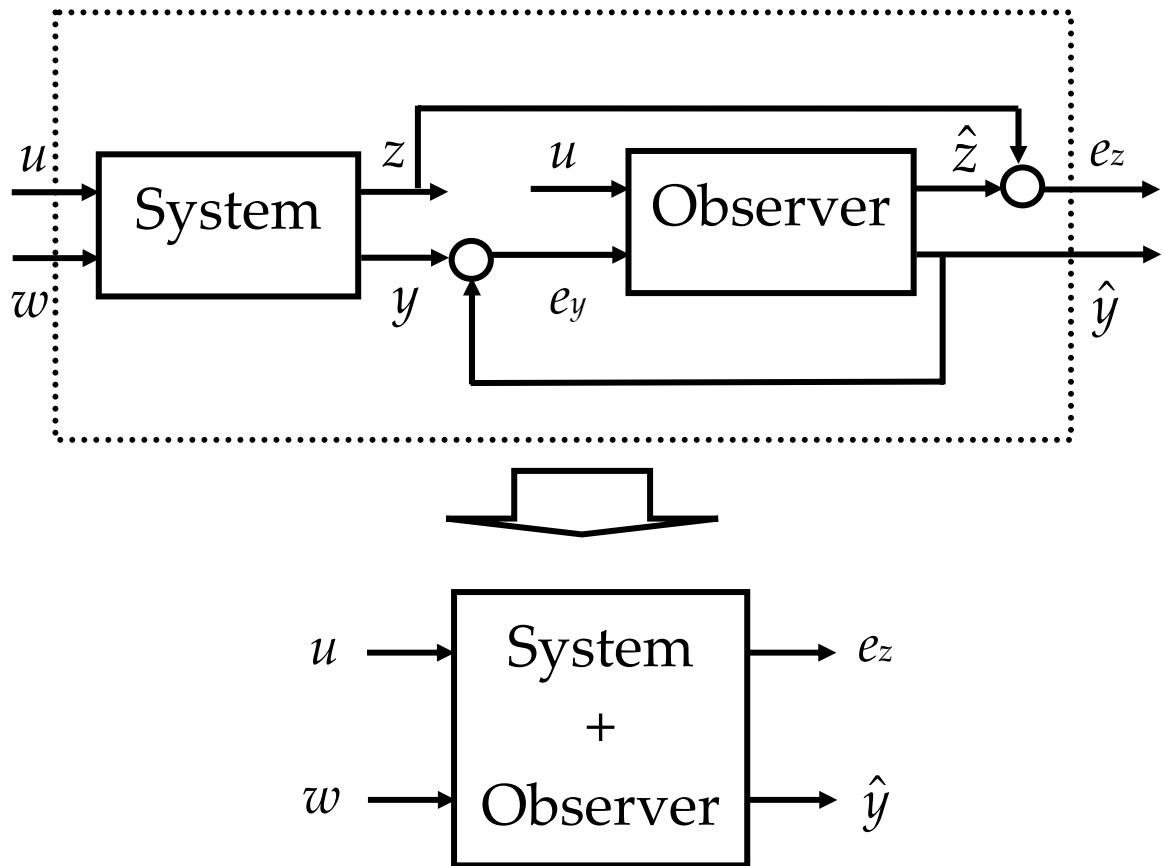


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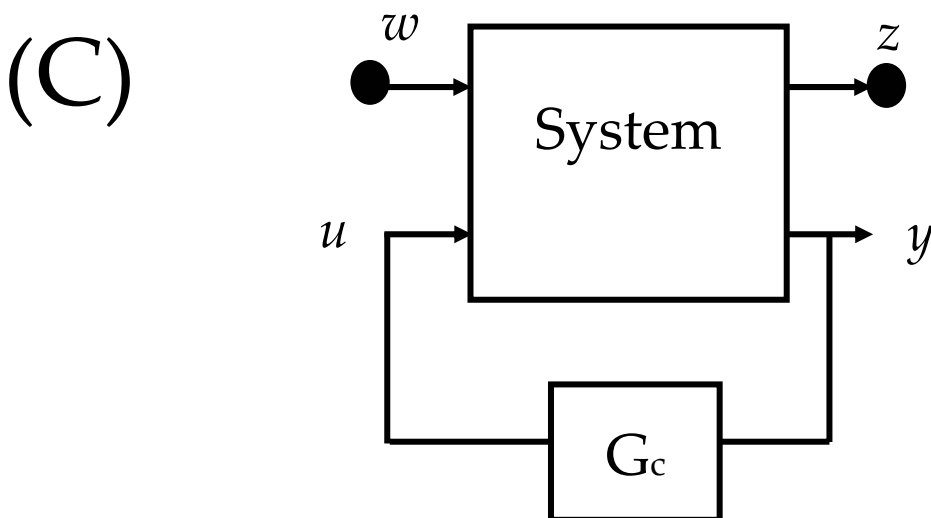
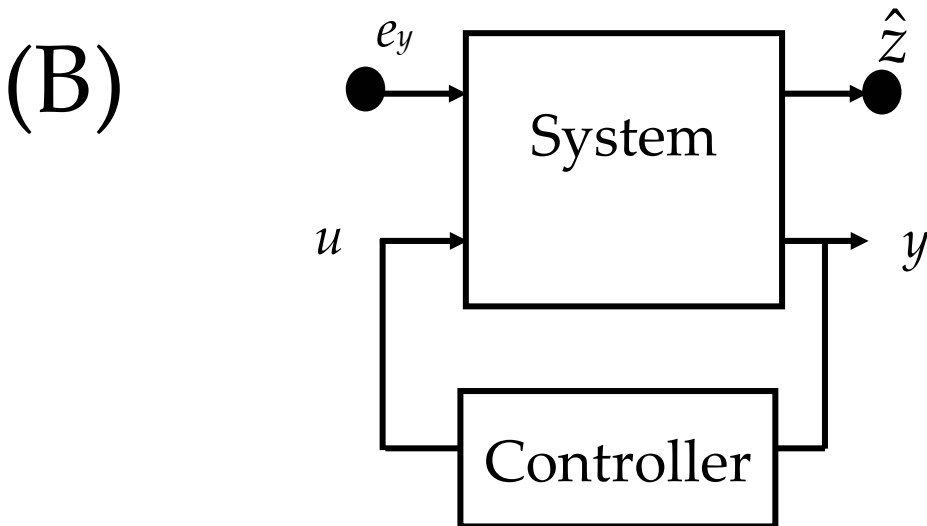
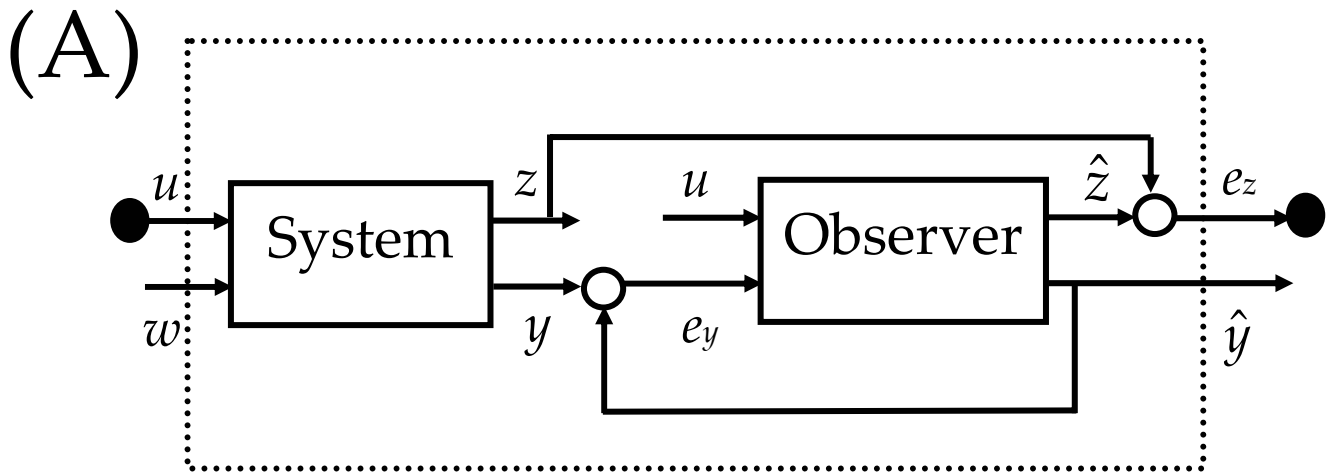


Figure 5

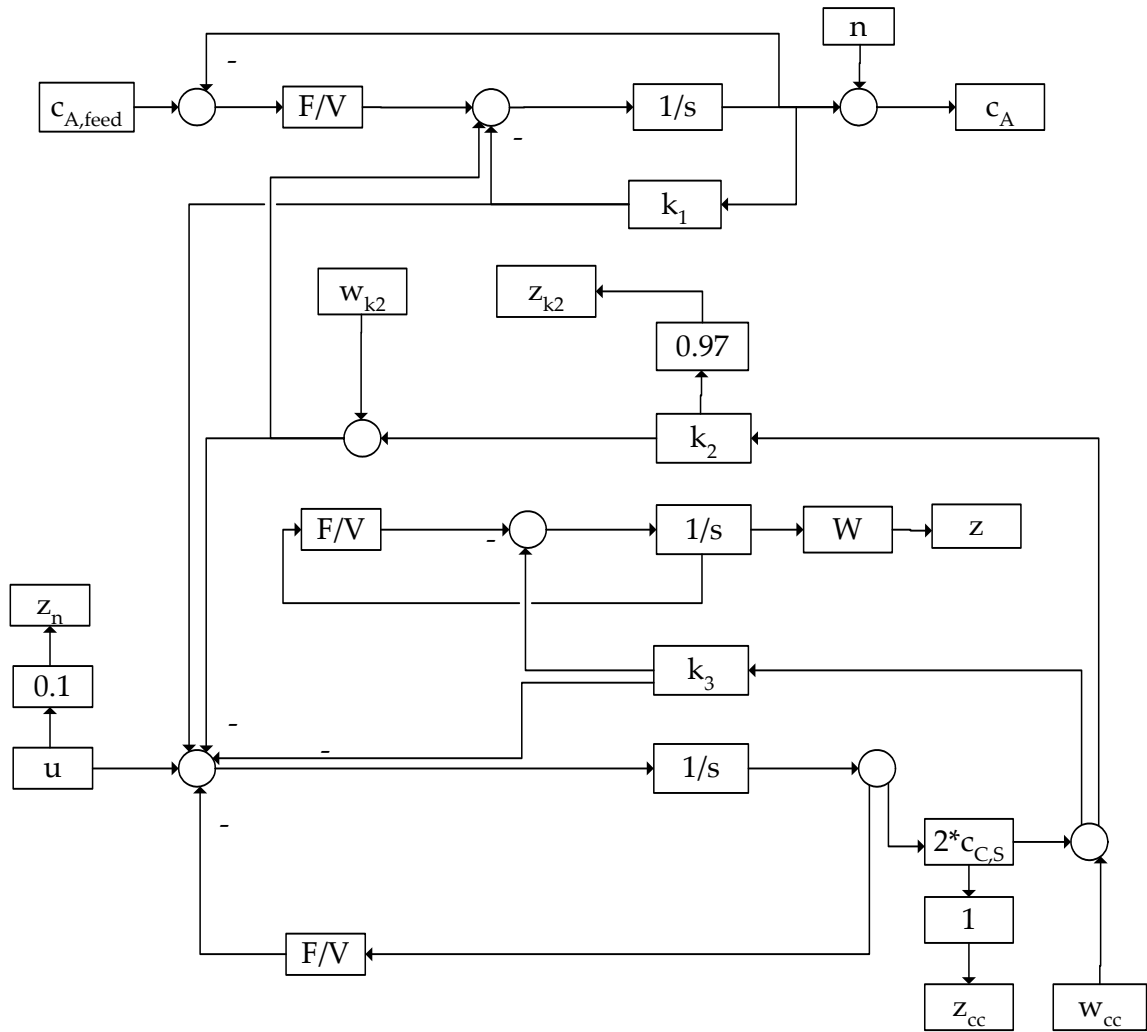


Figure 6

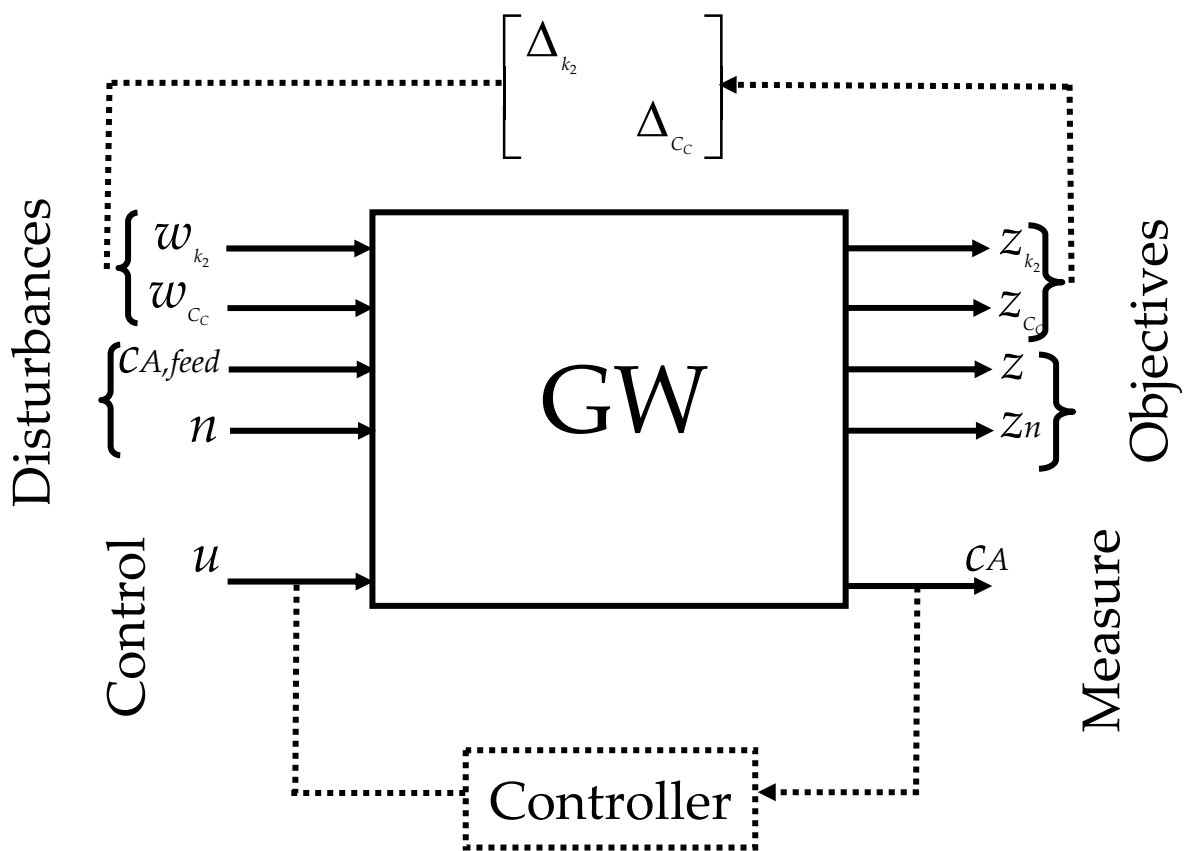


Figure 7

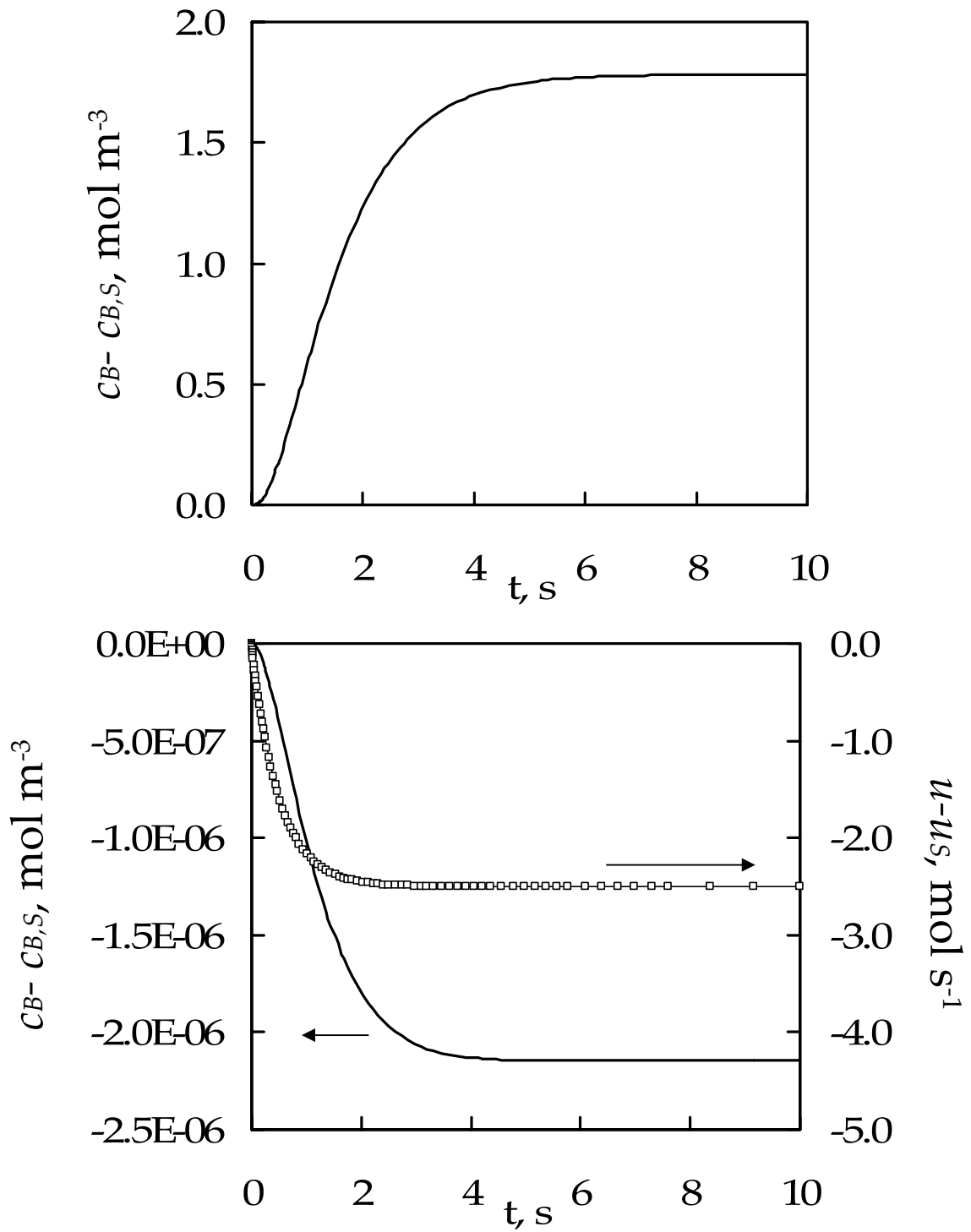


Figure 8

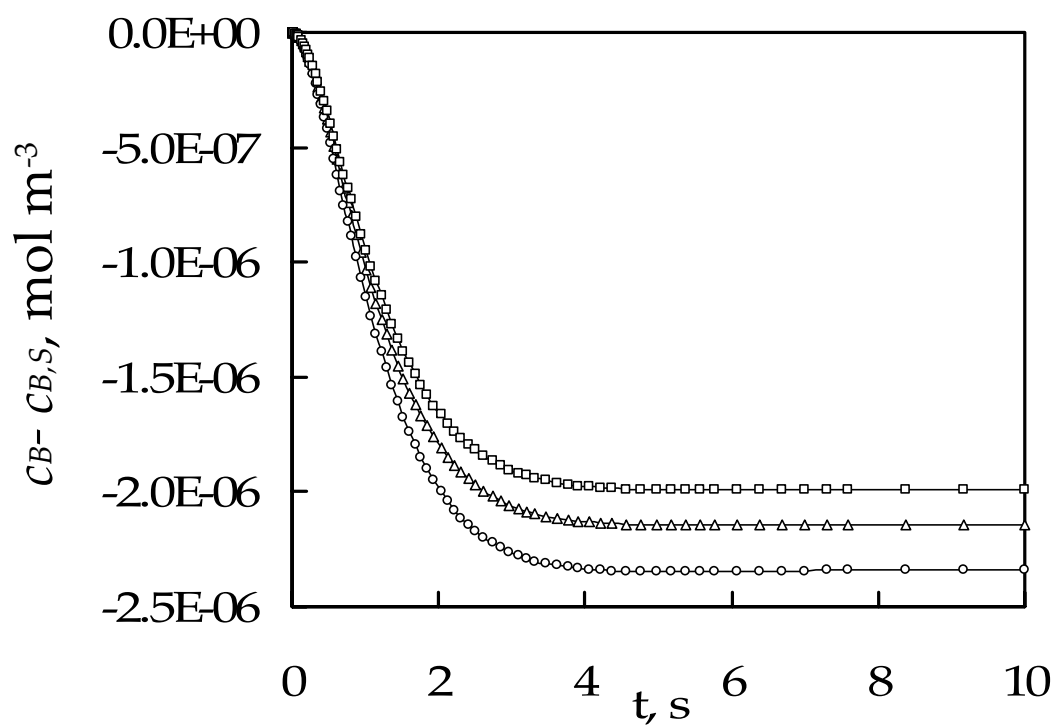


Figure 9

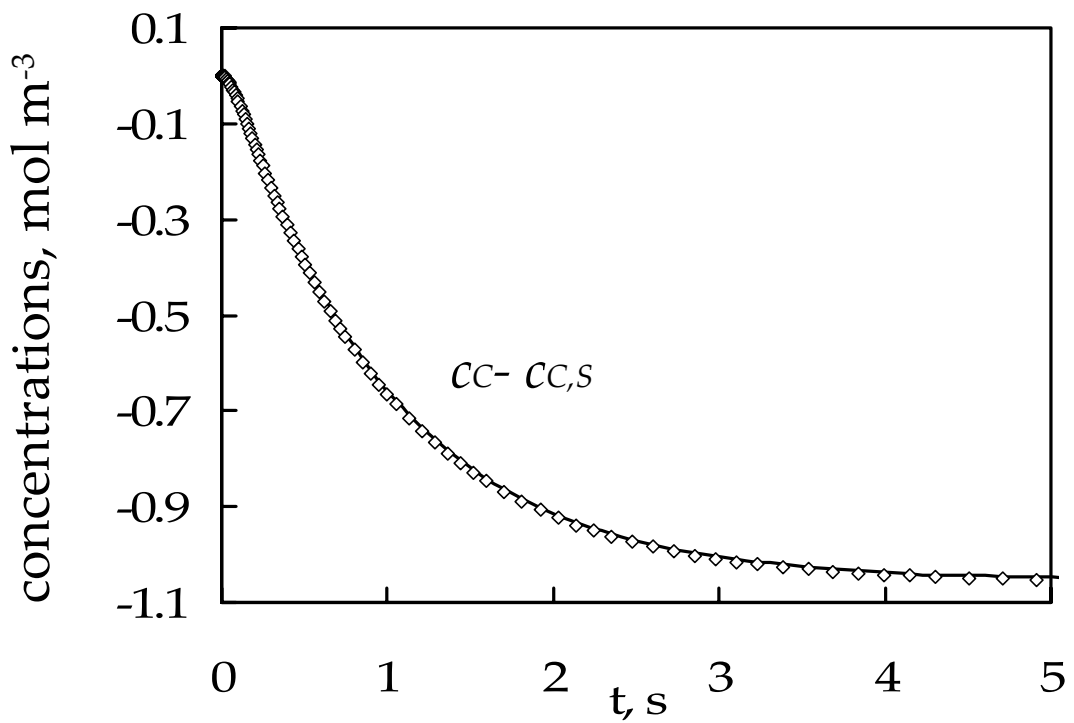
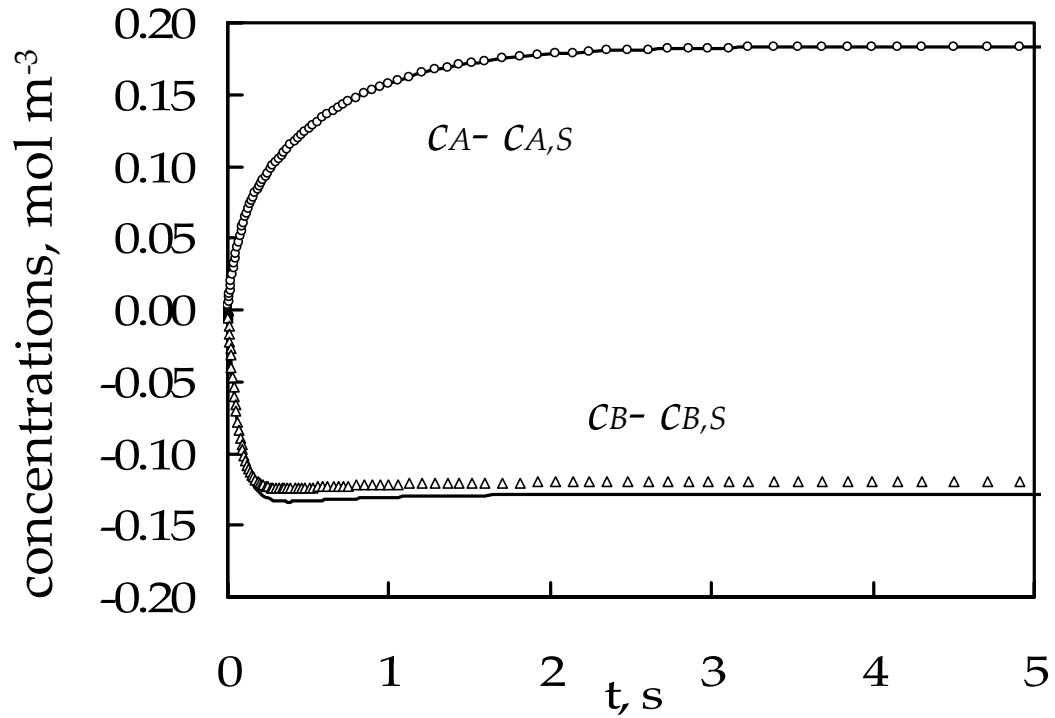


Figure 10

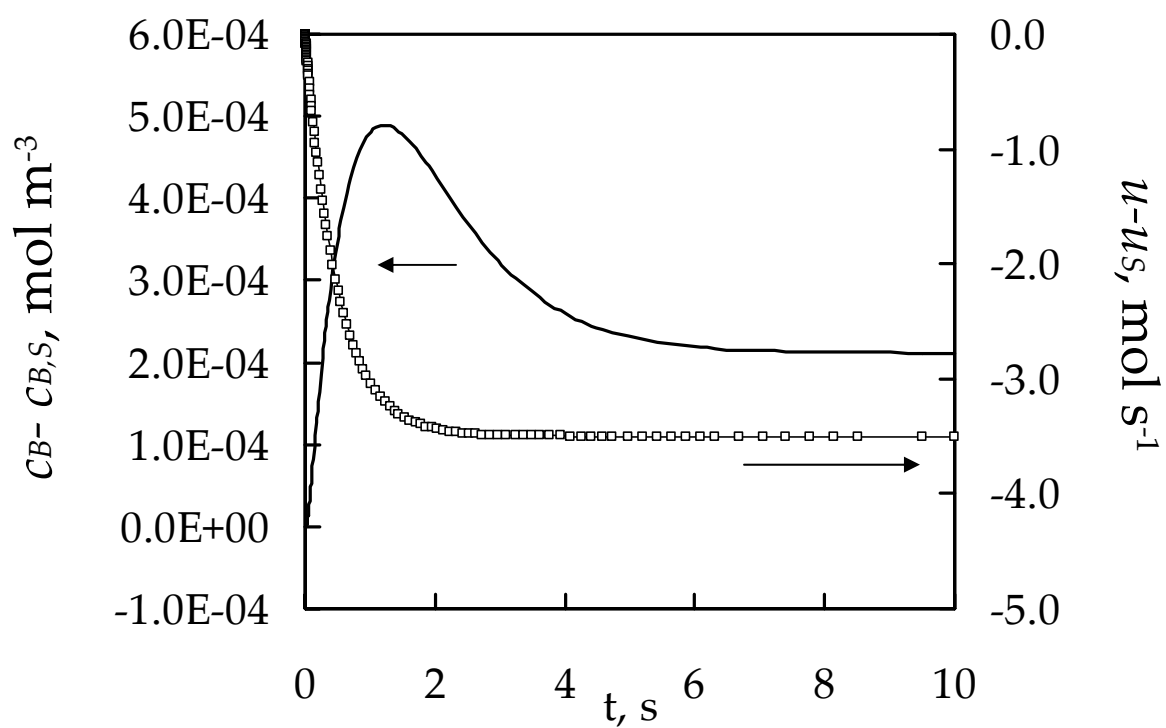




Table 1

input → output ↓	$C_{A,feed}$	$n$	$w_{k_2}$	$w_{C_C}$	$u$
$z$	$\frac{4.35(s+2000)}{(s+10)(s+1)(s+15.3121)(s+1.60788)}$	0	$\frac{-4.35(s+1)(s+2000)}{(s+10)(s+1)(s+15.3121)(s+1.60788)}$	$\frac{2.5(s+1)(s+2)(s+2000)}{(s+10)(s+1)(s+15.3121)(s+1.60788)}$	$\frac{4.35(s+2)(s+2000)}{(s+10)(s+1)(s+15.3121)(s+1.60788)}$
$z_n$	0	0	0	0	0.1
$z_{k_2}$	$\frac{5.0634}{(s+15.3121)(s+1.60788)}$	0	$\frac{-5.0634(s+1)}{(s+15.3121)(s+1.60788)}$	$\frac{2.91(s+1)(s+2)}{(s+15.3121)(s+1.60788)}$	$\frac{5.0634(s+2)}{(s+15.3121)(s+1.60788)}$
$z_{C_C}$	$\frac{1.74}{(s+15.3121)(s+1.60788)}$	0	$\frac{-1.74(s+1)}{(s+15.3121)(s+1.60788)}$	$\frac{-13.92(s+1.625)}{(s+15.3121)(s+1.60788)}$	$\frac{1.74(s+2)}{(s+15.3121)(s+1.60788)}$
$y$	$\frac{s+14.92}{(s+15.3121)(s+1.60788)}$	1	$\frac{s+9.7}{(s+15.3121)(s+1.60788)}$	$\frac{3(s+1)}{(s+15.3121)(s+1.60788)}$	$\frac{5.22}{(s+15.3121)(s+1.60788)}$